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Analytic Moduli Spaces as Orbit Spaces

By

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Introduction

In this paper we formulate an abstract quotient theorem for convergent maps of PO-spaces (4.3), being inspired by the work [6] of I. F. Donin. This theorem, which is proved in detail in [5], gives a criterion for certain quotient groupoids over the category of germs of analytic spaces to have a semiuniversal deformation. With this result in hands, one can solve many local moduli problems in analytic geometry in a unified manner, see [3, 4] and [5]. Let us finally remark that we use the notions and notations introduced in [2] without any further comment.

§ 1. Formal Maps

1.1. In the following K denotes a fixed commutative $\boldsymbol{Q}$-algebra. For a K-module F and finite family E_i, $i\in I$, of K-modules the K-multilinear maps $\varphi:\prod_{i\in I}E_i\to F$ form a K-module $\mathrm{Mult}_K(E_i, i\in I; F)$. In case $I=\{1, \cdots, p\}$ we write $\mathrm{Mult}_K(E_1, \cdots, E_p; F)$ instead of $\mathrm{Mult}_K(E_i, i\in I; F)$.

For two K-modules E and F and $p\in \boldsymbol{N}$ let $\mathrm{Mult}_p(E, F)=\mathrm{Mult}_{K,p}(E, F)$ denote the K-module $\mathrm{Mult}_K(E_1, \cdots, E_p; F)$ with $E_i:=E$ for $1\leqq i\leqq p$, and let $\mathrm{Hom}_p(E, F)=\mathrm{Hom}_{K,p}(E, F)$ be the image of the K-linear map $\mathrm{Mult}_p(E, F)\to F^E$ sending φ to $\varphi\Delta^{(p)}$; here $\Delta^{(p)}: E\to E^p$ is the diagonal map. The elements of the K-module $F[\![E]\!]:=F_K[\![E]\!]:=\prod_{p\in \boldsymbol{N}}\mathrm{Hom}_p(E, F)$ are called *formal power series* on E with values in F. For a power series $u=\sum_{p\in \boldsymbol{N}}u_p$ from $F[\![E]\!]$ we denote by $\mathrm{T}(u):=u_1$ the *tangent map* of u. Let $F[\![E]\!]_+$ be the submodule of $F[\![E]\!]$ consisting of all power series without constant term.

Suppose now that E is of the form $E=E_1\times\cdots\times E_k$ with K-modules $E_1, \cdots, E_k$. For an element $\nu=(\nu_1, \cdots, \nu_k)$ of $\boldsymbol{N}^k$ we put $E_\nu:=E_1^{\nu_1}\times\cdots\times E_k^{\nu_k}$ for abbreviation, and denote by $\mathrm{Mult}_\nu(E, F)=\mathrm{Mult}_{K,\nu}(E, F)$ the K-module consisting

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of all $|\nu|$-linear maps $\varphi: E_\nu \to F$. Further let $\mathrm{Hom}_\nu(E, F) = \mathrm{Hom}_{K,\nu}(E, F)$ be the image of the K-linear map $\mathrm{Mult}_\nu(E, F) \to F^E$ sending φ to $\varphi\Delta^{(\nu)}$; here $\Delta^{(\nu)}: E \to E_\nu$ is the "multidiagonal" map. Then every formal power series u from $F[\![E]\!]$ has a unique representation $u = \sum_{\nu\in N^k} u_\nu$ with elements u_ν of $\mathrm{Hom}_\nu(E, F)$.

1.2. By a *punctured K-module* we understand a pair $(E, 0)$ consisting of a K-module E and its origin. Let $(E, 0)$ and $(F, 0)$ be two punctured K-modules. A *formal map* $u: (E, 0) \to (F, 0)$ is a formal power series from $F[\![E]\!]_+$.

Endowed with the formal maps as morphisms (and the composition of formal power series as composition of morphisms), the punctured K-modules form a category (For_K) with products. For a punctured K-module $(E, 0)$ we denote by $(E, 0)^\cdot$ the corresponding contravariant set valued functor on (For_K). Then a formal map u from $(E, 0)$ to $(F, 0)$ induces a morphism of functors from $(E, 0)^\cdot$ to $(F, 0)^\cdot$ being denoted by $\dot{u}$.

