Publ. RIMS, Kyoto Univ.
23 (1987), 553-558

Local Moduli for Strictly Pseudoconvex Spaces

By

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Introduction

Let X be a strictly pseudoconvex complex space with exceptional subvariety $E \subseteq X$ and $\dim_C(T^1(X, \mathcal{O}_X)) < \infty$. In this paper we give an outline of a proof for the existence of a convergent formally semiuniversal deformation of the germ (X, E). If E is analytically thin in X, any such deformation is in fact semiuniversal in the convergent sense. More generally, we formulate a similar result for strictly pseudoconvex graded complex spaces (2.2), containing also the solution of the local moduli problem for coherent analytic sheaves on strictly pseudoconvex spaces. Detailed proofs are given in [7].

§1. Strictly Pseudoconvex Graded Complex Spaces

1.1. Let X be a graded complex space in the sense of [6] 1.4. Then $X^{an} := \mathrm{Specan}(\mathcal{O}_X)$ is a complex space over X_0, and we have a natural flat morphism $p_X : X^{an} \to X$ of C-ringed spaces. To the canonical surjection from $\mathcal{O}_X$ onto $(\mathcal{O}_X)_0$ there corresponds a section $i_X : X_0 \to X^{an}$ of the structure morphism of X^{an}, whose image will be identified with X_0.

Let $f : X \to Y$ be a morphism of graded complex spaces. Then f induces a map $f^{an} : X^{an} \to Y^{an}$. We call f a *Stein map,* if the underlying holomorphic map $f_0 : X_0 \to Y_0$ is Stein. This is obviously true if and only if f^{an} is Stein. f is called *proper*, if f_0 is proper and if $\mathcal{O}_X$ is a finite algebra over $(\mathcal{O}_X)_0 \otimes_{(\mathcal{O}_Y)_0} \mathcal{O}_Y$. If moreover f_0 has finite fibres, we shall say that f is *finite.* One easily shows that f is proper resp. finite if and only if f^{an} is proper resp. finite.

Further f is called *holomorphically convex* resp. *strictly pseudoconvex,* if f^{an} is holomorphically convex resp. strictly pseudoconvex. For $Y = \mathrm{Spec}(C)$ we obtain the notion of a *holomorphically convex* resp. *strictly pseudoconvex* graded complex space.

Communicated by M. Kashiwara, November 26, 1986.
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Example 1.2. Let X_0 be a strictly pseudoconvex complex space with exceptional subvariety E_0 and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_{X_0}$-module. Then $X:=(X_0, \mathrm{S}_{\mathcal{O}_{X_0}}(\mathcal{L}))$ is a strictly pseudoconvex graded complex space, and $i_X(E_0)$ is the exceptional subvariety of $X^{\mathrm{an}}=\mathbf{V}(\mathcal{L})$.

1.3. Let $X\xrightarrow{g}S$ be a holomorphically convex map of graded complex spaces. Then there exists a graded complex space Y over S and a proper S-morphism $f: X\to Y$ such that $f^{\mathrm{an}}: X^{\mathrm{an}}\to Y^{\mathrm{an}}$ is the Remmert quotient of X^{an} relative Y^{an} (in the sense of [10]). Moreover the equality $f_*(\mathcal{O}_X)=\mathcal{O}_Y$ holds, and $Y\to S$ is a Stein morphism. The pair (f, Y) is determined uniquely up to a unique isomorphism and is called the *Remmert quotient of X relative S*.

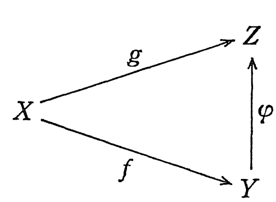
Let $E\subseteqq X^{\mathrm{an}}$ be the exceptional subvariety of X^{an} relative S^{an}. If S is a complex space, the following properties hold: (a) E is contained in X_0. (b) $f^{-1}(f(E))=E$. (c) E is the exceptional subvariety of X_0 relative S. We then also call E the *exceptional subvariety of X relative S*.

§ 2. Versal Deformations of Strictly Pseudoconvex Graded Complex Spaces

2.1. Let X be a strictly pseudoconvex graded complex space, $c\in \boldsymbol{N}$ a natural number, and suppose $K\subseteqq X$ is a compact analytic set containing the exceptional subvariety of X. With these assumptions and notations the following proposition holds.

