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An Aspect of Differentiable Measures on $\mathbb{R}^\infty$

By

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Introduction

In this paper we shall study differentiable probability measures μ on the usual Borel σ-field $\mathfrak{B}(\boldsymbol{R}^\infty)$ on $\boldsymbol{R}^\infty$ which is the countable direct-product on the real line $\boldsymbol{R}$. First in Section 1 we prepare some basic tools for our later discussions and discuss general properties of these measures. In Section 2 we consider measures of product-type, $\mu=\prod_{n=1}^{\infty}\mu_n$ and investigate the set D_μ of all differentiable shifts of μ. And using these results, we characterize measures μ such that $D_\mu \supseteqq l^2$. If μ_n $(n=1, 2\cdots)$ is the same measure, then μ is said to be a stationary product measure. In Section 3 we take up stationary product measures. It will turn out that D_μ is an Orlicz sequence space. Lastly in Section 4 we consider a relation of differentiability and quasi-invariance of μ.

§ 1. Preliminary Discussions

Let μ be a probability measure on the measure space $(\boldsymbol{R}^\infty, \mathfrak{B}(\boldsymbol{R}^\infty))$. For $a=(a_n)\in \boldsymbol{R}^\infty$ if $\lim_{t\to 0} t^{-1}\{\mu(E+ta)-\mu(E)\}\equiv \partial_a\mu(E)$ exists for all $E\in \mathfrak{B}(\boldsymbol{R}^\infty)$, then μ is said to be differentiable in the a-direction, or a is said to be a differentiable shift for μ. The set of all differentiable shifts will be denoted by D_μ. It is remarkable that the above pointwise limit can be taken the place of the total variation norm of signed measures. (See, [5]) It is not hard to see that $\frac{d}{dt}\mu(E+ta)=\partial_a\mu(E+ta)$ for all $t\in \boldsymbol{R}$, $\partial_a\mu=0$ if and only if $a=0$, D_μ is a linear subspace of $\boldsymbol{R}^\infty$, $\partial_{\alpha a+\beta b}\mu=\alpha\partial_a\mu+\beta\partial_b\mu$ for all

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$a, b \in D_\mu$ and $\alpha, \beta \in \boldsymbol{R}$, and $\partial_a \mu$ is absolutely continuous with respect to μ. Put for $a \in D_\mu$ $\|a\|_\mu \equiv \int \left| \frac{d\partial_a \mu}{d\mu}(x) \right| d\mu(x) = \left\| \frac{d\partial_a \mu}{d\mu} \right\|_{L^1_\mu}$. Then $\|\cdot\|_\mu$ is a norm on D_μ. Let $p_n: x=(x_n) \in \boldsymbol{R}^\infty \mapsto (x_1, \cdots, x_n) \in \boldsymbol{R}^n$ and $q_n: x=(x_n) \in \boldsymbol{R}^\infty \mapsto x_n \in \boldsymbol{R}$, and put $\mu_n(E) = \mu(q_n^{-1}(E))$ for all Borel sets E of $\boldsymbol{R}$. Then for any $a=(a_n) \in D_\mu$ $\partial_a \mu(q_n^{-1}(E)) = \lim_{t \to 0} t^{-1}\{\mu_n(E+ta_n) - \mu_n(E)\} = a_n \lim_{t \to 0} t^{-1}\{\mu_n(E+t) - \mu_n(E)\}$. Thus μ_n is differentiable and we have $\|a\|_\mu \geqq \|\partial \mu_n\| |a_n|$.

Theorem 1.1. *$(D_\mu, \|\cdot\|_\mu)$ is a Banach space of cotype 2 and the normed topology is stronger than the product topology on $\boldsymbol{R}^\infty$.*

Proof. The last assertion is a direct consequence of the above inequality. Let $\{a^{(k)}\}_k$ be a Cauchy sequence in $(D_\mu, \|\cdot\|_\mu)$. Then $a^{(k)}$ converges to $a \in \boldsymbol{R}^\infty$ in the product topology and $\{\partial_{a^{(k)}} \mu\}_k$ is a Cauchy sequence in the total variation norm of signed measures. It follows from Theorem 3.1 in [5] that $a \in D_\mu$ and $\|\partial_{a^{(k)}} \mu - \partial_a \mu\|_{\text{tot}} \longrightarrow 0 (k \longrightarrow \infty)$. Since $(D_\mu, \|\cdot\|_\mu)$ is isomorphic to a subspace of L^1_μ, it is of cotype 2. Q.E.D.

Lemma 1.1. *Let (X, τ) be a topological linear space such that X is a subset of $\boldsymbol{R}^\infty$, the vector topology τ is stronger than the product topology on $\boldsymbol{R}^\infty$ and τ is metrizable with d such that (X, d) is a complete metric space. Then if it holds $X \subseteqq D_\mu$ or $D_\mu \subseteqq X$ for some μ, the injection is continuous in either case.*

Proof. It is a consequence of closed graph theorem and Theorem 1.1. Q.E.D.

Put $\Phi = l^p (1 \leqq p < \infty)$ or c_0 and $e_n = (0, \cdots, 0, \overset{(n)}{1}, 0 \cdots)$; canonical base of Φ. And let μ be a Φ-differentiable measure. (That is, $D_\mu \supseteqq \Phi$.) Then by the above lemma, there exists some constant K such that $\int \left| \frac{d\partial_a \mu}{d\mu}(x) \right| d\mu(x) \leqq K\|a\|_\Phi$. Hence putting $\frac{d\partial_{e_n} \mu}{d\mu}(x) \equiv \rho_n(x)$, we have $\int \left| \frac{d\partial_a \mu}{d\mu}(x) - \sum_{n=1}^N a_n \rho_n(x) \right| d\mu(x) \longrightarrow 0 (N \longrightarrow \infty)$ for all $a=(a_n) \in \Phi$. Especially, $\sum_{n=1}^\infty a_n \int_B \rho_n(x) d\mu(x)$ converges to $\partial_a \mu(B)$ for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$. First we consider the case $p=1$.

Theorem 1.2. *Let μ be a probability measure on $\mathfrak{B}(\boldsymbol{R}^\infty)$ such that $D_\mu \supset \{x=(x_n) \in \boldsymbol{R}^\infty | x_n = 0$ except finite numbers of n's.$\} \equiv \boldsymbol{R}_0^\infty$. Then $D_\mu \supseteqq l^1 \Longleftrightarrow$*

$\sup_n \int \left| \frac{d\partial_{e_n}\mu}{d\mu}(x) \right| d\mu(x) < \infty.$

Proof. If $D_\mu \supseteqq l^1$, then by the inequality above we have $\int |\rho_n(x)| d\mu(x) \leqq K\|e_n\|_{l^1} = K$. Conversely if this condition is satisfied, $\{\sum_{n=1}^N a_n\rho_n(x)\}_N$ is a Cauchy sequence in L^1_μ for all $a=(a_n)\in l^1$. So we have $a\in D_\mu$ in virtue of Theorem 1.1. Q.E.D.

Now let $1<p<\infty$, $p^{-1}+q^{-1}=1$ and consider an l^p-or c_0-differentiable measures μ. Then an l^q- or l^1-valued set function T_μ: $B\in\mathfrak{B}(\boldsymbol{R}^\infty)\to (\int_B \rho_n(x)d\mu(x))_n$ is defined.

Lemma 1.2. *For any $\varepsilon>0$, there exists some $\delta>0$ such that $\mu(E)\leqq\delta$ implies $\|T_\mu(E)\|<\varepsilon$.*

Proof. Put $d(A, B)=\mu(A\ominus B)$ for all $A, B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. Identifying A with B if $\mu(A\ominus B)=0$, we have a complete metric space $(\mathfrak{B}(\boldsymbol{R}^\infty), d)$. From the absolute continuity of indefinite integral, $T_n(B)\equiv(\int_B \rho_1(x)d\mu(x), \cdots, \int_B \rho_n(x)d\mu(x), 0, 0, \cdots)$ is continuous on $(\mathfrak{B}(\boldsymbol{R}^\infty), d)$. Since $\lim_n T_n(B)= T_\mu(B)$, so a continuous point B_0 of T exists in virtue of Baire's theorem. Thus for any $\varepsilon>0$ there exists some $\delta>0$ such that $\mu(B\ominus B_0)\leqq\delta$ implies $\|T_\mu(B)-T_\mu(B_0)\|<2^{-1}\varepsilon$. Now let $\mu(E)\leqq\delta$. Put $B_1=E\cup B_0$ and $B_2= B_0\cap E^c$. Then we have $\mu(B_1-B_0)\leqq\delta$, $\mu(B_2-B_0)\leqq\delta$ and $E=B_1\cap B_2{}^c$. Hence $\|T_\mu(E)\|=\|T_\mu(B_1)\ominus T_\mu(B_2)\|\leqq\|T_\mu(B_1)\ominus T_\mu(B_0)\|+\|T_\mu(B_0)-T_\mu(B_2)\| <\varepsilon$. Q.E.D.

Theorem 1.3. *Let μ be an l^p- or c_0-differentiable measure. Then T_μ is an l^q- or l^1-valued vector measure on $(\boldsymbol{R}^\infty, \mathfrak{B}(\boldsymbol{R}^\infty))$, and it is absolutely continuous with μ.*

Proof follows directly from the above lemma. Q.E.D.

