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An Unbounded Generalization of the Tomita-Takesaki Theory II

By

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Abstract

An unbounded generalization of the fundamental concepts of the Tomita-Takesaki theory such as modular automorphism groups and Radon-Nikodym derivatives is considered.

§ 1. Introduction

In this paper we continue our study of an unbounded generalization of the Tomita-Takesaki theory begun in a previous paper [14].

The Tomita-Takesaki theory shows that the vector state ω_{ξ_0} defined by a cyclic and separating vector ξ_0 for a von Neumann algebra satisfies the KMS-condition with respect to the modular automorphism group $\{\sigma_t^{\xi_0}\}$. To extend these results to unbounded operator algebras, we define the notions of modular vectors, standard vectors and standard systems for a closed O_p^*-algebra $(\mathscr{M}, \mathscr{D})$. Using the unbounded Tomita-Takesaki theory developed in a previous paper [14], we show that if ξ_0 is a modular vector for $(\mathscr{M}, \mathscr{D})$ then a one-parameter group $\{\sigma_t^{\xi_0}\}$ of $*$-automorphisms of an unbounded bicommutant $(\mathscr{M}/\mathscr{D}_{\xi_0})''_{wc}$ of the O_p^*-algebra $\mathscr{M}/\mathscr{D}_{\xi_0}$ on a dense subspace $\mathscr{D}_{\xi_0}$ of $\mathscr{D}$ is defined, and the vector state ω_{ξ_0} on $(\mathscr{M}/\mathscr{D}_{\xi_0})''_{wc}$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$.

We next apply the unitary Radon-Nikodym cocycle introduced by Connes [3] to the unbounded case. Let $(\mathscr{M}, \mathscr{D})$ be a closed O_p^*-algebra and a pair (ξ_1, ξ_2) of vectors in $\mathscr{D}$ be strongly cyclic for $\mathscr{M}$ and separating for the usual commutant $\mathscr{M}'' \equiv (\mathscr{M}'_w)'$ of the weak commutant $\mathscr{M}'_w$ of $\mathscr{M}$. Connes showed that the modular automorphism

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groups $\{\sigma_t^{\xi_1}\}$ and $\{\sigma_t^{\xi_2}\}$ of the von Neumann algebra $\mathscr{M}''$ satisfy the relation: $\sigma_t^{\xi_1}(A)=(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t\sigma_t^{\xi_2}(A)\ (D\omega''_{\xi_1}:D\omega''_{\xi_2})_t^*$ for all $t\in\mathbf{R}$ and $A\in\mathscr{M}''$, where $(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t$ is the unitary Radon-Nikodym cocycle for the vector state ω''_{ξ_1} of $\mathscr{M}''$ relative to the vector state ω''_{ξ_2} of $\mathscr{M}''$. To extend the above result to the O_p^*-algebra $(\mathscr{M}, \mathscr{D})$, we have to consider the following problems:

1. *the extension of the modular automorphism groups* $\{\sigma_t^{\xi_1}\}$ *and* $\{\sigma_t^{\xi_2}\}$ *of* $\mathscr{M}''$ *to the* O_p^**-algebra* $(\mathscr{M}, \mathscr{D})$;

2. *the invariance of domains under the unitary Radon-Nikodym cocycle* $(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t$.

With this view, we define the following notion: A pair (ξ_1, ξ_2) is said to be relative modular for $(\mathscr{M}, \mathscr{D})$ if there exists a subspace $\mathscr{E}$ of $\mathscr{D}$ such that $\xi_1, \xi_2\in\mathscr{E}$, $\mathscr{M}\mathscr{E}=\mathscr{E}$, $\Delta_{\xi_1}''^{it}\mathscr{E}=\mathscr{E}$ and $\Delta_{\xi_2}''^{it}\mathscr{E}=\mathscr{E}$ for all $t\in\mathbf{R}$, where Δ''_{ξ_1} and Δ''_{ξ_2} are modular operators of the left Hilbert algebras $\mathscr{M}''\xi_1$ and $\mathscr{M}''\xi_2$, respectively. Let (ξ_1, ξ_2) be relative modular for $(\mathscr{M}, \mathscr{D})$. We denote by $\mathscr{D}_{\xi_1\xi_2}$ the maximal subspace in the set of the above subspaces $\mathscr{E}$ of $\mathscr{D}$, denote by $(\mathscr{M}/\mathscr{D}_{\xi_1\xi_2})''_{wc}$ an unbounded bicommutant of the O_p^*-algebra $(\mathscr{M}/\mathscr{D}_{\xi_1\xi_2}, \mathscr{D}_{\xi_1\xi_2})$ and put $\sigma_t^{\xi_1}(X)=\Delta_{\xi_1}''^{it}X\Delta_{\xi_1}''^{-it}$ and $\sigma_t^{\xi_2}(X)=\Delta_{\xi_2}''^{it}X\Delta_{\xi_2}''^{-it}$ for $t\in\mathbf{R}$ and $X\in(\mathscr{M}/\mathscr{D}_{\xi_1\xi_2})''_{wc}$. We show that the closed O_p^*-algebra $(\mathscr{M}/\mathscr{D}_{\xi_1\xi_2})''_{wc}$ contains $(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t/\mathscr{D}_{\xi_1\xi_2}$ for all $t\in\mathbf{R}$, and $\{\sigma_t^{\xi_1}\}$ and $\{\sigma_t^{\xi_2}\}$ are one-parameter groups of $*$-automorphisms of $(\mathscr{M}/\mathscr{D}_{\xi_1\xi_2})''_{wc}$ which satisfy the relation: $\sigma_t^{\xi_1}(X)\xi=(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t\,\sigma_t^{\xi_2}(X)\,(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t^*\xi$ for all $t\in\mathbf{R}$, $X\in(\mathscr{M}/\mathscr{D}_{\xi_1\xi_2})''_{wc}$ and $\xi\in\mathscr{D}_{\xi_1\xi_2}$.

We study Radon-Nikodym theorems and Lebesgue decomposition theorems for O_p^*-algebras. Radon-Nikodym theorems for von Neumann algebras have been investigated in detail [1, 3, 6, 19, 24, 28, 32]. In particular, in [19] Kosaki recently defined the notions of absolute continuity and singularity for normal forms on a von Neumann algebra $\mathscr{M}_0$ with a cyclic and separating vector ξ_0, and established a Lebesgue decomposition theorem. Further, he characterized strongly ω_{ξ_0}-absolutely continuous (called ω_{ξ_0}-absolutely continuous by Kosaki) forms and ω_{ξ_0}-singular forms using the Tomita-Takesaki theory (modular operators, relative modular operators, unitary Radon-Nikodym cocycles etc).

On the other hand, in the case of O_p^*-algebras the study in this direction seems to be hardly done except for [8, 13, 16]. The difficulties in the case of O_p^*-algebras exist in the points that σ-weakly continuous positive linear functional on an O_p^*-algebra $\mathscr{M}$ is not necessarily a vector state and a pathological relation between the O_p^*-algebra $\mathscr{M}$ and the von Neumann algebra $\mathscr{M}''$ occures frequently.

In [8] Gudder defined the notion of strongly absolute continuity which is stronger than one of classical absolute continuity, and tried to obtain a Radon-Nikodym theorem for a $*$-algebra with no additional assumptions. Further, he defined the notion of singularity, and established a Lebesgue decomposition theorem in the Banach $*$-algebra case. After that, developing Gudder's results, in [13, 16] we obtained the following: Speaking roughly, a positive linear functional ϕ on a closed O_p^*-algebra $(\mathscr{M}, \mathscr{D})$ with a strongly cyclic vector ξ_0 is decomposed into the sum: $\phi=\phi_c+\phi_s$, where ϕ_c is a strongly ω_{ξ_0}-absolutely continuous part of ϕ and ϕ_s is a ω_{ξ_0}-singular part of ϕ; and ϕ is strongly ω_{ξ_0}-absolutely continuous if and only if $\phi=\phi_c$ if and only if ϕ is represented as $\phi=\omega_{H'\xi_0}$ for some positive self-adjoint operator H' affiliated with $\mathscr{M}'$ such that $\xi_0\in\mathscr{D}(H')$ and $H'\xi_0\in\mathscr{D}$. However, we didn't know whether the strongly ω_{ξ_0}-absolutely continuous part ϕ_c of ϕ in the above Lebesgue decomposition theorem is maximal, or not.

In Section 4 we show that Gudder's definitions of absolute continuity and singularity are identical with Kosaki's definitions, respectively, and apply Kosaki's results to the case of O_p^*-algebras. In particular, we obtain that a strongly ω_{ξ_0}-absolutely continuous part ϕ_c in our Lebesgue decomposition theorem is maximal in the set of strongly ω_{ξ_0}-absolutely continuous parts of ϕ. Further, using an unbounded generalization of the Tomita-Takesaki theory developed in a previous paper [14] and Section 3, we generalize the Radon-Nikodym theorem of Pedersen and Takesaki [24] to the unbounded case.

In the case of O_p^*-algebras satisfying the von Neumann density type theorem, somewhat of the pathological facts for O_p^*-algebras are omitted, and so in Section 5 we obtain more detailed results for the Radon-Nikodym theorems, and further apply these results to the spatial theory for O_p^*-algebras.

In Section 6 we first investigate the absolute continuity and the singularity of concrete positive linear functionals on the O_p^*-algebra $\mathscr{P}\left(-i\frac{d}{dt}\right)$ generated by the differential operator $-i\frac{d}{dt}$, and next characterize positive linear functionals on the canonical algebra $\mathscr{A}$ for one degree of freedom which are invariant with respect to the one-parameter group $\{\Delta^{it}_{\{e^{-n\beta}\}}\}_{t\in\boldsymbol{R}}$ of $*$-automorphisms of $\mathscr{A}$ defined by [10], and finally by modifying Kosaki's examples [19] for von Neumann algebras we construct some concrete examples of positive linear functionals on the maximal O_p^*-algebra $\mathscr{L}^{\dagger}(\mathscr{S}(\mathbb{R}))$ on the Schwartz space $\mathscr{S}(\mathbb{R})$ which show that the sum of singular positive linear functionals need not be singular, the strongly absolute continuity is not hereditary and the Lebesgue decomposition is not necessarily unique.

§2. Preliminaries

In this section we review some of the definitions and the basic properties about O_p^*-algebras and refer to [7, 9, 15, 16, 20, 23, 25, 29] for further details.

Let $\mathscr{D}$ be a pre-Hilbert space with inner product $(\ |\ )$ and $\mathfrak{H}(\mathscr{D})$ be the Hilbert space obtained by the completion of $\mathscr{D}$. We denote by $\mathscr{C}^{\dagger}(\mathscr{D}, \mathfrak{H}(\mathscr{D}))$ the set of all linear operators X such that $\mathscr{D}(X)\cap\mathscr{D}(X^*)\supset\mathscr{D}$, and define a subset $\mathscr{L}^{\dagger}(\mathscr{D})$ of $\mathscr{C}^{\dagger}(\mathscr{D}, \mathfrak{H}(\mathscr{D}))$ by

$$\mathscr{L}^{\dagger}(\mathscr{D})=\{X\in\mathscr{C}^{\dagger}(\mathscr{D}, \mathfrak{H}(\mathscr{D}))\,;\ \mathscr{D}(X)=\mathscr{D},\ X\mathscr{D}\subset\mathscr{D},\ X^*\mathscr{D}\subset\mathscr{D}\}.$$

Then $\mathscr{C}^{\dagger}(\mathscr{D}, \mathfrak{H}(\mathscr{D}))$ is a $*$-invariant vector space with the usual operations and the adjoint X^*, and $\mathscr{L}^{\dagger}(\mathscr{D})$ is a $*$-algebra with involution $X^{\dagger}=X^*/\mathscr{D}$. A $*$-subalgebra $\mathscr{M}$ of $\mathscr{L}^{\dagger}(\mathscr{D})$ is said to be an O_p^*-algebra on $\mathscr{D}$. We here treat with only O_p^*-algebras with identity operator I. An O_p^*-algebra $\mathscr{M}$ on $\mathscr{D}$ is also denoted by $(\mathscr{M}, \mathscr{D})$.

Let $(\mathscr{M}, \mathscr{D})$ be an O_p^*-algebra. A locally convex topology on $\mathscr{D}$ defined by a family $\{\|\cdot\|_X; X\in\mathscr{M}\}$ of seminorms:

$$\|\xi\|_X=\|\xi\|+\|X\xi\|,\qquad \xi\in\mathscr{D}$$

is said to be the induced topology on $\mathscr{D}$, which is denoted by $t_{\mathscr{M}}$. If $(\mathscr{D}, t_{\mathscr{M}})$ is complete, then $(\mathscr{M}, \mathscr{D})$ is said to be closed. It follows from ([25] Lemma 2.6) that for each O_p^*-algebra $(\mathscr{M}, \mathscr{D})$ there exists a closed O_p^*-algebra $(\tilde{\mathscr{M}}, \tilde{\mathscr{D}})$ which is the smallest closed extension of $(\mathscr{M}, \mathscr{D})$, which is said to be the closure of $(\mathscr{M}, \mathscr{D})$. A vector ξ_0 in $\mathscr{D}$ is said to be cyclic (resp. strongly cyclic) for $\mathscr{M}$ if $\mathscr{M}\xi_0$ is dense in $\mathfrak{H}(\mathscr{D})$ (resp. $(\mathscr{D}, t_{\mathscr{M}})$). If $\mathscr{D}=\bigcap_{X\in\mathscr{M}}\mathscr{D}(X^*)$, then $(\mathscr{M}, \mathscr{D})$ is said to be self-adjoint.

We define some locally convex topologies on an O_p^*-algebra $(\mathscr{M}, \mathscr{D})$. Locally convex topologies on $\mathscr{C}^\dagger(\mathscr{D}, \mathfrak{H}(\mathscr{D}))$ defined by systems $\{P_{\xi,\eta}(\cdot)\,;\,\xi,\eta\in\mathscr{D}\}$, $\{P_\xi(\cdot)\,;\,\xi\in\mathscr{D}\}$ and $\{P_\xi^*(\cdot)\,;\,\xi\in\mathscr{D}\}$ of seminorms:

$$P_{\xi,\eta}(X)=|(X\xi\,|\,\eta)|,\;\; P_\xi(X)=\|X\xi\|,\;\; P_\xi^*(X)=\|X\xi\|+\|X^*\xi\|$$

are said to be a weak topology, a strong topology and a strong* topology, which are denoted by t_w, t_s and t_s^*, respectively. To introduce σ-weak, σ-strong, σ-strong* topologies on $\mathscr{M}$, we define an O_p^*-algebra $([\mathscr{M}], \mathscr{D}^\infty(\mathscr{M}))$ as follows:

$$\mathscr{D}^\infty(\mathscr{M})=\{\{\xi_k\}\subset\mathscr{D}\,;\,\sum_{k=1}^{\infty}\|X\xi_k\|^2<\infty \text{ for all } X\in\mathscr{M}\}\,;$$

$$[X]\{\xi_k\}=\{X\xi_k\},\;X\in\mathscr{M},\;\{\xi_k\}\in\mathscr{D}^\infty(\mathscr{M})\,;$$

$$[\mathscr{M}]=\{[X]\,;\,X\in\mathscr{M}\}.$$

The weakest locally convex topology on $\mathscr{M}$ such that the map $X\to[X]$ of $\mathscr{M}$ into $(\mathscr{C}^\dagger(\mathscr{D}^\infty(\mathscr{M}), \mathfrak{H}(\mathscr{D})^\infty), t_w)$ (resp. $(\mathscr{C}^\dagger(\mathscr{D}^\infty(\mathscr{M}), \mathfrak{H}(\mathscr{D})^\infty), t_s)$, $(\mathscr{C}^\dagger(\mathscr{D}^\infty(\mathscr{M}), \mathfrak{H}(\mathscr{D})^\infty), t_s^*)$) is said to be a σ-weak (resp. σ-strong, σ-strong*) topology for $\mathscr{M}$, which is denoted by $t_{\sigma w}^{\mathscr{M}}$ (resp. $t_{\sigma s}^{\mathscr{M}}$, $t_{\sigma s}^{*\mathscr{M}}$), where $\mathfrak{H}(\mathscr{D})^\infty$ is the direct sum of the Hilbert spaces $\mathfrak{H}_n=\mathfrak{H}(\mathscr{D})$ for $n=1, 2, \ldots$.