1.3. A *formal group* over K is a punctured K-module $G = (G, 0)$ endowed with formal maps $m_G: G \times G \to G$ and $j_G: G \to G$ such that $\dot{G}$ is a group valued functor with respect to $\dot{m}_G: \dot{G} \times \dot{G} \to \dot{G}$, the inversion mapping being $\dot{j}_G$.

Let G be a formal group over K and let $(E, 0)$ be a punctured K-module. An *operation of G on* $(E, 0)$ is a formal map $\omega: G \times (E, 0) \to (E, 0)$ such that $\dot{\omega}$ is an operation of the group valued functor $\dot{G}$ on the set valued functor $(E, 0)^\cdot$. Then the composition of ω and $(\mathrm{id}_G, 0): G \to G \times (E, 0)$ is called the *orbital map* of ω.

Let $(E, 0)$ and $(F, 0)$ be two punctured K-modules, on which G operates. A formal map $u: (E, 0) \to (F, 0)$ is said to be *G-equivariant*, if $\dot{u}$ is equivariant with respect to the action of $\dot{G}$. We say that G operates with *fixed point* on $(F, 0)$, if the canonical map $0 \to (F, 0)$ is G-equivariant.

§ 2. Convergent Maps of PO-Spaces

2.1. Let E be a $\boldsymbol{C}$-vector space. A *pseudo-norm* on E is a mapping $\|\cdot\|: E \to \boldsymbol{R}_+ \cup \{+\infty\}$ with the following properties: (1) For any α from $\boldsymbol{C}$ and any x from E such that $\|x\| < \infty$ we have $\|\alpha x\| = |\alpha|\,\|x\|$. (2) $\|x+y\| \leqq \|x\| + \|y\|$ for any pair x, y of elements of E.

2.2. Let now K be a commutative $\boldsymbol{C}$-algebra and let E, F be two K-modules. We suppose that for every integer $p \in \boldsymbol{N}$ we are given a pseudo-norm $|\cdot| = |\cdot|_p$ resp. $\|\cdot\| = \|\cdot\|_p$ on $\mathrm{Mult}_p(E, F) = \mathrm{Mult}_{K,p}(E, F)$ resp. $\mathrm{Hom}_p(E, F) = \mathrm{Hom}_{K,p}(E, F)$ such that $|\cdot|_p = \|\cdot\|_p$ for $p = 0, 1$. If $t > 0$ is a real number and $u = \sum_p u_p$ a formal power series from $F[\![E]\!]$, we put

$$|u|_t:=\sum_{p\in N}|\tilde{u}_p|t^p$$

and

$$\|u\|_t:=\sum_{p\in N}\|u_p\|t^p;$$

here $\tilde{u}_p$ denotes the symmetric p-linear map corresponding to u_p. We say that u is *convergent with respect to* $\|\cdot\|$ resp. *strongly convergent with respect to* $|\cdot|$, if there exists a $t>0$ such that $\|u\|_t<\infty$ resp. $|u|_t<\infty$. Clearly $|\cdot|_t$ and $\|\cdot\|_t$ are pseudo-norms on $F[\![E]\!]$ for every $t>0$.

2.3. Let G be another K-module and suppose moreover that for every integer $p\in N$ we are given a pseudo-norm $|\cdot|=|\cdot|_p$ on $\mathrm{Mult}_p(F, G)$ and $\mathrm{Mult}_p(E, G)$ and a pseudo-norm $\|\cdot\|=\|\cdot\|_p$ on $\mathrm{Hom}_p(F, G)$ and $\mathrm{Hom}_p(E, G)$ such that $|\cdot|_p=\|\cdot\|_p$ for $p=0, 1$. Furthermore we assume that there exists a real number $\alpha\geqq 1$ with the following property: If ψ is an element of $\mathrm{Mult}_p(F, G)$ such that $|\psi|<\infty$ and if $u_i\in\mathrm{Hom}_{q_i}(E, F)$ and $\varphi_i\in\mathrm{Mult}_{q_i}(E, F)$ are elements such that $\|u_i\|<\infty$, $|\varphi_i|<\infty$, $1\leqq i\leqq p$, then in case $p\geqq 1$ and $q_i\geqq 1$ the estimates

$$\|\psi\circ(u_1, \cdots, u_p)\|\leqq\alpha^{q_1+\cdots+q_p-1}|\psi|\,\|u_1\|\cdots\|u_p\|, \tag{2.3.1}$$

$$|\psi\circ(\varphi_1\times\cdots\times\varphi_p)|\leqq\alpha^{q_1+\cdots+q_p-1}|\psi|\,|\varphi_1|\cdots|\varphi_p| \tag{2.3.2}$$

hold. The following (easy) proposition shows in particular, that in these circumstances the convergence notion introduced in 2.2 is stable under composition.

