

Local Moduli Spaces in Analytic Geometry

By

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Introduction

In this paper we formulate a general existence theorem for local moduli spaces for maps of graded complex spaces (3.2). From this result, which is proved in detail in [6], we obtain easily the existence of semiuniversal deformations of compact complex spaces, isolated singularities, coherent analytic sheaves with compact supports or isolated singularities, principal bundles over compact complex spaces and the existence of the Douady space and so on. Further applications are given in [5].

§ 1. Graded Complex Spaces

1.1. Let A be a commutative ring. By a (positively) graded A -ringed space we understand an A -ringed space $X=(X, \mathcal{O}_X)$ such that the structure sheaf of X is an \mathbb{N} -graded A -algebra $\mathcal{O}_X = \coprod_{i \in \mathbb{N}} \mathcal{O}_X(i)$. Then $\Gamma(U, \mathcal{O}_X) := \coprod_{i \in \mathbb{N}} \Gamma(U, \mathcal{O}_X(i))$ is a graded A -algebra for every subset U of X . In an obvious way one defines the notion of a morphism of graded A -ringed spaces.

Examples 1.2. (1) Let X be a graded A -ringed space and c an element of $\mathbb{N} \cup \{\infty\}$. If we put $(\mathcal{O}_X)_{\langle \leq c \rangle} := \coprod_{i \leq c} \mathcal{O}_X(i)$, then $X_{\langle \leq c \rangle} := (X, (\mathcal{O}_X)_{\langle \leq c \rangle})$ is again a graded A -ringed space, which is functorial in X . We will use the abbreviation $X_0 := X_{\langle \leq 0 \rangle}$.

(2) Let X be an A -ringed space and \mathcal{F} an \mathcal{O}_X -module. Then the trivial extension $X[\mathcal{F}]$ of X by \mathcal{F} is a graded A -ringed space in a natural way with $(\mathcal{O}_{X[\mathcal{F}]})_0 = \mathcal{O}_X$ and $(\mathcal{O}_{X[\mathcal{F}]})_1 = \mathcal{F}$.

(3) Let X be an A -ringed space and $r=(r_1, \dots, r_m)$ an element of \mathbb{N}_+^m . There exists a unique graduation of the polynomial algebra $\mathcal{O}_X[w_1, \dots, w_m]$ such that the variable w_j has degree r_j , for $1 \leq j \leq m$. Hence $(X, \mathcal{O}_X[w_1, \dots, w_m])$

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is a graded A -ringed space being denoted by $X(m; r)$.

1.3. Let X be a graded A -ringed space. By a *graded \mathcal{O}_X -module* we understand a module \mathcal{F} over \mathcal{O}_X endowed with a \mathbf{Z} -graduation $\mathcal{F} = \coprod_{i \in \mathbf{Z}} \mathcal{F}_i$ being compatible with the graduation of \mathcal{O}_X . Then $\mathcal{F}_{(\geq c)} := \coprod_{i \geq c} \mathcal{F}_i$ resp. $\mathcal{F}_{(\leq c)} := \coprod_{i \leq c} \mathcal{F}_i$ is a graded submodule resp. quotient module of \mathcal{F} for any element c of $\mathbf{Z} \cup \{-\infty\}$.

1.4. A *graded complex space* is a graded \mathbf{C} -ringed space X such that the following properties hold: (1) $(X, (\mathcal{O}_X)_0)$ is a complex space. (2) \mathcal{O}_X is an $(\mathcal{O}_X)_0$ -algebra of finite presentation. (3) The homogeneous components $(\mathcal{O}_X)_i$ are coherent modules over $(\mathcal{O}_X)_0$.

The graded complex spaces form a full subcategory of the category of all graded \mathbf{C} -ringed spaces. It's easy to see that this category has finite fibre products.

By a *graded pseudocomplex space* we understand a graded \mathbf{C} -ringed space Z being locally isomorphic to a space of the form $(T, \mathcal{O}_X|_T)$, where X is a graded complex space and T is a closed subset of X . The graded pseudocomplex spaces again form a category with finite fibre products.

Examples 1.5. (1) Let X be a graded complex space and c an element of $\mathbf{N} \cup \{\infty\}$. Then $X_{(\leq c)}$ is again a graded complex space, which in case $c \in \mathbf{N}$ is a complex space at the same time.