We define commutants of an O_p^*-algebra $(\mathscr{M}, \mathscr{D})$ as follows:

$$\mathscr{M}'_w=\{C\in\mathscr{B}(\mathfrak{H}(\mathscr{D}))\,;\,(CX\xi\,|\,\eta)=(C\xi\,|\,X^\dagger\eta) \text{ for each } \xi,\eta\in\mathscr{D} \text{ and } X\in\mathscr{M}\},$$

where $\mathscr{B}(\mathfrak{H}(\mathscr{D}))$ is the set of all bounded linear operators on $\mathfrak{H}(\mathscr{D})$;

$$\mathscr{M}'_\sigma=\{S\in\mathscr{C}^\dagger(\mathscr{D}, \mathfrak{H}(\mathscr{D}))\,;\,(X\xi\,|\,S\eta)=(S^*\xi\,|\,X^\dagger\eta) \text{ for each } \xi,\eta\in\mathscr{D} \text{ and } X\in\mathscr{M}\}\,;$$

$$\mathscr{M}'_c=\mathscr{M}'_\sigma\cap\mathscr{L}^\dagger(\mathscr{D}).$$

Then $\mathscr{M}'_w$ (simply, $\mathscr{M}'$) is a $*$-invariant weakly closed subspace of $\mathscr{B}(\mathfrak{H}(\mathscr{D}))$, but it is not necessarily an algebra [9, 15, 25]. If $(\mathscr{M}, \mathscr{D})$ is self-adjoint, then $\mathscr{M}'\mathscr{D}=\mathscr{D}$, which implies $\mathscr{M}'$ is an algebra; and the converses don't necessarily hold. But, if $\mathscr{M}'$ is an algebra, then there exists a closed O_p^*-algebra $(\hat{\mathscr{M}}, \hat{\mathscr{D}})$ which is the smallest extension of $(\mathscr{M}, \mathscr{D})$ satisfying $\hat{\mathscr{M}}'=\mathscr{M}'$ and $\mathscr{M}'\hat{\mathscr{D}}=\hat{\mathscr{D}}$ [16]. This result is a particular case of Proposition 5. 5 in the Schmüdgen paper [29]. $\mathscr{M}'_\sigma$ is a strongly* closed subspace of $\mathscr{C}^\dagger(\mathscr{D}, \mathfrak{H}(\mathscr{D}))$ whose bounded part is identical with $\mathscr{M}'$; and $\mathscr{M}'_c$ is an O_p^*-algebra on $\mathscr{D}$. We next define bicommutants of $\mathscr{M}$ as follows:

$$\mathscr{M}''\equiv(\mathscr{M}'_w)'=\{A\in\mathscr{B}(\mathfrak{H}(\mathscr{D}));\quad AC=CA \text{ for each } C\in\mathscr{M}'\}$$

$$\mathscr{M}''_{w\sigma}=\{X\in\mathscr{C}^\dagger(\mathscr{D}, \mathfrak{H}(\mathscr{D}));\ (CX\xi|\eta)=(C\xi|X^*\eta) \text{ for each } \xi, \eta\in\mathscr{D} \text{ and } C\in\mathscr{M}'\},$$

$$\mathscr{M}''_{wc}=\mathscr{M}''_{w\sigma}\cap\mathscr{L}^\dagger(\mathscr{D}).$$

Then $\mathscr{M}''$ is a von Neumann algebra on $\mathfrak{H}(\mathscr{D})$, but $(\mathscr{M}'')'$ is not necessarily identical with $\mathscr{M}'$. If $\mathscr{M}'$ is an algebra, then $(\mathscr{M}'')'=\mathscr{M}'$. $\mathscr{M}''_{w\sigma}$ is a strongly* closed $*$-invariant subspace of $\mathscr{C}^\dagger(\mathscr{D}, \mathfrak{H}(\mathscr{D}))$ containing $\mathscr{M}\cup\mathscr{M}''$ whose bounded part is identical with $\mathscr{M}''$; and $\mathscr{M}''_{wc}$ is an O_p^*-algebra on $\mathscr{D}$, which equals

$$\mathscr{R}(\mathscr{M}'', \mathscr{D})\equiv\{X\in\mathscr{L}^\dagger(\mathscr{D});\ \bar{X} \text{ is affiliated with } \mathscr{M}''\}$$

if $\mathscr{M}'\mathscr{D}=\mathscr{D}$. Further, $\mathscr{M}'$ is an algebra if and only if the closure $\overline{\mathscr{M}''}^{t_s^*}$ of $\mathscr{M}''$ in $(\mathscr{C}^\dagger(\mathscr{D}, \mathfrak{H}(\mathscr{D})), t_s^*)$ equals $\mathscr{M}''_{w\sigma}$ if and only if $\overline{\mathscr{M}''}^{t_s^*}\cap\mathscr{L}^\dagger(\mathscr{D})=\mathscr{M}''_{wc}$ [16].

A closed O_p^*-algebra $(\mathscr{M}, \mathscr{D})$ is said to be a generalized von Neumann algebra if $\mathscr{M}'\mathscr{D}=\mathscr{D}$ and $\mathscr{M}=\mathscr{M}''_{wc}$. If $(\mathscr{M}, \mathscr{D})$ is a closed ${}_{\iota}O_p^*$-algebra such that $\mathscr{M}'\mathscr{D}=\mathscr{D}$, then $\mathscr{M}''_{wc}$ is a generalized von Neumann algebra.

Let $\mathscr{A}$ be a $*$-algebra. A $*$-homomorphism π of $\mathscr{A}$ onto an O_p^*-algebra on a dense subspace $\mathscr{D}(\pi)$ in a Hilbert space $\mathfrak{H}(\pi)$ is said to be a $*$-representation of $\mathscr{A}$ in $\mathfrak{H}_\pi$ with domain $\mathscr{D}(\pi)$. Let π be a $*$-representation of $\mathscr{A}$. We put

$$\mathscr{D}(\tilde{\pi})=\bigcap_{x\in\mathscr{A}}\mathscr{D}(\overline{\pi(x)}),\ \tilde{\pi}(x)\xi=\overline{\pi(x)}\xi,\ x\in\mathscr{A},\ \xi\in\mathscr{D}(\tilde{\pi});$$

$$\mathscr{D}(\pi^*)=\bigcap_{x\in\mathscr{A}}\mathscr{D}(\pi(x)^*),\ \pi^*(x)\xi=\pi(x^*)^*\xi,\ x\in\mathscr{A},\ \xi\in\mathscr{D}(\pi^*).$$

Then $\tilde{\pi}$ is a closed $*$-representation of $\mathscr{A}$ which is the smallest closed extension of π, which is said to be the closure of π, and π^* is a closed representation of $\mathscr{A}$, but it is not necessarily a $*$-representation [9, 15, 25]. A $*$-representation π of $\mathscr{A}$ is said to be closed (resp. self-adjoint) if $\pi=\tilde{\pi}$ (resp. $\pi=\pi^*$); that is, the O_p^*-algebra $(\pi(\mathscr{A}), \mathscr{D}(\pi))$ is closed (resp. self-adjoint).

Let ϕ be a positive linear functional on a $*$-algebra $\mathscr{A}$. It is easily shown that $\mathscr{N}_\phi=\{x\in\mathscr{A};\ \phi(x^*x)=0\}$ is a left ideal in $\mathscr{A}$. For each $x\in\mathscr{A}$ we denote by $\lambda_\phi(x)$ the coset of $\mathscr{A}/\mathscr{N}_\phi$ which contains x, and define an inner product $(\ |\ )$ on $\lambda_\phi(\mathscr{A})$ by

$$(\lambda_\phi(x)\,|\,\lambda_\phi(y))=\phi(y^*x),\qquad x,y\in\mathscr{A}.$$

Let $\mathfrak{H}_\phi$ be the Hilbert space which is completion of the pre-Hilbert space $\lambda_\phi(\mathscr{A})$, and π_ϕ be the closure of a $*$-representation π_ϕ^0 of $\mathscr{A}$ defined by

$$\pi_\phi^0(x)\lambda_\phi(y)=\lambda_\phi(xy),\ x,\ y\in\mathscr{A}.$$

The triple $(\pi_\phi, \lambda_\phi, \mathfrak{H}_\phi)$ is said to be the GNS-construction for ϕ.

§3. Modular Vectors and Relative Modular Vectors

In this section we first apply the unbounded Tomita-Takesaki theory developed in a previous paper [14] to the case of a closed O_p^*-algebra with a strongly cyclic and separating vector.

Throughout this section let $(\mathscr{M}, \mathscr{D})$ be a closed O_p^*-algebra such that $\mathscr{M}'\mathscr{D}=\mathscr{D}$ and a vector ξ_0 in $\mathscr{D}$ be cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$. Since $\mathscr{M}'\mathscr{D}=\mathscr{D}$, it follows that $\bar{X}$ is affiliated with $\mathscr{M}''$ for each $X\in\mathscr{M}$, which implies that ξ_0 is a cyclic vector for $\mathscr{M}''$, so that $\mathscr{M}''\xi_0$ is an achieved left Hilbert algebra in $\mathfrak{H}(\mathscr{D})$ equipped with the multiplication $(A\xi_0)(B\xi_0)=AB\xi_0$ and the involution $A\xi_0\to A^*\xi_0$. Let S''_{ξ_0} be the closure of the involution $A\xi_0\to A^*\xi_0$ and

$$S''_{\xi_0}=J''_{\xi_0}\Delta''^{1/2}_{\xi_0}$$

be the polar decomposition of S''_{ξ_0}. The fundamental theorem of Tomita

$$\begin{aligned}(3.1)\qquad & J''_{\xi_0}\mathscr{M}''J''_{\xi_0}=\mathscr{M}',\\ & \Delta''^{it}_{\xi_0}\mathscr{M}''\Delta''^{-it}_{\xi_0}=\mathscr{M}'',\quad \Delta''^{it}_{\xi_0}\mathscr{M}'\Delta''^{-it}_{\xi_0}=\mathscr{M}',\quad t\in\mathbb{R}\end{aligned}$$

is obtained. Further, $\mathscr{M}\xi_0$ possesses the structure of an unbounded generalization of left Hilbert algebras; that is, $\mathscr{M}\xi_0$ is a dense subspace in $\mathfrak{H}(\mathscr{D})$ and a $*$-algebra with the multiplication $(X\xi_0)(Y\xi_0)=XY\xi_0$ and the closable involution $X\xi_0\to X^\dagger\xi_0$. Let S_{ξ_0} be the closure of the involution $X\xi_0\to X^\dagger\xi_0$ and

$$S_{\xi_0}=J_{\xi_0}\Delta_{\xi_0}^{1/2}$$

be the polar decomposition of S_{ξ_0}. Then, $S_{\xi_0}\subset S''_{\xi_0}$, but they don't necessarily equal. To extend (3. 1) to the unbounded left Hilbert algebra $\mathscr{M}\xi_0$, we introduce the following notions:

Definition 3. 1. A vector ξ_0 in $\mathscr{D}$ is said to be modular for $(\mathscr{M}, \mathscr{D})$ if the following conditions hold:

(1) ξ_0 is strongly cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$;

(2) there exists a subspace $\mathscr{E}$ of $\mathscr{D}$ such that $\mathscr{M}\xi_0\subset\mathscr{E}\subset\mathscr{D}$, $\mathscr{M}\mathscr{E}=\mathscr{E}$ and $\Delta_{\xi_0}''^{it}\mathscr{E}=\mathscr{E}$ for all $t\in\mathbf{R}$.

A modular vector ξ_0 for $(\mathscr{M}, \mathscr{D})$ is said to be standard if $S''_{\xi_0}=S_{\xi_0}$.

A positive linear functional ϕ on a $*$-algebra $\mathscr{A}$ with identity e is said to be modular (resp. standard) if $\lambda_\phi(e)$ is a modular (resp. standard) vector for the O_p^*-algebra $(\pi_\phi(\mathscr{A}), \mathscr{D}(\pi_\phi))$.

Let ξ_0 be a modular vector for $(\mathscr{M}, \mathscr{D})$. Put

$$\mathscr{D}_{\xi_0}=\bigcup_{\mathscr{E}\in\mathscr{F}}\mathscr{E},$$

$$\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})=\{X\in\mathscr{L}^\dagger(\mathscr{D}_{\xi_0});\ \bar{X} \textit{ is affiliated with } \mathscr{M}''\},$$

where $\mathscr{F}$ is the set of all subspaces $\mathscr{E}$ of $\mathscr{D}$ satisfying (1) and (2) of Definition 3. 1. Then $\mathscr{D}_{\xi_0}$ is the largest element of $\mathscr{F}$.

By ([14] Theorem 3. 3) we have the following

Theorem 3. 2. *Suppose ξ_0 is a modular vector for $(\mathscr{M}, \mathscr{D})$. Then the following statements hold.*

(1) *$\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$ is a generalized von Neumann algebra on $\mathscr{D}_{\xi_0}$, which equals the bicommutant $(\mathscr{M}/\mathscr{D}_{\xi_0})''_{wc}$ of the O_p^*-algebra $(\mathscr{M}/\mathscr{D}_{\xi_0}, \mathscr{D}_{\xi_0})$. In particular, if $(\mathscr{M}, \mathscr{D})$ is self-adjoint, then so is $\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$.*

(2) *Put*

$$\sigma_t^{\xi_0}(X)=\Delta_{\xi_0}''^{it}X\Delta_{\xi_0}''^{-it}, \quad X\in\mathscr{M}, \quad t\in\mathbf{R}.$$

Then $\{\sigma_t^{\xi_0}\}_{t\in\mathbf{R}}$ is a one-parameter group of $*$-automorphisms of $\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$.

(3) *The positive linear functional ω_{ξ_0} on $\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$ defined by*

$$\omega_{\xi_0}(X)=(X\xi_0|\xi_0), \quad X\in\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$$

satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$; *that is, for each* $X, Y \in \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$ *there exists a function* $f_{X,Y}$ *in* $A(0, 1)$ *such that*

$$f_{X,Y}(t) = \omega_{\xi_0}(\sigma_t^{\xi_0}(X)Y) \text{ and } f_{X,Y}(t+i) = \omega_{\xi_0}(Y\sigma_t^{\xi_0}(X))$$

for all $t \in \mathbb{R}$, *where* $A(0, 1)$ *is the set of all complex-valued functions, bounded and continuous on* $0 \leqq I_m z \leqq 1$ *and analytic in the interior.*

Definition 3.3. A system $(\mathscr{M}, \mathscr{D}, \xi_0)$ is said to be standard if the following conditions hold:

(1) $(\mathscr{M}, \mathscr{D})$ is a generalized von Neumann algebra;

(2) a vector ξ_0 in $\mathscr{D}$ is cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$;

(3) $\Delta_{\xi_0}''^{it}\mathscr{D} = \mathscr{D}$ for all $t \in \mathbb{R}$.

A standard system $(\mathscr{M}, \mathscr{D}, \xi_0)$ is said to be full if ξ_0 is a strongly cyclic vector for $\mathscr{M}$.

Lemma 3.4. (1) *Suppose* $(\mathscr{M}, \mathscr{D}, \xi_0)$ *is a standard system. Then* $\{\sigma_t^{\xi_0}\}$ *is a one-parameter group of* $*$*-automorphisms of* $\mathscr{M}$ *and* ω_{ξ_0} *is a standard positive linear functional on* $\mathscr{M}$ *which satisfies the KMS-condition with respect to* $\{\sigma_t^{\xi_0}\}$.

(2) *Suppose* $(\mathscr{M}, \mathscr{D}, \xi_0)$ *is a full standard system. Then* ξ_0 *is a standard vector for* $(\mathscr{M}, \mathscr{D})$ *with* $\mathscr{D}_{\xi_0} = \mathscr{D}$.

Proof. (1) It is clear that $\{\sigma_t^{\xi_0}\}$ is a one-parameter group of $*$-automorphisms of $\mathscr{M}$, which implies

$$\Delta_{\xi_0}''^{it}\overline{\mathscr{M}\xi_0}^{t_\mathscr{M}} = \overline{\mathscr{M}\xi_0}^{t_\mathscr{M}}$$

for all $t \in \mathbb{R}$, where $\overline{\mathscr{M}\xi_0}^{t_\mathscr{M}}$ denote the closure of $\mathscr{M}\xi_0$ relative to the induced topology $t_\mathscr{M}$. Hence, ω_{ξ_0} is a modular positive linear functional on $\mathscr{M}$ with $\mathscr{D}_{\omega_{\xi_0}} = \mathscr{D}(\pi_{\omega_{\xi_0}})$. Further, it follows from ([14] Lemma 3.8) that $\Delta_{\xi_0}''^{it} = \Delta_{\xi_0}^{it}$ for all $t \in \mathbb{R}$, which implies ω_{ξ_0} is standard.