Theorem 2.2. *If the $\boldsymbol{C}$-vector space $\overline{\mathrm{Def}}^{\mathrm{gr},\geqq c}_{(X,K)}(D)$ of isomorphism classes of deformations of (X, K) in $\mathrm{Def}^{\mathrm{gr},\geqq c}_{(X,K)}$ over the double point D is of finite dimension, the germ (X, K) has a convergent formally semiuniversal deformation in the groupoid $\mathrm{Def}^{\mathrm{gr},\geqq c}_{(X,K)}$ over* (Gan).

Sketch of proof. Let $f: X\to Y$ be the Remmert quotient of X and $L\subseteqq Y$ be the finite set $f(K)$. Then, after shrinking Y as a neighborhood of L if necessary, there exists a commutative diagram



of graded complex spaces with the following properties:

(1) Z is a "number space" $\boldsymbol{C}^{n'}(n''; r)$.

(2) φ is a finite map, and the cokernel of the algebra homomorphism from $\Gamma'(Z, \mathcal{O}_Z)$ to $\Gamma'(Y, \mathcal{O}_Y)$ associated with φ is a finite dimensional $\boldsymbol{C}$-vector space.

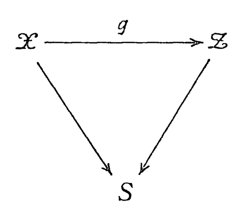
(3) The canonical functor $\mathrm{Def}_X^{\mathrm{gr}, \geqq c} \to \mathrm{Def}_{(X,K)}^{\mathrm{gr}, \geqq c}$ is minimally formally smooth.

(4) If $\bar{X} \to \bar{S} = (\bar{S}, \bar{s})$ denotes the formal semiuniversal deformation of X in $\mathrm{Def}_X^{\mathrm{gr}, \geqq c}$, there exists an $\bar{S}$-morphism $\bar{g}: \bar{X} \to \bar{S} \times Z$ such that $\bar{g}(\bar{s}) = g$ and $\bar{g}_{(\leqq c-1)} = (g_{(\leqq c-1)})_{\bar{S}}$ in case $c \geqq 1$ holds.

Then, if we put $G := g(K) \subseteq Z$ for abbreviation, the canonical functor

$$\mathrm{Def}_{(g,G)}^{\mathrm{gr}, \geqq c} \longrightarrow \mathrm{Def}_{(X, g^{-1}(G))}^{\mathrm{gr}, \geqq c}$$

is surjective, and the $\boldsymbol{C}$-vector space $\overline{\mathrm{Def}}_{(g,G)}^{\mathrm{gr}, \geqq c}(D)$ is finite dimensional. Now [6] 3.2 implies that (g, G) has a convergent semiuniversal deformation



in $\mathrm{Def}_{(g,G)}^{\mathrm{gr}, \geqq c}$. Then $\mathfrak{X} \to S$ is a complete deformation of $(X, g^{-1}(G))$ in $\mathrm{Def}_{(X, g^{-1}(G))}^{\mathrm{gr}, \geqq c}$. By restricting to K, it defines a complete deformation of (X, K) in $\mathrm{Def}_{(X,K)}^{\mathrm{gr}, \geqq c}$. Now the assertion follows from [5], 2.7 and 2.3.

From 2.2, applied to a *complex* space X and $c=0$, and 3.3 combined with [4], 3.2 and 3.4, we get the following result, which I obtained already in 1983:

Theorem 2.3. *Let X be a strictly pseudoconvex complex space and suppose $K \subseteq X$ is a compact analytic set containing the exceptional subvariety of X. If $\overline{\mathrm{Def}}_{(X,K)}(D)$ is finite dimensional, we have:*

(a) *The germ (X, K) has a convergent formally semiuniversal deformation.*

(b) *If the exceptional subvariety of X is analytically thin in X, then any such deformation of (X, K) is semiuniversal.*

Now we turn to deformations of coherent sheaves on strictly pseudoconvex spaces. Special cases of the following theorem were proved in [14, 15].

Theorem 2.4. *Let X be a strictly pseudoconvex complex space and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that $T := \mathrm{Supp}(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}))$ is compact. Then:*

(a) *$\mathcal{F}$ has a convergent formally semiuniversal deformation $\mathcal{G}$ over a germ S.*

(b) *If $K \subseteq X$ is a strictly pseudoconvex compact set such that $T \subseteq K$, then*

$(\mathcal{Q}, K)$ is a semiuniversal deformation of $(\mathcal{F}, K)$.[1]

This follows easily from 2.2, applied to $X[\mathcal{F}]$ and $c=1$.

