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Hodge Spectral Sequence and Symmetry on Compact Kähler Spaces

By

Takeo OHSAWA*

Introduction

For every complex manifold M, there exists a canonical spectral sequence which abuts to the de Rham cohomology of M. It consists of the set of C^∞ differential forms on M, and the complex exterior derivatives $\bar{\partial}$ and ∂ of type $(0,1)$ and $(1,0)$, respectively, and its E_1-term is defined to be Ker $\bar{\partial}/\mathrm{Im}\,\bar{\partial}$. This will be referred to as the Hodge spectral sequence on M, after the celebrated result of W. Hodge [4].

Hodge's theorem states that the Hodge spectral sequence degenerates at E_1 and that $E_1^{p,q}(M)\cong E_1^{q,p}(M)$ if M is a compact Kähler manifold. Here $E_1^{p,q}(M)$ denotes the (p,q)-component of the E_1-term.

The purpose of the present note is to study an analogue of Hodge spectral sequences on compact complex spaces within the spirit of the previous note [7], where we considered the spaces which admit only isolated singularities.

Our main result is as follows.

Theorem 1 *Let X be a compact Kähler space of pure dimension and let Y be an analytic subset of X containing the singular locus of X. Then, the Hodge spectral sequence on $X\backslash Y$ degenerates for the total degrees less than* codim $Y-1$ *at the E_1-term. Moreover, $E_1^{p,q}(X\backslash Y)\cong E_1^{q,p}(X\backslash Y)$ for* $p+q<$ codim $Y-1$.

In order to understand the symmetry $E_1^{p,q}(X\backslash Y)\cong E_1^{q,p}(X\backslash Y)$, we shall also prove the following.

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* Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606 Japan.

Theorem 2 *Let X and Y be as above, and let (E, h) be a flat Hermitian vector bundle over $X\backslash Y$. Then, $H^{p,q}(X\backslash Y, E)\cong H^{q,p}(X\backslash Y, E^*)$, for $p+q<\mathrm{codim}\ Y-1$. Here $H^{p,q}$ denotes the cohomology of type (p, q) in the sense of Dolbeault and E^* denotes the dual bundle of E.*

For the proof of the above mentioned results, an L^2-version of Andreotti-Grauert's vanishing theorem on q-complete spaces is necessary which is to be proved in §2 by using a new L^2-estimate obtained in [8].

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§ 1. Preliminaries

Definition A (reduced) complex space X together with the following data $\{U_j, \varphi_j\}_{j\in A}$ is called a Kähler space.

1) A is a set of indices.
2) $\{U_j\}_{j\in A}$ is an open covering of X.
3) φ_j is a C^∞ strictly plurisubharmonic function on U_j.
4) $\varphi_j-\varphi_k$ is pluriharmonic on $U_j\cap U_k$.

Given a Kähler space X, one attaches a Kähler metric on the complement of the singular locus by $\partial\bar\partial\varphi_j$, which is globally well defined by condition 4).

Let X be a compact Kähler space of pure dimension n with singular locus Z, and let Y be an analytic subset of X containing Z. We shall denote by ds^2 the prescribed Kähler metric on $X\backslash Z$.

Let $\{Y_\alpha\}_{\alpha=0}^m$ be a partition of Y into subsets satisfying the following properties.

i) $\bar{Y}_\alpha$ are pure dimensional analytic subsets of Y.
ii) $Y_{\alpha+1}\subset\bar{Y}_\alpha$.
iii) $\dim \bar{Y}_\alpha=m-\alpha$.
iv) The reduced structures of Y_α are regular.

Such a partition (a stratification of Y) always exists, since the singular loci of complex analytic spaces are analytic subsets.

As a complex manifold Y_α has a holomorphic coordinate patch. In other words, for each point $y\in Y_\alpha$ one can find a Stein open neighbourhood U in Y_α and a biholomorphic map from U onto a domain in $\mathbb{C}^{m-\alpha}$. Since every holomorphic function on U is holomorphically extendable to a neighbourhood of U in X, it follows that U is a holomorphic neighbourhood retract in X. Therefore, Y_α can be covered by Stein open subsets, each of which has a Stein neighbourhood, say V, with a holomorphic embedding into a domain of some complex number space $\mathbb{C}^N$ such that the image of $V\cap Y_\alpha$ is contained in a linear subspace of dimension $m-\alpha$. Identifying V as a subspace of $\mathbb{C}^N$, one sees that the restrictions of linear functions vanishing on $V\cap Y_\alpha$, say $z_1,\dots,z_{N-m+\alpha}$, generate the ideal of holomorphic functions vanishing on $V\cap Y_\alpha$ in the ring of holomorphic functions on V. One associates to V a possibly smaller Stein open set

$$W:=\left\{x\in V;\ \sum_{\nu=1}^{N-m+\alpha}|z_\nu(x)|^2<\frac{1}{2}\right\}$$

and define a plurisubharmonic function ϕ_W on W by

$$\phi_W(x):=-\ln(-\ln\|z'(x)\|^2).$$

Here we put $z':=(z_1,\dots,z_{N-m+\alpha})$ and $\|z'(x)\|^2:=\sum_{\nu=1}^{N-m+\alpha}|z_\nu(x)|^2$.