Let μ be an $l^p-(1<p<\infty)$ differentiable measure. $l^q(1<q<\infty)$ has Radon-Nikodim property. So if T_μ has bounded variation, sup $\{\sum_{B\in\pi}\|T_\mu(B)\|_{l^q}$: π partition of $\boldsymbol{R}^\infty$ into a finite number of pairwise disjoint Borel subsets$\}<\infty$, then there exists a function F in $L^1(\boldsymbol{R}^\infty, \mathfrak{B}(\boldsymbol{R}^\infty), \mu, l^q)$ such that $T_\mu(B)=\int_B F(x)d\mu(x)$ for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. From the definition of

T_μ, we have $F(x)=\sum_{n=1}^\infty \rho_n(x)e_n$ for μ-a.e.x. Hence $F\in L^1 \Longleftrightarrow \int(\sum_{n=1}^\infty |\rho_n(x)|^q)^{1/q}d\mu(x)<\infty$. Settling these arguments,

Theorem 1.4. *Let $D_\mu \supset \boldsymbol{R}_0^\infty$ and $1<p<\infty$. Then $D_\mu \supseteqq l^p$ and the l^q-valued vector measure T_μ has bounded variation if and only if* $\int\left(\sum_{n=1}^\infty \left|\frac{d\partial_{e_n}\mu}{d\mu}(x)\right|^q\right)^{1/q} d\mu(x)<\infty$.

Proof. It is nothing to prove "if part". Conversely if the last inequality holds, then $\int\left|\sum_{n=1}^\infty a_n\rho_n(x)\right|d\mu(x)\leqq \|a\|_{l^p}\int(\sum_{n=1}^\infty|\rho_n(x)|^q)^{1/q}d\mu(x)$, and $\|T_\mu(B)\|_{l^q}\leqq \int_B(\sum_{n=1}^\infty|\rho_n(x)|^q)^{1/q}d\mu(x)$ for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. So $D_\mu\supseteqq l^p$ holds and T_μ has a bounded variation. Q.E.D.

Remark 1. In general the bounded variation condition is not satisfied. For example canonical Gaussian measure G on $(\boldsymbol{R}^\infty, \mathfrak{B}(\boldsymbol{R}^\infty))$, being the product of 1-dimensional Gaussian measure $dg(t)=(2\pi)^{-1/2}\exp(-2^{-1}t^2)dt$, is l^2-differentiable. (See [5] or later arguments.) However $\int(\sum_{n=1}^\infty \rho_n(x)^2)^{1/2}dG(X)=\int(\sum_{n=1}^\infty x_n{}^2)^{1/2}dG(x)=\infty$.

§ 2. Measures of Product Type

In this section we consider measures μ of product type consisting of 1-dimensional probability measures μ_n on $(\boldsymbol{R}, \mathfrak{B}(\boldsymbol{R}))$. Then μ is $\boldsymbol{R}_0^\infty$-differentiable if and only if all the μ_n are differentiable on $\boldsymbol{R}$. Further by Theorem 7.1 in [5], it is equivalent that each μ_n has a density $f_n(t)$ with the Lebesgue measure dt on $\boldsymbol{R}$ and that $f_n'(t)$ (in distribution sense) belongs to $L^1_{dt}(\boldsymbol{R})$. Consequently we may assume that $f_n(t)$ is an absolutely continuous function with the derivative $f_n'(t)\in L^1_{dt}(\boldsymbol{R})$. From now on we always assume that μ is of such type, $d\mu(x)=\otimes_n f_n(x_n)dx_n$ and $D_\mu\supset\boldsymbol{R}_0^\infty$, and we put $\phi_n(t)=f_n'(t)/f_n(t)$. Now consider a conditional expectation $\mathrm{Exp}(F|\mathfrak{B}_n)(x)$ of $F\in L^1_\mu(\boldsymbol{R}^\infty)$ relative to the sub-σ-field $\mathfrak{B}_n=p_n{}^{-1}(\mathfrak{B}(\boldsymbol{R}^n))$. If $a=(a_n)\in D_\mu$, $\frac{d\partial_a\mu}{d\mu}(x)=\lim_{N\to\infty}\mathrm{Exp}\left(\frac{d\partial_a\mu}{d\mu}\middle|\mathfrak{B}_N\right)(x)=\lim_{N\to\infty}\sum_{n=1}^N a_nf_n'(x_n)/f_n(x_n)$ in L^1_μ.

Hence $\{\sum_{n=1}^N a_n\phi_n(x_n)\}$ forms a Cauchy sequence in L^1_μ and $\|a\|_\mu=\int|\sum_{n=1}^\infty$

$a_n\phi_n(x_n)|d\mu(x)$.

Theorem 2.1. *For* $d\mu(x)=\otimes_n f_n(x_n)dx_n$ *and* $D_\mu \supset \mathbb{R}_0^\infty$, *besides the conclusion of Theorem* 1.1, D_μ *contains* $\mathbb{R}_0^\infty$ *densely and the canonical base* $e_n=(0, \cdots, 0, \overset{(n)}{1}, 0, \cdots)$ $(n=1, 2, \cdots)$ *forms an unconditional base of* $(D_\mu, \|\cdot\|_\mu)$.

Proof. It may be only to prove the unconditionality. Let $\{\varepsilon_n(\omega)\}$ be a Bernoulli sequence. They are independent identically distributed random variables on a probability space (Ω, P) such that $P(\varepsilon_n(\omega)=\pm 1)=2^{-1}$. Now $\phi_n(x_n)$ $(n=1, 2, \cdots)$ are independent and $\int \phi_n(x_n)d\mu(x)=0$. Thus for any $a=(a_n)\in \mathbb{R}^\infty$, $\{\sum_{n=1}^N a_n\phi_n(x_n)\}$ is a martingale and it follows that $\int |\sum_{n\in N_1} a_n\phi_n(x_n)|d\mu(x) \leqq \int |\sum_{n\in N_2} a_n\phi_n(x_n)|d\mu(x)$ for finite subsets $N_1 \subseteqq N_2$ of natural numbers. Hence for all $\omega\in\Omega$ and $a=(a_n)\in\mathbb{R}^\infty$,

$$2^{-1}\int |\textstyle\sum_{n=1}^N a_n\varepsilon_n(\omega)\phi_n(x_n)|d\mu(x) \leqq \int |\sum_{n=1}^N a_n\phi_n(x_n)|d\mu(x)$$
$$\leqq 2\int |\textstyle\sum_{n=1}^N a_n\varepsilon_n(\omega)\phi_n(x_n)|d\mu(x)\,.$$

The unconditional constant of the base is less than 2. Q.E.D.

Corollary. $(a_n)\in D_\mu \Longleftrightarrow (\varepsilon_n(\omega)a_n)\in D_\mu$ *for all* $\omega\in\Omega$.

Proposition 2.1. *For* $d\mu(x)=\otimes_n f_n(x_n)dx_n$ *and* $D_\mu\supset\mathbb{R}_0^\infty$,

$$a=(a_n)\in D_\mu \Longleftrightarrow \sup_N \int |\textstyle\sum_{n=1}^N a_n\phi_n(x_n)|d\mu(x)<\infty\,.$$

Proof. $(\Longrightarrow)$ It is obvious by the preceding arguments.

$(\Longleftarrow)$ Using Ottaviani's inequality for independent random variables, after some calculations we can derive that

$$\int \sup_N |\textstyle\sum_{n=1}^N a_n\phi_n(x_n)|d\mu(x) \leqq 12\sup_N \int |\sum_{n=1}^N a_n\phi_n(x_n)|d\mu(x)<\infty\,.$$

Hence $\{\sum_{n=1}^N a_n\phi_n(x_n)\}_N$ is a uniformly integral martingale. It follows from martingale theory that they form a Cauchy sequence in $L_\mu^1(\mathbb{R}^\infty)$, and so we have $a\in D_\mu$. Q.E.D.

These arguments are collected as a following theorem.

Theorem 2.2. *For* $d\mu(x)=\otimes_n f_n(x_n)dx_n$ *and* $D_\mu\supset\mathbb{R}_0^\infty$, *the following are all equivalent.*

(1) $a=(a_n)\in D_\mu$.

(2) $\{\sum_{n=1}^{N} a_n\phi_n(x_n)\}_N$ *forms a Cauchy sequence in* L_μ^1.

(3) $^\forall B\in\mathfrak{B}(\boldsymbol{R}^\infty)$, $\sum_{n=1}^{\infty} a_n\int_B \phi_n(x_n)d\mu(x)$ *converges.*

(4) $^\forall B\in\mathfrak{B}(\boldsymbol{R}^\infty)$, $\sup_N |\sum_{n=1}^{N} a_n\int_B \phi_n(x_n)d\mu(x)|<\infty$.

(5) $\sup_N \int |\sum_{n=1}^{N} a_n\phi_n(x_n)|d\mu(x)<\infty$.

Proof. It is clear from what we have discussed. Q.E.D.

Theorem 2.3. *For* $d\mu(x)=\otimes_n f_n(x_n)dx_n$ *and* $D_\mu\supset\boldsymbol{R}_0^\infty$, *the following are all equivalent.*

(1) $c_0\subset D_\mu$

(2) $l^\infty\subset D_\mu$

(3) $\sup_N \int |\sum_{n=1}^{N} \phi_n(x_n)|d\mu(x)=S<\infty$.

