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Pseudo Runge-Kutta Processes

By

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§ 0. Introduction

In this paper we shall study numerical methods for ordinary differential equations of the initial value problem:

$$(0.1)\qquad \begin{cases} y'=f(x, y) \\ y(x_0)=y_0 \end{cases}.$$

We shall assume that the function $f(x, y)$ satisfies the following conditions:

(A) $f(x, y)$ is defined and continuous in the strip

$$\Omega=\{(x, y);\ a\leq x\leq b, |y|<\infty\}.$$

(B) There exists a constant L such that for any x with $a\leq x\leq b$ and any two numbers y_1 and y_2

$$|f(x, y_1)-f(x, y_2)|\leq L|y_1-y_2|.$$

Under the assumptions (A) and (B), we know that the initial value problem (0. 1) has a unique solution. In addition to the first order scalar equation (0. 1), it is possible to consider a system of equations or an equivalent high order single equation. In this paper we consider only (0. 1) because the numerical formulas for such a system are almost similar to those of the scalar equation (0. 1).

Consider the sequence of points x_n defined by $x_n=x_0+nh$, $n=1, 2, \cdots$. The parameter h, which will always be regarded as a constant, is called the step length. Let y_n be an approximation to the theoretical solution $y(x_n)$ at x_n and we set $f_n=f(x_n, y_n)$.

The general r-stage Runge-Kutta method is defined by

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$$
\begin{aligned}
(0.2) \qquad & y_{n+1}=y_n+h\Phi(x_n,y_n;h),\\
& \Phi(x_n,y_n;h)=\sum_{i=1}^{r}w_ik_i,\\
& k_1=f(x_n,y_n),\\
& k_i=f(x_n+a_ih,y_n+h\sum_{j=1}^{r-1}b_{ij}k_j),\\
& a_i=\sum_{j=1}^{i-1}b_{ij}, \quad (i=2,3,\cdots,p).
\end{aligned}
$$

This method was first proposed by Runge [20] and subsequently developed by Heun [12] and Kutta [13]. According to the choice of the stage number r and the parameters a_i, b_{ij}, w_i, we shall derive various methods. In [1], [2], [3] and [4]. Butcher has proved the following results for Runge-Kutta method (0. 2):

$$
\begin{aligned}
& p_1(r)=r \quad (r=1,2,3,4),\\
& p_1(5)=4,\\
& p_1(6)=5,\\
& p_1(7)=6,\\
& p_1(8)=6,\\
& p_1(9)=7,\\
& p_1(10)=7,\\
& p_1(11)=8,\\
& p_1(r)=r-2 \quad (12\leq r),
\end{aligned}
$$

where $p_1(r)$ denotes the highest order that can be attained by the r-stage method (0. 2) above.

As we can see, the r-stage Runge-Kutta method (0. 2) requires r functional evaluations per step. We shall look for other Runge-Kutta type methods which have the same order as (0. 2), but which requires fewer functional evaluations than (0. 2). Such methods have been discussed by Rosen [19], Ceschino, Kuntzmann [7] and many others. For instance, Byrne and Lambert [6] have defined the following two-step Runge-Kutta method:

$$
(0.3) \qquad y_{n+1}=y_n+h\sum_{j=1}^{r}w_{0j}k_j+h\sum_{j=1}^{r}w_{1j}k_j,
$$

where

$$
\begin{aligned}
& k_{01}=f(x_{n-1},y_{n-1}),\\
& k_{0j}=f(x_{n-1}+a_jh,\ y_{n-1}+h\sum_{l=0}^{j-1}b_{jl}k_{0l}) \quad (j=2,3,\cdots,r),
\end{aligned}
$$

$$k_{11}=f(x_n, y_n),$$
$$k_{1j}=f(x_n+a_jh,\ y_n+h\sum_{l=0}^{j-1}b_{jl}k_{1l})\quad (j=2, 3, \cdots, r),$$
$$a_i=\sum_{j=1}^{i-1}b_{ij},\quad (i=2, 3, \cdots, r).$$

In [6], [7] it is shown that the two-step Runge-Kutta method (0.3) has order

$$p_1(r)=r+1\quad (r=2, 3, 4).$$

Costabile [8] has also proposed the following pseudo-Runge-Kutta method:

$$(0.4)\qquad \begin{aligned} y_{n+1}&=y_n+h\sum_{i=0}^{r}w_ik_i,\\ k_0&=f(x_{n-1}, y_{n-1}),\\ k_1&=f(x_n, y_n),\\ k_i&=f(x_n+a_ih, y_n+h\sum_{j=0}^{i}b_{ij}k_j). \end{aligned}$$

This method has the same order as that of the two-step Runge-Kutta method (0.3) in the same stage. Therefore, the two-step Runge-Kutta method (0.3) and the pseudo-Runge-Kutta method (0.4) require fewer functional evaluations than Runge-Kutta method (0.2). However computational experiments indicate that the local accuracy of the two methods (0.3) and (0.4) is frequently inferior to that of the Runge-Kutta method (0.2).

To improve this defect, we [14], [15] have proposed another pseudo-Runge-Kutta method. Our method is defined by

$$(0.5)\qquad \begin{aligned} y_{n+1}&=v_1y_{n-1}+v_2y_n+h\Phi(x_{n-1}, x_n, y_{n-1}, y_n; h),\\ \Phi(x_{n-1}, x_n, y_{n-1}, y_n;\ h)&=h\sum_{i=0}^{r}w_ik_i,\\ k_0&=f(x_{n-1}, y_{n-1}),\quad k_1=f(x_n, y_n),\\ k_i&=f(x_n+a_ih,\ y_n+b_i(y_n-y_{n-1})+h\sum_{j=0}^{i-1}b_{ij}k_j),\\ a_i&=b_i+\sum_{j=0}^{i-1}b_{ij},\quad 0<a_i\le 1,\quad (i=2, 3, \cdots, r). \end{aligned}$$

The method (0.5) has the following order:

$$p_1(r)=r+2\quad (r=2, 3, 4).$$

In comparing our method and other three methods (0.2), (0.3) and (0.4) in the same order, our method requires fewer functional evaluations than the other methods and has almost the same accuracy as Runge-Kutta method (0. 2). Therefore our method is more economical than the other three methods.

The methods discussed above are all explicit. The main advantage of explicit methods is that they give numerical solutions explicitly at each step. However, they are inadequate for the stiff problem [10]. A drawback of the classical Runge-Kutta method (0, 2) for the stiff problem can be overcome by introducing stable methods.

J. C. Butcher [1], [3] was the first who considered an implicit Runge-Kutta method, which is A-stable. The general r-stage implicit Runge-Kutta method is defined by:

$$
\begin{aligned}
(0.6) \qquad & y_{n+1}=y_n+h\Phi(x_n,y_n;h),\\
& \Phi(x_n,y_n;h)=\sum_{i=1}^{r} w_i k_i,\\
& k_i=f(x_n+a_i h,\ y_n+h\sum_{i=2}^{r} b_{ij}k_j),\\
& a_i=\sum_{j=1}^{r} b_{ij}\,(i=2,3,4,\cdots,r).
\end{aligned}
$$

It is possible to consider some other implicit pseudo-Runge-Kutta methods in a way similar to that of the implicit Runge-Kutta method (0. 6). Our r-stage implicit pseudo-Runge-Kutta method is:

$$
\begin{aligned}
(0.7) \qquad & y_{n+1}=v_1 y_{n-1}+v_2 y_n+h\Phi(x_{n-1},x_n,y_{n-1},y_n;h),\\
& \Phi(x_{n-1},x_n,y_{n-1},y_n;h)=\sum_{i=0}^{r} w_i k_i,\\
& k_0=f(x_{n-1},y_{n-1}),\ k_1=f(x_n,y_n),\\
& k_i=f(x_n+a_i h, y_n+b_i(y_n-y_{n-1})+h\sum_{j=0}^{r} b_{ij}k_j),\\
& a_i=b_i+\sum_{j=0}^{r} b_{ij} \quad (i=2,3,\cdots,r).
\end{aligned}
$$

It will be seen that it is equivalent to certain implicit Runge-Kutta methods in some special cases. For example, if we take $r=2$, $v_1=w_0=w_1=b_2=b_{20}=b_{21}=0$, $v_2=1$, $w_2=0.5$, $a_2=0.5$, $b_{22}=0.5$ in (0. 7), then the method (0. 7) becomes the second order Gauss-type implicit method, and if $r=3$, $v_2=1$, $v_1=w_0=0$, $w_2=w_3=0.5$, $a_2=0.5-\sqrt{3}/6$, $a_3=0.5-\sqrt{3}/6$, $b_2=b_{20}=0$, $b_{21}=0.25$, $b_{22}=0.25-\sqrt{3}/6$, $b_3=b_{30}=0$, $b_{31}=0.25-\sqrt{3}/6$ and $b_{32}=0.25$, then the method (0. 7) becomes an implicit Runge-Kutta method of order 4. In contrast with explicit methods, implicit methods increase their attainable orders. In [3], Butcher proved for his method (0.6), which is A-stable, the following result:

$$p_1(r)=2(r-1) \quad (r=2,3,4,\cdots).$$

Whereas our method (0.7) has

$$p_1(r)=r+3 \quad (r=2,3).$$

As we have mentioned a little about the stiff problem, it is important to derive A-stable methods. In this paper we derive A-stable methods for our method (0.7) when $r=2$ and 3. In these cases, we get

$$p_2(2)=2,$$
$$p_2(3)=4,$$

where $p_2(r)$ is the order that can be attained by an r-stage A-stable method.