(2) Let X be a complex space. If we provide \mathcal{O}_X with the trivial graduation, X becomes a graded complex space. In this way we consider the category of complex spaces as a full subcategory of the category of graded complex spaces.

(3) Let X be a complex space and $r = (r_1, \dots, r_m)$ a tuple from \mathbf{N}_+^m . Then $X(m; r)$ is a graded complex space. In particular $\mathbf{C}^n(m; r)$ is a graded complex space.

1.6. Let Z be a fixed graded complex space. We then define a groupoid D_Z over the category (Gan) of germs of analytic spaces as follows: An object of D_Z over a space germ $S = (S, 0)$ is a triple $f = (\mathcal{X} \xrightarrow{f} \mathcal{Y}, K, L)$ consisting of S -flat graded complex spaces \mathcal{X}, \mathcal{Y} over $S \times Z$, closed subsets $K \subseteq \mathcal{X}(0), L \subseteq \mathcal{Y}(0)$ and an $S \times Z$ -morphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ such that $f(K) \subseteq L$. The morphisms in D_Z are defined in an obvious way.

For an object $f = (\mathcal{X} \xrightarrow{f} \mathcal{Y}, K, L)$ of $D_Z(S)$, let $f(0) = (\mathcal{X}(0) \xrightarrow{f(0)} \mathcal{Y}(0), K, L)$ denote the object of $D_Z(\text{Spec}(\mathbf{C}))$ obtained from f by base change via the in-

clusion of $\text{Spec}(C)$ into $(S, 0)$. Further we have for any $c \in \mathbb{N}$ a natural functor

$$(1.6.1) \quad \leq c : D_Z \longrightarrow D_{Z(\leq c)}$$

of groupoids over (Gan) sending an object $(\mathcal{X} \xrightarrow{f} \mathcal{Y}, K, L)$ to the object $(\mathcal{X}_{(\leq c)} \xrightarrow{f_{(\leq c)}} \mathcal{Y}_{(\leq c)}, K, L)$.

1.7. Let now $f = (X \xrightarrow{f} Y, K, L)$ be an object of $D_Z(\text{Spec}(C))$. Then

$$\text{Def}_{\{f, K, L\}/Z}^{\text{gr}} := (D_Z)_f$$

is again a groupoid over (Gan) such that $\text{Def}_{\{f, K, L\}/Z}^{\text{gr}}(\text{Spec}(C))$ consists of one object (up to isomorphism) admitting only one automorphism. We call the objects of that groupoid (*graded deformations of the map germ* $(f, K, L) : (X, K) \rightarrow (Y, L)$ *relative* Z). Such a deformation over a space germ $S = (S, 0)$ is thus an object $f = (\mathcal{X} \xrightarrow{f} \mathcal{Y}, K, L)$ of D_Z over S endowed with an isomorphism $f \xrightarrow{\sim} f(0)$ in $D_Z(\text{Spec}(C))$. The functor (1.6.1) induces a functor

$$(1.7.1) \quad \text{Def}_{\{f, K, L\}/Z}^{\text{gr}} \longrightarrow \text{Def}_{\{f_{(\leq c)}, K, L\}/Z(\leq c)}^{\text{gr}}.$$

The fibre of this functor is a groupoid over (Gan) , which we denote by

$$\text{Def}_{\{f, K, L\}/Z}^{\text{gr}, \leq c \leq 1}.$$

An object of this groupoid over a space germ S consists of a deformation f of $(f, K, L)/Z$ over S endowed with an isomorphism $(f_{(\leq c)})_S \xrightarrow{\sim} f_{(\leq c)}$. In such a deformation, the homogeneous components of degree $\leq c$ of f are thus deformed in a trivial way. Finally we put $\text{Def}_{\{f, K, L\}/Z}^{\text{gr}, \leq 0} := \text{Def}_{\{f, K, L\}/Z}^{\text{gr}}$.