Proof. (1)⇒(3). By Lemma 1.1 there exists some constant $S>0$ such that $\int|\sum_{n=1}^{N} a_n\phi_n(x_n)|d\mu(x)\leqq S\,\mathrm{Max}\{|a_n|\,|\,n=1,\cdots,N\}$ for all N and $(a_n)\in\boldsymbol{R}^N$. Thus (3) is obtained by putting $a_n=1$ $(n=1,\cdots,N)$.
(3)⇒(2). Consider a point $(t_n)\in\boldsymbol{R}^N$ such that $|t_n|\leqq 1$ $(n=1,\cdots,N)$. Then (t_n) is a convex combination of points $(\varepsilon_n(\omega))\in\boldsymbol{R}^N$ running ω on Ω. Thus,

$$\int|\sum_{n=1}^{N} t_n\phi_n(x_n)|d\mu(x)\leqq\mathop{\mathrm{Max}}_{\omega\in\Omega}\int|\sum_{n=1}^{N}\varepsilon_n(\omega)\phi_n(x_n)|d\mu(x)$$

$$\leqq 2\int|\sum_{n=1}^{N}\phi_n(x_n)|d\mu(x)\leqq 2S.$$

It follows from (5) of Theorem 2.2 that $l^\infty\subset D_\mu$. "(2)⇒(1)" is obvious.
Q.E.D.

Theorem 2.4. *For* $d\mu(x)=\otimes_n f_n(x_n)dx_n$ *and* $D_\mu\supset\boldsymbol{R}_0^\infty$, $D_\mu\supseteqq l^1\Longleftrightarrow\sup_n\int_{-\infty}^{\infty}|f_n'(t)|dt<\infty$.

Proof. As $\frac{d\partial_{e_n}\mu}{d\mu}(x)=\phi_n(x_n)$, this is a restatement of Theorem 1.2.

Q. E. D.

Remark 2. Put $M=\sup_n \int_{-\infty}^{\infty} |f_n'(t)|dt$. Then $|f_n(b)-f_n(a)| \leqq \int_a^b |f_n'(t)|dt \leqq M$. Since $\lim_{R\to\infty} f_n(R)=0$ for all n, $|f_n(a)| \leqq M$ for all n and $a \in \boldsymbol{R}$.

To investigate the structure of D_μ in more detail, we wish to discuss it in probabilistic terms.

Lemma 2.1. *Let (Ω, P) be a probability space and $X(\omega)$ be in $L_P^1(\Omega)$. Then $\int_\Omega \int_0^1 \{1-\cos(uX(\omega))\}u^{-2}dudP(\omega) \leqq B\int_\Omega |X(\omega)|dP(\omega)$, where $B=\sup_t |\int_0^t \sin u/u \, du|$.*

Proof. Take any ε such that $0<\varepsilon<1$. Integrating by parts we have $\int_\Omega \int_\varepsilon^1 \{1-\cos(uX(\omega))\}u^{-2}dudP(\omega) = -\int_\Omega \{1-\cos(X(\omega))\}dP(\omega)+\varepsilon^{-1}\int_\Omega \{1-\cos(\varepsilon X(\omega))\}dP(\omega)+\int_\Omega \left\{\int_{\varepsilon X(\omega)}^{X(\omega)} \sin u/u \, du\right\}X(\omega)dP(\omega)$. Using $1-\cos \alpha \leqq |\alpha|$, we apply bounded convergence theorem to the second term in the above equality as $\varepsilon \to +0$. Then $\int_\Omega \int_0^1 \{1-\cos(uX(\omega))\}u^{-2}dudP(\omega) \leqq B\int_\Omega |X(\omega)|dP(\omega)$. Q.E.D.

Lemma 2.2. *Let $X(\omega) \in L_P^1(\Omega)$ and put for $\alpha \in \boldsymbol{R}$,*

$$N_X(\alpha)=\int_\Omega \int_0^1 \{1-\cos(\alpha uX(\omega))\}u^{-2}dudP(\omega).$$

(If $\int_\Omega |X(\omega)|dP(\omega)=0$, then $N_X \equiv 0$. So we assume that $\int_\Omega |X(\omega)|dP(\omega) \neq 0$.) Then

(1) $N_X(\alpha)$ is differentiable for $\alpha \neq 0$ and $N_X'(\alpha)>0$ for $\alpha>0$.
(2) $N_X(t\alpha) \leqq (4t^2+|t|)N_X(\alpha)$ for all t and α.

Proof. Integrating by substitution, we can rewrite as $N_X(\alpha)=\alpha\int_\Omega \int_0^\alpha \{1-\cos(uX(\omega))\}u^{-2}dudP(\omega)$. It follows that $N_X'(\alpha)=\int_\Omega \int_0^\alpha \{1-\cos(uX(\omega))\}u^{-2}dudP(\omega)+\alpha^{-1}\int_\Omega \{1-\cos(\alpha X(\omega))\}dP(\omega)>0$ for $\alpha>0$. It is easy to see from the above form of N_X that $N_X(t\alpha) \leqq |t|N_X(\alpha)$ for $|t| \leqq 1$. Now let $|t|>1$ and choose $n=0, 1, \cdots, \cdots$ such that $2^n<|t| \leqq 2^{n+1}$. Since $1-\cos(2^k\alpha)$

$\leqq 2^{2k}(1-\cos \alpha)$ for all α and $k=0, 1, \cdot\cdot, \cdot\cdot$, it follows that $N_X(t\alpha)\leqq N_X(2^{n+1}\alpha)\leqq 4\cdot 2^{2n}N_X(\alpha)\leqq 4t^2N_X(\alpha)$. Q.E.D.

By (2) of the above lemma we can define a function $M_X(\alpha)$ on $\boldsymbol{R}$ $M_X(\alpha)=\int_{-\infty}^{\infty}N_X(t\alpha)dg(t)=\int_{\Omega}\int_0^1\{1-\exp(-2^{-1}\alpha^2u^2X^2(\omega))\}u^{-2}dudP(\omega)$.

Lemma 2.3. *Under the same assumption on* $X(\omega)$ *as in Lemma* 2.2,

(1) $k'N_X(\alpha)\leqq M_X(\alpha)\leqq kN_X(\alpha)$ for $\alpha\in\boldsymbol{R}$, where $k=\int_{-\infty}^{\infty}(4t^2+|t|)dg(t)$ and $k'=g(|t|\geqq 1)$.

(2) $M_X(0)=0$, $\lim_{\alpha\to\infty}M_X(\alpha)=\infty$ and M_X is a strictly increasing function on $[0, \infty)$.

(3) For $\alpha\neq 0$, $M_X''(\alpha)$ exists and $M_X''(\alpha)>0$.

(4) M_X satisfies (Δ_2)-condition at 0,

$$(\Delta_2);\ \lim_{\alpha\to+0}\alpha M_X'(\alpha)/M_X(\alpha)<\infty\,.$$

Proof. Integrating both sides of (2) of Lemma 2.2 with $dg(t)$, $M_X(\alpha)\leqq kN_X(\alpha)$ for $\alpha\in\boldsymbol{R}$. On the other hand, $M_X(\alpha)=\int_{-\infty}^{\infty}N_X(t\alpha)dg(t)\geqq \int_{|t|\geqq 1}N_X(t\alpha)dg(t)\geqq k'N_X(\alpha)$. Since

$$M_X(\alpha)=\alpha\int_{\Omega}\int_0^{\alpha}\{1-\exp(-2^{-1}u^2X^2(\omega))\}u^{-2}dudP(\omega), \text{ it follows that}$$

$$\lim_{\alpha\to\infty}M_X(\alpha)=\infty,\ M_X'(\alpha)=\int_{\Omega}\int_0^{\alpha}\{1-\exp(-2^{-1}u^2X^2(\omega))\}u^{-2}dudP(\omega)$$

$$+\alpha^{-1}\int_{\Omega}\{1-\exp(-2^{-1}\alpha^2X^2(\omega))\}dP(\omega)> 0 \text{ for } \alpha>0 \text{ and}$$

$M_X''(\alpha)=\int_{\Omega}\exp(-2^{-1}\alpha^2X^2(\omega))X^2(\omega)dP(\omega)>0$, for $\alpha\neq 0$. Lastly for $\alpha>0$, $\alpha M_X'(\alpha)=M_X(\alpha)+\int_{\Omega}\{1-\exp(-2^{-1}\alpha^2X^2(\omega))\}dP(\omega)\leqq 2M_X(\alpha)$. Hence the (Δ_2)-condition at 0 is satisfied. Q.E.D.

Remark 3. An Orlicz function M is a continuous non-decreasing and convex function defined on $t\geqq 0$ such that $M(0)=0$ and $\lim_{t\to\infty}M(t)=\infty$. If $M(t)=0$ for some $t>0$, M is said to be a degenerate Orlicz function. By Lemma 2.3, M_X is a non-degenerate Orlicz function satisfying (Δ_2)-condition at 0. This condition is equivalent with that for every positive number $Q>0$, $\overline{\lim}_{t\to+0}M(Qt)/M(t)<\infty$. (See, [1].)

Theorem 2.5. *Let $\{X_n(\omega)\}$ be a sequence of independent random variables on (Ω, P) such that $X_n \in L_P^1(\Omega)$ and $\int_\Omega X_n(\omega)dP(\omega)=0$, $(n=1, \cdot\cdot, \cdot\cdot)$. Then the following are all equivalent.*

(1) $\sum_{n=1}^\infty X_n(\omega)$ *converges in* $L_P^1(\Omega)$.

(2) $\sum_{n=1}^\infty \int_\Omega \int_0^1 \{1-\cos(uX_n(\omega))\} u^{-2} du dP(\omega) = \sum_{n=1}^\infty N_{X_n}(1) < \infty$.

