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A Study on New Muller's Method

By

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Abstract

When we compute a root of equation $F(X)=0$, **Muller's Method** uses three initial approximations X_0, X_1, and X_2 and determines the next approximation X_3 by the intersection of the X-axis with the parabola through $(X_0, F(X_0))$, $(X_1, F(X_1))$, and $(X_2, F(X_2))$. The procedure is repeated successively to improve the approximate solution of an equation $F(X)=0$.

Suppose a continuous function F, defined on the interval $[X_0, X_1]$ is given, with $F(X_0)$ and $F(X_1)$ being opposite signs. In our **New-Muller's Method** we choose X_0 and X_1 as the ends of the interval and take another initial approximation X_2 as the mid-point of X_0, X_1 and new approximation X_3 is the intersection of the X-axis with a quadratic curve through $(X_0, F(X_0))$, $(X_2, F(X_2))$, and $(X_1, F(X_1))$. This method is proposed to improve the rate of convergence and calculate faster for reducing the interval. Let us call this method **New-Muller's Method** in this paper.

§ 1. New-Muller's Method

Let $F(X)$ be continuous function that has a root in an interval $[X_0, X_1]$.

Beginning with the initial approximations X_0 and X_1 under the condition $F(X_0)\cdot F(X_1)<0$ an intermediate initial approximation is taken as $X_2=(X_0+X_1)/2$.

Then X_3 is the intersection of the X-axis with a quadratic curve $G(X)=0$ through three point $(X_0, F(X_0))$, $(X_2, F(X_2))$, and $(X_1, F(X_1))$.

Next we determine the closed interval I which includes the solution of $F(X)=0$, as follow;

1) If $F(X_3)F(X_0)<0 \Longrightarrow I_0=[X_0, X_3]$
1) If $F(X_3)F(X_1)<0 \Longrightarrow I_0=[X_3, X_1]$

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1) If $F(X_3)F(X_2)<0$

a) $X_3<X_2 \Longrightarrow I_0=[X_3, X_2]$

b) $X_3>X_2 \Longrightarrow I_0=[X_2, X_3]$

The closed interval I_0 is determined as above. Note that the length of I_0 is diminished less than half of the original interval in any case. To find the second approximation, we put newly the lower bound of I_0 as X_0, and upper bound of I_0 as X_1. We repeat above **New-Muller's Method** to find the nearer root of the quadratic equation whose curve passes through the last three points, and this root of the quadratic equation is included in closed interval. Now we select a contracted closed interval I described as above.

We continue the process and have a nested closed intervals $I_0, I_1 \ldots.$ We repeat it until the interval shrinks sufficiently near the solution.

Definition 1-1) We take X_0 and X_1 such that $F(X_0)$ and $F(X_1)$ have the opposite signs. X_2 is another intermediate initial approximation determined by the initial approximations X_0, X_1; that is, $X_2=(X_0+X_1)/2$.

Definition 1-2) The closed interval determined by **New-Muller's Method** is called **New-Muller's** restricted closed interval.

Although there may occur several cases according to the signatures of $F(X_n)$, it is not difficult to show that the following theorem holds in any cases.

Theorem 1-1) *Let $F(X)$ be continuous on the closed interval $[X_n, X_{n+1}]$. We apply the above procedure to have the sequence of intervals I_n, where $n=0, 1, 2, \ldots$. Then* **New-Muller's Method** *gives a Cantor's sequence of the nested closed interval, and if $n\to\infty$ then $I_n \to R$. (R is a root of $F(X)=0$.)*

§2. Method of Calculation

When the initial approximations X_0 and X_1 satisfies $F(X_0)\cdot F(X_1)<0$ another initial value is $X_2=(X_0+X_1)/2$. Let us put the quadratic polynomial that passes through the three points $(X_0, F(X_0))$ $(X_2, F(X_2))$, and $(X_1, F(X_1))$ to be

$$G(X) = a(X - X_1)^2 + b(X - X_1) + c.$$

From the above conditions we have

$$F(X_0) = a(X_0 - X_1)^2 + b(X_0 - X_1) + c$$
$$F(X_2) = a(X_2 - X_1)^2 + b(X_2 - X_1) + c$$
$$F(X_1) = a0 + b0 + c$$

and we obtain

$$a = \frac{2[F(X_0) - F(X_1)] - 4[F(X_2) - F(X_1)]}{(X_0 - X_1)^2}$$
$$b = \frac{4[F(X_2) - F(X_1)] - [F(X_0) - F(X_1)]}{X_0 - X_1}$$
$$c = F(X_1)$$

To find out the root X which is $G(X) = 0$, we use the formular for the quadratic equation. We modify the formular as follows to avoid the error arised by subtracting two close numbers:

$$X_3 = X_1 - 2c / (b + sign(b)\,(b^2 - 4ac)^{1/2})$$

Here $sign(b)$ is determined by the same method as Muller's Method.

Next initial approximation X are obtained as described in §1. This process is repeated continuously until a satisfactory solution is found.