(2) This follows from (1).

Suppose ξ_0 is a modular vector for $(\mathscr{M}, \mathscr{D})$. By Theorem 3.2, $(\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0}), \mathscr{D}_{\xi_0}, \xi_0)$ is a standard system, but it is not necessarily full.

Lemma 3.5. *Suppose H is a positive self-adjoint operator in $\mathfrak{H}(\mathscr{D})$ affiliated with $\mathscr{M}'\cap\mathscr{M}''$ such that $\xi_0\in\mathscr{D}(H)$ and $H\xi_0\in\mathscr{D}$. Then the following statements hold.*

(1) *Suppose ξ_0 is a modular vector for $(\mathscr{M},\mathscr{D})$. Then $H\xi_0\in\mathscr{D}_{\xi_0}$ and the positive linear functional $\omega_{H\xi_0}$ on $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_0})$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. Further, suppose H is non-singular. Then $(\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_0}),\mathscr{D}_{\xi_0},H\xi_0)$ is a standard system with $S''_{H\xi_0}=S''_{\xi_0}$.*

(2) *Suppose $(\mathscr{M},\mathscr{D},\xi_0)$ is a standard system and H is non-singular. Then $(\mathscr{M},\mathscr{D},H\xi_0)$ is a standard system. In particular, if $(\mathscr{M},\mathscr{D},\xi_0)$ is full, then so is $(\mathscr{M},\mathscr{D},H\xi_0)$.*

Proof. (1) Since $\Delta''^{it}_{\xi_0}H\xi_0=H\xi_0$ for all $t\in\mathbf{R}$ and $\mathscr{D}_{\xi_0}$ is maximal, it follows that $H\xi_0\in\mathscr{D}_{\xi_0}$, so that the positive linear functional $\omega_{H\xi_0}$ on $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_0})$ is well-defined. By ([32] Theorem 15.4) the normal form $\omega''_{H\xi_0}$ on the von Neumann algebra $\mathscr{M}''$ defined by

$$\omega''_{H\xi_0}(A)=(AH\xi_0|H\xi_0),\quad A\in\mathscr{M}''$$

satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. Hence, for each $A,B\in\mathscr{M}''$ there exists a function $f_{A,B}\in A(0,1)$ such that

$$f_{A,B}(t)=\omega''_{H\xi_0}(\sigma_t^{\xi_0}(A)B),\ f_{A,B}(t+i)=\omega''_{H\xi_0}(B\sigma_t^{\xi_0}(A))$$

for all $t\in\mathbf{R}$. Since $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_0})''=\mathscr{M}''$ and $\mathscr{M}'\mathscr{D}_{\xi_0}=\mathscr{D}_{\xi_0}$, it follows that for each $X,Y\in\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_0})$ there exist sequences $\{A_n\}$, $\{B_n\}$ in $\mathscr{M}''$ such that $\lim_{n\to\infty}A_nH\xi_0=XH\xi_0$, $\lim_{n\to\infty}A_n^*H\xi_0=X^\dagger H\xi_0$, $\lim_{n\to\infty}B_nH\xi_0=YH\xi_0$ and $\lim_{n\to\infty}B_n^*H\xi_0=Y^\dagger H\xi_0$. Then, since we have

$$\sup_{t\in\mathbf{R}}|f_{A_n,B_n}(t)-(\sigma_t^{\xi_0}(X)YH\xi_0|H\xi_0)|=0,$$

$$\sup_{t\in\mathbf{R}}|f_{A_n,B_n}(t+i)-(Y\sigma_t^{\xi_0}(X)H\xi_0|H\xi_0)|=0,$$

it follows that there exists a function $f_{X,Y}\in A(0,1)$ such that

$$f_{X,Y}(t)=\omega_{H\xi_0}(\sigma_t^{\xi_0}(X)Y),\quad f_{X,Y}(t+i)=\omega_{H\xi_0}(Y\sigma_t^{\xi_0}(X))$$

for all $t\in\mathbf{R}$; that is, $\omega_{H\xi_0}$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$.

Suppose H is non-singular. Then it is clear that $H\xi_0$ is cyclic and separating for $\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})''=\mathscr{M}''$. Let $H=\int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of H and put

$$H_n=\int_0^n \lambda dE(\lambda),\ K_n=\int_{1/n}^n \frac{1}{\lambda} dE(\lambda),\ E_n=\int_{1/n}^n dE(\lambda),\ n\in\mathbb{N}.$$

Since $H_n, K_n, E_n\in\mathscr{M}'\cap\mathscr{M}''$, it follows that their restrictions to $\mathscr{D}_{\xi_0}$ are contained in $\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$. Since H is non-singular, it follows that $\{E_n\}$ converges strongly to I, which implies

$$\lim_{n\to\infty} K_n X H\xi_0=\lim_{n\to\infty} E_n X\xi_0=X\xi_0$$

for each $X\in\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$, so that $H\xi_0$ is cyclic for $\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$. Further, we have

$$\lim_{n\to\infty} AK_nH\xi_0=\lim_{n\to\infty} E_nA\xi_0=A\xi_0,$$
$$\lim_{n\to\infty} K_nA^*H\xi_0=\lim_{n\to\infty} K_nHA^*\xi_0=A^*\xi_0,$$
$$\lim_{n\to\infty} AH_n\xi_0=AH\xi_0,\ \lim_{n\to\infty} A^*H_n\xi_0=A^*H\xi_0$$

for each $A\in\mathscr{M}''$. Hence, $S''_{H\xi_0}=S''_{\xi_0}$, and so $\Delta''^{it}_{H\xi_0}\mathscr{D}_{\xi_0}=\Delta''^{it}_{\xi_0}\mathscr{D}_{\xi_0}=\mathscr{D}_{\xi_0}$ for all $t\in\mathbb{R}$. Thus $(\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0}), \mathscr{D}_{\xi_0}, H\xi_0)$ is a standard system.

(2) It follows from (1) that if $(\mathscr{M}, \mathscr{D}, \xi_0)$ is a standard system, then so is $(\mathscr{M}, \mathscr{D}, H\xi_0)$. Suppose $(\mathscr{M}, \mathscr{D}, \xi_0)$ is full. For each $X\in\mathscr{M}$ we have

$$\lim_{n\to\infty} K_nXH\xi_0=X\xi_0 \text{ and } \lim_{n\to\infty} YK_nXH\xi_0=YX\xi_0$$

for each $Y\in\mathscr{M}$. Hence, $H\xi_0$ is a strongly cyclic vector for $\mathscr{M}$. Thus, $(\mathscr{M}, \mathscr{D}, H\xi_0)$ is full.

To apply the unitary Radon-Nikodym cocycle introduced by Connes [3] to unbounded operator algebras, we define the following notion.

Definition 3.6. Let $(\mathscr{M}, \mathscr{D})$ be a closed O_p^*-algebra. A pair (ξ_1, ξ_2) of vectors in $\mathscr{D}$ is said to be relative modular for $(\mathscr{M}, \mathscr{D})$ if the following conditions hold:

(1) ξ_1 and ξ_2 are strongly cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$;

(2) there exists a subspace $\mathscr{E}$ of $\mathscr{D}$ such that

(a) $\xi_1, \xi_2\in\mathscr{E}$;

(b) $\mathscr{M}\mathscr{E}=\mathscr{E}$;

(c) $\Delta_{\xi_1}^{\prime\prime it}\mathscr{E}=\mathscr{E}$ and $\Delta_{\xi_2}^{\prime\prime it}\mathscr{E}=\mathscr{E}$

for all $t\in\mathbb{R}$.

Lemma 3.7. *Let $(\mathscr{M},\mathscr{D})$ be a closed O_p^*-algebra such that $\mathscr{M}'\mathscr{D}=\mathscr{D}$ and a pair (ξ_1,ξ_2) in $\mathscr{D}$ be relative modular for $(\mathscr{M},\mathscr{D})$. Then the following statements hold.*

(1) *Put*

$$\mathscr{D}_{\xi_1\xi_2}=\bigcup_{\mathscr{E}\in\mathscr{F}}\mathscr{E},$$

where $\mathscr{F}$ is the set of all subspaces $\mathscr{E}$ of $\mathscr{D}$ satisfying (a), (b) *and* (c) *of Definition* 3.6. *Then $\mathscr{D}_{\xi_1\xi_2}$ is maximal in $\mathscr{F}$.*

(2) *ξ_1 and ξ_2 are modular vectors for $(\mathscr{M},\mathscr{D})$ satisfying $\mathscr{D}_{\xi_1\xi_2}\subset\mathscr{D}_{\xi_1}\cap\mathscr{D}_{\xi_2}$.*

(3) $\mathscr{M}'\mathscr{D}_{\xi_1\xi_2}=\mathscr{D}_{\xi_1\xi_2}$.

(4) *Put*

$$\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})=\{X\in\mathscr{L}^\dagger(\mathscr{D}_{\xi_1\xi_2})\,;\ \bar{X}\text{ is affiliated with }\mathscr{M}''\}.$$

Then $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$ is a generalized von Neumann algebra on $\mathscr{D}_{\xi_1\xi_2}$ such that $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})'=\mathscr{M}'$. In particular, if $(\mathscr{M},\mathscr{D})$ is self-adjoint, then $(\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2}),\mathscr{D}_{\xi_1\xi_2})$ is self-adjoint.

(5) *Put*

$$\sigma_t^{\xi_1}(X)=\Delta_{\xi_1}^{\prime\prime it}X\Delta_{\xi_1}^{\prime\prime -it},\quad \sigma_t^{\xi_2}(X)=\Delta_{\xi_2}^{\prime\prime it}X\Delta_{\xi_2}^{\prime\prime -it}$$

for $X\in\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$ and $t\in\mathbb{R}$. Then $\{\sigma_t^{\xi_1}\}_{t\in\mathbb{R}}$ and $\{\sigma_t^{\xi_2}\}_{t\in\mathbb{R}}$ are one-parameter groups of $*$-automorphisms of the generalized von Neumann algebra $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$.

(6) *$(\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2}),\mathscr{D}_{\xi_1\xi_2},\xi_1)$ and $(\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2}),\mathscr{D}_{\xi_1\xi_2},\xi_2)$ are standard systems.*

Proof. The statements (1) and (2) are trivial.

(3) It is easily shown that the subspace generated by $\mathscr{M}'\mathscr{D}_{\xi_1\xi_2}$ satisfies the conditions (1), (2) and (3) of Definition 3.6. Since $\mathscr{D}_{\xi_1\xi_2}$ is maximal, we have $\mathscr{M}'\mathscr{D}_{\xi_1\xi_2}=\mathscr{D}_{\xi_1\xi_2}$.

(4) Since $\mathscr{M}\xi_1\subset\mathscr{D}_{\xi_1\xi_2}\subset\mathscr{D}$, we have $(\mathscr{M}/\mathscr{D}_{\xi_1\xi_2})'=\mathscr{M}'$. It hence

follows from (3) that $\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2})$ is an O_p^*-algebra on $\mathscr{D}_{\xi_1\xi_2}$ containing $\mathscr{M}/\mathscr{D}_{\xi_1\xi_2}$ such that

$$(3.2)\qquad \begin{cases} \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2})' = \mathscr{M}', \\ \Delta_{\xi_1}''^{it} \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2}) \Delta_{\xi_1}''^{-it} = \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2}), \\ \Delta_{\xi_2}''^{it} \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2}) \Delta_{\xi_2}''^{-it} = \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2}), \ t \in \mathbb{R}. \end{cases}$$

Put

$$\tilde{\mathscr{D}}_{\xi_1\xi_2} = \bigcap_{X \in \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2})} \mathscr{D}(\bar{X}), \ \mathscr{D}^*_{\xi_1\xi_2} = \bigcap_{X \in \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2})} \mathscr{D}(X^*).$$

Then it is shown that $\tilde{\mathscr{D}}_{\xi_1\xi_2}$ is an element of $\mathscr{F}$. Since $\mathscr{D}_{\xi_1\xi_2}$ is maximal, it follows that $\tilde{\mathscr{D}}_{\xi_1\xi_2} = \mathscr{D}_{\xi_1\xi_2}$; that is, $(\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2}), \mathscr{D}_{\xi_1\xi_2})$ is closed. Thus, $(\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2}), \mathscr{D}_{\xi_1\xi_2})$ is a generalized von Neumann algebra. Suppose $(\mathscr{M}, \mathscr{D})$ is self-adjoint. Then it is shown that $\mathscr{D}^*_{\xi_1\xi_2}$ is an element of $\mathscr{F}$, which implies $(\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2}), \mathscr{D}_{\xi_1\xi_2})$ is self-adjoint.

(5) This follows from (3. 2)

(6) This follows from (3) and (4).

Let $(\mathscr{M}, \mathscr{D})$ be a closed O_p^*-algebra such that $\mathscr{M}'\mathscr{D} = \mathscr{D}$ and vectors ξ_1 and ξ_2 in $\mathscr{D}$ be strongly cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$. Let $\mathfrak{H}_4$ be a four-dimensional Hilbert space with an orthogonal basis $\{\eta_{ij}\}_{i,j=1,2}$ and $\mathscr{F}_2$ be a 2×2-matrix algebra generated by the matrices E_{ij} which are defined by $E_{ij}\eta_{kl} = \delta_{jk}\eta_{il}$. Then we have the following

Lemma 3. 8. *$\mathscr{M}\otimes\mathscr{F}_2$ is a closed O_p^*-algebra on $\mathscr{D}\otimes\mathfrak{H}_4$ such that $(\mathscr{M}\otimes\mathscr{F}_2)'(\mathscr{D}\otimes\mathfrak{H}_4) = \mathscr{D}\otimes\mathfrak{H}_4$, and a vector $\Omega_{\xi_1\xi_2} \equiv \xi_1\otimes\eta_{11} + \xi_2\otimes\eta_{22}$ in $\mathscr{D}\otimes\mathfrak{H}_4$ is strongly cyclic for $\mathscr{M}\otimes\mathscr{F}_2$ and separating for $(\mathscr{M}\otimes\mathscr{F}_2)''$.*

Theorem 3. 9. *Let $(\mathscr{M}, \mathscr{D})$ be a closed O_p^*-algebra such that $\mathscr{M}'\mathscr{D} = \mathscr{D}$, and vectors ξ_1 and ξ_2 in $\mathscr{D}$ be strongly cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$. Then the following statements hold.*

I. *A pair (ξ_1, ξ_2) in $\mathscr{D}$ is relative modular for $(\mathscr{M}, \mathscr{D})$ if and only if $\Omega_{\xi_1\xi_2}$ is a modular vector for $(\mathscr{M}\otimes\mathscr{F}_2, \mathscr{D}\otimes\mathfrak{H}_4)$. In this case, $\mathscr{D}_{\Omega_{\xi_1\xi_2}} = \mathscr{D}_{\xi_1\xi_2}\otimes\mathfrak{H}_4$.*

II. *Suppose that (ξ_1, ξ_2) is relative modular for $(\mathscr{M}, \mathscr{D})$. Then*

(1) $(D\omega''_{\xi_1}: D\omega''_{\xi_2})_t/\mathscr{D}_{\xi_1\xi_2} \in \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2})$ *for all* $t \in \mathbf{R}$, *where* $(D\omega''_{\xi_1}: D\omega''_{\xi_2})_t$ *denotes the unitary Radon-Nikodym cocycle of the normal form* ω''_{ξ_1} *of* $\mathscr{M}''$ *relative to the normal form* ω''_{ξ_2} *of* $\mathscr{M}''$;

(2) $\sigma_t^{\xi_1}(X)\xi = (D\omega''_{\xi_1}: D\omega''_{\xi_2})_t \sigma_t^{\xi_2}(X)(D\omega''_{\xi_1}: D\omega''_{\xi_2})_t^* \xi$ *for all* $t \in \mathbf{R}$, $X \in \mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_1\xi_2})$ *and* $\xi \in \mathscr{D}_{\xi_1\xi_2}$.