In case $K = f^{-1}(L)$ we write (f, L) resp. $\text{Def}_{\{f, L\}/Z}^{\text{gr}, \leq a}$ instead of (f, K, L) resp. $\text{Def}_{\{f, K, L\}/Z}^{\text{gr}, \leq a}$ for $a \in \mathbb{N}$. If $K = \emptyset$ is empty, we put $\text{Def}_{\{f, L\}/Z}^{\text{gr}, \leq a} := \text{Def}_{\{f, K, L\}/Z}^{\text{gr}, \leq a}$ and $\text{Def}_{\{f, L\}/Z}^{\text{gr}, \leq a} := \text{Def}_{\{f, \bar{Y}\}/Z}^{\text{gr}, \leq a}$ for $a \in \mathbb{N}$. Finally we suppress in case $Z = \text{Spec}(C)$ the index Z in the notations introduced above.

If $f : X \rightarrow Y$ is a map of complex spaces over a complex space Z , we shall consider f as a Z -morphism of graded complex spaces with trivial graduation and omit the index "gr" in the above groupoids.

§ 2. Resolvents

2.1. Let X be a graded complex space. By a *quasipolyhedron* for X we mean a triple (P, φ, E) consisting of a compact subset P of X , a closed embedding $\varphi : U \rightarrow V$ of graded complex spaces, where U is an open subspace of X and V is an open subspace of a number space $\mathbb{C}^{n'}(n''; r)$, and an open special polycylinder $E \subseteq \mathbb{C}^{n'}(n''; r)$ with center x_0 and $\bar{E} \subseteq V$, such that $\varphi^{-1}(\bar{E}) = P$ holds.

If $\varphi^{-1}(x_0)$ is non-empty, we say that (P, φ, E) is a *polyhedron* for X and

call the unique point $x \in P$ such that $\varphi(x) = x_0$ the *center* of (P, φ, E) . In the following we write P instead of (P, φ, E) and denote U resp. V resp. φ resp. E by $U\langle P \rangle$ resp. $V\langle P \rangle$ resp. $\varphi\langle P \rangle$ resp. $E\langle P \rangle$.

2.2. Let now $f: X \rightarrow Y$ be a morphism of graded complex spaces. A *quasipolyhedron for f* (or for X relative Y) is a pair (P, Q) consisting of a quasipolyhedron P resp. Q for X resp. Y having the following properties: (1) $f(U\langle P \rangle) \subseteq U\langle Q \rangle$. (2) $V\langle P \rangle$ is of the form $V\langle P \rangle = V\langle Q \rangle \times W$ with an open subspace W of a number space $\mathbb{C}^u(l''; s)$, and the diagram

$$\begin{CD} U\langle P \rangle @>\varphi\langle P \rangle>> V\langle P \rangle = V\langle Q \rangle \times W \\ @Vf|U\langle P \rangle \downarrow VVV @VV\text{pr}_1 V \\ U\langle Q \rangle @>\varphi\langle Q \rangle>> V\langle Q \rangle \end{CD}$$

commutes. (3) $E\langle P \rangle$ is of the form $E\langle P \rangle = E\langle Q \rangle \times F$ with a polycylinder F in $\mathbb{C}^u(l''; s)$. If P and Q are moreover polyhedra, we call (P, Q) a *polyhedron for f* (or for X relative Y).

For any element t of $]0, 1]$, the pair $((\varphi\langle P \rangle, E\langle Q \rangle \times F^{(t)}), Q)$ is again a quasipolyhedron for f , which we call the *shrinking* of (P, Q) with respect to t , and denote by $(P, Q)^{(t)}$. Suppose that Q is a polyhedron with center y and put $U := U\langle P \rangle \cap X(y)$ and $P(y) := P \cap X(y)$. Then $\varphi\langle P \rangle$ induces a closed embedding $\varphi: U \rightarrow W$, and $(P(y), \varphi, F)$ is a quasipolyhedron for the fibre $X(y)$, which we denote by $(P, Q)(y)$.

2.3. Let now $\mathcal{P} = (P_i, Q)_{i \in I}$ be a family of polyhedra for $f: X \rightarrow Y$ over the same base polyhedron Q . The nerve of the family $(P_i)_{i \in I}$ of compact subsets of X is a simplicial scheme over I , which we denote by $\text{Ner}(\mathcal{P})$ and call the *nerve* of \mathcal{P} . The subset $|\mathcal{P}| := \bigcup_{i \in I} P_i$ of X is called the *support* of \mathcal{P} .