We now outline the organization of this paper. In section 1, we discuss the attainable order with 2-, 3- and 4-stage methods (0.5). In section 2, we discuss the attainable order with 2 and 3 stage implicit methods (0.7). In section 3, we analyze a stability of the implicit method (0.7). Finally, in section 4, we give some numerical examples, which show some useful properties of our method. We have not discussed the selection of parameters appearing in the method (0.7) and other related results. These results will appear soon.

§ 1. Explicit Method

§ 1.1 Non-Existence of Order 5 with $r=2$

To investigate the attainable order of pseudo-Runge-Kutta method (0.5), we are required to expand the formula (0.5) as its Taylor series about the point (x_n, y_n), where we assume that $y(x_n)=y_n$. Details can be found in [18], so shall be omitted.

We check whether it is possible for the method (0.5) to have order 5 with 2-stage, by investigating order conditions in two cases when w_2 is equal to zero and otherwise.

Case 1.1: $w_2\neq 0$. We need the following order conditions.

$$\frac{(-1)^j v_1}{j}+\sum_{i=0}^{2} a_i^{j-1} w_i=\frac{1}{j} \quad (j=1,2,\cdots,5), \tag{1.1}$$

$$a_2^j=(-1)^{j+1}\{b_0+jb_{20}\} \quad (j=2,3,4), \tag{1.2}$$

with $a_0=-1$, $a_1=0$.

The equation (1.2) implies that

$$a_2(a_2+1)^2=0.$$

So we have

$$a_2=-1, 0. \tag{1.3}$$

On the other hand, for $a_2=-1, 0$ the equation (1.1) has no solution. Indeed, if we define

$$D_1=\begin{pmatrix} \frac{1}{2} & -1 & a_2 \\ -\frac{1}{3} & 1 & a_2^2 \\ \frac{1}{4} & -1 & a_2^3 \end{pmatrix},\quad U_1=\begin{pmatrix} v_1 \\ w_0 \\ w_2 \end{pmatrix} \text{ and } V_1=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{pmatrix},$$

$$\tilde{D}_1=(D_1, V_1),$$

then (1.1) can be expressed as

$$D_1U_1=V_1.$$

In the case (1.3), we have

$$\mathrm{Rank}(D_1)=2,\ \mathrm{Rank}(\tilde{D}_1)=3.$$

Thus the equations (1.1) and (1.2) have no solution.

Case 1.2: $w_2=0$.

Put

$$D_2=\begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{3} & 1 \\ \frac{1}{4} & -1 \end{pmatrix},\quad U_2=\begin{pmatrix} v_1 \\ w_0 \\ 0 \end{pmatrix} \text{ and } V_2=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{pmatrix}.$$

$$\tilde{D}_2=(D_2, V_2).$$

Then we have

$$\mathrm{Rank}(D_2)=2 \text{ and } \mathrm{Rank}(\tilde{D}_2)=3.$$

Thus the equations (1.1) and (1.2) have no solution.

§ 1.2. Non-Existence of Order 6 with r=3

By going through the same procedure as in the case when $r=2$, we check whether it is possible for the method (0.5) to have order 6 with 3-stage. We consider two cases (A): $a_2 \neq a_3$ and (B) $a_2 = a_3$.

(A): $a_2 \neq a_3$.

So we consider order conditions in the following four cases.

Case 2.1: $w_2 \neq 0$, $w_3 \neq 0$.

We need the following order conditions:

$$(1.4)\qquad \frac{(-1)^j v_1}{j} + \sum_{i=0}^{3} a_i^{j-1} w_i = \frac{1}{j} \quad (j=1, 2, \cdots, 6),$$

$$(1.5)\qquad a_2^j = (-1)^{j+1}\{b_2 + j b_{20}\},$$

$$(1.6)\qquad a_3^j = (-1)^{j+1} b_3 + j\{(-1)^{j+1} b_{30} + a_2^{j-1} b_{32}\} \qquad (j=2, 3, 4).$$

The equations (1.5) and (1.6) imply that

$$(1.7)\qquad a_2 = -1, 0 \text{ and } a_3 = -1, 0.$$

If we define

$$D_3(a_2, a_3) = \begin{pmatrix} \frac{1}{2} & -1 & a_2 a_3 \\ -\frac{1}{3} & 1 & a_2^2 a_3^2 \\ \frac{1}{4} & -1 & a_2^3 a_3^3 \\ -\frac{1}{5} & 1 & a_2^4 a_3^4 \\ \frac{1}{6} & -1 & a_2^5 a_3^5 \end{pmatrix}, \quad U_3 = \begin{pmatrix} v_1 \\ w_0 \\ w_2 \\ w_3 \end{pmatrix} \text{ and } V_3 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix}.$$

$$\tilde{D}_3 = (D_3(a_2, a_3), V_3),$$

then the equation (1.4) can be expresses as

$$D_3(a_2, a_3) U_3 = V_3.$$

In the case (1.7), we have

$$\text{Rank}(D_3(a_2, a_3)) = 2 \text{ and } \text{Rank}(\tilde{D}_3) = 3.$$

Thus the equations (1.4), (1.5) and (1.6) have no solution.

Case 2.2: $w_2 \neq 0$, $w_3 = 0$.

In this case, we need the condition (1. 4) with $w_3=0$. Define

$$D_4(a_2)=\begin{pmatrix} \frac{1}{2} & -1 & a_2 \\ -\frac{1}{3} & 1 & a_2^2 \\ \frac{1}{4} & -1 & a_2^3 \\ -\frac{1}{5} & -1 & a_2^4 \end{pmatrix},\ D_5(a_2)=\begin{pmatrix} -\frac{1}{3} & 1 & a_2^2 \\ \frac{1}{4} & -1 & a_2^3 \\ -\frac{1}{5} & 1 & a_2^4 \\ \frac{1}{6} & -1 & a_2^5 \end{pmatrix},$$

$$V_4=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \end{pmatrix},\ V_5=\begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix},\ U_4=\begin{pmatrix} v_1 \\ w_0 \\ w_2 \\ -1 \end{pmatrix}.$$

$$\tilde{D}_4(a_2)=(D_4(a_2),\ V_4),\ \text{and}\ \tilde{D}_5(a_2)=(D_5(a_2),\ V_5).$$

Then we need the conditions $\det(\tilde{D}_4(a_2))=\det(\tilde{D}_5(a_2))=0$. Since

$$\det(\tilde{D}_4(a_2))=-\frac{1}{30}a_2(a_2+1)(5a_2^2-a_2-2)=0,$$

$$\det(\tilde{D}_5(a_2))=-\frac{1}{90}a_2^2(a_2+1)(6a_2^2-a_2-3)=0,$$

we have

$$a_2=-1,0.$$

However, for $a_2=-1,0$ we have

$$\mathrm{Rank}(D_4(a_2))=2\ \text{and}\ \mathrm{Rank}(\tilde{D}_4(a_2))=3.$$

Thus the equation (1. 4) has no solution.

Case 2. 3: $w_2=0$, $w_3\neq 0$. In this case, we need the condition (1. 4) with $w_2=0$. Then we have

$$\det(\tilde{D}_4(a_3))=0,\ \text{and}\ \det\ ((\tilde{D}_5(a_3))=0,$$

which imply that $a_3=-1,0$, thus we can prove the non-existence of solutions of (1. 4) in a way similar to that of Case 2. 2.

Case 2. 4: $w_2=w_3=0$. In this case, as we have seen in Case 1. 2, the equation (1. 4) has no solution. Now let us go on to

(B) : $a_2=a_3$.

In this case, we need the order condition (1. 4) with $a_2=a_3$. If we define

$$U_5=\begin{pmatrix} v_1 \\ w_0 \\ w_2+w_3 \\ -1 \end{pmatrix},$$

then the equation (1. 4) can be expressed as

$$\tilde{D}_4(a_2)U_5=0 \text{ and } \tilde{D}_5(a_2)U_5=0.$$

Therefore we get $a_2=-1, 0$. As we have seen in Case 2. 2, the equation (1. 4) has no solution.