Lemma 2.4. *Let the assumptions and notations be as in* 2.3. *Further let* u *resp.* v *be a formal power series from* $F[\![E]\!]_+$ *resp.* $G[\![F]\!]$ *and let* $t, s>0$ *be real numbers. Then:*

(1) *In case* $\|u\|_{\alpha t}\leqq s$ *we have* $\|vu\|_t\leqq|v|_s$.
(2) *In case* $|u|_{\alpha t}\leqq s$ *we have* $|vu|_t\leqq|v|_s$.

Example 2.5. Let $E=(E, \|\cdot\|)$ and $F=(F, \|\cdot\|)$ be two pseudo-normed K-modules. For an element φ resp. u of $\mathrm{Mult}_p(E, F)$ resp. $\mathrm{Hom}_p(E, F)$ we put

$$|\varphi|:=\sup\{\|\varphi(x_1, \cdots, x_p)\|: x_i\in E \text{ such that } \|x_i\|\leqq 1\},$$

$$\|u\|:=\sup\{\|u(x)\|: x\in E \text{ such that } \|x\|\leqq 1\}.$$

Obviously $\varphi\mapsto|\varphi|$ resp. $u\mapsto\|u\|$ is a pseudo-norm on $\mathrm{Mult}_p(E, F)$ resp. $\mathrm{Hom}_p(E, F)$. If $G=(G, \|\cdot\|)$ is a third pseudo-normed K-module, then these pseudo-norms satisfy the estimates (2.3.1) and (2.3.2) with $\alpha=1$.

2.6. Let $K=C$ and let E, F be two PO-spaces and λ an element of $]0, 1[$. Applying the construction from 2.5 to the semi-normed vector spaces $(E, \|\cdot\|_\lambda)$ and $(F, \|\cdot\|_\lambda)$ over C, we obtain a pseudo-norm $|\cdot|_\lambda$ resp. $\|\cdot\|_\lambda$ on $\mathrm{Mult}_p(E, F)$ resp. $\mathrm{Hom}_p(E, F)$ for every $p\in N$. We now fix an element ε of $]0, 1]$. Then

$$|\cdot|^0 := |\cdot|^{0,\varepsilon} := \sup\{|\cdot|_\lambda : \lambda \in [1-\varepsilon, 1[\}$$

resp.

$$\|\cdot\|^0 := \|\cdot\|^{0,\varepsilon} := \sup\{\|\cdot\|_\lambda : \lambda \in [1-\varepsilon, 1[\}$$

is a pseudo-norm on $\mathrm{Mult}_p(E, F)$ resp. $\mathrm{Hom}_p(E, F)$ again for every p. A formal power series $u = \sum_p u_p$ from $F[\![E]\!]$ is called *strictly convergent* (with respect to ε), if u is convergent with respect to $\|\cdot\|^0$. In this case u induces for every element λ of $[1-\varepsilon, 1[$ in a natural way a convergent power series $\bar{u}_\lambda = \sum_p (\bar{u}_p)_\lambda$ from $\bar{F}_\lambda[\![\bar{E}_\lambda]\!]$.

2.7. Now we introduce a construction, which integrates into the frame given by 2.2 and 2.3, but which is not a special case of 2.5.

Let E, F be two PO-spaces, $p \in \boldsymbol{N}_+$ and let λ, λ' be two elements of $]0, 1[$ such that $\lambda' \leqq \lambda$. For a form φ from $\mathrm{Mult}_p(E, F) = \mathrm{Mult}_{C,p}(E, F)$ resp. a polynomial u from $\mathrm{Hom}_p(E, F) = \mathrm{Hom}_{C,p}(E, F)$ let $|\varphi|_{(\lambda,\lambda')}$ resp. $\|u\|_{(\lambda,\lambda')}$ be the supremum of the numbers $\|\varphi(x_1, \cdots, x_p)\|_{\lambda'}$ resp. $\|u(x)\|_{\lambda'}$, where x_i resp. x runs through the set of elements of E such that $\|x_i\|_\lambda \leqq 1$ resp. $\|x\|_\lambda \leqq 1$. Then $|\cdot|_{(\lambda,\lambda')}$ resp. $\|\cdot\|_{(\lambda,\lambda')}$ is a pseudo-norm on $\mathrm{Mult}_p(E, F)$ resp. $\mathrm{Hom}_p(E, F)$. If ε is a fixed element of $]0, 1]$, then