Proof. I. Suppose (ξ_1, ξ_2) is relative modular for $(\mathscr{M}, \mathscr{D})$. Since $\xi_1, \xi_2 \in \mathscr{D}_{\xi_1\xi_2}$ and $\mathscr{M}\mathscr{D}_{\xi_1\xi_2} = \mathscr{D}_{\xi_1\xi_2}$, it follows that $\Omega_{\xi_1\xi_2} \in \mathscr{D}_{\xi_1\xi_2} \otimes \mathfrak{H}_4$ and $(\mathscr{M} \otimes \mathscr{F}_2)(\mathscr{D}_{\xi_1\xi_2} \otimes \mathfrak{H}_4) = \mathscr{D}_{\xi_1\xi_2} \otimes \mathfrak{H}_4$. To show $\Delta''^{it}_{\Omega_{\xi_1\xi_2}}(\mathscr{D}_{\xi_1\xi_2} \otimes \mathfrak{H}_4) = \mathscr{D}_{\xi_1\xi_2} \otimes \mathfrak{H}_4$ for all $t \in \mathbf{R}$, we here state about the definition and the basic properties of the relative modular operators [2]. Let ξ and η be cyclic and separating vectors for the von Neumann algebra $\mathscr{M}''$. Let $S''_{\xi\eta}$ denote the closure of the conjugate linear operator on $\mathscr{M}''\eta$ defined by

$$S''_{\xi\eta} A\eta = A^*\xi, \quad A \in \mathscr{M}''$$

and let

$$S''_{\xi\eta} = J''_{\xi\eta} \Delta''^{1/2}_{\xi\eta}$$

denote the polar decomposition of $S''_{\xi\eta}$. The positive selfadjoint operator $\Delta''_{\xi\eta} = S''^*_{\xi\eta} S''_{\xi\eta}$ is called the relative modular operator of ξ and η. The relative modular operators satisfy the following properties [2]:

(3.3) $\quad \Delta''^{it}_{\xi\eta} A \Delta''^{-it}_{\xi\eta} = \sigma_t^{\xi}(A), \quad A \in \mathscr{M}'', \quad t \in \mathbf{R};$

(3.4) $\quad \Delta''^{it}_{\xi\eta} \Delta''^{-it}_{\xi} \in \mathscr{M}', \quad t \in \mathbf{R};$

(3.5) $\quad (D\omega''_{\xi}: D\omega''_{\eta})_t = \Delta''^{it}_{\xi\zeta} \Delta''^{-it}_{\eta\zeta}, \quad t \in \mathbf{R}$

for each cyclic and separating vector ζ for $\mathscr{M}''$. By (3.4) and Lemma 3.7 we have

$$\begin{aligned} \Delta''^{it}_{\xi_1\xi_2} \mathscr{D}_{\xi_1\xi_2} &= \Delta''^{it}_{\xi_1\xi_2} \Delta''^{-it}_{\xi_1} \Delta''^{it}_{\xi_1} \mathscr{D}_{\xi_1\xi_2} \\ &\subset \mathscr{M}' \mathscr{D}_{\xi_1\xi_2} = \mathscr{D}_{\xi_1\xi_2}, \\ \Delta''^{it}_{\xi_2\xi_1} \mathscr{D}_{\xi_1\xi_2} &\subset \mathscr{D}_{\xi_1\xi_2}, \quad t \in \mathbf{R}. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} &\Delta''^{it}_{\Omega_{\xi_1\xi_2}}(\zeta_1 \otimes \eta_{11} + \zeta_2 \otimes \eta_{21} + \zeta_3 \otimes \eta_{12} + \zeta_4 \otimes \eta_{22}) \\ &= \Delta''^{it}_{\xi_1} \zeta_1 \otimes \eta_{11} + \Delta''^{it}_{\xi_2\xi_1} \zeta_2 \otimes \eta_{21} + \Delta''^{it}_{\xi_1\xi_2} \zeta_3 \otimes \eta_{12} + \Delta''^{it}_{\xi_2} \zeta_4 \otimes \eta_{22} \end{aligned}$$

for all $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathscr{D}_{\xi_1\xi_2}$ and $t \in \mathbf{R}$, it follows from (3.6) that

$$\Delta''^{it}_{\Omega_{\xi_1\xi_2}}(\mathscr{D}_{\xi_1\xi_2}\otimes\mathfrak{H}_4)=\mathscr{D}_{\xi_1\xi_2}\otimes\mathfrak{H}_4, \qquad t\in\mathbb{R},$$

which implies that $\Omega_{\xi_1\xi_2}$ is a modular vector for $(\mathscr{M}\otimes\mathscr{F}_2, \mathscr{D}_{\xi_1\xi_2}\otimes\mathfrak{H}_4)$ with

$$\mathscr{D}_{\xi_1\xi_2}\otimes\mathfrak{H}_4\subset\mathscr{D}_{\Omega_{\xi_1\xi_2}}. \tag{3.7}$$

Suppose $\Omega_{\xi_1\xi_2}$ is a modular vector for $(\mathscr{M}\otimes\mathscr{F}_2, \mathscr{D}\otimes\mathfrak{H}_4)$. Put

$$\mathscr{E}=\{\zeta_1\in\mathscr{D}\,;\ \zeta_1\otimes\eta_{11}+\zeta_2\otimes\eta_{21}+\zeta_3\otimes\eta_{12}+\zeta_4\otimes\eta_{22}\in\mathscr{D}_{\Omega_{\xi_1\xi_2}}\}.$$

Identifying

$$\zeta=\zeta_1\otimes\eta_{11}+\zeta_2\otimes\eta_{21}+\zeta_3\otimes\eta_{12}+\zeta_4\otimes\eta_{22}\in\mathfrak{H}\otimes\mathfrak{H}_4$$

with $\zeta=(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\in\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H}\oplus\mathfrak{H}$, every element $X=\sum_{i,j=1}^{2}X_{ij}\otimes E_{ij}\in\mathscr{M}\otimes\mathscr{F}_2$ is represented as the following matrix

$$X=\begin{pmatrix} X_{11} & X_{12} & 0 & 0\\ X_{21} & X_{22} & 0 & 0\\ 0 & 0 & X_{11} & X_{12}\\ 0 & 0 & X_{21} & X_{22}\end{pmatrix}.$$

Further, it is clear that

$$(\mathscr{M}\otimes\mathscr{F}_2)'=\left\{\begin{pmatrix} C_{11} & 0 & C_{12} & 0\\ 0 & C_{11} & 0 & C_{12}\\ C_{21} & 0 & C_{22} & 0\\ 0 & C_{21} & 0 & C_{22}\end{pmatrix};\ C_{ij}\in\mathscr{M}',\ i,j=1,2\right\}.$$

Since $(\mathscr{M}\otimes\mathscr{F}_2)\mathscr{D}_{\Omega_{\xi_1\xi_2}}=\mathscr{D}_{\Omega_{\xi_1\xi_2}}$ and $(\mathscr{M}\otimes\mathscr{F}_2)'\mathscr{D}_{\Omega_{\xi_1\xi_2}}=\mathscr{D}_{\Omega_{\xi_1\xi_2}}$, it follows that

$$\zeta_i\in\mathscr{E} \qquad (i=1,2,3,4) \tag{3.8}$$

for each $\zeta=(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\in\mathscr{D}_{\Omega_{\xi_1\xi_2}}$, which implies that $\xi_1,\xi_2\in\mathscr{E}$, $\mathscr{M}\mathscr{E}=\mathscr{E}$, and $\Delta''^{it}_{\xi_1}\mathscr{E}=\mathscr{E}$, and $\Delta''^{it}_{\xi_2}\mathscr{E}=\mathscr{E}$ for all $t\in\mathbb{R}$, so that (ξ_1,ξ_2) is relative modular for $(\mathscr{M},\mathscr{D})$ with $\mathscr{E}\subset\mathscr{D}_{\xi_1\xi_2}$. Hence, by (3.7) and (3.8) we have

$$\mathscr{D}_{\xi_1\xi_2}\otimes\mathfrak{H}_4\subset\mathscr{D}_{\Omega_{\xi_1\xi_2}}\subset\mathscr{E}\otimes\mathfrak{H}_4\subset\mathscr{D}_{\xi_1\xi_2}\otimes\mathfrak{H}_4.$$

II. By (3.5) we have

$$(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t=\Delta''^{it}_{\xi_1}\Delta''^{-it}_{\xi_2\xi_1}=\Delta''^{it}_{\xi_1\xi_2}\Delta''^{-it}_{\xi_2}, \qquad t\in\mathbb{R}.$$

It hence follows that

$$\begin{aligned}(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t\mathscr{D}_{\xi_1\xi_2}&=\Delta''^{it}_{\xi_1}\Delta''^{-it}_{\xi_2\xi_1}\mathscr{D}_{\xi_1\xi_2} &&\text{(by 3.6)}\\&=\Delta''^{it}_{\xi_1}\mathscr{D}_{\xi_1\xi_2}\\&=\mathscr{D}_{\xi_1\xi_2}\end{aligned}$$

and

$$\begin{aligned}&(D\omega''_{\xi_1}:D\omega''_{\xi_2})_t\sigma^{\xi_2}_t(X)(D\omega''_{\xi_1}:D\omega''_{\xi_2})^*_t\xi\\&=\Delta''^{it}_{\xi_1\xi_2}\Delta''^{-it}_{\xi_2}\Delta''^{it}_{\xi_2}X\Delta''^{-it}_{\xi_2}\Delta''^{it}_{\xi_2}\Delta''^{-it}_{\xi_1\xi_2}\xi\\&=\Delta''^{it}_{\xi_1\xi_2}X\Delta''^{-it}_{\xi_1\xi_2}\xi &&\text{(by 3.3)}\\&=\sigma^{\xi_1}_t(X)\xi\end{aligned}$$

for all $t\in\mathbf{R}$, $X\in\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$ and $\xi\in\mathscr{D}_{\xi_1\xi_2}$. This completes the proof.

By Theorem 3.9 we have the following

Corollary 3.10. *Suppose $(\mathscr{M},\mathscr{D},\xi_0)$ and $(\mathscr{M},\mathscr{D},\xi_1)$ are full standard systems. Then (ξ_0,ξ_1) is relative modular for $(\mathscr{M},\mathscr{D})$, $(D\omega''_{\xi_1}:D\omega''_{\xi_0})_t/\mathscr{D}\in\mathscr{M}$ for all $t\in\mathbf{R}$ and*

$$\sigma^{\xi_1}_t(X)\zeta=(D\omega''_{\xi_1}:D\omega''_{\xi_0})_t\sigma^{\xi_0}_t(X)(D\omega''_{\xi_1}:D\omega''_{\xi_0})^*_t\zeta$$

for all $t\in\mathbf{R}$, $X\in\mathscr{M}$ and $\zeta\in\mathscr{D}$.

Poposition 3.11. *Let $(\mathscr{M},\mathscr{D})$ be a closed O^*_p-algebra such that $\mathscr{M}'\mathscr{D}=\mathscr{D}$ and a pair (ξ_1,ξ_2) of vectors in $\mathscr{D}$ be relative modular for $(\mathscr{M},\mathscr{D})$. Then the following statements are equivalent.*

(1) *The positive linear functional ω_{ξ_1} on the generalized von Neumann algebra $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$ is $\{\sigma^{\xi_2}_t\}$-invariant.*

(2) *The positive linear functional ω_{ξ_2} on $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$ is $\{\sigma^{\xi_1}_t\}$-invariant.*

(3) *$\{(D\omega''_{\xi_2}:D\omega''_{\xi_1})_t\}_{t\in\mathbf{R}}$ is a strongly continuous one-parameter group of unitary operators in $\mathscr{M}''^{\xi_1}_\sigma\cap\mathscr{M}''^{\xi_2}_\sigma$, where $\mathscr{M}''^{\xi_i}_\sigma$ denotes the fixed-point algebra of $\{\sigma^{\xi_i}_t\}$ in $\mathscr{M}''$ $(i=1,2)$.*

Proof. (1)$\Rightarrow$(3) It follows from Theorem 3.9 and the $\{\sigma_t^{\xi_2}\}$-invariance of ω_{ξ_1} that

$$(X\xi_1|\xi_1)=(\sigma_t^{\xi_2}(X)\xi_1|\xi_1)$$
$$=((D\omega''_{\xi_2}:D\omega''_{\xi_1})_t\sigma_t^{\xi_1}(X)(D\omega''_{\xi_2}:D\omega''_{\xi_1})_t^*\xi_1|\xi_1)$$

for each $X\in\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$ and $t\in\mathbf{R}$, which implies by

$$(D\omega''_{\xi_2}:D\omega''_{\xi_1})_t/\mathscr{D}_{\xi_1\xi_2}\in\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$$

that

$$((D\omega''_{\xi_2}:D\omega''_{\xi_1})_{-t}\xi_1|X^\dagger\xi_1)=((D\omega''_{\xi_2}:D\omega''_{\xi_1})_{-t}X\xi_1|\xi_1) \tag{3.9}$$

for all $X\in\mathscr{M}$ and $t\in\mathbf{R}$. Since $(\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2}),\mathscr{D}_{\xi_1\xi_2},\xi_1)$ is a standard system and $\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})''=\mathscr{M}''$ by Lemma 3.7, it follows from Lemma 3.4 (1) and (3.9) that

$$((D\omega''_{\xi_2}:D\omega''_{\xi_1})_{-t}\xi_1|A^*\xi_1)=((D\omega''_{\xi_2}:D\omega''_{\xi_1})_{-t}A\xi_1|\xi_1)$$

for all $A\in\mathscr{M}''$ and $t\in\mathbf{R}$, which implies the normal form ω''_{ξ_1} on $\mathscr{M}''$ is $\{\sigma_t^{\xi_2}\}$-invariant, so that the statement (3) follows from ([31] Corollary 10.28).

(3)$\Rightarrow$(1) By ([31] Corollary 10.28) we have

$$\omega''_{\xi_1}(\sigma_t^{\xi_2}(A))=\omega''_{\xi_1}(A)$$

for all $A\in\mathscr{M}''$ and $t\in\mathbf{R}$, which implies

$$\omega_{\xi_1}(\sigma_t^{\xi_2}(X))=\omega_{\xi_1}(X)$$

for all $X\in\mathscr{R}(\mathscr{M}'',\mathscr{D}_{\xi_1\xi_2})$ and $t\in\mathbf{R}$.

Similarly, the equivalence of (2) and (3) is shown.

Proposition 3.12. *Let $(\mathscr{M},\mathscr{D})$ be a closed O_p^*-algebra such that $\mathscr{M}'\mathscr{D}=\mathscr{D}$, and a vector η_0 in $\mathscr{D}$ be strongly cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$.*

I. *Suppose η_0 is tracial; that is,*

$$(XY\eta_0|\eta_0)=(YX\eta_0|\eta_0)$$

for each X, $Y\in\mathscr{M}$. Then the following statements hold.

(1) *η_0 is a standard vector for $(\mathscr{M},\mathscr{D})$ with $\Delta''_{\eta_0}=I$.*

(2) *Suppose ξ is a modular vector for $(\mathscr{M},\mathscr{D})$ such that $\eta_0\in\mathscr{D}_\xi$.*

Then, a pair (ξ, η_0) *is relative modular for* $(\mathscr{M}, \mathscr{D})$ *with* $\mathscr{D}_{\xi\eta_0} = \mathscr{D}_\xi$, *and* $\{(D\omega''_\xi : D\omega''_{\eta_0})_t\}_{t\in\mathbf{R}}$ *is a strongly continuous one-parameter group of unitary operators, which satisfies*

$$(D\omega''_\xi : D\omega''_{\eta_0})_t \mathscr{D}_\xi = \mathscr{D}_\xi \text{ and}$$
$$\sigma_t^\xi(X)\zeta = (D\omega''_\xi : D\omega''_{\eta_0})_t X (D\omega''_\xi : D\omega''_{\eta_0})_t^* \zeta$$

for each $t\in\mathbf{R}$, $X\in\mathscr{R}(\mathscr{M}'', \mathscr{D}_\xi)$ *and* $\zeta\in\mathscr{D}_\xi$.