§ 1. 3. Non-Existence of Order 7 with r=4

Proceeding as before, we check whether it is possible for the method (0. 5) to have order 7 with 4-stage. We consider the two cases (A) : $a_i \neq a_j (i \neq j)$ and (B) : $a_i=a_j (i \neq j)$.

(A) : $a_i \neq a_j$ $(i \neq j)$.

Let us consider the order conditions in the following eight cases.

Case 3. 1 : $w_2 \neq 0$, $w_3 \neq 0$ and $w_4 \neq 0$. In this case, we need the following order conditions:

(1. 8) $$\frac{(-1)^j v_1}{j}+\sum_{i=0}^{4} a_i^{j-1} w_i=\frac{1}{j} \quad (j=1, 2, \cdots, 7),$$

(1. 9) $$a_2^j=(-1)^{j+1}\{b_2+jb_{20}\},$$

(1. 10) $$a_3^j=(-1)^{j+1}b_3+j\{(-1)^{j+1}b_{30}+a_2^{j-1}b_{32}\},$$

(1. 11) $$a_4^j=(-1)^{j+1}b_4+j\{(-1)^{j+1}b_{40}+a_2^{j-1}b_{42}+a_3^{j-1}b_{43}\} \ (j=2, 3, 4).$$

The equations (1. 9), (1. 10) and (1. 11) imply that

(1. 12) $$a_2=-1, 0, \ a_3=-1, 0 \text{ and } a_4=-1, 0.$$

Define

$$D_6=\begin{pmatrix} \frac{1}{2} & -1 & a_2a_3a_4 \\ -\frac{1}{3} & 1 & a_2^2a_3^2a_4^2 \\ \frac{1}{4} & -1 & a_2^3a_3^3a_4^3 \\ -\frac{1}{5} & 1 & a_2^4a_3^4a_4^4 \\ \frac{1}{6} & -1 & a_2^5a_3^5a_4^5 \end{pmatrix},\quad U_6=\begin{pmatrix} v_1 \\ w_0 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \text{ and } V_6=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix}.$$

Then the equation (1. 8) can be expressed as

$$D_6U_6=V_6 .$$

In the case (1. 12), we have

$$\operatorname{Rank}(D_6)=2,\ \operatorname{Rank}(\tilde{D}_6)=3,$$

where $\tilde{D}_6=(D_6,\ V_6)$.

Therefore, the equations (1. 8), (1. 9), (1. 10) and (1. 11) have no solution.

Case 3. 2: $w_2=0$, $w_3\neq 0$ and $w_4\neq 0$. In the case $b_{32}w_3+b_{42}w_4\neq 0$, we need the same conditions as that of case 3. 1. And in the case $b_{32}w_3+b_{42}w_4=0$, we need the conditions (1. 8), (1. 10) and (1. 11) with $b_{32}=b_{42}=0$, which imply $a_3=-1, 0$ and $a_4=-1, 0$. Thus we can prove the non-existence of the solution of (1. 8) in a way similar to that of Case 2. 2.

Case 3. 3: $w_2\neq 0$, $w_3=0$ and $w_4\neq 0$. In this case, we need the following condition, in addition to (1. 8):

(1. 13) $$a_2^i=(-1)^{i+1}\{b_2+ib_{20}\}\quad (i=2, 3, 4, 5).$$

The equation (1. 13) implies that $a_2=0$. Therefore, as we have considered in Case 2. 2, there are no solutions satisfying (1. 8).

Case 3. 4: $w_2=0$, $w_3=0$ and $w_4\neq 0$. In this case, we need the same order conditions as that of Case 2. 2. Therefore we can prove the non-existence of solutions of (1. 8).

Case 3. 5: $w_2\neq 0$, $w_3\neq 0$ and $w_4=0$. In this case, we need the same conditions as that of the case $r=3$ (see Section 1. 2). Thus we cannot get order 7.

Case 3. 6: $w_2=0$, $w_3\neq 0$ and $w_4=0$. This is equivalent to Case 2. 2.

Case 3. 7: $w_2\neq 0$, $w_3=0$ and $w_4=0$. This is equivalent to Case

2. 2.

Case 3. 8: $w_2=0$, $w_3=0$ and $w_4=0$. In this case, as we have considered in Case 1. 2, the equation (1. 8) has no solution. Now let us go on to

(B): $a_i=a_j \quad (i\neq j)$.

Let us consider the order conditions in the six cases.

Case 3. 9: $b_{32}b_{43}w_4\neq 0$. In this case, we need the conditions (1. 8) and (1. 9) without assuming $a_i=a_j(i\neq j)$. Therefore we can prove the non-existence of solutions of (1. 8) and (1. 9) in a way similar to that of Case 2. 2.

Case 3. 10: $w_4=0$. This is equivalent to Case 3. 6.

Case 3. 11: $b_{43}=0, b_{32}w_3+b_{42}w_4\neq 0$. In this case, we need the condition (1. 13) without assuming $a_i=a_j(i\neq j)$, and we get $a_2=0$. Therefore we can prove the non-existence of solutions of (1. 8) in a way similar to that of Case 2. 2.

Case 3. 12: $b_{43}=0, b_{32}w_3+b_{42}w_4=0$. In this case, if we assume $a_2=a_3$, then we need the condition (1. 11) with $b_{42}=b_{43}=0$, which implies $a_4=-1, 0$. If we assume $a_2=a_3$, then we need the condition (1. 10) with $b_{32}=0$ which implies $a_3=-1, 0$, and if we assume $a_3=a_4$, then we need the condition (1. 9) which implies $a_2=-1. 0$. Therefore we can prove the non-existence of solutions of (1. 8) in a way similar to that of Case 2. 2.

Case 3. 13: $b_{32}=0$. In this case, if we assume $a_2=a_3$, then we need the condition (1. 11) with $b_{42}=b_{43}=0$, $j=3, 4, 5$, which implies $a_4=-1$, 0. If we assume $a_2=a_4$, then we need the condition (1. 10) with $b_{32}=0$ which implies $a_3=-1, 0$. If we assume $a_3=a_4$, then we need the condition (1. 9) with which implies $a_2=-1, 0$. Therefore we can prove the non-existence of solutions of (1. 8) in a way similar to that of Case 2. 2.

Case 3. 14: $a_2=a_3=a_4$. In this case, we can prove the non-existence of solutions (1. 8) in a way similar to that of Case 2. 2. Consequently we can conclude the following result.

Theorem. *The attainable order is* 4, 5 *and* 6 *for the explicit pseudo-Runge-Kutta method* (0. 5) *of* 2, 3 *and* 4 *stage respectively.*

§ 2. Implicit Method

In this section, we consider the r-stage implicit pseudo-Runge-Kutta method (0.7). The functions k_i $(i=2, 3, \cdots, r)$ are no longer defined explicitly but by the set of $r-1$ implicit equations. We assume that the solution of each of $r-1$ implicit equations. We assume that the solution of each function k_i may be expressed in the form:

$$k_i=f(x_n, y_n)+\sum_{j=1}^{p} c_j h^j+0(h^{p+1}) \qquad (i=2, 3, \cdots, r),$$

and

$$y_n=y(x_n).$$

§ 2.1. Non-Existence of Order 6 with r=2

In [16], we have already proved the existence of order 5 with 2-stage. We now check whether there exists order 6 with 2-stage, by investigating the order conditions in the two cases when w_2 is equal zero and otherwise.