Suppose that a point y in Y_α belongs to the polar sets of two such functions ϕ_W and $\phi_{W'}$ (i.e. $\phi_W(y)=\phi_{W'}(y)=-\infty$). Then, there exists a neighbourhood $\Omega\ni y$ and a constant C such that

$$|\exp(-\phi_W)-\exp(-\phi_{W'})|<C \quad \text{on } \Omega\backslash Y. \tag{1}$$

In fact, this follows from that z_i are generators of the ideal sheaf of $V\cap Y_\alpha$.

Now let $\mathscr{I}_\alpha$ be the ideal sheaf of $\overline{Y}_\alpha$ in the structure sheaf $\mathcal{O}_X$ of X. Then, for each point $y\in\overline{Y}_\alpha$ there exists a neighbourhood U_y in X and finitely many holomorphic functions $f_1,\dots,f_m (m=m(y))$ which generate the stalks of $\mathscr{I}_\alpha$ at every point of U_y (cf. [3]). Then we put $W_y:=\left\{x\in U_y;\ \sum_{j=1}^{m}|f_j|^2<\frac{1}{2}\right\}$ and $\phi_y:=-\ln(-\ln\|f\|^2)$, where $\|f\|^2:=\sum_{j=1}^{m}|f_j|^2$.

Let $\{W_k\}$ be a finite system of such Stein open subsets of X whose union contains $\bar{Y}_\alpha$, where we put $W_k = W_{y_k}$, and let ϕ_k be the associated plurisubharmonic functions on W_k defined as above. Such a system $\{W_k, \phi_k\}$ shall be referred to as a *polarized cover along* $\bar{Y}_\alpha$. Suppose that $y \in W_k \cap W_l$, Then, by the same reasoning as above, one sees that there exists a neighbourhood $\Omega \ni y$ and a constant C such that

$$(1') \qquad |\exp(-\phi_k) - \exp(-\phi_l)| < C \quad \text{on } \Omega \backslash \bar{Y}_\alpha.$$

Let $\{W_k, \phi_k\}$ be a polarized cover along $\bar{Y}_\alpha$ and let $\{\rho_k, \rho\}$ be a C^∞ partition of unity associated to the covering $\{W_k, X \backslash \bar{Y}_\alpha\}$ of X such that $\rho_k \geq 0$. Namely, ρ_k is a system of nonnegative C^∞ functions on X such that supp $\rho_k \Subset W_k$ and $\sum \rho_k \equiv 1$ on a neighbourhood of $\bar{Y}_\alpha$, say W_α, and $\rho := 1 - \sum \rho_k$.

We put $\phi_\alpha := \sum \rho_k \phi_k$. Then we have

$$(2) \qquad \begin{aligned} \partial\bar{\partial}\phi_\alpha &= \sum \partial\rho_k \bar{\partial}\phi_k + \sum \partial\phi_k \bar{\partial}\rho_k + \sum \phi_k \partial\bar{\partial}\rho_k + \sum \rho_k \partial\bar{\partial}\phi_k \\ &= \sum \partial\rho_k \bar{\partial}\phi_k - \sum (\partial \sum \rho_l) \bar{\partial}\phi_k + \sum \partial\phi_k \bar{\partial}\rho_k - \sum \partial\phi_k (\bar{\partial} \sum \rho_l) \\ &\quad + \sum \phi_k \partial\bar{\partial}\rho_k - \sum \phi_k (\partial\bar{\partial} \sum \rho_l) + \sum \rho_k \partial\bar{\partial}\phi_k \\ &= \sum_{k,l} \partial\rho_k (\bar{\partial}\phi_k - \bar{\partial}\phi_l) + \sum_{k,l} (\partial\phi_k - \partial\phi_l) \bar{\partial}\rho_k + \sum_{k,l} (\phi_k - \phi_l) \partial\bar{\partial}\rho_k \\ &\quad + \sum \rho_k \partial\bar{\partial}\phi_k, \end{aligned}$$

on $W_\alpha \backslash \bar{Y}_\alpha$.

We are going to estimate the eigenvalues of $\partial\bar{\partial}\phi_\alpha$.

