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Undecidability of Free Pseudo-Complemented Semilattices

By

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Abstract

Decision problem for the first order theory of free objects in equational classes of algebras was investigated for groups (Malcev [10]), semigroups (Quine [12]), commutative semigroups (Mostowski [11]), distributive lattices (Ershov [6]) and several varieties of rings (Lavrov [9]). Recently this question was solved for all varieties of Hilbert algebras and distributive pseudo-complemented lattices (see [7], [8]). In this paper we prove that the theory of all finitely generated free pseudo-complemented semilattices is undecidable.

By a *pseudo-complemented semilattice* (*pcs* for short) we mean an algebra $\mathfrak{A}=\langle A;\wedge,\neg,0\rangle$ of type $\langle 2,1,0\rangle$ such that $\langle A;\wedge,0\rangle$ is a meet semilattice with the smallest element 0 and the unary operation $\neg$ is defined by

$$a\wedge x=0 \quad \text{iff} \quad x\leqq\neg a.$$

The class ***PCS*** of all *pcs* form a variety whose only non-trivial subvariety ***B*** (of Boolean algebras) is definable, relatively to ***PCS***, by the identity

$$\neg\neg x=x.$$

An element a of a *pcs* is *regular* if $\neg\neg a=a$. It is known that regular elements are exactly of the form $\neg b$.

These facts and the basic arithmetic of *pcs* can be found in [2]. For the main concepts in universal algebra the reader is referred to [5].

Now we recall Balbes' [1] description of finitely generated free *pcs*.

Let $n=\{0,\cdots,n-1\}$ be an arbitrary natural number. For $S\subset n$ let $\mathfrak{B}_S$ denote the *pcs* obtained from the lattice 2^S of all subsets of S by adjoining a new smallest element 0_S. By $\mathfrak{L}(n)$ we mean the direct product $\prod_{S\subset n}\mathfrak{B}_S$.

For every subset $A\cup\{i\}$ of n let us define two elements of $\mathfrak{L}(n)$ by putting

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$$\alpha_i(S)=\begin{cases} S-\{i\} & \text{if } i\in S, \\ 0_S & \text{otherwise}, \end{cases} \qquad \text{for all } S\subset n, \tag{1}$$

and

$$\beta_A=\neg\left(\bigwedge_{i\in A}\alpha_i\wedge\bigwedge_{j\notin A}\neg\alpha_j\right). \tag{2}$$

From [1] we know that

$$\beta_A(S)=\begin{cases} S & \text{if } S\neq A, \\ 0_S & \text{if } S=A. \end{cases} \tag{3}$$

The following Theorem due to R. Balbes [1] describes finitely generated *free pcs'*.

Theorem 1. *The n-freely generated pseudo-complemented semilattice is isomorphic to a subalgebra* $\mathfrak{PS}(n)$ *of* $\mathfrak{S}(n)$, *(freely) generated by the set* $\{\alpha_i : i<n\}$. *Every element* γ *of* $\mathfrak{PS}(n)$ *can be represented in the form*

$$\gamma=\bigwedge_{i\in C}\alpha_i\wedge\gamma^r, \tag{4}$$

for some $C\subset n$ *and some regular element* γ^r *of* $\mathfrak{PS}(n)$.

Using this Theorem we can give the first order characterization of free generators in $\mathfrak{PS}(n)$. An element a of a *pcs* $\mathfrak{A}$ is said to be *preregular* if a is not regular but every $b>a$ is regular.

Corollary 2. *The only preregular elements in free pseudo-complemented semilattice are its free generators,*

Proof. First we prove that all α_i are preregular. Of course they are not regular, as $\boldsymbol{PCS}\neq\boldsymbol{B}$. Now, let $\gamma=\bigwedge_{i\in C}\alpha_i\wedge\gamma^r$ be essentially larger than α_j. Then $\alpha_j\leqq\alpha_i$ for all $i\in C$, which is impossible for $i\neq j$ as $\{\alpha_i : i<n\}$ freely generates $\mathfrak{PS}(n)$. Thus $C\subset\{j\}$. If $C=\{j\}$ then $\gamma=\alpha_j\wedge\gamma^r$, which leads to the contradiction $\alpha_j<\gamma\leqq\alpha_j$. Thus $C=\emptyset$, and consequently $\gamma=\gamma^r$ is regular.