(3) $\sum_{n=1}^\infty \int_\Omega \int_0^1 \{1-\exp(-2^{-1}u^2 X_n^2(\omega))\} u^{-2} du dP(\omega) = \sum_{n=1}^\infty M_{X_n}(1) < \infty$.

Proof. Equivalence of (2) and (3) is obvious by Lemma 2.3. Before the rest of the proofs we wish to state some remarks. Let $\{\varepsilon_n(\omega')\}$ be a Bernoulli sequence on a probability space (Ω', P'). Then

$$\int_\Omega \int_{\Omega'} \{1-\cos(\textstyle\sum_n \varepsilon_n(\omega')X_n(\omega))\} dP(\omega)dP'(\omega')$$

$$= \int_\Omega \int_{\Omega'} \{1-\exp(i\textstyle\sum_n \varepsilon_n(\omega')X_n(\omega))\} dP(\omega)dP'(\omega')$$

$$= \int_\Omega \{1-\textstyle\prod_n \cos(X_n(\omega))\} dP(\omega)$$

$$= 1-\textstyle\prod_n \int_\Omega \cos(X_n(\omega))dP(\omega) \leqq \sum_n \{1-\int_\Omega \cos(X_n(\omega))dP(\omega)\}.$$

Conversely if $1-\int_\Omega \cos(X_n(\omega))dP(\omega) \equiv a_n < 2^{-1}$ for all n and $1-\int_\Omega \int_{\Omega'} \cos(\sum_n \varepsilon_n(\omega')X_n(\omega))dP(\omega)dP'(\omega') < 2^{-1}$, then using $K_2 \leqq -\log u/(1-u) \leqq K_1$ for $|1-u| \leqq 2^{-1}$ (K_1 and K_2 are suitable positive constants), we have $\sum_n a_n \leqq -K_2^{-1} \sum_n \log(1-a_n) = -K_2^{-1} \log \prod_n (1-a_n) \leqq K_1 K_2^{-1} \{1-\int_\Omega \int_{\Omega'} \cos(\sum_n \varepsilon_n(\omega')X_n(\omega))dP(\omega)dP'(\omega')\}$. Next put $H(t) = \int_0^1 (1-\cos t\xi) d\xi = 1 - \sin t/t$. Then $H \geqq 0$ and $H(t) \geqq 2^{-1}$, if $|t| \geqq 2$. Lastly we notice that $\int_\Omega |X(\omega)| dP(\omega) = \int_0^\infty P(|X(\omega)| \geqq t)dt$ holds for all $\mathbb{R}$-valued measurable functions.

Proof of (1)⇒(2). Put $M = \sup_N \int_\Omega \int_{\Omega'} |\sum_{n=1}^N \varepsilon_n(\omega')X_n(\omega)| dP(\omega)dP'(\omega')$ and take τ such that $0 < \tau \leqq (2M)^{-1}$. Then for all $0 \leqq u \leqq 1$, $\int_\Omega \int_{\Omega'} \{1-\cos$

$(\tau u\sum_{n=1}^{N}\varepsilon_n(\omega')X_n(\omega))\}dP(\omega)dP'(\omega')\leqq Mu\tau\leqq 2^{-1}$ and $\int_\Omega\{1-\cos(\tau uX_n(\omega))\}dP(\omega)\leqq Mu\tau\leqq 2^{-1}$ for all n. Hence we have $\sum_{n=1}^{N}N_{X_n}(\tau)=\sum_{n=1}^{N}\int_\Omega\int_0^1\{1-\cos(\tau uX_n(\omega))\}u^{-2}dudP(\omega)\leqq K_2^{-1}K_1\int_\Omega\int_{\Omega'}\int_0^1\{1-\cos(\tau u\sum_{n=1}^{N}\varepsilon_n(\omega')X_n(\omega))\}u^{-2}dudP(\omega)dP'(\omega')\leqq K_2^{-1}K_1B\tau\int_\Omega\int_{\Omega'}|\sum_{n=1}^{N}\varepsilon_n(\omega')X_n(\omega)|dP(\omega)dP(\omega')\leqq K_2^{-1}K_1BM\tau$, where B is the constant in Lemma 2.1. Since $N_{X_n}(1)\leqq 2^{2k}N_{X_n}(\tau)$ for k such that $2^{-k}\leqq\tau$, it follows that $\sum_{n=1}^{\infty}N_{X_n}(1)<\infty$.

Proof of (2)⇒(1). $\int_1^\infty P\times P'(|\sum_{n=1}^{N}\varepsilon_n(\omega')X_n(\omega)|\geqq R)dR$

$$\leqq 2\int_1^\infty\int_\Omega\int_{\Omega'}H(2R^{-1}\textstyle\sum_{n=1}^{N}\varepsilon_n(\omega')X_n(\omega))dP(\omega)dP'(\omega')dR$$

$$=2\int_0^1\int_1^\infty\int_\Omega\int_{\Omega'}\{1-\cos(2R^{-1}\xi\textstyle\sum_{n=1}^{N}\varepsilon_n(\omega')X_n(\omega))\}dP(\omega)dP'(\omega')dRd\xi$$

$$\leqq 2\textstyle\sum_{n=1}^{N}\displaystyle\int_0^1\int_0^1\int_\Omega\{1-\cos(2u\xi X_n(\omega))\}u^{-2}dudP(\omega)d\xi$$

$$\leqq 2\textstyle\sum_{n=1}^{N}\displaystyle\int_0^1 N_{X_n}(2\xi)d\xi\leqq 2\textstyle\sum_{n=1}^{N}N_{X_n}(2)\leqq 8\sum_{n=1}^{\infty}N_{X_n}(1)\,.$$

It follows that $\sup_N\int_\Omega|\sum_{n=1}^{N}X_n(\omega)|dP(\omega)\leqq 2\{1+8\sum_{n=1}^{\infty}N_{X_n}(1)\}<\infty$. As we have seen in the proof of Proposition 2.1, $L_P^1(\Omega)$-convergence of $\{\sum_{n=1}^{N}X_n(\omega)\}$ is equivalent to $\sup_N\int|\sum_{n=1}^{N}X_n(\omega)|dP(\omega)<\infty$ for independent random variables $X_n(\omega)$ with mean 0. Thus we have (2)⇒(1). Q.E.D.

From Theorem 2.2 and Theorem 2.5 we have.

Theorem 2.6. *For $d\mu(x)=\otimes_n f_n(x_n)dx_n$ and $D_\mu\supset \boldsymbol{R}_0^\infty$, the following are all equivalent.*

(1) $a=(a_n)\in D_\mu$.

(2) *For* $N_n(\alpha)\equiv\int_{R^\infty}\int_0^1\{1-\cos(\alpha u\phi_n(x_n))\}u^{-2}dud\mu(x)(n=1,\cdots)$,

$\sum_{n=1}^{\infty}N_n(a_n)<\infty$.

(3) *For* $M_n(\alpha)\equiv\int_{R^\infty}\int_0^1\{1-\exp(-2^{-1}\alpha^2u^2\phi_n^2(x_n))\}u^{-2}dud\mu(x)$ $(n=1,\cdots)$,

$\sum_{n=1}^{\infty}M_n(a_n)<\infty$.

Remark 4. In [1], modular sequence spaces are stated as follows. Let $\{M_n\}$ be a sequence of Orlicz functions. The space $l_{\{M_n\}}$ is the Banach space of all sequences $a=(a_n)$ with $\sum_{n=1}^{\infty} M_n(\rho^{-1}|a_n|)<\infty$ for some $\rho>0$, equipped with the norm $\|a\|=\inf\{\rho|\sum_{n=1}^{\infty} M_n(\rho^{-1}|a_n|)\leqq 1\}$. The space $l_{\{M_n\}}$ is called a modular sequence space. If every M_n is the same Orlicz function M, then $l_{\{M_n\}}=l_M$ is called an Orlicz sequence space. Theorem 2.6 shows that D_μ for μ of product type is a modular sequence space. Moreover if μ is a stationary product with f, $d\mu(x)=\otimes_n f(x_n)dx_n$, then D_μ is an Orlicz sequence space l_{M_Φ}, $\phi=f'/f$.

Theorem 2.7. *For $d\mu(x)=\otimes_n f_n(x_n)dx_n$ and $D_\mu\supset \dot{\mathbf{R}}_0^\infty$, the following are all equivalent.*

(1) $D_\mu \supseteqq l^2$.

(2) *There exists $(\delta_n)\in l^2$ such that*

$$\sup_n \int_{\mathbf{R}^\infty}\int_0^1 \phi_n^2(x_n)\exp(-\delta_n^2\phi_n^2(x_n)u^2)du\,d\mu(x)\equiv M<\infty.$$

(3) *There exists $(\delta_n)\in D_\mu$ such that the same inequality as in* (2) *holds.*

Proof. (2)⇒(3).

Put $F(t)=\int_0^1\{1-\exp(-u^2t)\}u^{-2}du/t\int_0^1\exp(-u^2t)du$ for $0\leqq t<\infty$.