$$|\cdot|^1 := |\cdot|^{1,\varepsilon} := \sup\{(\lambda-\lambda')^{p-1}|\cdot|_{(\lambda,\lambda')} : \lambda, \lambda' \in [1-\varepsilon, 1[, \lambda' < \lambda\}$$

resp.

$$\|\cdot\|^1 := \|\cdot\|^{1,\varepsilon} := \sup\{(\lambda-\lambda')^{p-1}\|\cdot\|_{(\lambda,\lambda')} : \lambda, \lambda' \in [1-\varepsilon, 1[, \lambda' < \lambda\}$$

is a pseudo-norm on $\mathrm{Mult}_p(E, F)$ resp. $\mathrm{Hom}_p(E, F)$ again. For y from $\mathrm{Mult}_0(E, F) = \mathrm{Hom}_0(E, F) = F$ we put $|y|^1 := \|y\|^1 := \|y\|^0$. One can show that these pseudo-norms satisfy the estimates (2.3.1) and (2.3.2) with $\alpha = 2$. A power series u from $F[\![E]\!]$ is called *convergent of type* $(1; -1)$ (with respect to ε), if u is convergent with respect to the pseudo-norms $\|\cdot\|^{1,\varepsilon}$. For example, a power series which is strictly convergent in the sense of 2.6 is convergent of type $(1; -1)$.

2.8. For fixed ε from $]0, 1]$, we denote by $(\mathrm{Con}^\varepsilon)$ the category consisting of the punctured PO-spaces, the morphisms being those formal maps $u : (E, 0) \to (F, 0)$, which are convergent of type $(1; -1)$ with respect to ε. If $\varepsilon = 1/2$, we write (Con) instead of $(\mathrm{Con}^\varepsilon)$. A morphism $u : (E, 0) \to (F, 0)$ in $(\mathrm{Con}^\varepsilon)$ will be called *direct*, if its tangent map $\mathrm{T}(u) : E \to F$ splits in $\mathrm{PO}_\varepsilon(\boldsymbol{C})$.

2.9. Let K be a commutative $\boldsymbol{C}$-algebra, $E_1, \cdots, E_k$, F a sequence of K-modules, and put $E := E_1 \times \cdots \times E_k$. We suppose that for every ν from $\boldsymbol{N}^k$ we are given a pseudo-norm $\|\cdot\|_\nu$ on $\mathrm{Hom}_{K,\nu}(E, F)$. If t is a k-tuple from $(\boldsymbol{R}_+^*)^k$ and $u = \sum_{\nu \in \boldsymbol{N}^k} u_\nu$ a formal power series from $F[\![E]\!]$, we put

$$\|u\|_t := \sum_{\nu \in \boldsymbol{N}^k} \|u_\nu\| t^\nu .$$

Of course u is called *convergent* with respect to the pseudo-norms $\|\cdot\|_\nu$, if there exists a t such that $\|u\|_t<\infty$.

2.10. Let $E_1, \cdots, E_k, F$ be a sequence of PO-spaces, $E:=E_1\times\cdots\times E_k$ and let $\delta=(\delta_1, \cdots, \delta_k)$ be an element of $\{\pm1\}^k$ and ν a k-tuple from N^k such that $|\nu|\geqq1$. Further let λ, λ' be two elements of $]0, 1[$ with $\lambda'\leqq\lambda$, and let $(\lambda_1, \cdots, \lambda_k)$ denote the tuple given by $\lambda_i:=\lambda$ resp. $\lambda_i:=\lambda'$ if $\delta_i=1$ resp. $\delta_i=-1$. For a homogeneous polynomial u from $\mathrm{Hom}_\nu(E, F)=\mathrm{Hom}_{C,\nu}(E, F)$ let $\|u\|_{\delta,(\lambda,\lambda')}$ be the supremum of the numbers $\|u(x)\|_{\lambda'}$, where $x=(x_1, \cdots, x_k)$ is an arbitrary element of $E=E_1\times\cdots\times E_k$ such that $\|x_i\|_{\lambda_i}\leqq1$ for $1\leqq i\leqq k$. Then $\|\cdot\|_{\delta,(\lambda,\lambda')}$ is a pseudo-norm on $\mathrm{Hom}_\nu(E, F)$. If ε is a fixed number from $]0, 1]$, then