Case I: $w_2\neq 0$. We need the following order conditions:

$$\frac{(-1)^i v_1}{i}+\sum_{j=0}^{2} a_j^{i-1} w_j=\frac{1}{i} \qquad (i=1, 2, \cdots, 6). \tag{2.1}$$

Let us define

$$D_7=\begin{pmatrix} \frac{1}{2} & -1 & a_2 \\ -\frac{1}{3} & 1 & a_2^2 \\ \frac{1}{4} & -1 & a_2^3 \end{pmatrix}, \quad U_7=\begin{pmatrix} v_1 \\ w_0 \\ w_2 \end{pmatrix} \text{ and } V_7=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{pmatrix},$$

$$D_8=\begin{pmatrix} \frac{1}{2} & -1 & a_2 & \frac{1}{2} \\ -\frac{1}{3} & 1 & a_2^2 & \frac{1}{3} \\ \frac{1}{4} & -1 & a_2^3 & \frac{1}{4} \\ -\frac{1}{5} & 1 & a_2^4 & \frac{1}{5} \end{pmatrix}, \quad D_9=\begin{pmatrix} -\frac{1}{3} & 1 & a_2^2 & \frac{1}{3} \\ \frac{1}{4} & -1 & a_2^3 & \frac{1}{4} \\ -\frac{1}{5} & 1 & a_2^4 & \frac{1}{5} \\ \frac{1}{6} & -1 & a_2^5 & \frac{1}{6} \end{pmatrix}, \quad U_8=\begin{pmatrix} v_1 \\ w_0 \\ w_2 \\ -1 \end{pmatrix},$$

$$\tilde{D}_7 = (D_7, V_7).$$

Then the equation (2. 1) can be expressed as

$$D_7 U_7 = V_7,\ D_8 U_8 = 0 \text{ and } D_9 U_8 = 0.$$

And we have

$$\det(D_8) = -\frac{1}{30} a_2 (a_2+1)(5a_2^2 - a_2 - 2), \tag{2. 2}$$

$$\det(D_9) = -\frac{1}{90} a_2^2 (a_2+1)(6a_2^2 - a_2 - 3).$$

Therefore, in order to have the solution of (2. 1) we have

$$a_2 = -1, 0.$$

However, for $a_2 = -1, 0$ we have

$$\mathrm{Rank}(D_7) = 2, \text{ and } \mathrm{Rank}(\tilde{D}_7) = 3.$$

Thus the equation (2. 1) has no solution.

Case II: $w_2 = 0$. In this case, we need the condition (2. 1) with $w_2 = 0$, as we have seen in Case 1. 1, the equation (2. 1) has no solution.

§ 2. 2. Non-Existence of Order 7 with r=3

In [15], we have already proved the existence of order 6 with 3-stage. Proceeding as before, we check whether there exists order 7 with 3-stage, by investigating the order conditions on the two cases when w_3 is equal zero and otherwise.

Case I: $w_3 \neq 0$. In this case, we need the following order conditions:

$$\frac{(-1)^j v_1}{j} + \sum_{i=0}^{3} a_i^{j-1} w_i = \frac{1}{j} \quad (j = 1, 2, \cdots, 7), \tag{2. 3}$$

$$\frac{(-1)^j v_1}{j} + \sum_{i=0}^{3} q_{ji} w_i = \frac{1}{j} \quad (j = 3, 4, \cdots, 7), \tag{2. 4}$$

with

$$q_{ij} = (-1)^{i+1} b_j + i\{(-1)^{i+1} b_{2j} + a_2^{i-1} b_{2j} + a_3^{i-1} b_{3j}\} \quad (i = 2, 3, \cdots, 7,\ j = 2, 3),$$
$$q_{i0} = (-1)^{i-1},\ q_{i1} = 0.$$

Define

$$D_{10}=\begin{pmatrix} \frac{1}{2} & -1 & a_2 & a_3 \\ -\frac{1}{3} & 1 & a_2^2 & a_3^2 \\ \frac{1}{4} & -1 & a_2^3 & a_3^3 \\ -\frac{1}{5} & 1 & a_2^4 & a_3^4 \\ \frac{1}{6} & -1 & a_2^5 & a_3^5 \end{pmatrix},\quad D_{11}=\begin{pmatrix} -1 & -2 & 2a_2 & 2a_3 & \frac{1}{3}+\frac{v_1}{3}-w_0 \\ 1 & 3 & 3a_2^2 & 3a_3^2 & \frac{1}{4}-\frac{v_1}{4}+w_0 \\ -1 & -4 & 4a_2^3 & 4a_3^3 & \frac{1}{5}+\frac{v_1}{5}-w_0 \\ 1 & 5 & 5a_2^4 & 5a_3^4 & \frac{1}{6}-\frac{v_1}{6}+w_0 \\ -1 & -6 & 6a_2^5 & 6a_3^5 & \frac{1}{7}+\frac{v_1}{7}-w_0 \end{pmatrix}$$

$$D_{12}=\begin{pmatrix} -\frac{1}{3} & 1 & a_2^2 & a_3^2 \\ \frac{1}{4} & -1 & a_2^3 & a_3^3 \\ -\frac{1}{5} & 1 & a_2^4 & a_3^4 \\ \frac{1}{6} & -1 & a_2^5 & a_3^5 \\ -\frac{1}{7} & 1 & a_2^6 & a_3^6 \end{pmatrix},\quad U_{11}=\begin{pmatrix} b_2w_2+b_3w_3 \\ b_{20}w_2+b_{30}w_3 \\ b_{22}w_2+b_{32}w_3 \\ b_{23}w_2+b_{33}w_3 \\ -1 \end{pmatrix},\quad U_{10}=\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix},\quad U_{12}=\begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \\ \frac{1}{7} \end{pmatrix},$$

$$U^t=(v_1,\ w_0,\ w_2,\ w_3,\ -1),\quad \tilde{D}_{10}=(D_{10},\ U_{10}),\quad \tilde{D}_{12}=(D_{12},\ U_{12}).$$

Then the equations (2. 3) and (2. 4) can be expressed as

$$\tilde{D}_{10}U=0,\quad D_{11}U_{11}=0 \text{ and } \tilde{D}_{12}U=0.$$

Therefore, in order to have the solutions of (2. 3) and (2. 4) we need $\det(\tilde{D}_{10})=0$, $\det(D_{11})=0$ and $\det(\tilde{D}_{12})=0$. And we have

$$\det(\tilde{D}_{10})=-\frac{1}{90}a_2a_3(a_2+1)(a_3+1)(a_2-a_3)g_1(a_2,a_3),$$

$$\det(D_{11})=-a_2^2a_3^2(a_2+1)(a_3+1)(a_2-a_3)\det(D_{13}),$$

$$\det(\tilde{D}_{12})=-\frac{1}{630}a_2^2a_3^2(a_2+1)(a_3+1)(a_2-a_3)g_2(a_2,a_3),$$

where

(2. 5) $$g_1(a_2,a_3)=15a_2^2a_3^2-3a_2a_3(a_2+a_3)-6(a_2+a_3)^2+14a_2a_3+a_2+a_3+3.$$

(2. 6) $$g_2(a_2,a_3)=42a_2^2a_3^2-7a_2a_3(a_2+a_3)-21(a_2^2+a_3^2)+4a_2a_3 +3(a_2+a_3)+12,$$

$$D_{13}=\begin{pmatrix} 1 & 0 & 3 & \dfrac{7+v_1}{12} \\ -0.4 & 4 & 0.8 & 0.8 \\ 1 & 5(a_2+a_3-1) & -1-5a_2a_3 & -\dfrac{2}{3} \\ -\dfrac{12}{7} & \begin{matrix} 6(a_2^2+a_2a_3+a_3^2 \\ -(a_2+a_3)+1) \end{matrix} & -6a_2a_3(a_2+a_3-1)+\dfrac{6}{7} & \dfrac{8}{7} \end{pmatrix}.$$

Then, from (2.5) and (2.6) we have

(2.7) $\qquad 21y^2-7xy+6y+x-3=0,$

(2.8) $\qquad y=\dfrac{21x^2-x-18}{7x+34},$

with $x=a_2+a_3$, $y=a_2a_3$. And from (2.3) we have

(2.9) $\qquad v_1=\dfrac{35x-50y-27}{10y+5x+3}.$

Let us define

(2.10) $\qquad T(x,y)=\det(D_{13}).$

Then using (2.8) and (2.9), the equation (2.10) can be expressed as

(2.11) $\qquad T(x,y)=\dfrac{g_3(x)}{35(7x+34)^3},$

with

(2.12) $\qquad g_3(x)=594247500x^6+500320380x^5-639473520x^4$
$\qquad\qquad -186267480x^3+394247760x^2-1548000x-32500800.$

From (2.5) and (2.8) we have

(2.13) $\qquad g_1(x)=41160x^5+175420x^4-169820x^3-227800x^2$
$\qquad\qquad +90800x-8160.$

Now we investigate the algebraic character of (2.12) and (2.13) by computing Sylvester's determinant $D(g_1(x), g_3(x))$;

$$\mathrm{Det}(g_1(x), g_3(x))=-31118006978871253249 08...........$$

Therefore the equations (2.12) and (2.13) have no common roots. Thus the common roots of the equations (2.5), (2.6) and (2.10) are

(2.14) $\qquad a_2=-1, 0,\ a_3=-1, 0$ and $a_2=a_3.$

However in the case (2.14),

$$\mathrm{Rank}(D_{10})=2,\ \mathrm{Rank}(\tilde{D}_{10})=3.$$

Thus, the equations (2. 3) and (2. 4) have no solution.

Case II: $w_3=0$. In this case, we need the order conditions (2. 3) with $w_3=0$. Therefore, as we have seen in §2. 1, the equation (2. 3) has no solution. Thus, we can conclude the following results.