Then $\int_0^1\{1-\exp(-u^2t)\}u^{-2}du$ and $t\int_0^1\exp(-u^2t)du$ regarded as functions of t are both $O(\sqrt{t})$ at $t=\infty$ and $O(t)$ at $t=0$. Hence some constant k exists such that $F(t)\leqq k$, and $\sum_{n=1}^{\infty} M_n(\sqrt{2}\delta_n)$

$$=\sum_{n=1}^{\infty}\int_{\mathbf{R}^\infty}\int_0^1\{1-\exp(-\delta_n^2u^2\phi_n^2(x_n))\}u^{-2}du\,d\mu(x)$$

$$\leqq k\sum_{n=1}^{\infty}\int_{\mathbf{R}^\infty}\int_0^1\delta_n^2\phi_n^2(x_n)\exp(-\delta_n^2u^2\phi_n^2(x_n))du\,d\mu(x)\leqq kM\sum_{n=1}^{\infty}\delta_n^2<\infty.$$

This shows $(\delta_n)\in D_\mu$.

(3)⇒(1). Let $a=(a_n)\in l^2$. Then $M_n(a_n)=$

$$\int_{\mathbf{R}^\infty}\int_0^1\{1-\exp(-2^{-1}a_n^2u^2\phi_n^2(x_n))\}\exp(-\delta_n^2u^2\phi_n^2(x_n))u^{-2}du\,d\mu(x)$$

$$\times\int_{\mathbf{R}^\infty}\int_0^1\{1-\exp(-2^{-1}a_n^2u^2\phi_n^2(x_n))\}\{1-\exp(-\delta_n^2u^2\phi_n^2(x_n))\}u^{-2}du\,d\mu(x)$$

$$\leqq 2^{-1}a_n^2\int_{\mathbf{R}^\infty}\int_0^1\phi_n^2(x_n)\exp(-\delta_n^2u^2\phi_n^2(x_n))du\,d\mu(x)+$$

$$+\int_{R^\infty}\int_0^1\{1-\exp(-\delta_n^2u^2\phi_n^2(x_n))\}u^{-2}dud\mu(x)\leqq 2^{-1}Ma_n^2+M_n(\sqrt{2}\delta_n)\,.$$

So we have $\sum_{n=1}^\infty M_n(a_n)\leqq 2^{-1}M\sum_{n=1}^\infty a_n^2+\sum_{n=1}^\infty M_n(\sqrt{2}\delta_n)<\infty$.
(1)⇒(3) This assertion follows from the next lemma.

Lemma 2.4. *Let $\{X_n(\omega)\}\subset L_P^1(\Omega)$ be a sequence of independent random variables on (Ω, P) and suppose that $\sum_{n=1}^\infty M_{X_n}(a_n)<\infty$ for all $a=(a_n)\in l^2$. Then there exists $(\delta_n)\in l^2$ such that*

$$\sup_n\int_\Omega\int_0^1 X_n^2(\omega)\exp(-\delta_n^2u^2X_n^2(\omega))dudP(\omega)<\infty\,.$$

Proof. For $M_{X_n}=M_n$, the modular sequence space $l_{\{M_n\}}$ includes l^2. It follows from closed grafh theorem that the injection $l^2\longrightarrow l_{\{M_n\}}$ is continuous. So there exists some constant $R>0$ such that $\sum_{n=1}^\infty M_n(\sqrt{2}a_n)\leqq R$ for $\|(a_n)\|_{l^2}\leqq 1$. From here we shall proceed in a similar manner with in Lemma 3.2 in [2]. Put $E_{n,t}=\{(u,\omega)|0\leqq u\leqq 1, |uX_n(\omega)|<t\}$, $F_{n,t}=\{(u,\omega)|0\leqq u\leqq 1, |uX_n(\omega)|\leqq t\}$ and $t_n=\inf\{t>0|\iint_{E_{n,t}}X_n^2(\omega)dudP(\omega)>2R\}$. If the above set is empty, we put $t_n=\infty$. Note that for all $t>0$, $\iint_{E_{n,t}}X_n^2(\omega)dudP(\omega)<\infty$, $\iint_{F_{n,t_n}}X_n^2(\omega)dudP(\omega)\geqq 2R$ and $\iint_{E_{n,t_n}}X_n^2(\omega)dudP(\omega)\leqq 2R$. Let $s_n=t_n^{-1}$ if $t_n<\infty$, and $s_n=0$ if $t_n=\infty$. Then for all N, $\sum_{n=1}^N s_n^2\leqq 1$. In fact suppose that it would be false for some N. Then since $1-e^{-t}\geqq(1-e^{-1})t$ for $0\leqq t\leqq 1$,

$$R\geqq\sum_{n=1}^N M_n(\sqrt{2}s_n(s_1^2+\cdots+s_N^2)^{-1/2})$$
$$=\sum_{n=1}^N\int_\Omega\int_0^1\{1-\exp(-s_n^2(s_1^2+\cdots+s_N^2)^{-1}u^2X_n^2(\omega))\}u^{-2}dudP(\omega)\,.$$

Thus we have

$$R\geqq(1-e^{-1})\sum_{n=1}^N s_n^2(s_1^2+\cdots+s_N^2)^{-1}\iint_{F_{n,t_n}}X_n^2(\omega)dudP(\omega)\geqq 2(1-e^{-1})R\,,$$

which is a contradiction. Now we have $(s_n)\in l^2$. Therefore if we define $D_n=\{(u.\omega)|0\leqq u\leqq 1, |uX_n(\omega)|\geqq t_n\}$ and $\delta_n^2=\iint_{D_n}u^{-2}dudP(\omega)$, then $\sum_{n=1}^\infty\delta_n^2\leqq(1-e^{-1})^{-1}\sum_{n=1}^\infty\int_\Omega\int_0^1\{1-\exp(-s_n^2u^2X_n^2(\omega))\}u^{-2}dudP(\omega)<\infty$. Finally, $\int_\Omega\int_0^1 X_n^2(\omega)\exp(-\delta_n^2u^2X_n^2(\omega))dudP(\omega)\leqq\iint_{E_{n,t_n}}X_n^2(\omega)dudP(\omega)$

$$+\delta_n^{-2}\iint_{D_n} u^{-2}dudP(\omega)\leqq 2R+1.$$ Q.E.D.

Remark 5. For $D_\mu\supseteqq l^2$, it is sufficient that $\sup_n\int\phi_n^2(x_n)d\mu(x)<\infty$ which follows directly from Theorem 2.7. However this is not the necessary condition. We will see in the following example that even if none of $\phi_n(x_n)$ belongs to $L^2_\mu(R^\infty)$, we may have $D_\mu\supseteqq l^2$.

Example 1. Take two functions $g_i(t)$ $(i=1,2)$ on R such that $\int_{-\infty}^\infty g_i(t)dt=\int_{-\infty}^\infty|g_i'(t)|dt=1$, $\mathrm{Car}(g_1)\cap\mathrm{Car}(g_2)=\phi$ and $\int_{-\infty}^\infty|g_1'(t)|^2/g_1(t)dt<\infty$, $\int_{-\infty}^\infty|g_2'(t)|^2/g_2(t)dt=\infty$. Then a measure ν_1 defined by $d\nu_1(x)=\otimes_n g_1(x_n)dx_n$ is l^2-differentiable by the above remark, and hence for each $a=(a_n)\in l^2$ $\sup_N\int|\sum_{n=1}^N a_n\psi_1(x_n)|d\nu_1(x)\equiv M(a)<\infty$, by Proposition 2.1, where $\psi_i(t)=g_i'(t)/g_i(t)$ $(i=1,2)$. Now take $(c_n)\in l^2$ such that $0<c_n<1$ $(n=1,\cdots)$ and put $\alpha_{n,1}=1-c_n$, $\alpha_{n,2}=c_n$. Then a function f_n defined by $f_n(t)=\alpha_{n,1}g_1(t)+\alpha_{n,2}g_2(t)$ we have $\phi_n(t)\equiv f_n'(t)/f_n(t)=\psi_1(t)+\psi_2(t)$, which follows from Car $(g_1)\cap$Car $(g_2)=\phi$. Hence $\int\phi_n^2(x_n)d\mu(x)=\alpha_{n,1}\int_{-\infty}^\infty\psi_1(t)^2dt+\alpha_{n,2}\int_{-\infty}^\infty\psi_2(t)^2dt=\infty$ for all n. On the other hand we have $D_\mu\supseteqq l^2$. For this we shall show that $\sup_N\int|\sum_{n=1}^N a_n\phi_n(x_n)|d\mu(x)<\infty$ for all $(a_n)\in l^2$. Now

$$\int|\sum_{n=1}^N a_n\phi_n(x_n)|d\mu(x)=\int_{R^N}|\sum_{n=1}^N a_n\phi_n(x_n)|f_1(x_1)\cdots f_N(x_N)dx_1\cdots dx_N$$

$$=\sum_{\substack{i_k=1,2\\1\leq k\leq N}}\alpha_{1,i_1}\alpha_{2,i_2}\cdots\alpha_{N,i_N}\int_{R^N}|\sum_{n=1}^N a_n\phi_n(x_n)|g_{i_1}(x_1)\cdots g_{i_N}(x_N)dx_1\cdots dx_N$$

$$\leqq\sum_{\substack{i_k=1,2\\1\leq k\leq N}}\alpha_{1,i_1}\cdots\alpha_{N,i_N}\int_{R^N}|\sum_{i_n=1}a_n\phi_n(x_n)|g_{i_1}(x_1)\cdots g_{i_N}(x_N)dx_1\cdots dx_N$$

$$+\sum_{\substack{i_k=1,2\\1\leq k\leq N}}\alpha_{1,i_1}\cdots\alpha_{N,i_N}\int_{R^N}\{\sum_{i_n=2}|a_n||\phi_n(x_n)|\}g_{i_1}(x_1)\cdots g_{i_N}(x_N)dx_1\cdots dx_N$$