$$\|\cdot\|_\delta^1:=\|\cdot\|_\delta^{1,\varepsilon}:=\sup\{(\lambda-\lambda')^{|\nu|-1}\|\cdot\|_{\delta,(\lambda,\lambda')}:\lambda, \lambda'\in[1-\varepsilon, 1[, \lambda'<\lambda\}$$

is a pseudo-norm on $\mathrm{Hom}_\nu(E, F)$ again. For $\nu=0$ let $\|\cdot\|_\delta^1:=\|\cdot\|_\delta^{1,\varepsilon}:=\|\cdot\|^1$ be the pseudo-norm on $\mathrm{Hom}_0(E, F)=F$ defined in 2.7. A formal power series $u=\sum_{\nu\in N^k}u_\nu$ from $F[\![E]\!]$ is called *convergent of type* $(\delta;-1)$ (with respect to ε), if u is convergent with respect to the pseudo-norms $\|\cdot\|_\delta^1$ (in the sense of 2.9). Note that u is then in particular convergent of type $(1;-1)$ in the sense of 2.7. If u is convergent of type $(\delta;-1)$, then u induces for every pair λ, λ' of elements of $[1-\varepsilon, 1[$ with $\lambda'<\lambda$ in a natural way a convergent power series $\bar{u}_{(\lambda,\lambda')}$ from $\bar{F}_{\lambda'}[\![(\bar{E}_1)_{\lambda_1}\times\cdots\times(\bar{E}_k)_{\lambda_k}]\!]$.

Remark 2.11. In an earlier version of this paper we worked with a weaker notion of convergence for formal maps of PO-spaces. The notion used here was suggested by S. Kosarew.

§3. PO-Lie Groups

3.1. Let $G=(G, 0)$ be a formal group over C in the sense of 1.3 such that G is a PO-space. G will be called a PO-*Lie group*, if the formal maps m_G and j_G are convergent (with respect to $\varepsilon=1/2$).

Let now G be a PO-Lie group, $\omega: G\times(E, 0)\to(E, 0)$ a convergent operation of G on a punctured PO-space $(E, 0)$ and ε an element of $]0, 1/2]$. We say that ω is *direct* (with respect to ε), if the corresponding orbital map $G\to(E, 0)$ is a direct morphism in $(\mathrm{Con}^\varepsilon)$.

Example 3.2. Let r be a tuple from $(R_+^*)^n$ and $U:=P(0;r)\subseteq C^n$ the open polycylinder of polyradius r with center 0 and $G:=\Gamma(U, \mathcal{O}_{C^n})^n$. Further let

$$m=m_G: (G, 0)\times(G, 0)\longrightarrow(G, 0)$$

be the formal map with series expansion $m=\sum_{p,q\in N}m_{p,q}$ given by $m_{p,1}(e, f):=(1/p!)D^p(f)\cdot e^p$ if $p\geqq0$, $m_{1,0}(e, f):=e$ and $m_{p,q}:=0$ otherwise. Then G is a formal group over $\boldsymbol{C}$ with respect to m_G. S. Kosarew has verified that m_G and j_G are convergent in the sense of 2.7. Hence G is even a PO-Lie group.

3.3. Let E be a PO-space and t_0 an element of $[0, 1[$. Then we denote by

$$(E, 0)\check{}_{t_0}: (\mathrm{Gan}) \longrightarrow (\mathrm{sets})$$

the functor on the category (Gan) of germs of (finite dimensional) analytic spaces, sending a germ $S=(S, 0)$ to the $\mathcal{O}_{S,0}$-module

$$(E, 0)\check{}_{t_0}(S) := \varinjlim_{\lambda > t_0} \mathrm{Hom}(S, (\bar{E}_\lambda, 0))\,;$$