For a tuple $t = (t_i)_{i \in I}$ of elements of $]0, 1]$, $\mathcal{P}^{(t)} := ((P_i, Q)^{(t_i)})_{i \in I}$ is again a family of polyhedra for f . If s is a number such that $t_i = s$ for all i , we shall write $\mathcal{P}^{(s)}$ instead of $\mathcal{P}^{(t)}$. If y denotes the center of Q , then $\mathcal{P}(y) := ((P_i, Q)(y))_{i \in I}$ is a family of polyhedra for the fibre $X(y)$ of f in y . Finally we put

$$n(\mathcal{P}) := \sup \{ \dim(V\langle Q \rangle) + \dim(W_j) + \sum_{i \in I} \dim(W_i) : j \in I \};$$

here W_i is the graded complex space such that $V\langle P_i \rangle = V\langle Q \rangle \times W_i$.

We say that \mathcal{P} satisfies the *polyhedral axiom*, if: (1) There exists a tuple $t = (t_i)_{i \in I}$ of elements from $]0, 1[$ such that $\text{Ner}(\mathcal{P}) = \text{Ner}(\mathcal{P}^{(t)}(y))$. (2) For any subset $\{i_0, i_1\}$ in $\text{Ner}(\mathcal{P})$, we have $P_{i_0} \cup P_{i_1} \subseteq U\langle P_{i_0} \rangle \cap U\langle P_{i_1} \rangle$.

2.4. Let $f : X \rightarrow Y$ be a morphism of graded complex spaces and $\mathcal{P} = (P_i, Q)_{i \in I}$ a finite family of polyhedra for f . Suppose that $V \langle P_i \rangle = V \langle Q \rangle \times W_i$, $E \langle P_i \rangle = E \langle Q \rangle \times F_i$ and $\varphi \langle P_i \rangle = (\rho_i, \tau_i)$ with $\rho_i := \varphi \langle Q \rangle \circ (f|U \langle P_i \rangle)$ and morphisms $\tau_i : U \langle P_i \rangle \rightarrow W_i$. For an element α of $\mathcal{N} := \text{Ner}(\mathcal{P})$ we abbreviate $U_\alpha := U_\alpha \langle \mathcal{P} \rangle := \bigcap_{i \in \alpha} U \langle P_i \rangle$, $W_\alpha := W_\alpha \langle \mathcal{P} \rangle := \prod_{i \in \alpha} W_i$, $V_\alpha := V_\alpha \langle \mathcal{P} \rangle = V \langle Q \rangle \times W_\alpha$,

$$\varphi_\alpha := \varphi_\alpha \langle \mathcal{P} \rangle := (\varphi \langle Q \rangle \circ (f|U_\alpha), (\tau_i|U_\alpha)_{i \in \alpha}) : U_\alpha \longrightarrow V_\alpha,$$

$F_\alpha := F_\alpha \langle \mathcal{P} \rangle := \prod_{i \in \alpha} F_i$, $E_\alpha := E_\alpha \langle \mathcal{P} \rangle := E \langle Q \rangle \times F_\alpha$ and $P_\alpha := \bigcap_{i \in \alpha} P_i \subseteq U_\alpha$. Then $((P_\alpha, \varphi_\alpha, E_\alpha), Q)$ is a quasipolyhedron for f . Further $U_* := U_* \langle \mathcal{P} \rangle := (U_\alpha)_{\alpha \in \mathcal{N}}$ resp. $V_* := V_* \langle \mathcal{P} \rangle := (V_\alpha)_{\alpha \in \mathcal{N}}$ is an \mathcal{N} -coobject in the category of graded complex spaces over $U \langle Q \rangle$ resp. $V \langle Q \rangle$, and $\varphi := (\varphi_\alpha)_{\alpha \in \mathcal{N}}$ is a closed embedding of U_* into V_* . Putting $\psi := \varphi \langle Q \rangle$, the canonical diagram

$$(2.4.1) \quad \begin{array}{ccc} U_* & \xrightarrow{\varphi} & V_* \\ \downarrow & & \downarrow \\ U \langle Q \rangle & \xrightarrow{\psi} & V \langle Q \rangle \end{array}$$