Theorem. *The attainable order is* 5 *and* 6 *for the implicit pseudo-Runge-Kutta method* (0. 7) *of* 2 *and* 3 *stage respectively.*

§ 3. Stability Analysis

In this section we attempt to derive A-stable methods. We define the stability of the numerical method (0. 7) in the following way. Let us apply the method (0. 7) to the test function $y'=\lambda y$ where λ is a complex constant with negative real, we have the following difference equation:

$$A_0 y_{n+1} - A_1 y_n - A_2 y_{n-1} = 0, \tag{3. 1}$$

where A_0, A_1 and A_2 are the function of λh, involving the coefficients a_i, b_i, b_{ij}, v_i, w_i. The numerical method (0. 7) is called A-stable if

(1) $|\gamma_i| \leq 1 \quad i=1, 2.$

(2) the root $|\gamma_i|=1$ is simple.

where γ are the characteristic roots of the difference equation (3. 1). If the method satisfies the conditions (1) and (2) for any negative real λ, it is said to be A_0-stable. If we impose the A-stability on the method (0. 7), we can obtain the highest orders of the A-stability for each stage r, using some results due to Wanner, Hairer and Norsett [23].

Proposition. *In the method* (0. 7) *with A-stability imposed on, the highest possible order of the A-stability is of order* $2(r-1)$ *for each stage* r.

Proof. The proof goes as follows (see [23, §6]):

Let

$$Q(z, R) = Q_0(z) R^k + Q_1(z) R^{k-1} + Q_2(z) R^{k-2} + \cdots + Q_k(z),$$

be a characteristic algebraic equation of multistep method, satisfying the following conditions;

(a) $Q(z, R)$ is irreducible, $Q_0(0)\neq 0$, $\frac{\partial Q}{\partial R}(0,1)\neq 0$,

(b) $\deg(Q_r)\leq J$, $(r=1,2,\cdots,k)$. Then G. Wanner, E. Hairer and S. P. Norsett [23, Theorem 2] showed that if $Q(z, R)$ is A-stable then the highest possible order is $2J$. In our case, as we will see later in Section 4. 1 and 4. 2, the characteristic equation of (0. 7) satisfies the conditions (a) and (b), thus we get the above proposition.

§ 3. 1. A-stable Method of Order 2 with 2-Stage

We apply the method (0. 7) with $r=2$ to the test function $y'=\lambda y$. This yields

$$y_{n+1}=A_1y_n+A_2y_{n-1}, \tag{3. 2}$$

where

$$A_1=\frac{1}{1-b_{22}\bar{h}}\{v_2+(w_1-b_{22}v_2+(1+b_2)w_2)\bar{h}+(-b_{22}w_1+b_{21}w_2)(\bar{h}^2)\},$$
$$A_2=\frac{1}{1-b_{22}\bar{h}}\{v_1+(w_0-b_{22}v_1-b_2w_2)\bar{h}+(-b_{22}w_0+b_{20}w_2)\bar{h}^2\},\quad (\bar{h}=\lambda h).$$

We see that the method (0. 7) is of order 2 if

$$(-1)^i\frac{v_1}{i}+\sum_{j=0}^{2}a_j^{i-1}w_j=\frac{1}{i}\quad (i=1,2) \tag{3. 3}$$

and furthermore we add the following conditions (3. 4) and (3. 5), under which the method (0. 7) becomes of order 3 if $\delta=0$:

$$(-1)\frac{v_1}{3}+\sum_{j=0}^{2}a_j^3w_j=\frac{1}{3}+\delta, \tag{3. 4}$$

$$\begin{aligned}&-0.5b_2-b_{20}+a_2b_{22}=0.5a_2^2,\\ &-b_{22}w_1+b_{21}w_2=0,\\ &-b_{22}w_0+b_{20}w_2=0.\end{aligned} \tag{3. 5}$$

Here we note that the condition (3. 5) is necessary for A_0-stability. From (3. 3), (3. 4) and (3. 5), we have

$$\begin{aligned}&v_1=5-6a_2(a_2+1)w_2+6\delta,\\ &w_0=2-a_2(3a_2+2)w_2+3\delta,\\ &w_1=4-(3a_2^2+4a_2+1)w_2+3\delta,\\ &v_2=1-v_1,\\ &b_2=3\{a_2(a_2+1)\}^2w_2-(3a_2^2+2a_2)-3\delta(a_2+a_2^2),\end{aligned} \tag{3. 6}$$

$$b_{2i}=0.5(a_2+a_2^2)w_i \quad (i=0, 1, 2).$$

With these values plugged in (3.2), the characteristic polynomial of (3.2) becomes

$$(3.7) \qquad (1-0.5z\bar{h})y_{n+1}-\left\{(6z-4-6\delta)+4\left(1-z+\frac{3}{4}\delta\right)\bar{h}\right\}y_n$$
$$-\{(5-6z+6\delta+(2-2.5z+3\delta)\bar{h}\}y_{n-1}=0,$$
$$z=a_2(a_2+1)w_2.$$

If we take $z=1$ and $\delta=\frac{1}{6}$, the difference equation (3.7) has characteristic roots:

$$r_1=0,$$
$$r_2=(1+0.5h)/(1-0.5h).$$

It follows that the method (0.7) of order 2 with 2-stage is A-stable. With the values $z=1$ and $\delta=\frac{1}{6}$, the constants (3.6) are given by

$$v_1=0,\ v_2=1,\ w_0=\frac{1}{2}-\frac{a_2}{a_2+1},\ w_1=\frac{3}{2}-\frac{1}{a_2},\ b_{21}=b_{22}=0.5,$$
$$b_2=\frac{1}{a_2\{a_2+1)}-\frac{7a_2^2+5a_2}{2}.$$

§ 3.2. A-stable Method of Order 4 with 3-Stage

By the same procedure as in the 2-stage, we discuss the A-stability of order 4 with 3-stage. Let us apply the method (0.7) with $r=3$ to the test function $y'=\lambda y$. Then we obtain the difference equation:

$$(3.8) \qquad A_0y_{n+1}=A_1y_n+A_2y_{n-1},$$

where

$$A_0=1+d_1\bar{h}+d_2\bar{h}^2,$$
$$A_1=v_1+(e_{11}+w_1)\bar{h}+e_{12}\bar{h}^2+e_{13}\bar{h}^3,$$
$$A_2=v_2+(e_{21}+w_0)\bar{h}+e_{22}\bar{h}^2+e_{23}\bar{h}^3,$$
$$e_{11}=(1+b_2)w_2+(1+b_3)w_3,$$
$$e_{12}=\{b_{21}-b_{33}(1+b_2)+b_{23}(1+b_3)\}w_2+\{b_{31}+b_{32}(1+b_2)-b_{22}(1+b_3)\}w_3,$$
$$e_{13}=(-b_{21}b_{33}+b_{23}b_{31})w_2+(-b_{22}b_{31}+b_{21}b_{32})w_3,$$
$$e_{21}=-(b_2w_2+b_3w_3),$$
$$e_{22}=(b_{20}+b_2b_{33}-b_{23}b_3)w_2+(b_{30}+b_{22}b_3-b_2b_{32})w_3,$$

$$e_{23}=(-b_{20}b_{33}+b_{23}b_{30})w_0+(-b_{22}b_{30}+b_{20}b_{32})w_3,$$
$$d_1=-(b_{22}+b_{33}),$$
$$d_2=b_{22}b_{33}-b_{23}b_{32}.$$

First, we consider the case when the method is of order 4. The conditions of order 4 with $r=3$ are given by

$$(3.9)\qquad (-1)^i\frac{v_1}{i}+\sum_{j=0}^{3}a_j^{i-1}w_j=\frac{1}{i}\quad (i=1,2,3,4)$$
$$(-1)^{k-1}\frac{b_j}{k}+\sum_{l=0}^{3}a_l^{k-1}b_{jl}=\frac{1}{k}a_j^k\quad (k=2,\ j=2,3),$$
$$\frac{1}{4!}v_1-\frac{1}{3!}w_0+\sum_{i=2}^{3}\left(-\frac{1}{3}b_i-\sum_{j=0}^{3}\frac{a_j^2b_{ij}}{2}\right)w_i=\frac{1}{4!}.$$

In addition (3.9), we consider the following conditions where (3.10) and (3.11) are the conditions of order 5.