Once for all, let $|\ |_k$ denote the length of the differential forms measured by $ds^2 + \partial\bar{\partial}\phi_k$. Then we have $|\partial\phi_k|_k \leq \sqrt{2}$, since $\phi_k = -\ln(-\ln\|f_k\|^2)$ for some vector f_k of holomorphic functions and

$$\begin{aligned} \partial\bar{\partial}\phi_k &= \frac{-\partial\bar{\partial} \ln\|f_k\|^2}{\ln\|f_k\|^2} + \frac{\partial \ln\|f_k\|^2 \bar{\partial} \ln\|f_k\|^2}{(\ln\|f_k\|^2)^2} \\ &\geq \partial\phi_k \bar{\partial}\phi_k. \end{aligned}$$

Let $K_{kl} \subset W_k \cap W_l$ be any compact subset. Then,

$$(3) \qquad C_{kl}^{-1}(ds^2 + \partial\bar{\partial}\phi_k) \leq ds^2 + \partial\bar{\partial}\phi_l \leq C_{kl}(ds^2 + \partial\bar{\partial}\phi_k)$$

on $K_{kl} \backslash \bar{Y}_\alpha$, where C_{kl} is a constant depending on K_{kl}. In particular we have

$$(4) \qquad |\partial\phi_k|_l \leq \sqrt{2C_{kl}} \quad \text{on } K_{kl} \backslash \bar{Y}_\alpha.$$

Proof of (3): We put $f_k = (a_1, \ldots, a_{m_k})$.

Then

$$\partial\bar{\partial}\phi_k = \frac{\sum_{\mu<\nu}(a_\mu\partial a_\nu - a_\nu\partial a_\mu)\overline{(a_\mu\partial a_\nu - a_\nu\partial a_\mu)}}{(-\ln\|f_k\|^2)^2\|f_k\|^4} + \frac{(\sum_\mu \bar{a}_\mu\partial a_\mu)(\sum_\nu a_\nu\bar{\partial}\bar{a}_\nu)}{(\ln\|f_k\|^2)^2\|f_k\|^4}. \tag{5}$$

Let $\phi_l = -\ln(-\ln\|f_l\|^2)$ and $f_l = (b_1, \dots, b_{m_l})$. Then

$$a_\mu = \sum_{j=1}^{m_l} u_{\mu j} b_j,\ 1 \leq \mu \leq m_k \tag{6}$$

for some holomorphic functions $u_{\mu j}$ on $W_k \cap W_l$.

Substituting (6) into (5) and applying the Cauchy-Schwartz inequality etc., we have

$$\begin{aligned}\partial\bar{\partial}\phi_k \geq C'_{kl} & \frac{\sum_{i<j}(b_i\partial b_j - b_j\partial b_i)\overline{(b_i\partial b_j - b_j\partial b_i)}}{(-\ln\|f_k\|^2)\|f_k\|^4} \\ & + \frac{(\sum_\mu\sum_{i,j} u_{\mu i}\bar{u}_{\mu j}\bar{b}_j\partial b_i)(\sum_\mu\sum_{i,j}\bar{u}_{\mu i}u_{\mu j}b_j\bar{\partial}\bar{b}_i)}{(\ln\|f_k\|^2)^2\|f_k\|^4} + O_{kl},\end{aligned} \tag{7}$$

on $K_{kl}\backslash\bar{Y}_\alpha$ for some constant C'_{kl}. Here O_{kl} has bounded length with respect to ds^2.

Note that $(\sum_{i=1}^m |\xi_i|^2)(\sum_{j=1}^m |\eta_j|^2) = \sum_{i<j}|\xi_i\eta_j - \xi_j\eta_i|^2 + |\sum_{i=1}^m \xi_i\bar{\eta}_i|^2$, for any complex numbers ξ_i and η_j, $1 \leq i, j \leq m$ (Lagrange's equality). Applying this equality to (7), we have

$$\begin{aligned}& \frac{(\sum_\mu\sum_{i,j} u_{\mu i}\bar{u}_{\mu j}\bar{b}_j\partial b_i)(\sum_\mu\sum_{i,j}\bar{u}_{\mu i}u_{\mu j}b_j\bar{\partial}\bar{b}_i)}{(\ln\|f_k\|^2)^2\|f_k\|^4} \\ & \leqq C\frac{\sum_{i,j}(b_i\partial b_j - b_j\partial b_i)\overline{(b_i\partial b_j - b_j\partial b_i)} + (\sum_i \bar{b}_i\partial b_i)(\sum_j b_j\bar{\partial}\bar{b}_j)}{(\ln\|f_k\|^2)^2\|f_k\|^4},\end{aligned} \tag{8}$$

on $K_{kl}\backslash\bar{Y}_\alpha$, for some constant C.