commutes. We abbreviate $F_* := F_* \langle \mathcal{P} \rangle := (F_\alpha)_{\alpha \in \mathcal{N}}$. Then $\bar{F}_* := (\bar{F}_\alpha)_{\alpha \in \mathcal{N}}$ and $P_* := P_* \langle \mathcal{P} \rangle := (P_\alpha)_{\alpha \in \mathcal{N}}$ are in a natural way \mathcal{N} -coobjects in the category of graded pseudocomplex spaces. With these notations we obtain from (2.4.1) the following canonical diagram

$$(2.4.2) \quad \begin{array}{ccc} P_* & \xrightarrow{\varphi'} & \bar{E} \langle Q \rangle \times \bar{F}_* \\ \downarrow f|P_* & & \downarrow \\ Q & \xrightarrow{\psi'} & \bar{E} \langle Q \rangle, \end{array}$$

where the horizontal arrows are closed embeddings.

2.5. Let the assumptions and notations be as in 2.4. By a *sheaf resolvent for f* (or *for X relative Y*) with respect to \mathcal{P} we understand a pair $(\mathcal{B}, \mathcal{R})$ consisting of a g -finite free graded DG-Algebra $\mathcal{B} = (\mathcal{B}, s)$ over $\mathcal{O}_{\bar{E} \langle Q \rangle}$ with $\mathcal{B}^0 = \mathcal{O}_{\bar{E} \langle Q \rangle}$ and a g -finite free graded DG-algebra $\mathcal{R} = (\mathcal{R}, s)$ over

$$\mathcal{B} \otimes_{\mathcal{O}_{\bar{E} \langle Q \rangle}} \mathcal{O}_{\bar{E} \langle Q \rangle \times \bar{F}_*}$$

with $\mathcal{R}^0 = \mathcal{O}_{\bar{E} \langle Q \rangle \times \bar{F}_*}$, such that \mathcal{B} resp. \mathcal{R} is a resolution of $\psi'_*(\mathcal{O}_Y|Q)$ resp. $\varphi'_*(\mathcal{O}_X|P_*)$. One can easily show that for a given \mathcal{P} there always exists a sheaf resolvent for f with respect to \mathcal{P} .

2.6. Let now X and Y be two graded complex spaces over a complex space Z and let $f : X \rightarrow Y$ be a Z -morphism. By a *polyhedron for f/Z* we understand a triple (P, Q, M) such that (P, Q) resp. (Q, M) is a polyhedron for f resp. for

Y relative Z . The phrases and notations introduced in 2.3 carry over in an obvious way to families of polyhedra for f/Z .

Let $\mathcal{P}=(P_i, Q, M)_{i \in I}$ be a finite family of polyhedra for f/Z . Then \mathcal{P} induces in a natural way a commutative diagram

$$\begin{array}{ccc} P_* & \xrightarrow{\varphi'} & \bar{E}\langle Q \rangle \times \bar{F}_* \\ \downarrow & \searrow \psi' & \downarrow \\ Q & \xrightarrow{\psi'} & \bar{E}\langle Q \rangle \\ \downarrow & \searrow \chi' & \downarrow \\ M & \xrightarrow{\chi'} & \bar{E}\langle M \rangle, \end{array}$$

see (2.4.2). By a *sheaf resolvent for f/Z with respect to \mathcal{P}* we understand a pair consisting of a sheaf resolvent $\mathcal{A} \rightarrow \mathcal{B}$ for Y relative Z with respect to (Q, M) and a sheaf resolvent $\mathcal{B} \rightarrow \mathcal{R}$ for f with respect to $(P_i, Q)_{i \in I}$.