II. *Conversely, suppose there exists a modular vector* ξ_0 *for* $(\mathscr{M}, \mathscr{D})$ *such that* $\eta_0\in\mathscr{D}_{\xi_0}$, $(D\omega''_{\xi_0} : D\omega''_{\eta_0})_t \mathscr{D}_{\xi_0} = \mathscr{D}_{\xi_0}$ *for each* $t\in\mathbf{R}$ *and*

$$\sigma_t^{\xi_0}(X)\zeta = (D\omega''_{\xi_0} : D\omega''_{\eta_0})_t X (D\omega''_{\xi_0} : D\omega''_{\eta_0})_t^* \zeta \tag{3.10}$$

for each $t\in\mathbf{R}$, $X\in\mathscr{M}$ *and* $\zeta\in\mathscr{D}_{\xi_0}$. *Then* η_0 *is a tracial vector.*

Proof. I. (1) Suppose η_0 is a tracial vector. Then it is easily shown that S_{η_0} equals the isometry J_{η_0}, and hence it follows from $S_{\eta_0}\subset S''_{\eta_0}$ that $S_{\eta_0} = S''_{\eta_0} = J_{\eta_0} = J''_{\eta_0}$. Hence, the statement (1) holds.

(2) Suppose ξ is a modular vector for $(\mathscr{M}, \mathscr{D})$ such that $\eta_0\in\mathscr{D}_\xi$. By (1), a pair (ξ, η_0) is relative modular for $(\mathscr{M}, \mathscr{D})$ with $\mathscr{D}_{\xi\eta_0} = \mathscr{D}_\xi$, and hence from Proposition 3.10 $\{(D\omega''_\xi : D\omega''_{\eta_0})_t\}_{t\in\mathbf{R}}$ is a strongly continuous one-parameter group of unitary operators, and further by Theorem 3.9

$$(D\omega''_\xi : D\omega''_{\eta_0})_t \mathscr{D}_\xi = \mathscr{D}_\xi,$$
$$\begin{aligned}\sigma_t^\xi(X)\zeta &= (D\omega''_\xi : D\omega''_{\eta_0})_t \sigma_t^{\eta_0}(X)\,(D\omega''_\xi : D\omega''_{\eta_0})_t^*\zeta \\ &= (D\omega''_\xi : D\omega''_{\eta_0})_t X (D\omega''_\xi : D\omega''_{\eta_0})_t^*\zeta\end{aligned}$$

for each $t\in\mathbf{R}$, $X\in\mathscr{R}(\mathscr{M}'', \mathscr{D}_\xi)$ and $\zeta\in\mathscr{D}_\xi$.

II. Since $(D\omega''_{\xi_0} : D\omega''_{\eta_0})_t \mathscr{D}_{\xi_0} = \mathscr{D}_{\xi_0}$ for each $t\in\mathbf{R}$, we have

$$\begin{aligned}\Delta''^{it}_{\eta_0} \mathscr{D}_{\xi_0} &= \Delta''^{it}_{\xi_0} (\Delta''^{-it}_{\xi_0} \Delta''^{it}_{\xi_0\eta_0}) (\Delta''^{-it}_{\xi_0\eta_0} \Delta''^{it}_{\eta_0}) \mathscr{D}_{\xi_0} \\ &\subset \Delta''^{it}_{\xi_0} \mathscr{M}' \mathscr{D}_{\xi_0} && \text{(by 3.4)} \\ &= \mathscr{D}_{\xi_0}\end{aligned}$$

for each $t\in\mathbf{R}$, which implies that the pair (ξ_0, η_0) is relative modular for $(\mathscr{M}, \mathscr{D})$ with $\mathscr{D}_{\xi_0\eta_0} = \mathscr{D}_{\xi_0}$. It hence follows from Theorem 3.9 that

$$\sigma_t^{\xi_0}(X)\zeta = (D\omega''_{\xi_0} : D\omega''_{\eta_0})_t \sigma_t^{\eta_0}(X)\,(D\omega''_{\xi_0} : D\omega''_{\eta_0})_t^* \zeta$$

for all $t \in \mathbb{R}$, $X \in \mathscr{M}$ and $\zeta \in \mathscr{D}_{\xi_0}$, which implies by (3.10)

$$\sigma_t^{\eta_0}(X)\zeta = X\zeta$$

for each $t \in \mathbb{R}$, $X \in \mathscr{M}$ and $\zeta \in \mathscr{D}_{\xi_0}$. Since the positive linear functional ω_{η_0} on $\mathscr{R}(\mathscr{M}'', \mathscr{D}_{\xi_0})$ satisfies the KMS-condition with respect to $\{\sigma_t^{\eta_0}\}$ by Theorem 3.2, for each $X, Y \in \mathscr{M}$ there exists a function $f_{X,Y}$ in $A(0, 1)$ such that

$$f_{X,Y}(t) = \omega_{\eta_0}(\sigma_t^{\eta_0}(X)\,Y) = \omega_{\eta_0}(XY),$$

$$f_{X,Y}(t+i) = \omega_{\eta_0}(Y\sigma_t^{\eta_0}(X)) = \omega_{\eta_0}(YX)$$

for all $t \in \mathbb{R}$, which implies

$$\omega_{\eta_0}(XY) = \omega_{\eta_0}(YX)$$

for each X, $Y \in \mathscr{M}$; that is, η_0 is a tracial vector. This completes the proof.

We give some concrete examples for standard systems and relative modular vectors.

(i) Let $\mathscr{M}_0$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$, T be a positive self-adjoint unbounded operator in $\mathfrak{H}$ affiliated with $\mathscr{M}_0$ and $\mathscr{D}^\infty(T) = \bigcap_{n=1}^{\infty} \mathscr{D}(T^n)$. Then the following statements hold.

(1) $\mathscr{R}(\mathscr{M}_0, \mathscr{D}^\infty(T)) = \overline{\mathscr{M}_0^{\mathscr{D}^\infty(T)}}^{t_s^* \text{ in } \mathscr{L}^\dagger(\mathscr{D}^\infty(T))}$, *where*

$$\mathscr{M}_0^{\mathscr{D}^\infty(T)} = \{A/\mathscr{D}^\infty(T)\,;\ A \in \mathscr{M}_0,\ A\mathscr{D}^\infty(T) \subset \mathscr{D}^\infty(T),\ A^*\mathscr{D}^\infty(T) \subset \mathscr{D}^\infty(T)\},$$

which are self-adjoint generalized von Neumann algebra containing $\{T^n\}_{n \in \mathbb{N}}$ *whose induced topology* $t_{\mathscr{R}(\mathscr{M}_0, \mathscr{D}^\infty(T))}$ *equals the Fréchet topology defined by the seminorms* $\{\|\cdot\|_n = \|T^n \cdot\|\,;\ n \in \mathbb{N}\}$.

(2) *Suppose* ξ_0 *is a cyclic and separating vector for* $\mathscr{M}_0$ *and* T *is affiliated with the fixed-point algebra* $\mathscr{M}_0^{\sigma^{\xi_0}}$ *of* $\{\sigma_t^{\xi_0}\}$ *in* $\mathscr{M}_0$ *such that* $\xi_0 \in \mathscr{D}^\infty(T)$. *Then* $(\mathscr{R}(\mathscr{M}_0, \mathscr{D}^\infty(T)), \mathscr{D}^\infty(T), \xi_0)$ *is a full standard system.*

(3) *Suppose* ξ_1 *and* ξ_2 *are cyclic and separating vectors for* $\mathscr{M}_0$ *and* T *is affiliated with* $\mathscr{M}_0^{\sigma^{\xi_1}} \cap \mathscr{M}_0^{\sigma^{\xi_2}}$ *such that* $\xi_1, \xi_2 \in \mathscr{D}^\infty(T)$. *Then* (ξ_1, ξ_2) *is*

relative modular for $(\mathscr{R}(\mathscr{M}_0, \mathscr{D}^\infty(T)), \mathscr{D}^\infty(T))$ *with* $\mathscr{D}_{\xi_1\xi_2}=\mathscr{D}^\infty(T)$. *By Theorem* 3.9, $\{\sigma_t^{\xi_1}\}$ *and* $\{\sigma_t^{\xi_2}\}$ *are one-parameter groups of* $*$*-automorphisms of* $\mathscr{R}(\mathscr{M}_0, \mathscr{D}^\infty(T))$, $(D\omega''_{\xi_1}: D\omega''_{\xi_2})_t/\mathscr{D}^\infty(T)\in\mathscr{R}(\mathscr{M}_0, \mathscr{D}^\infty(T))$ *for all* $t\in\mathbf{R}$ *and*

$$\sigma_t^{\xi_1}(X)\zeta=(D\omega''_{\xi_1}: D\omega''_{\xi_2})_t\sigma_t^{\xi_2}(X)(D\omega''_{\xi_1}: D\omega''_{\xi_2})_t^*\zeta$$

for all $t\in\mathbf{R}$, $X\in\mathscr{R}(\mathscr{M}_0, \mathscr{D}^\infty(T))$ *and* $\zeta\in\mathscr{D}^\infty(T)$.

(ii) Let $\mathscr{S}=\mathscr{S}(\mathbf{R})$ be the Schwartz space of infinitely differentiable rapidly decreasing functions and let

$$N=\sum_{n=0}^{\infty}(n+1)f_n\otimes\overline{f_n},$$

where $\{f_n\}$ is an orthonormal basis in the Hilbert space $L^2=L^2(\mathbf{R})$ contained in $\mathscr{S}$ consisting of the normalized Hermite functions. Then $\mathscr{S}=\mathscr{D}^\infty(N)$, and hence $\mathscr{L}^\dagger(\mathscr{S})$ is a selfadjoint O_p^*-algebra containing the inverse N of a positive Hilbert-Schmidt operator, which implies that a self-adjoint representation π of $\mathscr{L}^\dagger(\mathscr{S})$ on $L^2\otimes\overline{L^2}$ is defined by

$$\pi(X)T=XT, \qquad T\in\mathscr{S}\otimes\overline{L^2},$$

where $L^2\otimes\overline{L^2}$ denotes the Hilbert space of Hilbert-Schmidt operators on L^2 and $\mathscr{S}\otimes\overline{L^2}=\{T\in L^2\otimes\overline{L^2};\ TL^2\subset\mathscr{S}\}$. We put

$$\begin{aligned}\boldsymbol{s}_+=\{\{\alpha_n\};\ \alpha_n>0 \quad &\text{for} \quad n=0,1,2,\dots\\ &\text{and} \ \sup_n n^k|\alpha_n|<\infty \quad \text{for} \quad \text{each } k\in\mathbf{N}\},\\ \Omega_{\{\alpha_n\}}=\sum_{n=0}^{\infty}\alpha_n f_n\otimes\overline{f_n}, \qquad &\{\alpha_n\}\in\boldsymbol{s}_+.\end{aligned}$$

Then the following statements hold. The proofs follow from Section 5 in [14].

(1) $(\pi(\mathscr{L}^\dagger(\mathscr{S})), \mathscr{S}\otimes\overline{L^2}, \Omega_{\{\alpha_n\}})$ *is a full standard system for each* $\{\alpha_n\}\in\boldsymbol{s}_+$.

(2) *Every pair* $(\Omega_{\{\alpha_n\}}, \Omega_{\{\beta_n\}})$ *for* $\{\alpha_n\}$, $\{\beta_n\}\in\boldsymbol{s}_+$ *is relative modular for* $(\pi(\mathscr{L}^\dagger(\mathscr{S})), \mathscr{S}\otimes\overline{L^2})$ *with* $(\mathscr{S}\otimes\overline{L^2})_{\Omega_{\{\alpha_n\}}\Omega_{\{\beta_n\}}}=\mathscr{S}\otimes\overline{L^2}$.

(3) *Let* π_1 *be a self-adjoint representation of the canonical algebra* $\mathscr{A}$ *for one degree of freedom defined by*

$$\pi_1(x)=\pi(\pi_0(x)),\ x\in\mathscr{A},$$

where π_0 *denotes the Schrödinger representation of* $\mathscr{A}_0$. *Suppose* $\{\alpha_n\} \in \boldsymbol{s}_+$ *satisfies*

(3.11) $$0 < \alpha_n \leqq \gamma e^{-n\beta}, \quad n \in \mathbb{N}$$

for some $\beta > 0$ *and* $\gamma > 0$. *Then* $\Omega_{\{\alpha_n\}}$ *is a standard vector for* $(\pi_1(\mathscr{A}), \mathscr{S} \otimes \overline{L^2})$ *with* $(\mathscr{S} \otimes \overline{L^2})_{\Omega_{\{\alpha_n\}}} = \mathscr{S} \otimes \overline{L^2}$.

(4) *Suppose* $\{\alpha_n\}$, $\{\beta_n\} \in \boldsymbol{s}_+$ *satisfy the condition* (3.11). *Then* $(\Omega_{\{\alpha_n\}}, \Omega_{\{\beta_n\}})$ *is relative modular for* $(\pi_1(\mathscr{A}), \mathscr{S} \otimes \overline{L^2})$ *with* $(\mathscr{S} \otimes \overline{L^2})_{\Omega_{\{\alpha_n\}} \Omega_{\{\beta_n\}}} = \mathscr{S} \otimes \overline{L^2}$.

§ 4. Radon-Nikodym Theorems for O_p^*-algebras

In this section we study Radon-Nikodym theorems and Lebesque-decomposition theorems for O_p^*-algebras. We first investigate in more detail the Radon-Nikodym theorem and Lebesgue decomposition theorem obtained in [13, 16] with the help of Kosaki's results [19] for von Neumann algebras.

Let $(\mathscr{M}, \mathscr{D})$ be a closed O_p^*-algebra such that $\mathscr{M}'\mathscr{D} = \mathscr{D}$, ξ_0 be a strongly cyclic vector for $\mathscr{M}$ and let $\phi_0 = \omega_{\xi_0}$. For each positive linear functional ϕ on $\mathscr{M}$ we put

$$T_\phi^{\phi_0} X\xi_0 = \lambda_\phi(X), \quad X \in \mathscr{M}.$$

In accordance with the Gudder definition [8] and [13], we define the notions of ϕ_0-absolute continuity and ϕ_0-singularity, respectively as follows:

Definition 4.1. A positive linear functional ϕ on $\mathscr{M}$ is said to be ϕ_0-absolutely continuous if $T_\phi^{\phi_0}$ is a map; and ϕ is said to be strongly ϕ_0-absolutely continuous if $T_\phi^{\phi_0}$ is a closable map of $\mathfrak{H}(\mathscr{D})$ into $\mathfrak{H}_\phi$; and ϕ is said to be ϕ_0-dominated if $T_\phi^{\phi_0}$ is a continuous map. If for each $X \in \mathscr{M}$ there exists a sequence $\{X_n\}$ in $\mathscr{M}$ such that $\lim_{n\to\infty} \phi_0(X_n^\dagger X_n) = 0$ and $\lim_{n\to\infty} \phi((X_n - X)^\dagger (X_n - X)) = 0$, then ϕ is said to be ϕ_0-singular.

Remark 4.2. (1) The following statements hold immediately.

(a) *If* ϕ, ψ *are strongly* ϕ_0*-absolutely continuous, then so is* $\phi + \psi$.

(b) *If* $0 \leqq \psi \leqq \phi$ *and* ϕ *is* ϕ_0*-singular, then so is* ψ.

However, an analogous statement (a) (resp. (b)) for ϕ_0-singularity (resp. strongly ϕ_0-absolutely continuity) does not necessarily hold (Example 6.3).

(2) For normal forms on a von Neumann algebra with a cyclic and separating vector ξ_0 the notions of ϕ_0-absolute continuity and ϕ_0-singularity defined by Kosaki [19] are identical with the notions of strongly ϕ_0-absolute continuity and ϕ_0-singularity defined the above, respectively.