Since we have chosen W_k so that $\ln\|f_k\|^2 < -\ln 2$ on W_k, we have

$$\partial\bar{\partial}\phi_k \leq C'\partial\bar{\partial}\phi_l + O'_{kl} \quad \text{on } K_{kl}\backslash\bar{Y}_\alpha, \tag{9}$$

where C' is a constant and O'_{kl} is bounded with respect to ds^2. (3) follows from (9) immediately.

From (1′), (2), (3) and (4), we obtain

$$-A_\alpha ds^2 + \frac{1}{2}\sum \rho_k\partial\bar{\partial}\phi_k \leq \partial\bar{\partial}\phi_\alpha, \tag{10}$$

for sufficiently large $A_\alpha \geq 1$.

Thus we know that $Ads^2 + \partial\bar{\partial}\phi_\alpha$ is a metric on $X \backslash Y$ for any $A > A_\alpha$. Furthermore, let $\lambda_1^A \geq \ldots \geq \lambda_n^A$ be the eigenvalues of $\partial\bar{\partial}\phi_\alpha$ with respect to the metric $Ads^2 + \partial\bar{\partial}\phi_\alpha (A > A_\alpha)$. Then, from (10) one immediately sees that, for any $\varepsilon > 0$, there exists an $A > A_\alpha$ such that $\lambda_j^A > -\varepsilon$ for $n - \alpha < j$ on $X \backslash Y$. Moreover, (10) implies that at least $n - \alpha$ eigenvalues of $\partial\bar{\partial}\phi_\alpha$ with respect to ds^2 tend to $+\infty$ as one approaches to a point in Y_α (see (5) and recall Courant's mini-max principle). Hence, for any point $y \in Y_\alpha$ and $\varepsilon > 0$, one can choose a neighbourhood $\Omega \ni y$ in X so that $1 - \varepsilon < \lambda_j^A < 1 + \varepsilon$ for $1 \leq j \leq n - \alpha$ on $\Omega \backslash Y$.

Note that (4) implies $\partial\phi_\alpha \bar{\partial}\phi_\alpha < C(Ads^2 + \partial\bar{\partial}\phi_\alpha)$ for some $C > 0$.

For any positive number u we put $\phi_u := u \sum_{\alpha=0}^{m} \phi_\alpha$ and $ds_{A,u}^2 := Ads^2 + \partial\bar{\partial}\phi_u$. Then, $ds_{A,u}^2$ is a complete Kähler metric on $X \backslash Y$ whenever $A > u \sum_{\alpha=0}^{m} A_\alpha$.

Now we have the following.

Proposition 1.1 *Let (X, ds^2) be a compact Kähler space of pure dimension n and Y an analytic subset containing the singular locus of X. Then, for any $\varepsilon > 0$, there exist a complete Kähler metric ds_Y^2 on $X \backslash Y$, a proper C^∞ map $\phi : X \backslash Y \to (-\infty, 0]$ and a neighbourhood $W' \supset Y$ such that,*

(*) $\quad |\partial\phi|_Y^2 < \varepsilon$,

(**) $\quad |\partial\bar{\partial}\phi|_Y < 2n$,

(***) *The eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of $\partial\bar{\partial}\phi$ with respect to ds_Y^2 satisfy*
$1 - \varepsilon < \lambda_j < 1 + \varepsilon$ *for* $1 \leq j \leq \text{codim } Y$ *on* $W \backslash Y$,
$-\varepsilon < \lambda_j$ *for* $j > \text{codim } Y$ *on* $X \backslash Y$.

Here $|\ \ |_Y$ *denotes the length with respect to the metric* ds_Y^2.

Proof Let $A \gg 0$, $u \ll \frac{1}{A}$, and put $\phi = \phi_u$, $ds_Y^2 = ds_{A,u}^2$.

§ 2. Vanishing of the Local L^2-Cohomology

Let (M, ds^2) be a Hermitian manifold of dimension n, and let (E, h) be a Hermitian holomorphic vector bundle over M. For any C^∞ (1, 1)-form $G = i \sum G_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$ with $G_{\alpha\beta} = \bar{G}_{\beta\bar{\alpha}}$ on M, we define real-valued functions $\Gamma_{p,q}[G]$ by

$$\Gamma_{p,q}[G](x):=\min\{\sum_{\alpha=1}^{p}\lambda_{i_\alpha}(x)+\sum_{\beta=1}^{q}\lambda_{j_\beta}(x)-\sum_{k=1}^{n}\lambda_k(x);$$
$$\lambda_k(x)\ (1\leq k\leq n) \text{ are the eigenvalues of } G \text{ at } x,$$
$$1\leq i_1<\dots<i_p\leq n \text{ and } 1\leq j_1<\dots<j_q\leq n\}.$$

In terms of $\Gamma_{p,q}$ we shall state a sufficient condition for an à priori estimate for the operator $\bar\partial$. The L^2-norm for E-valued forms will be denoted by $\| \ \|_h$.