$$(3.10)\qquad -\frac{v_1}{5}+\sum_{j=0}^{3}a_j^4w_j=\frac{1}{5},$$
$$(3.11)\qquad (-1)^{k-1}\frac{b_j}{k}+\sum_{l=0}^{3}a_l^{k-1}b_{jl}=\frac{1}{k}a_j^k,\quad (k=3, j=2,3),$$
$$(3.12)\qquad -\frac{1}{5}v_1+w_0+\sum_{i=2}^{3}(-b_i+4\sum_{j=0}^{3}a_j^3b_{ij})w_i=\frac{1}{5}-\sum_{i=2}^{3}(4a_i^3+2a_i^2)w_i.$$

From (3.9), (3.10), (3.11) and (3.12) we have

$$(3.13)\qquad w_3=\frac{2(5a_2^2-a_2-2)}{a_3(a_3+1)(a_2-a_3)\{5a_2+3+5a_3(2a_2+1)\}},$$
$$w_2=\frac{12-6a_3(a_3+1)(2a_3+1)w_3}{6a_2(a_2+1)(2a_2+1)},$$
$$v_1=\frac{10a_2-7-12a_3(a_3+1)(a_2-a_3)w_3}{(2a_2+1)},$$
$$v_2=1-v_1,$$
$$w_0=0.5v_1+a_2w_2+a_3w_3-0.5,$$
$$w_1=1-(-v_1+w_0+w_2+w_3),$$
$$b_i=6a_2(a_2+1)b_{i2}+6a_3(a_3+1)b_{i3}-(3a_i^2+2a_i^3),$$
$$b_{i0}=-(3a_2^2+2a_2)b_{i2}-(3a_3^2+2a_3)b_{i3}+(a_i^2+a_i^3),$$
$$b_{i1}=-(3a_2^2+4a_2+1)b_{i2}-(3a_3^2+4a_3+1)b_{i3}+a_i(a_i+1)^2,$$
$$(i=2,3).$$

The most direct way for investigating A-stability conditions of the method (0.7) is usually the root-locus method, however we use the

Schur criterion. The stability conditions for the method (0. 7) with $r=3$ are

$$(3.14)\qquad \text{(A)}\quad |A_2|<|A_0|,$$
$$\text{(B)}\quad |\bar{A}_0A_1+A_2\bar{A}_1|<|A_2\bar{A}_2-A_0\bar{A}_0|.$$

First, let us analyze the A-stability of the method, where we take λ to be real in (3. 8). From the conditions (3. 13) and (3. 14), the following conditions are necessary

$$(3.15)\qquad (b_{22}b_{33}-b_{23}b_{32})w_1+(-b_{20}b_{33}+b_{23}b_{30})w_2+(-b_{22}b_{30}+b_{20}b_{32})w_3=0,$$

$$(3.16)\qquad (b_{22}b_{33}-b_{23}b_{32})w_0+(-b_{21}b_{33}+b_{23}b_{31})w_2+(-b_{22}b_{31}+b_{21}b_{32})w_3=0,$$

$$(3.17)\qquad |v_2+e_{21}+w_0\bar{h}+e_{22}\bar{h}^2|<|1+d_1\bar{h}+d_2\bar{h}^2|,$$

$$(3.18)\qquad |v_1+e_{11}+w_1\bar{h}+e_{12}\bar{h}^2|<|v_1+(d_1-e_{21}-w_0)\bar{h}+(d_2-e_{22})\bar{h}^2|.$$
$$(|v_2|<1).$$

We can rewrite the conditions (3. 15) and (3. 16) in the following clearer forms:

$$(3.19)\qquad \begin{aligned} d_{11}b_{23}+d_{12}b_{33}&=d_{13},\\ d_{21}b_{23}+d_{22}b_{33}&=d_{23},\end{aligned}$$

where

$$\begin{aligned}
d_{11}&=2b_{22}-(a_2^2+a_2^3)w_2,\\
d_{12}&=-2b_{32}+(a_3^2+a_3^3)w_3,\\
d_{13}&=\{(a_3^2+a_3^3)b_{22}-(a_2^2+a_2^3)b_{32}\}w_3,\\
d_{21}&=4b_{22}-a_2(a_2+1)^2w_2,\\
d_{22}&=-4b_{32}+a_3(a_3+1)^2w_2,\\
d_{23}&=\{a_3(a_3+1)^2b_{22}-a_2(a_2+1)^2b_{32}\}w_3.
\end{aligned}$$

Solving (3. 19), we have

$$(3.20)\qquad \Delta\left\{b_{23}-\frac{w_3}{w_2}b_{22}\right\}=0,$$

and

$$(3.21)\qquad \Delta\left\{b_{33}-\frac{w_3}{w_2}b_{32}\right\}=0,$$

where

$$\Delta=\{2a_3(1-a_3^2)b_{22}+2a_2(a_2^2-1)b_{32}+a_2a_3(a_2+1)(a_3+1)(a_3-a_2)w_2\}w_2.$$

We now investigate the stability conditions according as Δ is equal to zero or otherwise.

Case I: $\Delta \neq 0$.

From the A-stability conditions (3.17) and (3.18), we have

(3.22) $\quad e_{12}=0,\ e_{22}=0,\ |v_2|<1.$

From (3.21) we have

(3.23) $\quad b_{23}=\dfrac{w_3}{w_2}b_{22},$

(3.24) $\quad b_{33}=\dfrac{w_3}{w_2}b_{32}.$

By substituting (3.23) and (3.24) into (3.8) we obtain the following results:

(3.25) $\quad d_2=0,\ d_1=-\dfrac{1}{3},\ e_{11}+w_1=\dfrac{4}{3},\ d_1-e_{21}-w_0=-\dfrac{2}{3}.$

We see that the condition (3.25) contradicts the condition (3.17). Thus the method (0.7) with $r=3$ is unstable in the case $\Delta \neq 0$.

Case II: $\Delta=0$.

We use the maximum modulus principle for the analysis of stability conditions (A) and (B) in (3.14), and we use the following lemma, whose proof is not hard and so left for the reader.

Lemma. *Suppose that*

$$d_1<0,\ d_2>0 \ \textit{in}\ (3.8),$$

then we have

$$\frac{1}{A_0} \ \textit{is analytic for}\ \mathrm{Re}(\lambda)<0.$$

Thus the stability condition (A) in (3.14) is replaced by

(3.26) $\quad |A_2/A_0|<1 \quad \text{for } \mathrm{Re}(\lambda)=0,$

and

(3.27) $\quad A_2/A_0$ is analytic for $\mathrm{Re}(\lambda)=0.$

Similarly the condition (B) in (3.14) is replaced by

(3.28) $\quad |\bar{A}_0A_1+A_2\bar{A}_1|<\big||A_0|^2-|A_2|^2\big| \quad \text{for } \mathrm{Re}(\lambda)=0,$

(3.29) $\quad \{A_0A_1+A_2A_1\}/\{|A_0|^2-|A_2|^2\}$ is analytic for $\mathrm{Re}(\lambda)=0.$

We see that if the conditions (3.26) and (3.27) are satisfied, the condition (3.29) is also satisfied. From (3.26), we have

$$(3.30) \qquad (f_{23}^2-f_{13}^2)y^4+(f_{22}^2-2f_{21}f_{23}-f_{12}^2+2f_{11}f_{13})y^2+f_{21}^2-f_{11}^2>0,$$

and from (3.28) we have

$$(3.31) \qquad \left|\frac{q_1y^4+q_2y^2+q_3}{p_1y^4+p_2y^2+p_3}+i\frac{s_1y^2+s_2}{p_1y^4+p_2y^2+p_3}\,y\right|<1,$$

where

$$(3.32) \qquad \begin{aligned} &f_{11}=v_1,\\ &f_{12}=e_{21}+w_0+d_1v_2,\\ &f_{13}=e_{22}+d_2v_2+d_1w_0,\\ &f_{21}=1,\\ &f_{22}=d_1,\\ &f_{23}=d_2,\\ &g_{11}=v_1,\\ &g_{12}=e_{11}+w_1+d_1v_1,\\ &g_{13}=d_2v_1+e_{12}+d_1w_1,\\ &p_1=f_{23}^2-f_{13}^2,\\ &p_2=f_{22}^2-2f_{21}f_{23}+2f_{11}f_{13}-f_{12}^2,\\ &p_3=f_{21}^2-f_{11}^2,\\ &s_1=-(f_{23}g_{12}-f_{22}g_{13}+f_{12}g_{13}-f_{13}g_{12}),\\ &s_2=f_{21}g_{12}-f_{22}g_{11}+f_{12}g_{11}-f_{11}g_{12},\\ &q_1=f_{23}g_{13}+f_{13}g_{13},\\ &q_2=-(f_{23}g_{11}+f_{21}g_{13}-f_{22}g_{12}+f_{13}g_{11}+f_{11}g_{13}-f_{12}g_{12}),\\ &q_3=f_{21}g_{11}+f_{11}g_{11}. \end{aligned}$$