It is easily shown that bounded linear maps $T_{\phi_0}^{\phi_0+\phi}$ and $T_{\phi}^{\phi_0+\phi}$ defined by

$$T_{\phi_0}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)=X\xi_0,$$
$$T_{\phi}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)=\lambda_\phi(X), \quad X\in\mathscr{M}$$

satisfy

$$(4.1)\qquad \begin{aligned}&(T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi},\ (T_{\phi}^{\phi_0+\phi})^*T_{\phi}^{\phi_0+\phi}\in\pi_{\phi_0+\phi}(\mathscr{M})',\\ &(T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi}+(T_{\phi}^{\phi_0+\phi})^*T_{\phi}^{\phi_0+\phi}=I.\end{aligned}$$

Further, we have by (4.1)

$$\overline{\{(X\xi_0,\ \lambda_\phi(X));X\in\mathscr{M}\}}=\{(T_{\phi_0}^{\phi_0+\phi}\zeta,\ T_{\phi}^{\phi_0+\phi}\zeta);\ \zeta\in\mathfrak{H}_{\phi_0+\phi}\},$$

$$(4.2)\qquad C_p(T_{\phi}^{\phi_0})=\begin{pmatrix}T_{\phi_0}^{\phi_0+\phi}(T_{\phi_0}^{\phi_0+\phi})^* & T_{\phi_0}^{\phi_0+\phi}(T_{\phi}^{\phi_0+\phi})^*\\ T_{\phi}^{\phi_0+\phi}(T_{\phi_0}^{\phi_0+\phi})^* & T_{\phi}^{\phi_0+\phi}(T_{\phi}^{\phi_0+\phi})^*\end{pmatrix},$$

where $C_p(T_{\phi}^{\phi_0})$ denotes the projection from $\mathfrak{H}_{\phi_0+\phi}\oplus\mathfrak{H}_{\phi_0+\phi}$ onto

$$\overline{\{(X\xi_0,\ \lambda_\phi(X));\ X\in\mathscr{M}\}}.$$

Using these facts, in analogous with [19] we can characterize the notions of strongly ϕ_0-absolute continuity and ϕ_0-singularity by the maps $T_{\phi_0}^{\phi_0+\phi}$ and $T_{\phi}^{\phi_0+\phi}$ as follows:

Lemma 4.3. *Let ϕ be a positive linear functional on $\mathscr{M}$.*

I. *The following statements are equivalent.*

(1) *ϕ is strongly ϕ_0-absolutely continuous.*

(2) $T_{\phi_0}^{\phi_0+\phi}$ *is non-singular.*

In this case, $\mathscr{D}(\overline{T_\phi^{\phi_0}})=\mathscr{R}(T_{\phi_0}^{\phi_0+\phi})$, $\mathscr{R}(\overline{T_\phi^{\phi_0}})=\mathscr{R}(T_\phi^{\phi_0+\phi})$ *and*

$$\overline{T_\phi^{\phi_0}}=T_\phi^{\phi_0+\phi}(T_{\phi_0}^{\phi_0+\phi})^{-1}.$$

II. *The following statements are equivalent.*

(1) ϕ *is* ϕ_0*-singular.*

(2) $T_{\phi_0}^{\phi_0+\phi}$ *is a partial isometry.*

(2)′ $T_{\phi_0}^{\phi_0+\phi}(T_{\phi_0}^{\phi_0+\phi})^*=I_{\mathfrak{H}(\mathscr{D})}$.

(3) $T_\phi^{\phi_0+\phi}$ *is a partial isometry.*

(3)′ $T_\phi^{\phi_0+\phi}(T_\phi^{\phi_0+\phi})^*=I_{\mathfrak{H}_\phi}$.

(4) $\overline{\{(X\xi_0, \lambda_\phi(X))\,;\ X\in\mathscr{M}\}}=\mathfrak{H}(\mathscr{D})\oplus\mathfrak{H}_\phi$.

(5) $inf\ \{\phi_0(X^\dagger X)+\phi(Y^\dagger Y)\,;\ X, Y\in\mathscr{M}, X+Y=Z\}=0$

for each $Z\in\mathscr{M}$.

(5)′ $inf\ (\phi_0(X^\dagger X)+\phi(Y^\dagger Y)\,;\ X, Y\in\mathscr{M}, X+Y=I\}=0.$

We denote by $P(\mathscr{M})$ the set of all positive linear functionals on $\mathscr{M}$. Then, by an order relation $\phi\leqq\psi$ ($\phi(X^\dagger X)\leqq\psi(X^\dagger X)$ for each $X\in\mathscr{M}$) $(P(\mathscr{M}), \leqq)$ is an ordered set. We donote by $P(\mathscr{M}, \phi)$ the set of all elements ψ of $P(\mathscr{M})$ such that $\psi\leqq\phi$, and denote by $P_c^{\phi_0}(\mathscr{M}, \phi)$ (resp. $P_s^{\phi_0}(\mathscr{M}, \phi)$) the set of all strongly ϕ_0-absolutely continuous (resp. ϕ_0-singular) elements of $P(\mathscr{M}, \phi)$.

Lemma 4.4. *Suppose* ϕ *is a positive linear functional on* $\mathscr{M}$ *such that* $\pi_{\phi_0+\phi}(\mathscr{M})'$ *is a von Neumann algebra. Then the following statements hold.*

(1) *The isometry* U_ϕ *of* $\mathfrak{H}(\mathscr{D})$ *into* $\mathfrak{H}_{\phi_0+\phi}$ *defined by*

$$X\xi_0\longrightarrow((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}\lambda_{\phi_0+\phi}(X),\qquad X\in\mathscr{M}$$

satisfies

(4.3) $U_\phi^*\mathscr{D}(\pi_{\phi_0+\phi})\subset\mathscr{D}^*(\mathscr{M})$ *and* $X^*U_\phi^*\xi=$

$U_\phi^*\pi_{\phi_0+\phi}(X^\dagger)\xi$ *for each* $X\in\mathscr{M}$ *and* $\xi\in\mathscr{D}(\pi_{\phi_0+\phi})$.

(2) *A sequence* $\{H_n'^\phi\}$ *of positive operators on* $\mathfrak{H}(\mathscr{D})$ *defined by*

$$H_n'^{\phi}=U_\phi^*\left(\int_{1/n}^1 \lambda^{-1}(1-\lambda)\,dE(\lambda)\right)U_\phi, \qquad n\in N$$

satisfies

$$(4.4)\qquad \begin{array}{l}\{H_n'^{\phi}\}\subset\mathscr{M}',\ H_1'^{\phi}\leqq H_2'^{\phi}\leqq\dots \text{ and}\\ \lim_{n\to\infty}(H_n'^{\phi})^{1/2}X\xi_0 \text{ exists for each } X\in\mathscr{M},\end{array}$$

where $\int_0^1\lambda dE(\lambda)$ *is the spectral resolution of* $(T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi}$.

(3) *Put*

$$\phi_c(X)=\lim_{n\to\infty}(H_n'^{\phi}X\xi_0\,|\,\xi_0),$$
$$\phi_s(X)=(P_{\phi_0}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)\,|\,\lambda_{\phi_0+\phi}(I)), \qquad X\in\mathscr{M},$$

where $P_{\phi_0}^{\phi_0+\phi}$ *is the projection from* $\mathfrak{H}_{\phi_0+\phi}$ *onto* $\mathrm{Ker}(T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi}$. *Then* ϕ_c, $\phi_s\in P(\mathscr{M},\phi)$ *and* $\phi=\phi_c+\phi_s$.

Proof. (1) This is easily proved.

(2) Since $\pi_{\phi_0+\phi}(\mathscr{M})'$ is a von Neumann algebra, it follows that $K_n\equiv\int_{1/n}^1\lambda^{-1}(1-\lambda)\,dE(\lambda)\in\pi_{\phi_0+\phi}(\mathscr{M})'$ for $n\in\mathrm{N}$, which implies $H_n'^{\phi}\in\mathscr{M}'$ for $n\in\mathrm{N}$. Further, since $U_\phi U_\phi^*((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}=((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}$, it follows that $(H_n'^{\phi})^{1/2}=U_\phi^*K_n^{1/2}U_\phi$ for $n\in\mathrm{N}$, which implies that $H_1'^{\phi}\leqq H_2'^{\phi}\leq\dots$ and

$$\begin{aligned}
&\lim_{n,m\to\infty}\|(H_n'^{\phi})^{1/2}X\xi_0-(H_m'^{\phi})^{1/2}X\xi_0\|^2\\
&=\lim_{n,m\to\infty}\{((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi}K_n\lambda_{\phi_0+\phi}(X)\,|\,\lambda_{\phi_0+\phi}(X))\\
&\quad-(((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}K_n^{1/2}\lambda_{\phi_0+\phi}(X)\,|\,((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}K_m^{1/2}\lambda_{\phi_0+\phi}(X))\\
&\quad-(((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}K_m^{1/2}\lambda_{\phi_0+\phi}(X)\,|\,((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}K_n\lambda_{\phi_0+\phi}(X))\\
&\quad+((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi}K_m\lambda_{\phi_0+\phi}(X)\,|\,\lambda_{\phi_0+\phi}(X))\}\\
&=\lim_{n,m\to\infty}\{((T_\phi^{\phi_0+\phi})^*T_\phi^{\phi_0+\phi}(I-E(1/n))\lambda_{\phi_0+\phi}(X)\,|\,\lambda_{\phi_0+\phi}(X))\\
&\quad-(((T_\phi^{\phi_0+\phi})^*T_\phi^{\phi_0+\phi})^{1/2}(I-E(1/n))\lambda_{\phi_0+\phi}(X)\,|\,((T_\phi^{\phi_0+\phi})^*T_\phi^{\phi_0+\phi})^{1/2}\\
&\qquad\qquad\times(I-E(1/m))\lambda_{\phi_0+\phi}(X))\\
&\quad-(((T_\phi^{\phi_0+\phi})^*T_\phi^{\phi_0+\phi})^{1/2}(I-E(1/m))\lambda_{\phi_0+\phi}(X)\,|\,((T_\phi^{\phi_0+\phi})^*T_\phi^{\phi_0+\phi})^{1/2}\\
&\qquad\qquad\times(I-E(1/n))\lambda_{\phi_0+\phi}(X))
\end{aligned}$$

$$+((T_{\phi}^{\phi_0+\phi})^*T_{\phi}^{\phi_0+\phi}(I-E(1/m))\lambda_{\phi_0+\phi}(X)\,|\lambda_{\phi_0+\phi}(X))\}$$
$$=0$$

for each $X\in\mathscr{M}$, and hence $\lim_{n\to\infty}(H_n'^{\phi})^{1/2}X\xi_0$ exists for each $X\in\mathscr{M}$.

(3) This follows from the equality:

$$\begin{aligned}\lim_{n\to\infty}(H_n'^{\phi}X\xi_0\,|\,\xi_0) &= \lim_{n\to\infty}((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi}K_n\lambda_{\phi_0+\phi}(X)\,|\lambda_{\phi_0+\phi}(I))\\ &= \lim_{n\to\infty}((T_{\phi}^{\phi_0+\phi})^*T_{\phi}^{\phi_0+\phi}(I-E(1/n))\lambda_{\phi_0+\phi}(X)\,|\lambda_{\phi_0+\phi}(I))\\ &= ((T_{\phi}^{\phi_0+\phi})^*T_{\phi}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)\,|\lambda_{\phi_0+\phi}(I))\\ &\quad -((T_{\phi}^{\phi_0+\phi})^*T_{\phi}^{\phi_0+\phi}P_{\phi_0}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)\,|\lambda_{\phi_0+\phi}(I))\\ &= \phi(X)-((I-(T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})P_{\phi_0}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)\,|\lambda_{\phi_0+\phi}(I))\\ &= \phi(X)-(P_{\phi_0}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)\,|\lambda_{\phi_0+\phi}(I))\\ &= \phi(X)-\phi_s(X)\end{aligned}$$

for each $X\in\mathscr{M}$. This completes the proof.

By Lemma 4.2, Lemma 4.4 and ([16] Lemma 5.5) we have the following

Theorem 4.5. (**Radon-Nikodym theorem**) *Let $(\mathscr{M},\mathscr{D})$ be a closed O_p^*-algebra such that $\mathscr{M}'\mathscr{D}=\mathscr{D}$ and ξ_0 be a strongly cyclic vector for $\mathscr{M}$. Suppose ϕ is a positive linear functional on $\mathscr{M}$ such that $\pi_{\phi_0+\phi}(\mathscr{M})'$ is a von Neumann algebra. Then the following statements are equivalent.*

(1) *ϕ is strongly ϕ_0-absolutely continuous.*

(2) *$T_{\phi_0}^{\phi_0+\phi}$ is non-singular.*

(3) *ϕ is represented as*

$$\phi(X)=\lim_{n\to\infty}(H_n'X\xi_0\,|\,\xi_0),\qquad X\in\mathscr{M}$$

for some sequence $\{H_n'\}$ of positive operators in $\mathscr{M}'$ such that $H_1'\leqq H_2'\leqq\ldots$ and $\lim_{n\to\infty}H_n'^{1/2}X\xi_0$ exists for each $X\in\mathscr{M}$.

(4) *ϕ is represented as*

$$\phi(X)=(XH'\xi_0\,|\,H'\xi_0),\qquad X\in\mathscr{M}$$

for some positive self-adjoint operator H' affiliated with $\mathscr{M}'$ such that $\xi_0\in\mathscr{D}(H')$ and $H'\xi_0\in\mathscr{D}$.

Theorem 4.6. (Lebesgue-decomposition theorem) *Let* $(\mathcal{M}, \mathcal{D})$ *be a closed* O_p^**-algebra such that* $\mathcal{M}'\mathcal{D}=\mathcal{D}$ *and* ξ_0 *be a strongly cyclic vector for* $\mathcal{M}$. *Suppose* ϕ *is a positive linear functional on* $\mathcal{M}$ *such that* $\pi_{\phi_0+\phi}(\mathcal{M})'$ *is a von Neumann algebra. Then,* ϕ_c *is maximal in* $P_c^{\phi_0}(\mathcal{M}, \phi)$, $\phi_s \in P_s^{\phi_0}(\mathcal{M}, \phi)$ *and* $\phi=\phi_c+\phi_s$.

Proof. It follows from Lemma 4.4 and Theorem 4.5 that $\phi_c \in P_c^{\phi_0}(\mathcal{M}, \phi)$ and $\phi=\phi_c+\phi_s$. It is easily shown that $\phi_s \in P_s^{\phi_0}(\mathcal{M}, \phi)$. We show that ϕ_c is maximal in $P_c^{\phi_0}(\mathcal{M}, \phi)$. This is proved by analogy with ([19] Theorem 3.3). Take arbitrary $\psi \in P_c^{\phi_0}(\mathcal{M}, \phi)$. We denote by $T_{\phi_0+\psi}^{\phi_0+\phi}$ a bounded linear map of $\mathfrak{H}_{\phi_0+\phi}$ into $\mathfrak{H}_{\phi_0+\psi}$ defined by

$$\lambda_{\phi_0+\phi}(X) \longrightarrow \lambda_{\phi_0+\psi}(X).$$

Since ψ is strongly ϕ_0-absolutely continuous, it follows from Theorem 4.5 that $T_{\phi_0}^{\phi_0+\psi}$ is non-singular and $T_{\phi_0+\psi}^{\phi_0+\phi}=(T_{\phi_0}^{\phi_0+\psi})^{-1}T_{\phi_0}^{\phi_0+\phi}$. Hence, we have

$$\begin{aligned} T_{\phi_0+\psi}^{\phi_0+\phi}P_{\phi_0}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X) &= (T_{\phi_0}^{\phi_0+\psi})^{-1}T_{\phi_0}^{\phi_0+\phi}P_{\phi_0}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X) \\ &= 0 \end{aligned}$$

for each $X \in \mathcal{M}$, which implies

$$\begin{aligned} \psi(X^\dagger X)+\phi_0(X^\dagger X) &= \|\lambda_{\phi_0+\psi}(X)\|^2 \\ &= \|T_{\phi_0+\psi}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)\|^2 \\ &= \|T_{\phi_0+\psi}^{\phi_0+\phi}(I-P_{\phi_0}^{\phi_0+\phi})\lambda_{\phi_0+\phi}(X)\|^2 \\ &\leqq \|(I-P_{\phi_0}^{\phi_0+\phi})\lambda_{\phi_0+\phi}(X)\|^2 \\ &= \phi_c(X^\dagger X)+\phi_0(X^\dagger X) \end{aligned}$$

for each $X \in \mathcal{M}$. Hence, $\psi \leqq \phi_c$. This completes the proof.

Corollary 4.7. I. *Suppose* $\phi \in P(\mathcal{M})$ *satisfies* $\pi_{\phi_0+\phi}(\mathcal{M})'$ *is a von Neumann algebra. Then the following statements are equivalent.*

(1) ϕ *is* ϕ_0*-singular.*

(2) $\phi_c=0$.

(3) $P(\mathcal{M}, \phi) \cap P(\mathcal{M}, \phi_0)=\{0\}$.