Let ω be the fundamental form of ds^2 and Λ the adjoint of the multiplication $u\mapsto\omega\wedge u$. We denote by $\bar\partial_h^*$ the (L^2-) adjoint of the operator $\bar\partial$ wtih respect to the metrics ds^2 and h. The operator $\partial^*:=-\bar*\partial\bar*$ ($\bar*$: the conjugate after the Hodge's star) acts on E-valued forms and we denote by ∂_h the adjoint of ∂^* with respect to ds^2 and h. Then we have $[\bar\partial,\Lambda]=i\partial^*+T_1$ and $[\partial_h,\Lambda]=-i\bar\partial_h^*+T_2$, where $[\ ,\ ]$ denotes the Poisson bracket and $T_j(j=1,2)$ contain no differentiation (i.e. T_j are function-linear).

Let $\langle T_j\rangle$ denote the (L^2-) operator norms of T_j. Then, from the explicit expression of the operator T_1+T_2 in terms of $d\omega$ and other elementary operators like $\bar*$, Λ, etc. (cf. [5] appendix), we see that there exists a positive number β_n depending only on n such that $3\langle T_1\rangle^2+\langle T_2\rangle^2\leq\beta_n|d\omega|^2$. In what follows we fix such β_n.

Proposition 2.1 *Let F_1 be a C^∞ real-valued function on M and $h_1:=h\exp(-F_1)$. Let Θ be the curvature form of h. Suppose that there exists a C^∞ real-valued function F satisfying*

$$\Gamma_{p,q}[\partial\bar\partial(F+F_1)]\geq n|\Theta|+\beta_n|d\omega|^2+3|\partial F|^2+\varepsilon \tag{11}$$

for some $\varepsilon>0$. Then

$$\|\bar\partial u\|_{h_1}^2+\|\bar\partial_{h_1}^* u\|_{h_1}^2\geq\varepsilon\|u\|_{h_1}^2,$$

for any compactly supported E-valued $C^\infty(p,q)$-form u on M.

For the proof, see [8], Corollary 1.7.

Definition A Hermitian vector bundle (E,h) is said to be flat, if the operator $(\bar\partial+\partial_h)\circ(\bar\partial+\partial_h)$ is identically zero.

By the above definition, (E,h) is flat if and only if $\Theta\equiv0$.

In §1 we have constructed a metric ds^2 and a function ϕ satisfying several properties, from which we shall produce the functions F_1 and F as above. In particular, for flat vector bundles we have the following.

Proposition 2.2 *Let (N, ds_N^2) be a Kähler manifold of dimension n and let (E, h) be a flat Hermitian vector bundle over N. Suppose that there exist a positive integer r and a C^∞ real-valued function ϕ on N such that*

(i) $|\partial\phi|^2 < 1/12$.

(ii) $|\partial\bar\partial\phi| < 2n$.

(iii) *The eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of $\partial\bar\partial\phi$ satisfy*

$$1-\frac{1}{4n} < \lambda_j < 1+\frac{1}{4n} \quad \textit{for } 1 \leq j \leq r$$

$$-\frac{1}{4n} < \lambda_j \quad \textit{for } r < j.$$

Then, for any $A > 2^{16}\beta_n^2 n^4$ and $c \in \mathbf{R}$, the inequality (11) *is satisfied by $M = \{x \in N;\ \phi(x) < c\}$, $ds^2 = (A(c-\phi)^{-2}+1)\,ds_N^2 + 2A(c-\phi)^{-3}\partial\phi\bar\partial\phi$, $F = \phi$, $F_1 = A(c-\phi)^{-1}$ and $\varepsilon = 1/8$, for $p+q > 2n-r$.*

Proof Let $|\ \ |_A$ denote the length of the forms with respect to the metric ds^2. Let ω_N and ω be the fundamental forms of ds_N^2 and ds^2, respectively. Then, $d\omega = A(c-\phi)^{-3}d\phi \wedge (2\omega_N - i\partial\bar\partial\phi)$. We estimate $|d\omega|_A$ as follows.

First, from the definition of ds^2, $|d\phi|_A < 2A^{-1/2}(c-\phi)^{3/2}$ and $|\omega_N|_A < 2n(A(c-\phi)^{-2}+1)^{-1}$. Secondly, from (ii), $|\partial\bar\partial\phi|_A < 2n(A(c-\phi)^{-2}+1)^{-1}$.