The formula (3.30) may be rewritten as

$$(3.33) \qquad y^2\{z_1y^6+z_2y^4+z_3y^2+z_4\}>0,$$

with

$$\begin{aligned} &z_1=p_1^2-q_1^2,\\ &z_2=2(p_1p_2-q_1q_2)-s_1^2,\\ &z_3=2(p_1p_3-q_1q_3)+p_2^2-q_2^2-2s_1s_2,\\ &z_4=2p_3p_1-2q_3q_2-s_2^2. \end{aligned}$$

If we use the conditions (3.9), (3.15) and (3.16), then we see that

$$(3.34) \qquad z_3=0, \quad z_4=0.$$

We used REDUCE III to obtain (3.34), which requires rather complicated computation. Consequently A-stability conditions for the method (0.7) with (3.13), (3.19) and $\Delta=0$ are given by

(3. 35)
$$(f_{23}^2-f_{13}^2)y^4+(f_{22}^2-2f_{21}f_{23}-f_2^2+2f_{11}f_{13})y^2+f_{21}^2-f_{11}^2>0$$
$$f_{21}^2+(f_{22}^2-2f_{21}f_{23})y^2+f_{23}^2y^4\neq 0 \quad \text{for } y\neq 0,$$
$$y^2\{z_1y^2+z_2\}>0.$$

We find the suitable parameters a_2, a_3, and b_{23} by using computer, which satisfy (3. 35). These regions, where in particular we take $\rho(=b_{22}b_{33}-b_{23}b_{32})=0.83$ and $a_3=0.74$ respectively, are sketched in Figures I and II. If we take $a_2=0.7$, $a_3=0.74$ and $\rho=0.83$, whose points are in the stability region, then the constants in (0. 7) are given by:

$$v_1=\frac{125}{769},\ v_2=\frac{644}{769},\ w_0=\frac{50662}{1137351},\ w_1=\frac{98044}{199171},\ w_2=-\frac{500}{91511},$$
$$w_3=\frac{1562500}{2475411},\ b_2=-\frac{1235187911}{442175000},\ b_{20}=\frac{3895148891}{247618000},$$
$$b_{21}=\frac{13318668547}{910951248},\ b_{22}=\frac{25847383643}{19234612500},\ b_{23}=-\frac{119079316067}{114523325000},$$
$$b_3=\frac{1019782373527}{386903125000},\ b_{30}=-\frac{54331704350559}{3683317750000},\ b_{31}=\frac{3381538061}{247618000},$$
$$b_{32}=\frac{4156771908011}{3288676562500},\ b_{33}=\frac{2755200809427}{2708321875000}.$$

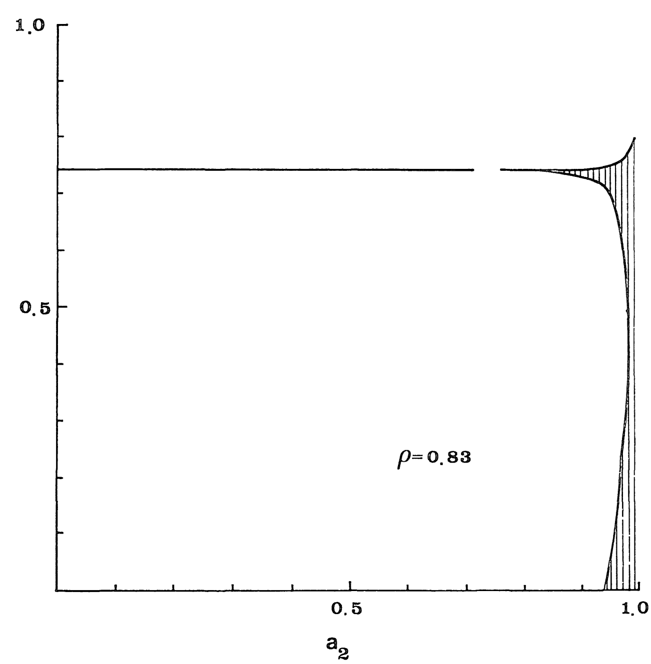

Figure (1) : The region (a_2, a_3) which satisfy A-stable condition (4. 32).

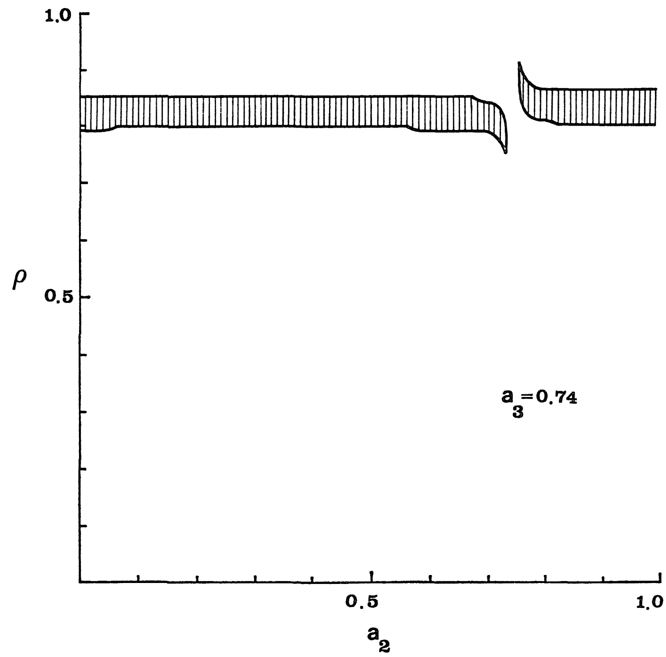

Figure (2) : The region (a_2, ρ) which satisfy A-stable condition (4. 32).

§ 4. Numerical Examples

In this section, we present some numerical results for the equations which have been often taken on in the literature of the numerical analysis. We use the following initial-value problems:

$$\text{I:}\quad \begin{cases} y' = -1000y,\ y(0) = 1, & y(x) = \exp(-1000x) \\ z' = y+1,\ z(0) = -0.0001, & z(x) = -0.001\exp(-1000x) + x. \end{cases}$$

$$\text{II:}\quad \begin{cases} y' = -10000y + 2z - 2\exp(-0.0001x) + 20000\exp(-x) \\ z' = -z + (0.9999)\exp(-0.0001x), \end{cases}$$

$$y(x) = 2\exp(-x) - \exp(-10000x)$$

$$z(x) = -\exp(-x) + \exp(-0.0001x).$$

The eigenvalues of Jacobian matrix of problems I and II are $(-1000,\ 0)$ and $(-10000,\ -1)$ respectively. In using the method (0. 7), it is necessary to solve a set of algebraic equations at each step to calculate an implicit function k_i. Of course, the evaluation of k_i requires some type of iterative procedure. We will discuss in detail those problems in another paper. We use the Newton-Raphson iteration method for obtaining an approximate real solution of an implicit function k_i, setting the initial approximation $k_i^{(0)}$ on the iterative processes by

$$k_i^{(0)}=f(x_n, y_n),$$

and we use the quantity:

$$E_{n+1}^{(i+1)}=|y_{n+1}^{(i+1)}-y_{n+1}^{(i)}|,$$

as control of the iteration number. The iteration is continued until $E_{n+1}^{(i+1)}$ become smaller than E^*, where E^* is a pre-assigned tolerance. The value y_1 necessary for the evaluation, when we use the method (0.7) of order 2, is computed by the implicit $R-K$ method of order 4, and the value y_1 necessary for the method (0. 7) of order 4 is computed by an implicit $R-K$ method of order 6. In solving an implicit function of $R-K$ method and the trapezoidal rule, we use the Newton-Raphson iteration. From Tables we can see that the advantage of our method over other methods lies in its accuracy. Computations are done in double precision arithmetic on the FACOM M-230 of Kyushu University.