II. *Suppose* $\phi, \psi \in P(\mathscr{M})$ *satisfies* $\pi_{\phi_0+\phi}(\mathscr{M})'$ *and* $\pi_{\phi_0+\psi}(\mathscr{M})'$ *are von Neumann algebras. Then the following statements hold.*

(1) $(\lambda\phi)_c = \lambda\phi_c$ *for* $\lambda \geqq 0$.

(2) *If* $0 \leqq \psi \leqq \phi$, *then* $\psi_c \leqq \phi_c$.

(3) *Further, if* $\pi_{\phi_0+\phi+\psi}(\mathscr{M})'$ *is a von Neumann algebra, then* $\phi_c + \psi_c \leqq (\phi+\psi)_c$.

Proof. I. (2)$\Rightarrow$(1) This is trivial.

(1)$\Rightarrow$(3) Take arbitrary $\psi \in P(\mathscr{M}, \phi) \cap P(\mathscr{M}, \phi_0)$. Since ϕ is ϕ_0-singular and $\psi \in P(\mathscr{M}, \phi)$, it follows from Remark 4. 2, (a) that ψ is ϕ_0-singular. On the other hand, ψ is strongly ϕ_0-absolutely continuous since $\psi \leqq \phi_0$. Hence, $\psi = 0$.

(3)$\Rightarrow$(2) By Theorem 4. 5 ϕ_c is represented as

$$\phi_c(X) = \lim_{n\to\infty} (H_n'^{\phi} X\xi_0 | \xi_0), \qquad X \in \mathscr{M}.$$

Then, it follows that for each $n \in \mathbb{N}$

$$\lambda\phi_c^n \in P(\mathscr{M}, \phi) \cap P(\mathscr{M}, \phi_0) = \{0\}$$

for some $\lambda > 0$, where

$$\phi_c^n(X) = (H_n'^{\phi} X\xi_0 | \xi_0), \qquad X \in \mathscr{M},$$

which implies $\phi_c = 0$.

II. This follows immediately from Theorem 4. 6.

Remark 4. 8. (1) In [13, 16] we have obtained the Lebesgue-decomposition theorem: $\phi_c \in P_c^{\phi_0}(\mathscr{M}, \phi)$, $\phi_s \in P_s^{\phi_0}(\mathscr{M}, \phi)$ and $\phi = \phi_c + \phi_s$. However, it did not know that ϕ_c is maximal in $P_c^{\phi_0}(\mathscr{M}, \phi)$. By Theorem 4. 6 this fact is true, but there exists a pathological fact that this Lebesgue decomposition is not unique in general (Example 6. 3).

(2) By Corollary 4. 7 the Kosaki definition of ϕ_0-singularity $P(\mathscr{M}, \phi) \cap P(\mathscr{M}, \phi_0) = \{0\}$ is identical with our definition of ϕ_0-singularity in the case $\pi_{\phi_0+\phi}(\mathscr{M})'$ is a von Neumann algebra.

We have treated with an unbounded generalization of the Tomita-Takesaki theory in [14] and Section 3, so that we now generalize the Radon-Nikodym theorem of Pedersen and Takesaki [24] to that for O_p^*-algebra.

Theorem 4.9. *Let* $(\mathscr{M}, \mathscr{D}, \xi_0)$ *be a standard system. Then the following statements hold.*

I. ϕ *is a* ϕ_0*-dominated positive linear functional on* $\mathscr{M}$ *which satisfies the KMS-condition with respect to* $\{\sigma_t^{\xi_0}\}$ *if and only if* ϕ *is represented as*

$$\phi(X) = (XH\xi_0 | H\xi_0), \quad X \in \mathscr{M}$$

for some positive operator H *in* $\mathscr{M}' \cap \mathscr{M}''$.

II. *The following statements are equivalent.*

(1) ϕ *is a strongly* ϕ_0*-absolutely continuous positive linear functional on* $\mathscr{M}$ *which satisfies the KMS-condition with respect to* $\{\sigma_t^{\xi_0}\}$ *such that* $\phi_0 + \phi$ *is standard.*

(2) ϕ *is represented as*

$$\phi(X) = \lim_{n\to\infty} (H_n X\xi_0 | \xi_0), \quad X \in \mathscr{M}$$

for some sequence $\{H_n\}$ *of positive operators in* $\mathscr{M}' \cap \mathscr{M}''$ *such that* $H_1 \leqq H_2 \leqq \dots$ *and* $\lim_{n\to\infty} H_n^{1/2} X\xi_0$ *exists for each* $X \in \mathscr{M}$.

(3) ϕ *is represented as*

$$\phi(X) = (XH\xi_0 | H\xi_0), \quad X \in \mathscr{M}$$

for some positive self-adjoint operator H *affiliated with* $\mathscr{M}' \cap \mathscr{M}''$ *such that* $\xi_0 \in \mathscr{D}(H)$ *and* $H\xi_0 \in \mathscr{D}$.

III. *Suppose* ϕ *is a positive linear functional on* $\mathscr{M}$ *which satisfies the KMS-condition with respect to* $\{\sigma_t^{\xi_0}\}$ *such that* $\phi_0 + \phi$ *is standard. Then, both the maximal strongly* ϕ_0*-absolutely continuous part* ϕ_c *and the* ϕ_0*-singular part* ϕ_s *of* ϕ *satisfy the KMS-condition with respect to* $\{\sigma_t^{\xi_0}\}$.

Proof. I. Since ϕ is ϕ_0-dominated, there exists a positive operator H in $\mathscr{M}'$ such that

$$\phi(X) = (XH\xi_0 | H\xi_0)$$

for all $X \in \mathscr{M}$. Put

$$\phi''(A) = (AH\xi_0 | H\xi_0), \quad A \in \mathscr{M}''.$$

Then ϕ'' is a normal form on the von Neumann algebra $\mathscr{M}''$ which satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. In fact, take arbitrary $A, B \in \mathscr{M}''$. Since $S''_{\xi_0} = S_{\xi_0}$, by Lemma 3.4 there exist sequences $\{X_n\}$ and $\{Y_n\}$ in $\mathscr{M}$ such that $\lim_{n\to\infty} X_n\xi_0 = A\xi_0$, $\lim_{n\to\infty} X_n^\dagger\xi_0 = A^*\xi_0$, $\lim_{n\to\infty} Y_n\xi_0$

$=B\xi_0$ and $\lim_{n\to\infty} Y_n^\dagger\xi_0=B^*\xi_0$. Since ϕ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$, there exists a sequence $\{f_{X_n,Y_n}\}$ in $A(0,1)$ such that

$$f_{X_n,Y_n}(t)=\phi(\sigma_t^{\xi_0}(X_n)Y_n)=(H^2Y_n\xi_0|\Delta_{\xi_0}^{it}X_n^\dagger\xi_0),$$

$$f_{X_n,Y_n}(t+i)=\phi(Y_n\sigma_t^{\xi_0}(X_n))=(H^2\Delta_{\xi_0}^{it}X_n\xi_0|Y_n^\dagger\xi_0)$$

for all $t\in\mathbb{R}$ and $n\in\mathbb{N}$, which implies that

$$\lim_{n\to\infty}\sup_{t\in R}|f_{X_n,Y_n}(t)-(H^2B\xi_0|\Delta_{\xi_0}^{it}A^*\xi_0)|=0,$$

$$\lim_{n\to\infty}\sup_{t\in R}|f_{X_n,Y_n}(t+i)-(H^2\Delta_{\xi_0}^{it}A\xi_0|B^*\xi_0)|=0.$$

Hence, there exists a function $f_{A,B}$ in $A(0, 1)$ such that

$$f_{A,B}(t)=(H^2B\xi_0|\Delta_{\xi_0}^{it}A^*\xi_0)=\phi''(\sigma_t^{\xi_0}(A)B),$$

$$f_{A,B}(t+i)=(H^2\Delta_{\xi_0}^{it}A\xi_0|B^*\xi_0)=\phi''(B\sigma_t^{\xi_0}(A))$$

for all $t\in\mathbb{R}$, which means that ϕ'' satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. It hence follows from ([32] Theorem 15. 4) that $H\in\mathcal{M}'\cap\mathcal{M}''$. The converse follows from Lemma 3. 5.

Suppose ϕ is a positive linear functional on $\mathcal{M}$ which satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$ such that $\phi_0+\phi$ is standard. Then it follows from Lemma 4. 4 that $H_n'^\phi\in\mathcal{M}'$ for $n\in\mathbb{N}$, $H_1'^\phi\leqq H_2'^\phi\leqq\ldots$, $\lim_{n\to\infty}(H_n'^\phi)^{1/2}X\xi$ exists for each $X\in\mathcal{M}$ and

$$\phi_c(X)=\lim_{n\to\infty}(H_n'^\phi X\xi_0|\xi_0),$$

$$\phi_s(X)=(P_{\phi_0}^{\phi_0+\phi}\lambda_{\phi_0+\phi}(X)|\lambda_{\phi_0+\phi}(I)),\ X\in\mathcal{M}.$$

Since $\phi_0+\phi$ is standard, it follows from the above I that

$$(T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi}\in\pi_{\phi_0+\phi}(\mathcal{M})'\cap\pi_{\phi_0+\phi}(\mathcal{M})''. \tag{4. 5}$$

We show $H_n'^\phi\in\mathcal{M}''$ for $n\in\mathbb{N}$. For each X, Y, $Z\in\mathcal{M}$ and $C\in\mathcal{M}'$ we have

$$\begin{aligned}
&(U_\phi CU_\phi^*((T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}\pi_{\phi_0+\phi}(X)\lambda_{\phi_0+\phi}(Y)|\lambda_{\phi_0+\phi}(Z))\\
&=(CXY\xi_0|U_\phi^*\lambda_{\phi_0+\phi}(Z))\\
&\qquad\qquad\qquad\text{(by 4. 3)}\\
&=(CY\xi_0|U_\phi^*\pi_{\phi_0+\phi}(X^\dagger)\lambda_{\phi_0+\phi}(Z))\\
&=(U_\phi CU_\phi^*(T_{\phi_0}^{\phi_0+\phi})^*T_{\phi_0}^{\phi_0+\phi})^{1/2}\lambda_{\phi_0+\phi}(Y)|\pi_{\phi_0+\phi}(X^\dagger)\lambda_{\phi_0+\phi}(Z)),
\end{aligned}$$

and hence $U_\phi C U_\phi^* ((T_{\phi_0}^{\phi_0+\phi})^* T_{\phi_0}^{\phi_0+\phi})^{1/2} \in \pi_{\phi_0+\phi}(\mathcal{M})'$, which implies

$$\begin{aligned} CH_n'^\phi X\xi_0 &= CU_\phi^* \left(\int_{1/n}^1 \lambda^{-1}(1-\lambda)\,dE(\lambda)\right) U_\phi X\xi_0 \\ &= U_\phi^*(U_\phi C U_\phi^*)\left(\int_{1/n}^1 \lambda^{-1}(1-\lambda)\,dE(\lambda)\right)((T_{\phi_0}^{\phi_0+\phi})^* T_{\phi_0}^{\phi_0+\phi})^{1/2}\lambda_{\phi_0+\phi}(X) \\ &= U_\phi^*(U_\phi C U_\phi^*)((T_{\phi_0}^{\phi_0+\phi})^* T_{\phi_0}^{\phi_0+\phi})^{1/2}\left(\int_{1/n}^1 \lambda^{-1}(1-\lambda)\,dE(\lambda)\right)\lambda_{\phi_0+\phi}(X)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by 4.5)} \\ &= U_\phi^*\left(\int_{1/n}^1 \lambda^{-1}(1-\lambda)\,dE(\lambda)\right)(U_\phi C U_\phi^*((T_{\phi_0}^{\phi_0+\phi})^* T_{\phi_0}^{\phi_0+\phi})^{1/2})\lambda_{\phi_0+\phi}(X) \\ &= H_n'^\phi C X\xi_0 \end{aligned}$$

for each $X \in \mathcal{M}$, $C \in \mathcal{M}'$ and $n \in \mathbb{N}$. Hence, $H_n'^\phi \in \mathcal{M}''$ for all $n \in \mathbb{N}$, which implies the implication (1)$\Rightarrow$(2) in II.

The implication (2)$\Rightarrow$(3) in II is similar to the proof of (2)$\Rightarrow$(3) in Theorem 4. 5.

We show the implication (3)$\Rightarrow$(1) in II. It is clear that ϕ is a strongly ϕ_0-absolutely continuous positive linear functional on $\mathcal{M}$. By Lemma 3. 5, (1), ϕ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. We note that $(1+H^2)^{1/2}$ is an invertible positive self-adjoint operator in $\mathfrak{H}(\mathcal{D})$ affiliated with $\mathcal{M}' \cap \mathcal{M}''$ such that $\mathcal{D}((1+H^2)^{1/2}) = \mathcal{D}(H) \supset \mathcal{M}\xi_0$ and $\phi_0 + \phi = \omega_{(1+H^2)^{1/2}\xi_0}$. It hence follows from Lemma 3. 4 (1) and Lemma 3. 5 (2) that $\phi_0 + \phi$ is standard.

In the above proof we have proved $H_n'^\phi \in \mathcal{M}' \cap \mathcal{M}''$ for $n \in \mathbb{N}$, and hence the statement III follows from the statement II. This completes the proof.

Let $(\mathcal{M}, \mathcal{D}, \xi_0)$ be a standard system. Then the following questions arise.

Question I. *Suppose ϕ is a positive linear functional on $\mathcal{M}$ which satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. Then, is ϕ automatically strongly ϕ_0-absolutely continuous?*

In Section 5 we shall state that the above question is affirmative in case that the O_p^*-algebra $(\mathcal{M}, \mathcal{D})$ satisfies the von Neumann density type theorem; that is, $[\mathcal{M}]''_{wo} = \overline{[\mathcal{M}]}^{t_s^*}$.

Question II. *Suppose ϕ is a positive linear functional on $\mathcal{M}$ which satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. Under what conditions is ϕ represented as*

$$\phi=\omega_{H\xi_0}$$

for some positive self-adjoint operator H affiliated with $\mathcal{M}'\cap\mathcal{M}''$ such that $\xi_0\in\mathcal{D}(H)$ and $H\xi_0\in\mathcal{D}$?

We here consider Question II.

1. *If ϕ is strongly ϕ_0-absolutely continuous and $\phi_0+\phi$ is standard, then Question II is affirmative (Theorem 4.9).*

However, it seems to be difficult to show directly that $\phi_0+\phi$ is standard, and so we consider when Question II is affirmative without the assumption of the standardness of $\phi_0+\phi$.

2. *Suppose ϕ is represented as*

$$\phi=\omega_{\xi},\ \xi\in\mathcal{D}$$

and the normal form ω''_{ξ} on the von Neumann algebra $\mathcal{M}''$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. Then Question II is affirmative.

In fact, by ([32] Theorem 15.4) there exists a positive self-adjoint operator H affiliated with $\mathcal{M}'\cap\mathcal{M}''$ such that $\xi_0\in\mathcal{D}(H)$ and

$$(A\xi\,|\,\xi)=(AH\xi_0\,|\,H\xi_0) \tag{4.6}$$

for all $A\in\mathcal{M}''$. Take an arbitrary $X\in\mathcal{M}$. Since $\mathcal{M}'\mathcal{D}=\mathcal{D}$, there is a sequence $\{X_n\}$ in $\mathcal{M}''$ such that $\lim_{n\to\infty} X_n\zeta=X\zeta$ for each $\zeta\in\mathcal{D}$. Then it follows from (4.6) and $\xi_0\in\mathcal{D}(H)$ that $\{X_n\xi_0\}\subset\mathcal{D}(H)$, $\lim_{n\to\infty} X_n\xi_0=X\xi_0$ and $\lim_{n,m\to\infty}\|HX_n\xi_0-HX_m\xi_0\|=\lim_{n,m\to\infty}\|X_n\xi-X_m\xi\|=0$, and hence $\mathcal{M}\xi_0\subset\mathcal{D}(H)$, and so $H\xi_0\in\mathcal{D}$ and $XH\xi_0=HX\xi_0$, which implies $\phi(X)=(XH\xi_0\,|\,H\xi_0)$ for all $X\in\mathcal{M}$.