Therefore,

$$\begin{aligned} |d\omega|_A &\leq A(c-\phi)^{-3}|d\phi|_A(2|\omega_N|_A + |\partial\bar\partial\phi|_A) \\ &< 6nA^{1/2}(c-\phi)^{-3/2}(A(c-\phi)^{-2}+1)^{-1}. \end{aligned}$$

Hence,

$$|d\omega|_A < 6nA^{1/2}(c-\phi)^{-3/2} < 6nA^{-1/4} \quad \text{if } A < (c-\phi)^2,$$

and

$$|d\omega|_A < 6nA^{-1/2}(c-\phi)^{1/2} \leq 6nA^{-1/4} \quad \text{if } A \geq (c-\phi)^2.$$

Thus, $\beta_n|d\omega|_A^2 \leq 36\beta_n n^2 A^{-1/2}$, so that

$$\beta_n |d\omega|_A^2 < \frac{1}{4} \quad \text{if } A > 2^{16}\beta_n^2 n^4. \tag{12}$$

To estimate the left hand side of the inequality, let $x \in M$ be any point and let L be the subspace of the complex tangent space of M at x spanned by the eigenvectors corresponding to $\lambda_1(x), \ldots, \lambda_r(x)$. Then, for any vector $v \in L$, one has, for $F = \phi$ and $F_1 = A(c-\phi)^{-1}$,

$$1 - \frac{1}{4n} < \frac{\langle \partial\bar{\partial}(F+F_1), v, \bar{v}\rangle}{|v|_A^2} < 1 + \frac{1}{4n}, \tag{13}$$

from (iii). Here $|v|_A$ denotes the length of v with respect to ds^2. Similarly, for any unit tangent vector w at x,

$$\langle \partial\bar{\partial}(F+F_1), w, \bar{w}\rangle > -\frac{1}{4n}. \tag{14}$$

Combining (13) and (14), we have

$$\Gamma_{p,q}(\partial\bar{\partial}(F+F_1)) > \frac{3}{4}, \quad \text{if } p+q > 2n-r. \tag{15}$$

From (i) we have

$$3|\partial F|_A^2 < \frac{1}{4}. \tag{16}$$

Combining (12), (15) and (16), we obtain the desired inequality for the flat bundle (E, h).

Applying Proposition 2.2 to the Kähler manifold $(W' \backslash Y, ds_Y^2)$ described in Proposition 1.1, we obtain the following.

Proposition 2.3 *Let (X, ds^2) be a compact Kähler space of pure dimension n and Y an analytic subset containing the singular locus of X. Then, there exists a C^∞ proper map $\phi: X\backslash Y \to (-\infty, 0]$ and $c \in \mathbb{R}$ (c; arbitrarily small) such that, for any compactly supported C^∞ (p,q)-form u on $W := \{x \in X\backslash Y;\ \phi(x) < c\}$ with values in a flat vector bundle (E, h) over $X\backslash Y$, the estimate*

$$\|\bar{\partial}u\|_{h_W}^2 + \|\bar{\partial}_{h_W}^* u\|_{h_W}^2 \geq \frac{1}{4}\|u\|_{h_W}^2$$

holds for $p+q > 2n - \mathrm{codim}\ Y$ with respect to the metrics

$$ds_W^2 = (A(c-\phi)^{-2}+1)\,ds_Y^2 + 2A(c-\phi)^{-3}\partial\phi\bar{\partial}\phi$$

and $h_W = h\exp(-A(c-\psi)^{-1})$, *where* $A > 2^{16}\beta_n^2 n^4$ *and* ds_Y^2 *is some* (*i.e. not arbitrary*) *complete Kähler metric on* $X\backslash Y$.

Since the above (W, ds_W^2) is a complete Hermitian manifold, Proposition 2.3 implies that the Hermitian bundle $(E|_W, h_W)$ is $W^{p,q}$-elliptic in the sense of Andreotti-Vesentini [2], if $p+q > 2n - \mathrm{codim}\, Y$.

Thus, in virtue of Andreotti-Vesentini's theorem, we have the following corollary to Proposition 2.3.

Corollary 2.4 *Under the above situation, let* f *be any* E*-valued* (p, q)*-form on* W *which is square integrable with respect to* ds_W^2 *and* h_W *and* $\bar\partial f = 0$ *in the sense of distribution. If* $p+q > 2n - \mathrm{codim}\ Y$, *then there exists an* E*-valued* $(p, q-1)$*-form* g *on* W, *square integrable with respect to* ds_W^2 *and* h_W *such that* $\bar\partial g = f$ *and* $\|g\|_{h_W} \leq 2\|f\|_{h_W}$.

§ 3. L^2 Cohomology and Harmonic Forms

Let (M, ds_M^2) be a Hermitian manifold of dimension n, and let (E, h) be a Hermitian vector bundle over M. We denote by $L^{p,q}(M, E)_h$ the set of square integrable E-valued (p, q)-forms on M with respect to ds_M^2 and h, and put

$$H_{(2)}^{p,q}(M, E)_h := \{f \in L^{p,q}(M, E)_h;\ \bar\partial f = 0\} / \{g \in L^{p,q}(M, E)_h;\ \exists u \in L^{p,q-1}(M, E)_h \text{ such that } g = \bar\partial u\}.$$

Here the derivatives are taken in the distribution sense.