here $\mathrm{Hom}(S, (\bar{E}_\lambda, 0))$ is the set of analytic map-germs $f: S \to \bar{E}_\lambda$ such that $f(0)=0$. Let now F be a second PO-space and $u: (E, 0) \to (F, 0)$ a morphism in (Con^{1-t_0}). Then u obviously induces a morphism

$$\check{u}_{t_0}: (E, 0)\check{}_{t_0} \longrightarrow (F, 0)\check{}_{t_0}$$

of functors. Since $(F, 0)\check{}_{t_0}(\mathrm{Spec}(\boldsymbol{C}))$ consists of exactly one element, the *fibre* of $\check{u}_{t_0}$ is a well defined subfunctor $W\check{}_{0t}$ of $(E, 0)\check{}_{t_0}$.

3.4. Let now G be a PO-Lie group and t_0 be a fixed element of $]1/2, 1[$. Then $G\check{}_{t_0} := (G, 0)\check{}_{t_0}$ is obviously a group-valued functor on (Gan). If $\omega: G \times (E, 0) \to (E, 0)$ is a convergent operation of G on a punctured PO-space $(E, 0)$, then $\check{\omega}_{t_0}$ is an operation of the group-valued functor $G\check{}_{t_0}$ on the set-valued functor $(E, 0)\check{}_{t_0}$.

Let $\eta: G \times (F, 0) \to (F, 0)$ be another convergent operation and let $u: (E, 0) \to (F, 0)$ be a G-equivariant convergent map. Then the functor morphism $\check{u}_{t_0}$ is obviously $G\check{}_{t_0}$-equivariant. If G operates with fixed point on $(F, 0)$, then the fibre $W\check{}_{t_0}$ of $\check{u}_{t_0}$ is invariant under the operation of $G\check{}_{t_0}$,

§4. The Quotient Theorem

4.1. Let $F: \mathcal{C} \to (\mathrm{sets})$ and $G: \mathcal{C} \to (\mathrm{groups})$ be contravariant functors on a category $\mathcal{C}$, and suppose given an operation of G on F. Then the *quotient groupoid* $F/G \xrightarrow{p} \mathcal{C}$ is defined as follows. The objects of F/G are the elements a of $F(S)$ with $S \in C$. If $a' \in F(S')$ and $a \in F(S)$ are two objects, then $\mathrm{Hom}_{F/G}(a', a)$ is the set of pairs (f, g) from $\mathrm{Hom}_C(S', S) \times G(S')$ such that $g \cdot F(f)(a) = a'$. Obviously $(\overline{F/G})(S) = F(S)/G(S)$ is the orbit space of $F(S)$ with respect to the operation of $G(S)$.

4.2. Suppose now given a PO-Lie group G operating on two punctured PO-spaces $(E, 0)$ and $(F, 0)$ such that G acts with fixed point on $(F, 0)$. Further let $u: (E, 0) \to (F, 0)$ be a strictly convergent G-equivariant morphism in (Con) and t_0 an element of $[1/2, 1[$. Then the group valued functor $\check{G_{t_0}}$ operates on the fibre $\check{W_{t_0}}$ of $\check{u}_{t_0}$, and the corresponding quotient groupoid $\check{W_{t_0}}/\check{G_{t_0}}$ over (Gan) satisfies Schlessinger's condition (S1′), see [1]. The following theorem gives a criterion for $\check{W_{t_0}}/\check{G}_{t_0}$ to have a semiuniversal deformation.

Theorem 4.3. *Let the assumptions and notations be as in* 4.2. *Moreover suppose that the following conditions hold:*

(1) *The operations* $G\times(E, 0)\xrightarrow{\omega}(E, 0)$ *and* $G\times(F, 0)\to(F, 0)$ *are convergent of type* $(-1, 1;-1)$, *and* E, F *are* $(1-t_0)$-*good.*

(2) ω *is direct with respect to* $1-t_0$.

(3) u *is direct with respect to* $1-t_0$.