2.7. Let $Z, f: X \rightarrow Y$ and $\mathcal{P}=(P_i, Q, M)_{i \in I}$ be as in 2.6. Further let $\mathcal{N} := \text{Ner}(\mathcal{P})$ be the nerve of \mathcal{P} , and let $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{R}$ be a sheaf resolvent for f/Z with respect to \mathcal{P} , and put $S := E\langle Q \rangle_0$. Moreover let c resp. d be a fixed number from \mathcal{N} resp. $\mathbf{Z}_{\leq 0}$. By ${}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}}$ resp. ${}^c\mathcal{M}_{\mathcal{B}|\mathcal{A}}$ resp. ${}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}|\mathcal{A}}$ we denote the DG -module over \mathcal{O}_S , whose sections over an open subset V of S are the compatible families of homomorphisms from

$$\prod_{\alpha \in \mathcal{N}} \text{Hom}_{\mathcal{R}_\alpha|V \times F_\alpha}(\Omega_{\mathcal{R}_\alpha|\mathcal{B}}|V \times F_\alpha, (\mathcal{R}_\alpha)_{(\geq c)}|V \times F_\alpha)$$

resp.

$$\text{Hom}_{\mathcal{B}|V}(\Omega_{\mathcal{B}|\mathcal{A}}|V, \mathcal{B}_{(\geq c)}|V)$$

resp.

$$\prod_{\alpha \in \mathcal{N}} \text{Hom}_{\mathcal{R}_\alpha|V \times F_\alpha}(\Omega_{\mathcal{R}_\alpha|\mathcal{A}}|V \times F_\alpha, (\mathcal{R}_\alpha)_{(\geq c)}|V \times F_\alpha) \times \Gamma(V, {}^c\mathcal{M}_{\mathcal{B}|\mathcal{A}}),$$

the differential being the map $u \mapsto [s, u]$. Further let $\mathcal{F}^d({}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}})$ resp. $\mathcal{F}^d({}^c\mathcal{M}_{\mathcal{B}|\mathcal{A}})$ resp. $\mathcal{F}^d({}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}|\mathcal{A}})$ be the subcomplex of ${}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}}$ resp. ${}^c\mathcal{M}_{\mathcal{B}|\mathcal{A}}$ resp. ${}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}|\mathcal{A}}$, whose sections over an open subset V of S are the families vanishing on the $\Omega_{\mathcal{R}|\mathcal{B}}^\nu|V \times F_*$ resp. $\Omega_{\mathcal{B}|\mathcal{A}}^\nu|V$ resp. on the $\Omega_{\mathcal{R}|\mathcal{A}}^\nu|V \times F_*$ and $\Omega_{\mathcal{B}|\mathcal{A}}^\nu|V$ for $\nu \geq d$. Then

$${}^{(c,d)}\mathcal{M}_{\mathcal{R}|\mathcal{B}} := {}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}} / \mathcal{F}^d({}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}}),$$

$${}^{(c,d)}\mathcal{M}_{\mathcal{B}|\mathcal{A}} := {}^c\mathcal{M}_{\mathcal{B}|\mathcal{A}} / \mathcal{F}^d({}^c\mathcal{M}_{\mathcal{B}|\mathcal{A}}),$$

$${}^{(c,d)}\mathcal{M}_{\mathcal{R}|\mathcal{B}|\mathcal{A}} := {}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}|\mathcal{A}} / \mathcal{F}^d({}^c\mathcal{M}_{\mathcal{R}|\mathcal{B}|\mathcal{A}})$$

are in a natural way complexes in $\mathbf{K}(\text{PO}_{1,1}(\mathcal{O}_S))$ having nuclear free components (see [3], 2.3, 3.4 and 3.6), and the canonical sequence

$$(2.7.1) \quad 0 \longrightarrow {}^{(c,d)}\mathcal{M}_{\mathcal{R}|\mathcal{B}} \longrightarrow {}^{(c,d)}\mathcal{M}_{\mathcal{R}|\mathcal{B}|\mathcal{A}} \longrightarrow {}^{(c,d)}\mathcal{M}_{\mathcal{B}|\mathcal{A}} \longrightarrow 0$$

of complexes is exact and splits naturally in $\mathbf{G}(\text{PO}_{1,1}(\mathcal{O}_S))$.

§3. Versal Deformations of Maps of Graded Complex Spaces

3.1. Let X and Y be two graded complex spaces over a graded complex space Z , $f: X \rightarrow Y$ a Z -morphism, $c \in \mathbb{N}$ a natural number, $T := \text{Supp}((\mathcal{O}_X)_{(\geq c)})$, $L \subseteq Y$ a closed subset and $K := f^{-1}(L) \cap T$.