$$\leqq\sum_{\substack{i_k=1,2\\1\leq k\leq N}}\alpha_{1,i_1}\cdots\alpha_{N,i_N}(M(a)+\sum_{i_n=2}|a_n|)=M(a)+\sum_{n=1}^N|a_n|\alpha_{n,2}$$

$$\leqq M(a)+\sum_{n=1}^\infty|a_n|c_n<\infty\,.$$

§ 3. Stationary Product Measures

In this section we shall consider stationary product measures with f, $d\mu(x)=\otimes_n f(x_n)dx_n$, where f is a density function and $f'\in L^1_{dt}(\boldsymbol{R})$. By Theorem 2.4 stationary product measures are l^1-differentiable. Put $f'/f=\phi$ and $M_\phi(a)=\int_{-\infty}^{\infty}\int_0^1\{1-\exp(-2^{-1}a^2u^2\phi^2(t))\}u^{-2}duf(t)dt$. Then $D_\mu=\{a=(a_n)|\sum_{n=1}^{\infty}M_\phi(a_n)<\infty\}$ is an Orlicz sequence space l_{M_ϕ} and the norm $\|\cdot\|_\mu$ is equivalent to the Orlicz norm, $\|(a_n)\|_{l_{M_\phi}}=\inf\{\rho>0|\sum_{n=1}^{\infty}M_\phi(\rho^{-1}a_n)\leqq 1\}$. (Note that "for some $\rho>0$" and "for all $\rho>0$" is equivalent for the convergence of $\sum_{n=1}^{\infty}M_\phi(\rho^{-1}a_n)$, since $M_\phi(2a)\leqq 4M_\phi(a)$ on $\boldsymbol{R}$.) Now $\sum_{n=1}^{\infty}M_\phi(a_n)<\infty$ implies $\sum_{n=1}^{\infty}\{1-\exp(-2^{-1}a_n{}^2u^2\phi^2(t))\}<\infty$ for du-a.e.u and for $f(t)dt$-a.e.t. This assures that $(a_n)\in l^2$ and hence $D_\mu\subseteqq l^2$. Let M be an Orlicz function and l_M be the Orlicz sequence space,

$$l_M=\{a=(a_n)|\textstyle\sum_{n=1}^{\infty}M(\rho^{-1}|a_n|)<\infty \text{ for some } \rho>0\}$$

$$\|(a_n)\|_{l_M}=\inf\{\rho>0|\textstyle\sum_{n=1}^{\infty}M(\rho^{-1}|a_n|)\leqq 1\}\,.$$

It is easy to deduce that M is degenerate if and only if $l_M=l^\infty$ and that if M is non degenerate, then M is strictly increasing. It follows that if $l_M\subseteqq D_\mu$ for some stationary product measure μ, then M must be strictly increasing.

Theorem 3.1. *For a stationary product measure μ with f, $D_\mu\supseteqq l_M\Longleftrightarrow$ there exist $\delta_0>0$ and $K>0$ such that $M(a)\geqq KM_\phi(a)$ on $[0, \delta_0]$.*

Proof. ($\Longleftarrow$) If $\sum_{n=1}^{\infty}M(\rho^{-1}|a_n|)<\infty$ for some $\rho>0$, then $\rho^{-1}|a_n|\leqq\delta_0$ for sufficiently large n, since M is strictly increasing in virtue of the assumption. Thus we have $\sum_{n=1}^{\infty}M_\phi(\rho^{-1}a_n)<\infty$.
($\Longrightarrow$) By the closed graph theorem the injection $l_M\longrightarrow l_{M_\phi}$ is continuous. It follows that there exists $R>0$ such that $\sum_{n=1}^{\infty}M(|a_n|)\leqq 1$ implies $\sum_{n=1}^{\infty}M_\phi(a_n)\leqq R$. Take $\rho_n(n=1, 2, \cdots)$ such that $M(\rho_n)=n^{-1}$. Then $\{\rho_n\}$ is strictly decreasing, $\lim_n\rho_n=0$ and $M_\phi(\rho_n)\leqq n^{-1}R$. Consider a on $(0, \rho_1]\equiv(0, \delta_0]$ and take k such that $\rho_{k+1}<a\leqq\rho_k$. Then $M(a)>M(\rho_{k+1})=(k+1)^{-1}\geqq k(k+1)^{-1}R^{-1}M_\phi(\rho_k)\geqq(2R)^{-1}M_\phi(a)$. Q. E. D.

Theorem 3.2. *There exists some constant A such that*

$$\left|\int_{-\infty}^{\infty}\{1-\exp(ia\phi(t))\}f(t)dt\right|\leqq A\int_{-\infty}^{\infty}\int_0^1\{1-\exp(-2^{-1}a^2u^2\phi^2(t))\}u^{-2}duf(t)dt$$

for all $\alpha \in \boldsymbol{R}$.

Proof. $\int_0^1 \{1-\exp(-2^{-1}tu^2)\}u^{-2}du$ regarded as a function of $t\in[0, \infty)$ is $O(\sqrt{t})$ at $t=\infty$ and $O(t)$ at $t=0$. While $|1+it-\exp(it)|$ is $O(t)$ at $t=\infty$ and $O(t^2)$ at $t=0$. It follows that there exists a constant $A>0$ such that $|1+it-\exp(it)|\leqq A\int_0^1 \{1-\exp(-2^{-1}t^2u^2)\}u^{-2}du$ for all $t\in[0, \infty)$. Replacing t by $\alpha\phi(t)$ and integrating both sides of the above inequality, we have $\int_{-\infty}^{\infty}|1+i\alpha\phi(t)-\exp(i\alpha\phi(t))|f(t)dt\leqq A\int_{-\infty}^{\infty}\int_0^1\{1-\exp(-2^{-1}\alpha^2\phi^2(t)u^2)\}u^{-2}duf(t)dt$. As $\int_{-\infty}^{\infty}\phi(t)f(t)dt=0$, we have reached the desired result. Q.E.D.

Corollary. *For a stationary product measure with* f, $D_\mu \supseteqq l_M \Longrightarrow$ *there exists* $A>0$ *such that*

$$\left|\int_{-\infty}^{\infty}\{1-\exp(i\alpha\phi(t))\}f(t)dt\right|\leqq AM(\alpha) \text{ for all } \alpha\in\boldsymbol{R}.$$

Proof. Using Theorem 3.1 and Theorem 3.2, there exists some constant A such that $\left|\int_{-\infty}^{\infty}\{1-\exp(i\alpha\phi(t))\}f(t)dt\right|\leqq AM(\alpha)$ for sufficiently small α. Note that M is strictly increasing and $\lim_{\alpha\to\infty}M(\alpha)=\infty$, while $\left|\int_{-\infty}^{\infty}\{1-\exp(i\alpha\phi(t))\}f(t)dt\right|\leqq 2$. Thus replacing A by a suitable constant if necessary, we may consider the above inequality holds for all $\alpha\in\boldsymbol{R}$.

Q.E.D.

Theorem 3.3. *If an Orlicz function* M *satisfies*
$(*)$ $\int_0^1 M(\alpha u)u^{-2}du\leqq BM(C\alpha)$ *for all* $\alpha\in\boldsymbol{R}$, *with some constants,* B, $C>0$, *Then the condition of Corollary of Theorem* 3.2 *is also a sufficient condition for* $D_\mu\supseteqq l_M$. *(Observe that* $M(\alpha)=|\alpha|^p(1<p\leqq 2)$ *satisfies* $(*)$.)

Proof. $N_\phi(\alpha)=\int_{-\infty}^{\infty}\int_0^1\{1-\cos(\alpha u\phi(t))\}u^{-2}duf(t)dt\leqq A\int_0^1 M(\alpha u)u^{-2}du$ $\leqq AB\, M(C\alpha)$. Q.E.D.

Now we shall consider l^p-differentiability. If $p>2$, this is impossible for such μ, and if $p=2$, a result is already obtained in [5] as the following theorem. (Using Theorem 2.7 we have another proof of this theorem.)

Theorem 3.4. *For a stationary product measure* μ *with* f,

$$D_\mu \supseteqq l^2 \Longleftrightarrow \int_{-\infty}^{\infty} \phi(t)^2 f(t)dt < \infty .$$

For $1<p\leqq 2$, $\int_{-\infty}^{\infty}\{1-\exp(ia\phi(t))\}f(t)dt = O(|a|^p)$ is necessary and sufficient for $D_\mu \supseteqq l^p$ by the above theorems. The following theorem is a weak version of this assertion but it is somewhat useful.

Theorem 3.5. *For a stationary product measure μ with f and for $1<p<2$,*

(1) $D_\mu \supseteqq l^p \Longrightarrow 0<{}^\forall q<p , \quad \int_{-\infty}^{\infty}|\phi(t)|^q f(t)dt<\infty .$

(2) $\int_{-\infty}^{\infty}|\phi(t)|^p f(t)dt<\infty \Longrightarrow D_\mu \supset l^p$ *but* $D_\mu \neq l^p$.