Let $L_{loc}^{p,q}(M, E)$ be the set of locally square integrable E-valued (p, q)-forms on M. We put

$$H^{p,q}(M, E) := \{f \in L_{loc}^{p,q}(M, E);\ \bar\partial f = 0\} / \{g \in L_{loc}^{p,q}(M, E);\ \exists u \in L_{loc}^{p,q-1}(M, E) \text{ such that } \bar\partial u = g\}.$$

Since the L^2-version of Dolbeault's Lemma is valid (cf. [6] or [9]), $H^{p,q}(M, E)$ is canonically isomorphic to the E-valued Dolbeault cohomology of type (p, q).

We put $\Box_h := \bar\partial\bar\partial_h^* + \bar\partial_h^*\bar\partial$ and $\overline{\Box}_h := \partial_h\partial^* + \partial^*\partial_h$. Clearly, $\overline{\Box}_h = *^{-1}\Box_h *$.

We put $\mathscr{H}^{p,q}(E)_h := \{f \in L^{p,q}(M, E)_h;\ \Box_h f = 0\}$.

If the metric ds_M^2 is Kählerian, one has $[\partial_h, \Lambda] = -i\bar\partial_h^*$ and $[\bar\partial, \Lambda] = i\partial^*$. Hence $[i(\bar\partial + \partial_h)(\bar\partial + \partial_h), \Lambda] = \bar\partial \cdot i[\partial_h, \Lambda] + i[\partial_h, \Lambda]\bar\partial + \partial_h \cdot i[\bar\partial, \Lambda] +$

$i[\bar{\partial}, \Lambda]\partial_h = \square_h - \bar{\square}_h$. If the bundle (E, h) is flat, then we have $\square_h = \bar{\square}_h$. Thus we obtain

Lemma 3.1 *Let (M, ds_M^2) be a Kähler manifold and (E, h) a flat Hermitian vector bundle over M. Then, $\mathscr{H}^{p,q}(E)_h \cong \mathscr{H}^{n-q,n-p}(E)_h$. Here the isomorphism is given by $f \mapsto *f$.*

Identifying h as a C^∞ section of Hom $(E, \bar{E}^*)$, we have $\partial_h = h^{-1} \circ \partial \circ h$. Therefore we obtain

Lemma 3.2 *Under the situation of Lemma 3.1, $\mathscr{H}^{p,q}(E)_h \cong \mathscr{H}^{q,p}(E^*)_{h^*}$. Here the isomorphism is given by $f \mapsto \overline{hf}$. $(h^* := {}^t h^{-1})$.*

The following is fundamental.

Proposition 3.3 *Let (M, ds_M^2) be a complete Hermitian manifold and (E, h) a Hermitian vector bundle over M. Then*

$$\mathscr{H}^{p,q}(E)_h = \{f \in L^{p,q}(M, E)_h;\ \bar{\partial} f = 0,\ \bar{\partial}_h^* f = 0\}.$$

Proof. See Andreotti-Vesentini [2].

Thus, if the metric ds_M^2 is complete, then we have an orthogonal decomposition:

$$L^{p,q}(M, E)_h = \mathscr{H}^{p,q}(E)_h \oplus \overline{R_{\bar{\partial}}^{p,q}(E)} \oplus \overline{R_{\bar{\partial}_h^*}^{p,q}(E)}.$$

Here $R_{\bar{\partial}}^{p,q}(E)$ (resp. $R_{\bar{\partial}_h^*}^{p,q}(E)$) denotes the range of $\bar{\partial}$ (resp. $\bar{\partial}_h^*$), and $\overline{R_{\bar{\partial}}^{p,q}(E)}$ (resp. $\overline{R_{\bar{\partial}_h^*}^{p,q}(E)}$) its closure.

From the above decomposition we obtain

$$H_{(2)}^{p,q}(M, E)_h \cong \mathscr{H}^{p,q}(E)_h, \tag{17}$$

if $R_{\bar{\partial}}^{p,q}(E)$ is closed (for instance it is the case when $H_{(2)}^{p,q}(M, E)_h$ is finite dimensional).

Combining Lemma 3.2 with (17), we have

Proposition 3.4 *Let (M, ds_M^2) be a complete Kähler manifold and (E, h) a flat Hermitian vector bundle over M. Suppose that* dim $H_{(2)}^{p,q}(M, E)_h < \infty$ *and* dim $H_{(2)}^{p,q}(M, E^*)_{h^*} < \infty$. *Then $H_{(2)}^{p,q}(M, E)_h \cong H_{(2)}^{q,p}(M, E^*)_{h^*}$.*

§ 4. Proof of Theorems

First we shall prove Theorem 2.