Theorem 3.2. *Let the assumptions and notations be as in 3.1. Moreover suppose that:*

- (1) T is proper over Y , and L is finite.
- (2) The \mathbb{C} -vector space $\overline{\text{Def}}_{\{f, \bar{k}, L\}/Z}^c(D)$ of isomorphism classes of deformations of (f, K, L) in $\text{Def}_{\{f, \bar{k}, L\}/Z}^c$ over the double point D is finite dimensional.

Then the germ $(f, K, L): (X, K) \rightarrow (Y, L)$ has a semiuniversal deformation in the groupoid $\text{Def}_{\{f, \bar{k}, L\}/Z}^c$ over (Gan) .

Sketch of proof. We may assume that $L = \{y\}$ consists of one point. Then there exists a family $\mathcal{P} = (P_i, Q, M)_{i \in I}$ of polyhedra for f/Z and an element z of $]0, 1[$ such that the following properties hold: (1) \mathcal{P} is finite and satisfies the polyhedral axiom. (2) y is the center of Q . (3) We have $\text{Ner}(\mathcal{P}) = \text{Ner}(\mathcal{P}^{(z)}(y))$. (4) $f^{-1}(Q) \cap T$ is contained in $|\mathcal{P}^{(z)}|$.

Let $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{R}$ be a sheaf resolvent for f/Z with respect to \mathcal{P} and $d \leq 0$ an integer with $d \leq -n(\mathcal{P}) - 7$. We put $S := E\langle Q \rangle_0$ and

$$\begin{aligned} E &:= \Gamma(S, {}^{(c, d)}\mathcal{M}_{\mathcal{R}|\mathcal{B}}), \\ F &:= \Gamma(S, {}^{(c, d)}\mathcal{M}_{\mathcal{R}|\mathcal{B}|\mathcal{A}}), \\ G &:= \Gamma(S, {}^{(c, d)}\mathcal{M}_{\mathcal{B}|\mathcal{A}}) \end{aligned}$$

for abbreviation, see 2.7. Then E, F and G are in a natural way complexes in $\mathbf{K}(\text{PO}_1(\mathbb{C}))$, and we have a canonical exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0,$$

which splits in $\mathbf{G}(\text{PO}_1(\mathbb{C}))$. If we can show that the complex $F^{(0, 2)}$ splits in $\mathbf{K}(\text{PO}(\mathbb{C}))$ (for a suitable choice of \mathcal{P}), then the assertion of 3.2 follows from the quotient theorem ([4] 4.3). For this purpose it suffices to prove the following:

- (a) The complex $E^{(0, 3)}$ splits in $\mathbf{K}(\text{PO}(\mathbb{C}))$.
- (b) The complex $G^{(-1, 2)}$ splits in $\mathbf{K}(\text{PO}(\mathbb{C}))$.
- (c) The coboundary maps $\delta^i: H^i(G) \rightarrow H^{i+1}(E)$ split in $\text{PO}(\mathbb{C})$ for $i = 0, 1$.

The properties (b) and (c) follow (for a suitable choice of Q and M) from the existence of privileged neighborhoods.

The property (a) is (for a suitable choice of \mathcal{P}) a consequence of the split-

ting criteria [3], 2.4, 4.2, 4.3 (1). Here the fundamental problem is the verification of the assumption (2) of loc.cit. 2.4. In order to do this, one first shows, that the property (a) is independent of the choice of the resolvent \mathcal{R} . This is the assertion of the theorem of the homotopy invariance of the (co)tangent complex, see [16] and [6] (III 4.15). The actual verification will be made then only for very special, so called *admissible* resolvents. Here the theorem on the "Nullhomotopie des Čech-Komplexes einer Auflösung" ([6] (II 5.3)) is an essential ingredient.

Remark 3.3. Generalizing the methods used here, one should be able to show that 3.2 remains valid, if one replaces the map $f: X \rightarrow Y$ by an n -chain

$$X^{(n)} \xrightarrow{f^{(n-1)}} X^{(n-1)} \longrightarrow \dots \longrightarrow X^{(2)} \xrightarrow{f^{(1)}} X^{(1)}$$

of graded complex spaces over Z (with $n \in \mathbb{N}_+$). Further 3.2 should be true also under suitable concavity assumptions on T . I hope to come back to this at another place.