Proof. (1) By the assumption, there exist $K>0$ and $R_0>0$ such that $M_\phi(R^{-1})\leqq KR^{-p}$ for $R\geqq R_0$. Now put $dm(t)=f(t)dt$. Then for $R\geqq R_0$, $m(|\phi(t)|\geqq R)\leqq \sqrt{e}(\sqrt{e}-1)^{-1}\int_{-\infty}^{\infty}\{1-\exp(-2^{-1}R^{-2}\phi^2(t))\}dm(t)\leqq \sqrt{e}(\sqrt{e}-1)^{-1}\int_{-\infty}^{\infty}\int_0^1\{1-\exp(-2^{-1}R^{-2}u^2\phi^2(t))\}u^{-2}dudm(t)=\sqrt{e}(\sqrt{e}-1)^{-1}M_\phi(R^{-1})\leqq \sqrt{e}(\sqrt{e}-1)^{-1}KR^{-p}$. It follows that $\int_{-\infty}^{\infty}|\phi(t)|^q f(t)dt\leqq R_0{}^q+\int_{R_0{}^q}^{\infty} m(|\phi(t)|\geqq R^{1/q})dR\leqq R_0{}^q+\sqrt{e}(\sqrt{e}-1)^{-1}K\int_{R_0{}^q}^{\infty} R^{-p/q}dR<\infty$. (2) Since $1-\exp(-2^{-1}t^2)\leqq K|t|^p$ with a suitable constant K, we have $M_\phi(a)=\int_{-\infty}^{\infty}\int_0^1\{1-\exp(-2^{-1}a^2u^2\phi^2(t))\}u^{-2}duf(t)dt\leqq K|a|^p\int_{-\infty}^{\infty}|\phi(t)|^p f(t)dt\int_0^1 u^{p-2}du$. Thus $l^p \subseteqq D_\mu$ by Theorem 3.1. If we would have $D_\mu=l^p$, then we proceed in the same way as in the proof of Theorem 3.1 and conclude that there exist $A>0$ and $\delta_0>0$ such that

$$Aa^p\leqq \int_{-\infty}^{\infty}\int_0^1\{1-\exp(-2^{-1}a^2u^2\phi^2(t))\}u^{-2}duf(t)dt , \quad \text{for } 0\leqq a\leqq \delta_0 .$$

Hence $A\leqq \int_{-\infty}^{\infty}\int_0^1\{1-\exp(-2^{-1}a^2u^2\phi^2(t))\}|a|^{-p}u^{-2}duf(t)dt$. However, since $\{1-\exp(-2^{-1}a^2u^2\phi^2(t))\}|a|^{-p}u^{-2}\leqq Ku^{p-2}|\phi(t)|^p$, we can apply Lebesgue bounded convergence theorem to get

$$\lim_{a\to 0}\int_{-\infty}^{\infty}\int_0^1\{1-\exp(-2^{-1}a^2u^2\phi^2(t))\}|a|^{-p}u^{-2}duf(t)dt=0 ,$$

which is a contradiction. Q.E.D.

Later we shall give examples of stationary product measures μ_p such that $D_{\mu_p}=l^p$ for $1<p<2$. However,

Theorem 3.6. *For stationary product measures μ, we always have $D_\mu \neq l^1$.*

Proof. If we would have $l^1=D_\mu$ for some μ, then there exists some constant $K>0$ such that $K|a|\leqq M_\phi(a)$ for sufficiently small $|a|$. It follows take $K|a|\leqq|a|\int_{-\infty}^{\infty}\int_0^{|a|}\{1-\exp(-2^{-1}u^2\phi^2(t))\}u^{-2}duf(t)dt$ and $K\leqq \lim_{a\to+0}$ $\int_{-\infty}^{\infty}\int_0^{a}\{1-\exp(-2^{-1}u^2\phi^2(t))\}u^{-2}duf(t)dt=0$. It is a contradiction. Q.E.D.

If the product type measure μ is not stationary, then "$l^1=D_\mu$" may occur, as seen in the following example.

Example 2. Take a sequence $\{\alpha_n\}$ such that $0<\alpha_n<1$ $(n=1,\cdots)$ $\sum_{n=1}^{\infty}(1-\alpha_n)<\infty$, and put $\beta_n=\alpha_n^{-1}(1-\alpha_n)$. Let $f_n(t)$ be an even function on $\mathbf{R}$ such that α_n on $[0,2^{-1}]$, $-\alpha_n\beta_n^{-1}(t-2^{-1})+\alpha_n$ on $[2^{-1},2^{-1}+\beta_n]$ and 0 on $[2^{-1}+\beta_n,\infty)$. Then $\int_{-\infty}^{\infty}f_n(t)dt=1$ and $\int_{-\infty}^{\infty}|f_n'(t)|dt=2\alpha_n$. Hence μ defined by $d\mu(x)=\otimes_n f_n(x_n)dx_n$ is l^1-differentiable. On the other hand if $a=(a_n)\in D_\mu$, then $\|\mu_{sa}-\mu\|_{tot}<2$ for sufficiently small $s>0$. It follows that for $sa\equiv b=(b_n)$, $\prod_{n=1}^{\infty}\int_{-\infty}^{\infty}\sqrt{f_n(t)}\sqrt{f_n(t+b_n)}\,dt>0$, which is equivalent to $\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}|\sqrt{f_n(t)}-\sqrt{f_n(t+b_n)}|^2dt<\infty\Leftrightarrow\sum_{n=1}^{\infty}|b_n|<\infty$. Thus we have $D_\mu=l^1$.

Remark 6. Let C_μ be the set of all continuous shifts, i.e. $C_\mu=\{a\in\mathbf{R}^\infty|\lim_{t\to0}\|\mu_{ta}-\mu\|_{tot}=0\}$. Then the proof of the second half is valid for $a\in C_\mu$. Thus in this case we have $C_\mu=D_\mu=l^1$. (We have $D_\mu\subseteqq C_\mu$ in general.)

Example 3. $(D_{\mu_p}=l^p,\ 1<p<2)$
Put $s=1-p^{-1}(0<s<2^{-1}\Leftrightarrow 1<p<2.)$ and define $f_p(t)=\sigma_p\exp(-|t|^s)$, where σ_p is the normalizing constant. Then

$$\phi_p(t)=f_p'(t)/f_p(t)=-s|t|^{s-1}sgn(t) \text{ and}$$

$$F_p(a)\equiv\int_{-\infty}^{\infty}(1-\exp(ia\phi_p(t))f_p(t)dt$$

$$=2\sigma_p\int_0^1\{1-\cos(sa|t|^{s-1})\}\exp(-|t|^s)dt$$

$$+2\sigma_p\int_1^{\infty}\{1-\cos(sa|t|^{s-1})\}\exp(-|t|^s)dt\,.$$

Some calculations derive that the first term and the second term of the right hand is $O(|a|^p)$ and $O(a^2)$ at $a=0$ respectively. So $F_p(a)=O(|a|^p)$ at $a=0$. It follows from Theorem 3.3 that we have $D_{\mu_p}=l^p$ for $d\mu_p(x)=\otimes_n f_p(x_n)dx_n$.

For our later discussions we investigate the set $T_\mu=\{a=(a_n)|\mu_a\simeq\mu\}$, where $\simeq$ means equivalence relation for absolute continuity. Put $g_p(\xi)=\int_{-\infty}^{\infty}\exp(2\pi i\xi t)\sqrt{f_p(t)}\,dt\equiv\mathfrak{F}(\sqrt{f_p})(\xi)$. Then it is well known (See, [2].) that $a\in T_{\mu_p}\Leftrightarrow\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\{1-\exp(2\pi i a_n\xi)\}|g_p(\xi)|^2d\xi<\infty$. Since $g_p(\xi)=O(|\xi|^{-(1+s)})$ at $|\xi|=\infty$ and $\int_{-\infty}^{\infty}\{1-\exp(ia\xi)\}|g_p(\xi)|^2d\xi=O(|a|^{3-2p^{-1}})$ at $a=0$ (Detailed calculations are omitted here.), it follows that $T_{\mu_p}=l^{3-2p^{-1}}\supsetneqq l^p=D_{\mu_p}$. (Since $3-2p^{-1}>p$ for $1<p<2$.) If $s=2^{-1}$, then we proceed in the same way as before. But in this case $\int_{-\infty}^{\infty}\{1-\exp(ia\phi_2(t))\}f_2(t)dt$ and $\int_{-\infty}^{\infty}\{1-\exp(ia\xi)\}|g_2(\xi)|^2d\xi$ are $O(a^2\log|a|)$ at $a=0$. Thus we have $T_{\mu_2}=D_{\mu_2}=\{a=(a_n)|\sum_{n=1}^{\infty}a_n^2(1+|\log|a_n||)<\infty\}$.

§ 4. Relation to Quasi-Invariance

Let μ be a probability measure on $\mathfrak{B}(\boldsymbol{R}^\infty)$. If for a subset $\Phi\subset\boldsymbol{R}^\infty$ we have $T_\mu\supseteqq\Phi$, then μ is said to be Φ-quasi-invariant. We wish to discuss the relation between T_μ, D_μ and C_μ. Note that we always have $C_\mu\supseteqq D_\mu$ and $C_\mu\supseteqq T_\mu^0\equiv\{a\in T_\mu|ta\in T_\mu \text{ for all } t\in\boldsymbol{R}\}$. (For example, see [3].)