§ 3. New-Muller's Algorithm

Suppose a continuous function F defined in the interval $[X_0, X_1]$, is given with $F(X_0)$ and $F(X_1)$ being opposite signs. This producess uses following algorithm.

Step 1 set $i = 1$

Step 2 while $i \leq N$ do step 3-9

Step 3 set $X_2 = (X_0 + X_1)/2$ (compute X)

Step 4 If $F(X_2) = 0$ or $(X_1 - X_0)/2 < \mathrm{EPS}$ then
OUTPUT(X_2): (Procedure completed successfully)
STOP

Step 5 set $h = X_1 - X_0$;
$S = F(X_0) - F(X_1)$;
$S = F(X_2) - F(X_1)$;
$a = (2S_1 - 4S_2)/h$;
$b = (4S_2 - S_1)/h$;

$c=F(X_1)$

Step 6 $D=(b-4ac)^{1/2}$

Step 7 If $|b-D|<|b+D|$ then set $E=b+D$

else set $E=b-D$

Step 8 Set $h1=-2c/E$

$X_3=X_2+h$

Step 9 If $|h|<$EPS then

OUTPUT(X_3); (Procedure completed successfully)

STOP

Step10 If $F(X_2)F(X_0)<0$

then set $X_1=X_2$ GO TO Step 2

Step11 If $F(X_3)F(X_2)<0$ and $X_2<X_3$

then set $X_0=X_2$, $X_1=X_3$ GO TO Step 2

Step12 If $F(X_3)F(X_1)<0$

then set $X_0=X_3$ GO TO Step 2

Step13 If $F(X_3)F(X_0)<0$

then set $X_1=X_3$ GO TO Step 2

Step14 If $F(X_3)F(X_2)<0$ and $X_3<X_2$

then set $X_0=X_3$, $X_1=X_2$ GO TO Step 2

Step15 If $F(X_2)F(X_3)<0$

then set $X_0=X_2$ GO TO Step 2

Examples;

1) $X^3-X-1=0$
2) $X^4-3X^3-X^2+2X+3=0$
3) $X^5-2X^4-4X^3+X^2+5X+3=0$
4) $X^6-8X^4-4X^3+7X^2+13X+6=0$
5) $X^7+X^6-8X^5-12X^4+3X^3+20X^2+19X+6=0$

The results are as follows by **Muller's Method** and **New Muller's**

Degree of Equation	Accuracy of last Root to be found		Times of Repeating	
	Muller	**New Muller**	**Muller**	**New Muller**
3	10^{-12}	10^{-12}	6	4
4	10^{-12}	10^{-12}	5	5
5	10^{-12}	10^{-12}	6	5
6	10^{-12}	10^{-12}	6	4

(Table a)

Method for above.

We have failed in being computed by Muller's Method as below **Table b.** (This **Table b** is a case of **Example 5**).)

Solved case with needed root	failed case
RUN-NUMBER=20	RUN-NUMBER=20
S(1)= 1.000	S(1)= 1.000
S(2)= 1.000	S(2)= 1.000
S(3)=− 8.000	S(3)=− 8.000
S(4)=−12.000	S(4)=−12.000
S(5)= 3.000	S(5)= 3.000
S(6)= 20.000	S(6)= 20.000
S(7)= 19.000	S(7)= 19.000
S(8)= 6.000	S(8)= 6.000
X0=1.5000000000000	X0=0. X1=0.50000000000000
X1=2.0000000000000	X2=1.0000000000000
X2=2.5000000000000	XANS =−0.18195349271674310376312178 1
XANS =1.4865575391975036634306661 65	XANS =−0.29768810779028233692855565 1
XANS =1.4803693433121029587873351 83	XANS =−0.59520560748653922322759513 0
XANS =1.4750500976250605633666168 61	XANS =−0.81273042361792215970694996 9
XANS =1.4749890380252160815288675 62	XANS =−0.68025082887470088577330784 6
XANS =1.4749890383334796698226369 926	XANS =−0.68602623290480940765334594 3
XANS =1.4749890383334796698226369 926	XANS =−0.68600293460265930889363517 1
1.47498903833480	XANS =−0.68600294823886027728576664 2
−1.7763568394003d−15	XANS =−0.68600294823886008854785245 6
	−0.68600294823886
	−1.1102230246252d−16

(**Table b**)

§ 4. Conclusion

New Muller's Method is different from Muller's Method in the following points:

1) New Muller's Method starts from two initial approximations X_0, X_1 and $X_2=(X_0+X_1)/2$ is used as an intermediate initial approximation.

2) As being shown at the **Table b**, when the initial approximation was not almost the approached value of the root we failed. But New Muller's Method is a otherwise. (By the intermediate value theorem, it is explicit.)

References

[1] Muller, D. E., *A method for solving algebraic equation using an automatic computer,* Mathematical Tables and other Aids to Computation, No. **10**, (1956), 208-215.
[2] Richard, D. L. and Burden, J., Douglas Faires *Numerical Analysis third edition*, Wadsworth, Inc., 1984, 66-77.
[3] Kadlanskv, I., *Set theory and Metric space*, Chelsea Publishing Company, 1977.