Problem I, $h=1/2^7$, $E^*=0.1E-7$, (M: number of iterations).
Absolute Error
(Im) Pseudo-Runge-Method

	2-stage			3-stage		
x	with (4.6) $a_2=0.5$, $z=1$, $\delta=1/6$			with (3.13) $a_2=0.70$, $a_3=0.74$, $\rho=0.83$		
	$y_n-y(x_n)$	$z_n-z(x_n)$	M	$y_n-y(x_n)$	$z_n-z(x_n)$	M
0.5	0.144E−15	0.116E−14	3	−0.328E−8	0.328E−11	4
1.0	0.0	0.244E−14	3	−0.328E−8	0.328E−11	4
5.0	0.0	0.117E−12	3	−0.765E−12	0.145E−12	4
10.0	0.0	0.262E−12	3	−0.815E−10	0.407E−12	4
15.0	0.0	0.406E−12	3	−0.236E−12	0.507E−12	4
20.0	0.0	0.225E−11	3	−0.246E−11	0.237E−11	4

Trapezoidal Rule

x	$y_n-y(x_n)$	$z_n-z(x_n)$	M
0.5	$-0.403E-8$	$0.807E-11$	5
1.0	$-0.141E-8$	$0.141E-11$	5
5.0	$-0.150E-8$	$0.311E-12$	5
10.0	$-0.236E-8$	$0.249E-12$	5
15.0	$-0.600E-8$	$0.413E-12$	6
20.0	$-0.345E-8$	$0.122E-12$	5

(Im) Runge-Method

x	Order-4			Order-6		
	$y_n-y(x_n)$	$z_n-z(x_n)$	M	$y_n-y(x_n)$	$z_n-z(x_n)$	M
0.5	$0.209E-9$	$-0.209E-12$	3	$0.311E-10$	$-0.306E-13$	5
1.0	$0.151E-8$	$-0.151E-11$	3	$0.325E-8$	$0.325E-11$	5
5.0	$0.220E-9$	$-0.163E-12$	3	$0.633E-9$	$-0.575E-12$	2
10.0	$0.167E-8$	$-0.155E-11$	2	$-0.207E-8$	$0.220E-11$	2
15.0	$0.336E-9$	$-0.167E-12$	4	$0.682E-8$	$-0.661E-11$	2
20.0	$0.279E-8$	$-0.169E-11$	2	$0.347E-10$	$0.111E-11$	5

Relative Error

(Im) Pseudo-Runge-Method

x	2-stage		3-stage	
	$\left\|\frac{y_n-y(x_n)}{y(x_n)}\right\|$	$\left\|\frac{z_n-z(x_n)}{z(x_n)}\right\|$	$\left\|\frac{y_n-y(x_n)}{y(x_n)}\right\|$	$\left\|\frac{z_n-z(x_n)}{z(x_n)}\right\|$
0.5		$0.229E-14$	$0.202E-2$	$0.647E-11$
1.0		$0.242E-14$		$0.352E-14$
5.0		$0.235E-13$		$0.289E-13$
10.0		$0.262E-13$		$0.407E-13$
15.0		$0.270E-13$		$0.338E-13$
20.0		$0.112E-12$		$0.118E-12$

(Im) Runge-Kutta Method

x	Trapezoidal Rule		Order 4		Order-6	
	$\left\|\frac{y_n-y(x_n)}{y(x_n)}\right\|$	$\left\|\frac{z_n-z(x_n)}{z(x_n)}\right\|$	$\left\|\frac{y_n-y(x_n)}{y(x_n)}\right\|$	$\left\|\frac{z_n-z(x_n)}{z(x_n)}\right\|$	$\left\|\frac{y_n-y(x_n)}{y(x_n)}\right\|$	$\left\|\frac{z_n-z(x_n)}{z(x_n)}\right\|$
0.5		$0.403E-11$	$0.341E+0$	$0.150E-11$	$0.221E-9$	$0.322E-11$
1.0		$0.141E-11$		$0.150E-11$		$0.322E-11$
5.0		$0.155E-11$		$0.326E-13$		$0.114E-12$
10.0		$0.249E-11$		$0.154E-12$		$0.220E-12$
15.0		$0.619E-11$		$0.111E-13$		$0.441E-12$
20.0		$0.459E-11$		$0.848E-13$		$0.559E-13$

Computational time

Pseudo-methods		Trapezoid	R-K (0.6)	
2-stage	3-stage		Order-4	Order-6
1.60 (sec)	1.81 (sec)	1.87 (sec)	1.68 (sec)	2.18 (sec)

Problem II, $h=1/2^7$, $E^*=0.1E-7$, (M: number of iterations).

Absolute Error

(Im) Pseudo-Runge-Method

x	2-stage with (3.6) $a_2=0.5$, $z=1.0$, $\delta=1/6$			3-stage with (3.13) $a_2=0.70$, $a_3=0.74$, $\rho=0.83$		
	$y_n-y(x_n)$	$z_n-z(x_n)$	M	$y_n-y(x_n)$	$z_n-z(x_n)$	M
0.5	$-0.277E-1$	$-0.153E-5$	3	$-0.721E-2$	$0.116E-9$	5
1.0	$-0.104E-2$	$-0.185E-5$	3	$-0.629E-4$	$0.141E-9$	5
5.0	$-0.560E-9$	$-0.170E-6$	3	$-0.149E-10$	$0.129E-10$	5
10.0	$-0.400E-11$	$-0.229E-8$	3	$-0.343E-13$	$0.176E-12$	5
15.0	$-0.285E-13$	$-0.231E-10$	3	$-0.569E-14$	$0.399E-14$	5
20.0	$-0.202E-15$	$-0.205E-12$	3	$-0.571E-15$	$0.227E-14$	5

Trapezoidal Rule

x	$y_n-y(x_n)$	$z_n-z(x_n)$	M
0.5	$0.279E-1$	$-0.154E-5$	21
1.0	$-0.779E-3$	$-0.187E-5$	21
5.0	$-0.257E-8$	$-0.171E-6$	14
10.0	$0.258E-8$	$-0.230E-8$	14
15.0	$0.223E-8$	$-0.233E-10$	13
20.0	$-0.732E-8$	$-0.210E-12$	11

(Im) Runge-Method

x	Order-4			Order-6		
	$y_n-y(x_n)$	$z_n-z(x_n)$	M	$y_n-y(x_n)$	$z_n-z(x_n)$	M
0.5	$-0.396E-4$	$0.155E-11$	4	$-0.213E-8$	$0.346E-15$	5
1.0	$-0.684E-7$	$0.188E-11$	4	$0.295E-11$	$0.555E-15$	5
5.0	$-0.121E-8$	$0.173E-12$	4	$0.522E-13$	$0.888E-15$	5
10.0	$-0.818E-11$	$0.313E-14$	4	$0.357E-15$	$0.832E-15$	5
15.0	$-0.465E-11$	$0.208E-15$	4	$-0.204E-16$	$0.832E-15$	5
20.0	$-0.860E-9$	$0.735E-15$	4	$-0.140E-11$	$0.666E-15$	5

Relative Error

(Im) Pseudo-Runge-Method

x	2-stage $\left\lvert\frac{y_n-y(x_n)}{y(x_n)}\right\rvert$	2-stage $\left\lvert\frac{z_n-z(x_n)}{z(x_n)}\right\rvert$	3-stage $\left\lvert\frac{y_n-y(x_n)}{y(x_n)}\right\rvert$	3-stage $\left\lvert\frac{z_n-z(x_n)}{z(x_n)}\right\rvert$
0.5	0.230E−1	0.384E−5	0.599E−2	0.292E−9
1.0	0.143E−2	0.292E−5	0.862E−4	0.222E−9
5.0	0.418E−7	0.171E−6	0.112E−8	0.130E−10
10.0	0.444E−7	0.229E−8	0.380E−9	0.176E−12
15.0	0.469E−7	0.231E−10	0.983E−8	0.400E−14
20.0	0.494E−7	0.205E−12	0.139E−6	0.228E−14

(Im) Runge-Kutta Method

x	Trapezoidal Rule $\left\lvert\frac{y_n-y(x_n)}{y(x_n)}\right\rvert$	Trapezoidal Rule $\left\lvert\frac{z_n-z(x_n)}{z(x_n)}\right\rvert$	Order 4 $\left\lvert\frac{y_n-y(x_n)}{y(x_n)}\right\rvert$	Order 4 $\left\lvert\frac{z_n-z(x_n)}{z(x_n)}\right\rvert$	Order-6 $\left\lvert\frac{y_n-y(x_n)}{y(x_n)}\right\rvert$	Order-6 $\left\lvert\frac{z_n-z(x_n)}{z(x_n)}\right\rvert$
0.5	0.230E−1	0.392E−5	0.329E−4	0.391E−11	0.177E−8	0.871E−15
1.0	0.105E−2	0.296E−5	0.937E−7	0.297E−11	0.404E−11	0.874E−15
5.0	0.190E−6	0.172E−6	0.908E−7	0.175E−12	0.391E−11	0.894E−15
10.0	0.282E−4	0.231E−8	0.908E−7	0.313E−14	0.396E−11	0.833E−15
15.0	0.364E−2	0.233E−10	0.766E−5	0.208E−15	0.336E−10	0.833E−15
20.0	0.177E+1	0.210E−11	0.210E+0	0.736E−15	0.344E−3	0.667E−15

Computational time

Pseudo-methods		Trapezoid	R-K(0.6)	
2-stage	3-stage		Order-2	Order-4
1.48(sec)	1.77(sec)	2.93(sec)	1.55(sec)	2.78(sec)

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