Theorem 4.1. *Let (X, τ) be a topological linear space such that X is a subset of $\boldsymbol{R}^\infty$, the vector topology τ is stronger than the product topology on $\boldsymbol{R}^\infty$ and τ is metrizable with d such that (X, d) is a complete metric space. If $X\cap T_\mu$ is dense in (X, d) and $C_\mu\supseteqq X$, then $T_\mu\supseteqq X$.*

Proof. Define a metric on C_μ such that $\delta(a, b)=\sup_{0\leq t\leq 1}\|\mu_{ta}-\mu_{tb}\|_{tot}$. Then (C_μ, δ) is a complete metric linear topological space whose topology is stronger than the product topology on $\boldsymbol{R}^\infty$. (For example, see [3].) Thus using closed graph theorem, the injection $X\mapsto C_\mu$ is continuous. Now take any $a\in X$. Then by the assumption a sequence $\{a_n\}\subset X\cap T_\mu$ exists such that $\lim_{n\to\infty} d(a, a_n)=0$, hence $\lim_{n\to\infty}\|\mu_a-\mu_{a_n}\|_{tot}=0$. Consequently, for $E\in\mathfrak{B}(\boldsymbol{R}^\infty)$, $\mu(E)=0\Rightarrow {}^\forall n$, $\mu_{a_n}(E)=0\Rightarrow\mu_a(E)=0$. Similarly, $\mu(E)=0$ implies that $\mu_{-a}(E)=0$.

Corollary 1. *Let μ be R_0^∞-quasi-invariant and (X, τ) satisfy the same assumption of Theorem 4.1. If X contains R_0^∞ densely, then $C_\mu \supseteqq X$ implies $T_\mu \supseteqq X$.*

Corollary 2. *If T_μ is a dense subset of (C_μ, δ), then $T_\mu^0 = C_\mu$.*

Proof. Put $X = C_\mu$ in Theorem 4.1. Then we have $T_\mu \supseteqq C_\mu$ and therefore $T_\mu = C_\mu$. Since C_μ is a linear space, so is T_μ and therefore $T_\mu^0 = T_\mu = C_\mu$. Q. E. D.

Now let $\mathfrak{B}^n$ be a minimal σ-field with which all the coordinate functions $x_{n+1}, x_{n+2}, \cdots$ are measurable and put $\mathfrak{B}^\infty = \bigcap_{n=1}^\infty \mathfrak{B}^n$. If R_0^∞-quasi-invariant measure μ takes only the values 1 or 0 on $\mathfrak{B}^\infty$, μ is said to be R_0^∞-ergodic.

Theorem 4.2. *If μ is R_0^∞-ergodic, then $T_\mu \supseteqq C_\mu$ and therefore $T_\mu^0 = C_\mu$.*

Proof. For $\mu_a \simeq \mu$, it is necessary and sufficient that $\mu = \mu_a$ on $\mathfrak{B}^\infty$. (See, [2].) Now let $a \in C_\mu$ and $B \in \mathfrak{B}^\infty$. Since $\mu(B + ta)$ is continuous for t and takes only the values 0 or 1, so it is a constant. Thus we have $\mu(B) = \mu(B + a)$. Q.E.D.

In general even if μ is R_0^∞-quasi-invariant, differentiability does not imply quasi-invariance and vice-versa.

Example 4. Let $f(t)$ be an even function on R such that $3(t-1)^2/2$ on $[0, 1]$ and 0 on $[1, \infty)$. Put $S; t \in R \longrightarrow t(1, 1, \cdot\cdot, 1, \cdot\cdot) = te \in l^\infty$ and define a measure ν on $\mathfrak{B}(R^\infty)$ such that $\nu(E) = \int_{S^{-1}(E)} f(t)dt$ for $E \in B(R^\infty)$. We convolute ν with $dG(x) = \otimes_n (2\pi)^{-1/2} \exp(-2^{-1} x_n^2) dx_n$, and thus $\nu * G = \mu$ is obtained. Since $D_G = l^2$ and $D_\nu \ni e$, so $D_\mu \supseteqq l^2 + Re$. Put $S \equiv \{x \in R^\infty | p(x) = \lim_n \frac{1}{n}(x_1 + \cdot\cdot + x_n)$; exists.$\}$. Then S is a linear space and $\mu(S) = 1$, which implies $T_\mu \cup C_\mu \subset S$. Observe that for $B \in \mathfrak{B}(R^\infty)$ and $E \in \mathfrak{B}(R^1)$,

$$\mu(B \cap p^{-1}(E)) = \int_E G(B - te) f(t) dt \text{ and}$$

$$\mu_a(B \cap p^{-1}(E)) = \int_E G(B - a - te) f(t) dt\,, \quad \text{if } p(a) = 0\,.$$

(Namely, $[G_{te}, p]$ is a canonical decomposition of μ as stated in [4].) From this some calculations derive that $\|\mu - \mu_a\|_{tot} = \|G - G_a\|_{tot}$, if $p(a) = 0$. Now let $a \in T_\mu$. Then $\mu_a(p^{-1}(E)) = \int_{E - p(a)} f(t) dt$ must be equivalent with $\mu(p^{-1}(E))$. Hence we have $p(a) = 0$. Moreover using the above two inequality for μ and μ_a, we can derive that G_{te} is equivalent with G_{te+a} for almost all t for the measure $f(t)dt$. (See, [4].) It follows from $T_G=C_G=l^2$ that $a\in l^2$ and thus $l^2=T_\mu$. (Reverse including relation directly follows from $T_G=l^2$.) Next let $a\in C_\mu$. Then for $b=a-p(a)e$, $0=\lim_{t\to 0}\|\mu-\mu_{tb}\|_{tot}=\lim_{t\to 0}\|G-G_{tb}\|_{tot}$. So we have $b\in l^2$, $a\in l^2+\boldsymbol{R}e$ and thus $l^2+\boldsymbol{R}e=C_\mu=D_\mu$.

Remark 7. If $\boldsymbol{R}_0^\infty$-differentiable measure μ is of product type, then $\boldsymbol{R}_0^\infty$ is a dense subset of D_μ. (See, Theorem 2.1.) However this does not hold in general. In fact the measure obtained in Example 4 is $\boldsymbol{R}_0^\infty$-differentiable and $\boldsymbol{R}_0^\infty$-quasi-invariant. However $D_\mu=l^2+\boldsymbol{R}e$ whose topology coincides with the product topology of l^2 and $\boldsymbol{R}$ does not contain $\boldsymbol{R}_0^\infty$ densely.

Remark 8. Theorem 4.2 assures that $D_\mu\subseteqq T_\mu$, but $D_\mu=T_\mu$ does not hold in general, as we have seen in Example 3 to the case $1<p<2$. Now we shall supply following examples to the case $p=1$ and $p=2$.

Example 5. $(T_\mu=l^1, D_\mu\neq l^1)$

Let $K_0(u)=\int_1^\infty \exp(-ut)(t^2-1)^{-1/2}dt\,(u>0)$ be modified Bessel function. Since $\lim_{u\to+0}K_0(u)=\infty$, K_0 is not bounded. Put $f(u)=4\pi^{-1}K_0^2(2\pi|u|)$. Then $\int_{-\infty}^\infty f(u)du=1$, $g(v)\equiv \mathcal{F}(\sqrt{f})(v)=\{\pi(1+v^2)\}^{-1/2}$ and $\int_{-\infty}^\infty \exp(ivt)|g(v)|^2dv=\exp(-|t|)$. It follows that a measure ν defined as the stationary product with f is l^1-quasi-invariant. (See, [2].) Now put $f_n(t)=f(t)$ if $f(t)\leqq n$, and $f_n(t)=n$ if $f(t)>n$. Then $\int_{-\infty}^\infty |f_n'(t)|dt<\infty$ for all n and $\lim_n\|f-f_n\|_{L^1}=0$. Thus taking a subsequence $\{n_k\}$ such that $\sum_{k=1}^\infty\|f-f_{n_k}\|_{L^1}<\infty$, ν is equivalent with μ defined as $d\mu(x)=\otimes_k f_{n_k}(x_k)dx_k$. (See, [2].) So we have $D_\mu\supset\boldsymbol{R}_0^\infty$ and $T_\mu=T_\nu=l^1$. However if it would $T_\mu=l^1$ hold, then by Remark 2 after Theorem 2.4 $f_{n_k}(t)(k=1,\cdots)$ must be uniformly bounded. Thus the same holds for f, but it contradicts to the unboundness of K_0.

Example 6. $(T_\mu=l^2, D_\mu\neq l^2)$

Let $(\alpha_n)\in l^1$ such that $0<\alpha_n<1$ $(n=1,\cdots)$ and (β_n) be a positive sequence such that $\lim_n\alpha_n\beta_n=\infty$. We take a non negative differentiable function $g_n(t)$ for each n satisfying $\int_{-\infty}^\infty g_n(t)dt=1$. $\int_{-\infty}^\infty|g_n'(t)|dt<\infty$ and $g_n(0)=\beta_n$. Now let $f_0(t)=(2\pi)^{-1/2}\exp(-2^{-1}t^2)$ and put $f_n(t)=(1-\alpha_n)f_0(t)+\alpha_n g_n(t)$ $(n=1,\cdots)$. Then $\|f_n-f_0\|_{L^1}=\alpha_n\|f_0-g_n\|_{L^1}\leqq 2\alpha_n$, so G which is the stationary product with f_0 is equivalent with μ defined as $d\mu(x)=\otimes_n g_n(x_n)dx_n$. Hence $T_\mu=T_G=l^2$. On the other hand $f_n(0)\geqq\alpha_n\beta_n\to\infty$ $(n\to\infty)$, so $\{f_n(t)\}$ is not uniformly bounded. It implies $D_\mu\not\supset l^1$.

Concerning the above example, we shall list a following fact whose proof follows directly from Remark 5 and Theorem 3.2 in [2].

Theorem 4.3. *If a product type measure μ is l^2-quasi-invariant, then there exists a product type measure ν such that $\nu\simeq\mu$ and $D_\nu\supseteqq l^2$.*

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