(4) *The tangent space* $(\overline{\check{W_{t_0}}/\check{G_{t_0}}})(D)$ *is finite dimensional.*[1]

Then the groupoid $W_{t_0}/\check{G}_{t_0}$ *has a semiuniversal deformation.*

Sketch of proof. By (3), the image I of $\mathrm{T}(u):E\to F$ is a direct summand of F in $\mathrm{PO}_{1-t_0}(C)$. Hence we can find a retraction $v:F\to I$ onto I in $\mathrm{PO}_{1-t_0}(C)$. For an element λ of $[t_0, 1[$ let W_λ resp. M_λ denote the fibre of $\bar{u}_\lambda:(\bar{E}_\lambda, 0)\to(\bar{F}_\lambda, 0)$ resp. $\bar{v}_\lambda\bar{u}_\lambda:(\bar{E}_\lambda, 0)\to(\bar{I}_\lambda, 0)$. Then $W=(W_\lambda)_{\lambda\in]t_0,1[}$ resp. $M=(M_\lambda)_{\lambda\in]t_0,1[}$ is a direct system of germs of Banach analytic spaces resp. manifolds suct that $W\subseteqq M$ and $\mathrm{T}(W_\lambda)=\mathrm{T}(M_\lambda)$ holds for all λ[2]. Moreover we have $\check{W_{t_0}}(S)=\varinjlim_{\lambda>t_0}\mathrm{Hom}(S, W_\lambda)$ for any germ S from (Gan). By (4), there exists an index $\lambda>t_0$ such that the canonical map from $\mathrm{T}(W_\lambda)$ to $(\overline{\check{W_{t_0}}/\check{G_{t_0}}})(D)$ is surjective. We choose a finite dimensional smooth subgerm $N\subseteqq M_\lambda$ for which the map

$$\mathrm{T}(N)\longrightarrow(\overline{\check{W_{t_0}}/\check{G_{t_0}}})(D)$$

is bijective. Then $Y:=N\cap W_\lambda$ is a subgerm such that $\mathrm{T}(Y)=\mathrm{T}(N)$. Moreover Y is finite dimensional by [7], 7.5, Prop. 7.

Let λ' be an arbitrary index such that $t_0<\lambda'<\lambda$. Then by (1) $\bar{\omega}_{(\lambda,\lambda')}$ induces an analytic map-germ from $(\bar{G}_{\lambda'}, 0)\times W_\lambda$ into $W_{\lambda'}$. Now one shows, using (2) and the propositions 4.4, 4.5 stated below combined with the inverse mapping theorem for analytic maps of Banach analytic manifolds, that the morphism

$$(\bar{G}_{\lambda'}, 0)\times Y\xrightarrow{\bar{\omega}_{(\lambda,\lambda')}}W_{\lambda'}$$

is smooth. From this one can easily conclude that the canonical functor $\rho:\dot{Y}\to\check{W_{t_0}}/\check{G_{t_0}}$ is minimally smooth[3]. Hence $\rho(\mathrm{id}_Y)\in(\check{W_{t_0}}/\check{G_{t_0}})(Y)$ is a semiuniversal deformation of $\check{W_{t_0}}/\check{G_{t_0}}$.

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[1] Here $D:=\mathrm{Spec}(C[\varepsilon])$ is the double point.

[2] For a germ Z of a (Banach) analytic space $\mathrm{T}(Z)$ denotes the tangent space in the distinguished point.

[3] If Z is a germ of an analytic space, Z denotes the functor on (Gan) defined by Z.

In the above proof, we made use of the following two propositions. Detailed proofs are given in [5], (II 12).

Lemma 4.4. *Let X be a finite dimensional complex space, let $i: F \to G$ be an injective continuous linear map of Banach spaces and let $f: X \to F$ be a holomorphic map. Then the subspaces $f^{-1}(0)$ and $(i \circ f)^{-1}(0)$ of X coincide.*

Lemma 4.5. *Let $X \to S$ be a map of germs of Banach analytic spaces of finite relative dimension, and let $Y \subseteq X$ be a subgerm of X which is S-anaflat.*[4] *Then if $X_0 = Y_0$ holds for the fibres in the distinguished point $0 \in S$, we already have $X = Y$.*

Remarks 4.6. (1) The proof of 4.3 presented here uses in an essential way the theory of Banach analytic spaces in the sense of Douady. In a forthcoming paper I will show that one can prove 4.3 using only power series techniques. As a byproduct, such a "finite dimensional" proof also gives more precise information on the structure of the base Y of the semiuniversal deformation of the quotient groupoid $W_{t_0}^{\vee}/\check{G}_{t_0}^{\vee}$.

(2) One can also show a relative version of the quotient theorem 4.3. Again, I will provide the details at another place.

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[4] In the sense of [7].