Remarks 2.5. (1) Let X be a strictly pseudoconvex complex space such that $T:=\mathrm{Supp}(\mathcal{T}^1(X, \mathcal{O}_X))$ is compact. Then I conjecture that one can find a convergent formally semiuniversal deformation $\mathfrak{X}$ of the whole space X with the following property: If $K\subseteqq X$ is any strictly pseudoconvex compact subset of X containing T, then the germ of $\mathfrak{X}$ with respect to K is a semiuniversal deformation of (X, K). In particular the thinness assumption in 2.3 (b) should be superfluous. See also 3.4.

(2) Let us mention that the result 2.3 has been announced already in the notes [13, 8] in different degrees of generality; but till now no proof has been published. In the special case where X is smooth of dimension $\geqq 4$ resp. 2, theorem 2.3 was shown in [1] resp. [12].

(3) The remarks [6], 3.3 and 4.8, carry over m. m. to the situation considered here.

§ 3. The Formal Principle for Deformations of Strictly Pseudoconvex Spaces

3.1. Let S be a complex space and $T\subseteqq S$ a closed subspace with infinitesimal neighborhoods S_n, $n\in \boldsymbol{N}$, and let $s\in T$ be a fixed point. For a complex space X over S and a closed subset K of the fibre X_s we denote by $(X|K)^\wedge$ the completion of the pseudocomplex space $X|K=(K, \mathcal{O}_X|K)$ along (the inverse image of) T. If Y is another complex space over S and L is a closed subset of Y_s and if $f: X\to Y$ is an S-morphism taking K to L, then

$$(f|(K, L))^\wedge : X\hat{|}K \longrightarrow Y\hat{|}L$$

denotes the completion along T of the S-morphism

$$f|(K, L) : X|K \longrightarrow Y|L$$

induced by f. With these notations, the following result holds.

Theorem 3.2. *Let $X\to S$ and $X'\to S$ be two strictly pseudoconvex maps of complex spaces such that the exceptional subvarieties $E\subseteqq X$ and $E'\subseteqq X'$ are analytically thin, and let $\bar{\varphi}: (X|E_s)^\wedge\to(X'|E'_s)^\wedge$ be an $\hat{S}$-morphism. Then, for any given natural number $n\in \boldsymbol{N}$, there exists an S-morphism $\varphi: X|E_s\to X'|E'_s$ such that $\varphi_{S_n}=\bar{\varphi}_{S_n}$.*

[1] By a *strictly pseudoconvex compact set* in X we mean a subset of X, which is the inverse image of a Stein compact set in the Remmert quotient of X.

Sketch of proof. We may assume E_s and E'_s being connected. Let $f: X \to Y$ resp. $f': X' \to Y'$ be the Remmert quotient of X resp. X' over S, and put $F := f(E)$ and $F' := f'(E')$. Then there exists a unique $\hat{S}$-morphism $\bar{\psi}$ such that the diagram

$$\begin{array}{ccc} X \hat{|} E_s & \xrightarrow{(f|(E_s, F_s))^\wedge} & Y \hat{|} F_s \\ \bar{\varphi} \downarrow & & \downarrow \bar{\psi} \\ X' \hat{|} E'_s & \xrightarrow[(f'|(E'_s, F'_s))^\wedge]{} & Y' \hat{|} F'_s \end{array}$$

commutes. Since E_s resp. E'_s is connected, $F_s = \{y\}$ resp. $F'_s = \{y'\}$ consists of exactly one point. By [2] the map f resp. f' is relatively algebraic after shrinking Y resp. Y' as a neighborhood of y resp. y'. Now one can accomplish the proof using Grothendieck's existence theorem ([9] (III 5.4.1), [11], Chap. V, Theorem 6.3), Artin's approximation theorem [3] and the relative GAGA theory.

Corollary 3.3. *Let X_0 be a strictly pseudoconvex complex space such that the exceptional subvariety E_0 of X_0 is analytically thin in X_0, and suppose $X \to S$, $X' \to S$ are two deformations of (X, E_0) over a space germ (S, s). Further let $\bar{\varphi} : (X|E_0)^\wedge \to (X'|E_0)^\wedge$ be an isomorphism of formal deformations and $n \in N$ a natural number. Then there exists an isomorphism $\varphi : X \to X'$ of deformations of (X_0, E_0) such that $\varphi_{(S_n, s)} = \bar{\varphi}_{(S_n, s)}$.*

Remark 3.4. I conjecture that the thinness assumptions in 3.2 and 3.3 are superfluous.

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Added in proof. V.P. Palamodov has informed me that he raised an assertion near to 2.3 as a problem in his joint seminar with B. Moishezon on Complex spaces in 1976/77.