Let X, Y, (E, h), etc. be as in Proposition 2. 3. We shall show that the natural homomorphism τ: $H_0^{p,q}(X\backslash Y, E) \to H_{(2)}^{p,q}(X\backslash Y, E)_h$ is isomorphism if $p+q>2n-\text{codim } Y+1$. Here $H_0^{p,q}$ denotes the cohomology with compact support and the L^2 cohomology $H_{(2)}^{p,q}$ is with respect to ds_Y^2.

Surjectivity: Let $[u] \in H_{(2)}^{p,q}(X\backslash Y, E)_h$, where $u \in L^{p,q}(X\backslash Y, E)_h$ and $\bar{\partial}u=0$. Clearly, $u|_W$ is square integrable, for any choice of W(or c), with respect to ds_W^2 and h_W. Hence, by Corollary 2. 4, one can find a $v \in L_{loc}^{p,q-1}(W, E)$, square integrable with respect to ds_W^2 and h_W, such that $\bar{\partial}v=u$. Since ds_W^2 is quasi-isometric to ds_Y^2 on a neighbourhood of Y, it follows immediately that u is represented by a compactly supported form, which completes the proof of the surjectivity.

Injectivity: Let $[w] \in H_0^{p,q}(X\backslash Y, E)$. If $\tau([w])=0$, then there exists an $f \in L^{p,q-1}(X\backslash Y, E)_h$ such that $\bar{\partial}f=w$. Since the support of w is compact, $\bar{\partial}f=0$ near Y. Hence, applying Corollary 2. 4, one can find a neighbourhood $W' \supset Y$ and an E-valued $(p, q-1)$-form g on $W'\backslash Y$ such that $\bar{\partial}g=f$ on $W'\backslash Y$, whence follows that $[w]=0$.

In virtue of Andreotti-Grauert's finiteness theorem (cf. [1]), dim $H^{p,q}(X\backslash Y, E)<\infty$ for $p+q<\text{codim } Y-1$. Hence, by Serre-Malgrange's duality

(18) $\quad \dim H_0^{p,q}(X\backslash Y, E^*)<\infty$, for $p+q>2n-\text{codim } Y+1$.

Similarly, we have

(19) $\quad \dim H_0^{q,p}(X\backslash Y, E)<\infty$, for $p+q>2n-\text{codim } Y+1$.

In view of the above isomorphism, we obtain the finite dimensionality of $H_{(2)}^{p,q}(X\backslash Y, E)_h$ and $H_{(2)}^{p,q}(X\backslash Y, E^*)_{h^*}$ for $p+q>2n-\text{codim } Y+1$. Thus, by Proposition 3. 4, we have $H_{(2)}^{p,q}(X\backslash Y, E)_h \cong H_{(2)}^{q,p}(X\backslash Y, E^*)_{h^*}$ for $p+q>2n-\text{codim } Y+1$, so that $H_0^{p,q}(X\backslash Y, E) \cong H_0^{q,p}(X\backslash Y, E^*)$ for $p+q>2n-\text{codim } Y+1$.

Hence, by the duality again we obtain

$$H^{p,q}(X\backslash Y, E) \cong H^{q,p}(X\backslash Y, E^*), \text{ for } p+q<\text{codim } Y-1,$$

which completes the proof of Theorem 2.

Proof of Theorem 1 $E_1^{p,q}(X\backslash Y)=E_\infty^{p,q}(X\backslash Y)$ if every cohomology class in $H^{p,q}(X\backslash Y)$ and $H^{p-1,q}(X\backslash Y)$ is represented by a d-closed form. This can be shown for $p+q<\text{codim } Y-1$ as follows.

First, taking the dual of the isomorphism τ: $H_0^{p,q}(X\backslash Y)\to H_{(2)}^{p,q}(X\backslash Y)$ we have $H_{(2)}^{p,q}(X\backslash Y)\cong H^{p,q}(X\backslash Y)$ for $p+q<\text{codim } Y-1$. (For the trivial bundle, (E, h) is not referred to.)

Therefore, from (17) $H^{p,q}(X\backslash Y)\cong\mathscr{H}^{p,q}$ for $p+q<\text{codim } Y-1$.

Since by the equality $\square=\overline{\square}$ combined with Proposition 3.3, every form in $\mathscr{H}^{p,q}$ is d-closed, the assertion is proved.

That $E_1^{p,q}(X\backslash Y)\cong E_1^{q,p}(X\backslash Y)$ for $p+q<\text{codim } Y-1$ is a corollary of Theorem 2.

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