Conversely, assume that $\gamma=\bigwedge_{i\in C}\alpha_i\wedge\gamma^r$ is a preregular element of $\mathfrak{PS}(n)$. Then C is non-empty. Moreover, C has not more than one element. Indeed, if i, j are two different elements of C, then $\gamma\leqq\alpha_i$ as well as $\gamma\leqq\alpha_j$. But neither α_i nor α_j is regular, which implies that $\alpha_i=\gamma=\alpha_j$. Therefore C has exactly one element, as claimed, and $\gamma=\alpha_j\wedge\gamma^r\leqq\alpha_j$ for some $j<n$. However the strong inequality $\gamma<\alpha_j$ is impossible, as α_j is not regular. Finally $\gamma=\alpha_j$, and we can finish the proof.

The proof of our undecidability result is based on the method of interpretation due to A. Tarski [14]. However we will need some modified version called by S. Burris and R. McKenzie [3] *interpretation by parameters and definable factor relations.* For details of this method (which will not be given here) the reader is referred to [3] or [13]. Now, we only recall that in a special case this method can be expressed as follows (see also [5]).

A class $\mathcal{P}$ of some partially ordered sets is said to be interpretable into a class $\mathcal{A}$ of some algebraic structures of type τ, if there are first order formulas:

$$\delta(x), \quad \varepsilon(x, y), \quad \rho(x, y),$$

of type τ, such that for every poset $\mathfrak{P}=\langle P, \leqq\rangle$ from $\mathcal{P}$, there is a structure $\mathfrak{A}\in\mathcal{A}$ for which, if we let

$$\begin{aligned} &A_\delta=\{a\in A:\ \mathfrak{A}\models\delta(a)\},\\ (5)\qquad &\Theta=\{\langle a, b\rangle\in A_\delta\times A_\delta:\ \mathfrak{A}\models\varepsilon(a, b)\},\\ &R=\{\langle a, b\rangle\in A_\delta\times A_\delta:\ \mathfrak{A}\models\rho(a, b)\},\end{aligned}$$

then Θ is an equivalence relation on A_δ, such that the quotient-set A_δ/Θ together with the relation

$$R/\Theta=\Theta\circ R\circ\Theta$$

form a poset isomorphic to $\mathfrak{P}$.

The power of the method of interpretation lies in the following Theorem, proof of which can be found in [3].

Theorem 3. *If a class $\mathcal{P}$ with hereditarily undecidable first order theory (i.e. every subtheory of* Th$(\mathcal{P})$ *is undecidable) is interpretable in $\mathcal{A}$ then $\mathcal{A}$ has (hereditarily) undecidable first order theory as well.*

By a partition lattice π_n we mean a lattice of all equivalence relations on arbitrary n-elements set. Ju. L. Ershov [6] and later S. Burris and H.P. Sankappanavar [4] proved the following

Theorem 4. *The class $\{\pi_n : n\geqq 1\}$ of finite partition lattices has hereditarily undecidable first order theory.*

Using above theorems we are able to prove the main result of this paper:

Theorem 5. *The first order theory of all finitely generated free pseudo-complemented semilattices is hereditarily undecidable.*

Proof. We will interpret $\{\pi_n : n\geqq 1\}$ into the class $\{\mathfrak{P}\mathfrak{s}(n) : n<\omega\}$ of all finitely generated pseudo-complemented semilattices. Actually we will show that π_n is isomorphic to some quotient of whole $\mathfrak{P}\mathfrak{s}(n)$, and that such quotients can be obtained in an uniform way.

From Corollary 2 we know that the formula

$$\sigma(u) \equiv u \neq \neg\neg u \quad \& \quad \forall x(x \wedge u = u \Rightarrow x = \neg\neg x \text{ or } x = u),$$

characterizes free generators in all nontrivial $\mathfrak{P}\mathfrak{s}(n)$. Denote by D_n the set of these generators, i.e. $D_n = \{\alpha_i : i < n\}$ in the convention of Theorem 1. Now we can see that for every fixed $\gamma \in \mathfrak{P}\mathfrak{s}(n)$,

$$\tilde{\gamma} = \{\langle \alpha, \beta \rangle \in D_n \times D_n : \neg\neg\alpha \wedge \gamma = \neg\neg\beta \wedge \gamma\}$$

is an equivalence relation on the set D_n. However it can happen that $\tilde{\gamma}_1 = \tilde{\gamma}_2$ for some $\gamma_1 \neq \gamma_2$. Using the formula

$$\varepsilon(x, y) \equiv \forall u\, \forall v\, \sigma(u) \;\&\; \sigma(v)$$
$$\Rightarrow (\neg\neg u \wedge x = \neg\neg v \wedge x \Leftrightarrow \neg\neg u \wedge y = \neg\neg v \wedge y)$$

we can identify the elements of $Ps(n)$ which give the same equivalence relation on D_n. It is clear that ε determines, in the sense of (5), the equivalence relation Θ on $Ps(n)$ and that $Ps(n)/\Theta$ can be treated as a poset of some equivalences on D_n with order given by

$$\rho(x, y) \equiv \forall u\, \forall v(\sigma(u) \;\&\; \sigma(v) \;\&\; \neg\neg u \wedge x = \neg\neg v \wedge x)$$
$$\Rightarrow \neg\neg u \wedge y = \neg\neg v \wedge y\,.$$

i.e. $\gamma_1/\Theta \leqq \gamma_2/\Theta$ iff $\mathfrak{P}\mathfrak{s}(n) \models \rho(\gamma_1, \gamma_2)$.