§ 4. Consequences

If we apply 3.2 to a map of complex spaces and $c=0$, we get the following result, which I obtained already in 1983.

Theorem 4.1. *Let X and Y be complex spaces over a complex space Z , $f: X \rightarrow Y$ a proper Z -morphism and let $L \subseteq Y$ be a finite subset. If the \mathbf{C} -vector space $\overline{\text{Def}}_{(f,L)/Z}(D)$ of isomorphism classes of deformations of (f, L) over the double point is finite dimensional, the germ $(f, L): (X, f^{-1}(L)) \rightarrow (Y, L)$ has a semiuniversal deformation in the groupoid $\text{Def}_{(f,L)/Z}$.*

Putting $Y=Z=\text{Spec}(\mathbf{C})$ resp. $X=\emptyset$ and $Z=\text{Spec}(\mathbf{C})$ in 4.1, we obtain the next two propositions.

Theorem 4.2 (Douady [10], Grauert [14], Forster-Knorr [12], Palamodov [17]). *Let X be a compact complex space. Then X has a semiuniversal deformation.*

Theorem 4.3 (Donin [7], Grauert [13], Pourcin [21]). *Let Y be a complex space and $L \subseteq Y$ a finite subset. If $\overline{\text{Def}}_{(Y,L)}(D)$ is finite dimensional, the germ (Y, L) has a semiuniversal deformation.*

The well known existence theorems for deformations of coherent sheaves are contained in 3.2, too:

Theorem 4.4 (Siu-Trautmann [23]). *Let X be a complex space and \mathcal{F} a*

coherent \mathcal{O}_X -module with compact support. Then \mathcal{F} has a semiuniversal deformation.

This follows from 3.2, applied to $X=X[\mathcal{F}]$, $Y=Z=\text{Spec}(\mathbb{C})$ and $c=1$.

Theorem 4.5 (Trautmann [24]). *Let Y be a complex space, \mathcal{G} a coherent \mathcal{O}_Y -module and let $L \subseteq Y$ be a finite subset such that $\text{Supp}(\mathcal{E} \otimes_{\mathcal{O}_Y}^L \mathcal{G}, \mathcal{G}) \subseteq L$. Then the germ (\mathcal{G}, L) has a semiuniversal deformation.*

This also follows from 3.2, applied to $X=\emptyset$, $Y=Y[\mathcal{G}]$, $Z=\text{Spec}(\mathbb{C})$ and $c=1$. —Using [2], one can deduce from 4.4 the following result:

Theorem 4.6 (Douady-Pourcin [9, 20]). *Let $f: X \rightarrow S$ be a separated holomorphic map and \mathcal{F} a coherent \mathcal{O}_X -module. Then the functor $\mathcal{Q}_{\mathcal{F}/X/S}$ of Grothendieck is representable by a separated complex space over S .*

In [19] it was shown, that the following theorem is a consequence of 4.4 and 4.6, and hence altogether also of 3.2.

Theorem 4.7. (Ponomarev [19], Donin [8]). *Let X be a compact complex space, G a complex Lie group and $P \rightarrow X$ a G -principal bundle over X . Then P has a semiuniversal deformation.*

Remarks 4.8. (1) In [6] we also show a local version of 3.2, containing as special cases as well the theorem of Retakh on versal deformations of analytic map germs, which was announced in [22] and proved in detail in [11], as the propositions 4.3, 4.5.

(2) Let S be a fixed analytic space germ. Then both theorem 3.2 and its corollaries remain m. m. true, if we replace the category (Gan) of analytic space germs by the category (Gan/ S) of space germs over S (compare also [4], 4.6 (2)). I will give the details of this generalization to the “relative case” at another place.

(3) In a subsequent paper I will show that one can construct versal deformations of complex spaces (and maps) with a group action using a suitable generalization of the methods employed in this paper. In this approach the description of equivariant deformations given in [15] plays an essential part.

(4) A fully satisfactory treatment of analytic deformation theory seems only be possible within the frame of the so-called *anticommutative (graded) complex spaces* (see [6], (VI 7)). These objects, which appear here only in disguise of the resolvents, are known in a somewhat more special form under the name “complex superspaces” since a while. I intend to give the details in

subsequent papers.

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