Now we show that every equivalence relation on D_n can be expressed in the form $\tilde{\gamma}$ for some $\gamma \in Ps(n)$. Let Σ be an equivalence relation on D_n with the corresponding partition $\mathfrak{R}$ of n. From (2) we know that the element $\gamma = \neg(\bigwedge\!\!\!\bigwedge_{A \in \mathfrak{R}} \beta_A)$ belongs to $\mathfrak{P}\mathfrak{s}(n)$, and by (3) we obtain

$$\gamma(S) = \begin{cases} S & \text{if } S \in \mathfrak{R}, \\ 0_S & \text{otherwise}. \end{cases} \tag{6}$$

By (1) we have

$$(\neg\neg\alpha_i)(S) = \begin{cases} S & \text{if } i \in S, \\ 0_S & \text{otherwise}, \end{cases}$$

which together with (6) gives

$$(\neg\neg\alpha_i \wedge \gamma)(S) = \begin{cases} S & \text{if } i \in S \in \mathfrak{R}, \\ 0_S & \text{otherwise}. \end{cases}$$

In particular $(\neg\neg\alpha_i \wedge \gamma)(S) = (\neg\neg\alpha_j \wedge \gamma)(S)$ for all $S \notin \mathfrak{R}$, and $i, j < n$.

To see that $\tilde{\gamma} = \Sigma$ let us write the following sequence of equivalent conditions:

$$\langle \alpha_i, \alpha_j \rangle \in \tilde{\gamma},$$

$$(\neg\neg\alpha_i \wedge \gamma)(S) = (\neg\neg\alpha_j \wedge \gamma)(S), \quad \text{for all } S \in \mathcal{R},$$

$$i \in S \text{ iff } j \in S, \quad \text{for all } S \in \mathcal{R},$$

$$i \in S \text{ and } j \in S, \quad \text{for some } S \in \mathcal{R}'_\omega,$$

$$\langle \alpha_i, \alpha_j \rangle \in \Sigma.$$

From the above considerations we know that for every $n \geqq 1$ the posets π_n and $Ps(n)/\Theta$ are isomorphic.

We have just shown that the formulas

$$\delta(x) \equiv x = x,$$

$$\varepsilon(x, y),$$

$$\rho(x, y)$$

define the required interpretation, and therefore our Theorem follows from Theorems 3 and 4.

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References

[1] Balbes, R., On free pseudo-complemented and relatively pseudo-complemented semilattices, *Fund. Math.*, **78** (1973), 119-131.
[2] Balbes, R. and Horn, A., Stone lattices, *Duke Math. J.*, **38** (1970), 537-547.
[3] Burris, S. and McKenzie, R., Decidability and Boolean Representation, *Mem. Amer. Math. Soc.*, **246** (1981).
[4] Burris, S. and Sankappanavar, H. P., Lattice-theoretic decision problems in universal algebra, *Algebra Universalis*, **5** (1975), 163-177.
[5] ———, ———, *A Course in Universal Algebra*, Springer Verlag, 1981.
[6] Ershov, Ju. L., New examples of undecidable theories (Russ.), *Algebra i Logika*, **5** (1966), 37-47.
[7] Idziak, P. M., Undecidability of free pseudocomplemented distributive lattices, manuscript 1986.
[8] ———, Undecidability of relatively free Hilbert algebras, manuscript 1986.
[9] Lavrov, I. A., The undecidability of the elementary theories of certain rings (Russ.), *Algebra i Logika*, **1** (1962), 39-45.
[10] Malcev, A. I., Axiomatizable classes of locally free algebras of certain types, (Russ.), *Sib. Mat. Zh.*, **3** (1962), 729-743.
[11] Mostowski, A., On direct products of theories, *J. Symb. Logic*, **17** (1952), 1-31.
[12] Quine, W. V., Concatenation as a basis for arithmetic, *J. Symb. Logic*, **11** (1946), 105-114.

[13] Rabin, M. O., Decidable Theories, *Handbook of Mathematical Logic*, J Barwise ed., North Holland, 1977, 595-629.
[14] Tarski, A., Mostowski, A. and Robinson, R. M., *Undecidable Theories*, North-Holland, Amsterdam, 1953.