3. *Suppose ϕ is strongly ϕ_0-absolutely continuous, $\pi_{\phi_0+\phi}(\mathcal{M})'$ is a von Neumann algebra and*

$$\phi(X^{\dagger}X)\leqq\gamma\{\phi_0(X^{\dagger}X)+\phi_0(XX^{\dagger})\},\ X\in\mathcal{M} \tag{4.7}$$

for some constant $\gamma>0$. Then Question II is affirmative.

In fact, by Theorem 4.5 ϕ is represented as

$$\phi=\omega_{H'\xi_0}$$

for some positive self-adjoint operator H' affiliated $\mathcal{M}'$ such that

$\xi_0 \in \mathscr{D}(H')$ and $H'\xi_0 \in \mathscr{D}$. Since $\omega_{H'\xi_0}$ is $\{\sigma_t^{\xi_0}\}$-invariant, we have

$$(4.8) \qquad (H'\Delta_{\xi_0}^{it}X\xi_0 | H'\Delta_{\xi_0}^{it}Y\xi_0) = (H'X\xi_0 | H'Y\xi_0)$$

for all $X, Y \in \mathscr{M}$. Take arbitrary $A \in \mathscr{M}''$. Since $S_{\xi_0}'' = S_{\xi_0}$, there exists a sequence $\{X_n\}$ in $\mathscr{M}$ such that $\lim_{n\to\infty} X_n\xi_0 = A\xi_0$ and $\lim_{n\to\infty} X_n^\dagger \xi_0 = A^*\xi_0$. By (4.7) and (4.8) we have

$$(4.9) \qquad \lim_{n\to\infty} H'X_n\xi_0 = H'A\xi_0 \text{ and } \lim_{n\to\infty} H'\Delta_{\xi_0}^{it}X_n\xi_0 = H'\Delta_{\xi_0}^{it}A\xi_0.$$

By (4.8) and (4.9) we have

$$\|H'\Delta_{\xi_0}^{it}A\xi_0\| = \|H'A\xi_0\|, \ (H'\Delta_{\xi_0}^{it}X\xi_0 | H'\Delta_{\xi_0}^{it}A\xi_0) = (H'X\xi_0 | H'A\xi_0)$$

for all $X \in \mathscr{M}$ and $A \in \mathscr{M}''$, which implies

$$(4.10) \qquad \|H'\Delta_{\xi_0}^{it}X\xi_0 - H'\Delta_{\xi_0}^{it}A\xi_0\| = \|H'X\xi_0 - H'A\xi_0\|$$

for all $X \in \mathscr{M}$ and $A \in \mathscr{M}''$. Since $\omega_{H'\xi_0}$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$, it follows from (4.9) and (4.10) that the normal form $\omega''_{H'\xi_0}$ on $\mathscr{M}''$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. By the above statement 2 Question II is affirmative.

4. *Suppose ϕ is represented as*

$$\phi = \omega_{H'\xi_0}$$

for some positive self-adjoint operator H' affiliated with $\mathscr{M}'$ such that $\xi_0 \in \mathscr{D}(H'^2)$ and $H'^2\xi_0 \in \mathscr{D}$. Then Question II is affirmative.

In fact, since $\mathscr{M}\xi_0 \subset \mathscr{D}(H'^2)$, $H'^2X\xi_0 = XH'^2\xi_0$ for each $X \in \mathscr{M}$ and $\omega_{H'\xi_0}$ is $\{\sigma_t^{\xi_0}\}$-invariant, it follows that

$$(4.11) \qquad H'^2\Delta_{\xi_0}^{it}X\xi_0 = \Delta_{\xi_0}^{it}H'^2X\xi_0$$

for all $X \in \mathscr{M} \cup \mathscr{M}''$ and $t \in \mathbf{R}$, which implies by $S_{\xi_0}'' = S_{\xi_0}$ that the normal form $\omega''_{H'\xi_0}$ on $\mathscr{M}''$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. By the statement 2 Question II is affirmative.

5. *Suppose $(\mathscr{M}, \mathscr{D}, \xi_0)$ is a full standard system and ϕ is strongly ϕ_0-absolutely continuous, $\pi_{\phi_0+\phi}(\mathscr{M})'$ is a von Neumann algebra and*

$$(4.12) \qquad \phi(X^\dagger X) \leqq \sum_{k=1}^{n} \phi_0(X^\dagger Y_k Y_k X), \ X \in \mathscr{M}$$

for some finite subset $\{Y_1, Y_2, \dots, Y_n\}$ of $\mathscr{M}$. Then Question II is affirmative.

In fact, ϕ is represented as

$$\phi=\omega_{H'\xi_0}$$

for some positive self-adjoint operator H' affiliated with $\mathscr{M}'$ such that $\xi_0\in\mathscr{D}(H')$ and $H'\xi_0\in\mathscr{D}$. Since ξ_0 is a strongly cyclic vector for $\mathscr{M}$, it follows from (4.12) that $\mathscr{D}\subset\mathscr{D}(H')$, which implies $\mathscr{M}\xi_0\subset\mathscr{D}(H'^2)$. Hence, by the statement 4, Question II is affirmative.

6. *Suppose the O_p^*-algebra $(\mathscr{M}, \mathscr{D})$ satisfies the von Neumann density type theorem and ϕ is σ-weakly continuous. Then Question II is affirmative (Theorem* 5.6).

We study a Radon-Nikodym theorem for $\{\sigma_t^{\xi_0}\}$-invariant positive linear functionals on $\mathscr{M}$. We denote by $\mathscr{M}''^{\sigma^{\xi_0}}$ and $\mathscr{M}'^{\sigma^{\xi_0}}$ the fixed point algebras of $\{\sigma_t^{\xi_0}\}$ in $\mathscr{M}''$ and $\mathscr{M}'$, respectively.

Theorem 4.10. *Let $(\mathscr{M}, \mathscr{D}, \xi_0)$ be a standard system.*

I. *The following statements are equivalent.*

(1) *ϕ is a ω_{ξ_0}-dominated, $\{\sigma_t^{\xi_0}\}$-invariant positive linear functional on $\mathscr{M}$.*

(2) *ϕ is represented as*

$$\phi=\omega_{H'\xi_0}$$

for some positive operator H' in $\mathscr{M}'^{\sigma^{\xi_0}}$.

(3) *ϕ is represented as*

$$\phi=\omega_{H\xi_0}$$

for some positive operator H in $\mathscr{M}''^{\sigma^{\xi_0}}$ such that $H\xi_0\in\mathscr{D}$.

In the following II and III, suppose ϕ is a positive linear functional on $\mathscr{M}$ such that $\phi_0+\phi\leqq\tau$ for some standard positive linear functional τ on $\mathscr{M}$ which satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$.

II. *Suppose ϕ is $\{\sigma_t^{\xi_0}\}$-invariant. Then ϕ is decomposed into the sum:*

$$\phi=\phi_c^\sigma+\phi_s^\sigma$$

where ϕ_c^σ is a strongly ϕ_0-absolutely continuous $\{\sigma_t^{\xi_0}\}$-invariant positive linear functional on $\mathscr{M}$ and ϕ_s^σ is a ϕ_0-singular, $\{\sigma_t^{\xi_0}\}$-invariant positive linear functional on $\mathscr{M}$. If ϕ is strongly ϕ_0-absolutely continuous, then $\phi=\phi_c^\sigma$; and if ϕ is ϕ_0-singular, then $\phi=\phi_s^\sigma$.

III. *The following statements are equivalent.*

(1) *ϕ is strongly ϕ_0-absolutely continuous and $\{\sigma_t^{\xi_0}\}$-invariant.*

(2) *ϕ is represented as*

$$\phi=\omega_{H'\xi_0}$$

for some positive self-adjoint operator H' affiliated with $\mathscr{M}'^{\sigma^{\xi_0}}$ such that $\xi_0\in\mathscr{D}(H')$ and $H'\xi_0\in\mathscr{D}$.

(3) *ϕ is represented as*

$$\phi=\omega_{H\xi_0}$$

for some positive self-adjoint operator H affiliated with $\mathscr{M}''^{\sigma^{\xi_0}}$ such that $\xi_0\in\mathscr{D}(H)$ and $H\xi_0\in\mathscr{D}$.

Proof. I. (1)$\Leftrightarrow$(2) This is trivial.

(2)$\Rightarrow$(3) Put

$$H=J_{\xi_0}H'J_{\xi_0}.$$

Then H is a positive operator in $\mathscr{M}''^{\sigma^{\xi_0}}$ satisfying $H\xi_0=H'\xi_0$, and hence $H\xi_0\in\mathscr{D}$ and $\phi=\omega_{H\xi_0}$.

(3)$\Rightarrow$(2) This is similar to the proof of (2)$\Rightarrow$(3).

II. Since τ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$, it follows from ([14] Lemma 3. 8) that

$$\Delta_\tau^{it}\lambda_\tau(X)=\lambda_\tau(\sigma_t^{\xi_0}(X)) \tag{4. 13}$$

for all $X\in\mathscr{M}$ and $t\in\mathbf{R}$. Since $\phi_0\leqq\tau$ and $\phi\leqq\tau$, there exist R, $K\in\pi_\tau(\mathscr{M})'$ such that $0\leqq R$, $K\leqq 1$ and

$$\phi_0(X)=(R\lambda_\tau(X)\,|\,\lambda_\tau(I)),\ \phi(X)=(K\lambda_\tau(X)\,|\,\lambda_\tau(I))$$

for $X\in\mathscr{M}$. Using (4. 13) and the standardness of τ, we can prove in the same way as in Theorem 4. 9 that the normal form on $\pi_\tau(\mathscr{M})''$:

$$A\rightarrow(RA\lambda_\tau(I)\,|\,\lambda_\tau(I))$$

satisfies the KMS-condition with respect to $\{\sigma_t^\tau\}$ and $A\rightarrow(KA\lambda_\tau(I)\,|\,\lambda_\tau(I))$ is $\{\sigma_t^\tau\}$-invariant. Hence, $R\in\pi_\tau(\mathscr{M})'\cap\pi_\tau(\mathscr{M})''$ and $K\in\pi_\tau(\mathscr{M})'^{\sigma^{\lambda_\tau(I)}}$. We donote by U the isometry of $\mathfrak{H}(\mathscr{D})$ into $\mathfrak{H}_\tau$ defined by:

$$UX\xi_0=R^{1/2}\lambda_\tau(X),\ X\in\mathscr{M}.$$

We now put

$$H'_n = U^*(\int_{1/n}^{1} 1/\lambda dE(\lambda))KU,\ n \in \mathbb{N},$$

where $R=\int_0^1 \lambda dE(\lambda)$ denotes the spectral resolutions of R. Since R and K commute, it follows that $\{H'_n\}$ is a sequence of positive operators $\mathcal{M}'^{\xi_0}_{\sigma}$ and $\lim_{n\to\infty} H'^{1/2}_n X\xi_0$ exists for each $X \in \mathcal{M}$. We here put

$$\phi^{\sigma}_c(X) = \lim_{n\to\infty} (H'_n X\xi_0 | \xi_0),$$
$$\phi^{\sigma}_s(X) = (KE(0)\lambda_\tau(X) | \lambda_\tau(I)),\ X \in \mathcal{M}.$$

Then it is easily shown that ϕ^{σ}_c is a strongly ϕ_0-absolutely continuous, $\{\sigma^{\xi_0}_t\}$-invariant positive linear functional on $\mathcal{M}$, ϕ^{σ}_s is a ϕ_0-singular, $\{\sigma^{\xi_0}_t\}$-invariant positive linear functional on $\mathcal{M}$ and $\phi = \phi^{\sigma}_c + \phi^{\sigma}_s$. Suppose ϕ is strongly ϕ_0-absolutely continuous. For each $X \in \mathcal{M}$ there is a sequence $\{X_n\}$ in $\mathcal{M}$ such that $\lim_{n\to\infty} \lambda_\tau(X_n) = E(0)\lambda_\tau(X)$.
Then we have

$$\lim_{n\to\infty} X_n\xi_0 = \lim_{n\to\infty} UR^{1/2}X_n\xi_0 = UR^{1/2}E(0)\lambda_\tau(X) = 0,$$
$$\lim_{n\to\infty} (\lambda_\phi(X_n) | \lambda_\phi(Y)) = \lim_{n\to\infty} (K\lambda_\tau(X_n) | \lambda_\tau(Y)) = (KE(0)\lambda_\tau(X) | \lambda_\tau(Y))$$

for each $Y \in \mathcal{M}$. Since $\phi \leqq \tau$ and ϕ is strongly ϕ_0-absolutely continuous, we have $\lim_{n\to\infty} \lambda_\phi(X_n) = 0$, and hence $(KE(0)\lambda_\tau(X) | \lambda_\tau(Y)) = 0$ for each $Y \in \mathcal{M}$. Hence, $KE(0)\lambda_\tau(X) = 0$; that is, $\phi^{\sigma}_s = 0$. Similarly, if ϕ is ϕ_0-singular, then $\phi = \phi^{\sigma}_s$.

III. (1)$\Leftrightarrow$(2) Using II, this is proved in similar to the proof of Theorem 4. 9.

(2)$\Rightarrow$(3) Put

$$H = J_{\xi_0} H' J_{\xi_0}.$$

Then H is a positive self-adjoint operator affiliated with $\mathcal{M}''^{\xi_0}_{\sigma}$ such that $\xi_0 \in \mathscr{D}(H)$, $H\xi_0 = H'\xi_0 \in \mathscr{D}$ and $\phi = \omega_{H\xi_0}$.

(3)$\Rightarrow$(2) This is similar to the proof of (2)$\Rightarrow$(3). This completes the proof.

Remark. We don't know whether ϕ^{σ}_c is maximal in the subset of $P^{\phi_0}_c(\mathcal{M}, \phi)$ of $\{\sigma^{\xi_0}_t\}$-invariant positive linear functionals or not.

§ 5. Radon-Nikodym Theorems for O_p^*-algebras Satisfying the von Neumann Density Type Theorem

Throughout this section, let $(\mathscr{M}, \mathscr{D})$ be a closed O_p^*-algebra such that $\mathscr{M}'\mathscr{D}=\mathscr{D}$ and $[\mathscr{M}]''_{w\sigma}=\overline{[\mathscr{M}]}^{t_s^*}$, and ξ_0 be a cyclic and separating vector for $\mathscr{M}''$. We denote by $\mathscr{M}_*^+$ the set of all positive linear functionals which are continuous relative to the σ-weak topology for $\mathscr{M}$, and denote by $\mathscr{P}_{\xi_0}^\natural$ the natural positive cone associated with $(\mathscr{M}'', \xi_0)$ [1, 4, 11].

Theorem 5.1. *Suppose $\phi \in \mathscr{M}_*^+$. Then there exists a unique vector ξ_ϕ in $\mathscr{P}_{\xi_0}^\natural \cap \mathscr{D}$ such that*

$$\phi(X)=(X\xi_\phi|\xi_\phi)$$

for all $X \in \mathscr{M}$.

Proof. By ([16] Lemma 5.2) there exists a vector ξ in $\mathscr{D}$ such that $\phi=\omega_\xi$. It hence follows from ([31] Theorem 10.25) that

$$(A\xi|\xi)=(A\xi_\phi|\xi_\phi),\ A\in\mathscr{M}'' \tag{5.1}$$

for a unique vector ξ_ϕ in $\mathscr{P}_{\xi_0}^\natural$. Take an arbitrary $X \in \mathscr{M}$. Let $(X^*\bar{X})^{1/2}=\int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of $(X^*\bar{X})^{1/2}$, and let $E_n=\int_0^n dE(\lambda)$ for $n\in\mathrm{N}$. Since $\mathscr{M}'\mathscr{D}=\mathscr{D}$, it follows that E_n, $\bar{X}E_n\in\mathscr{M}''$ for $n\in\mathrm{N}$. Hence, we have by (5.1)

$$\lim_{n\to\infty} E_n\xi_\phi=\xi_\phi \text{ and } \lim_{m,n\to\infty}||\bar{X}E_m\xi_\phi-\bar{X}E_n\xi_\phi|| =\lim_{m,n\to\infty}||\bar{X}E_m\xi-\bar{X}E_n\xi|| =0,$$

which implies $\xi_\phi\in\bigcap_{X\in\mathscr{M}}\mathscr{D}(\bar{X})=\mathscr{D}$ and $\phi=\omega_{\xi_\phi}$. Suppose $\phi=\omega_{\xi_1}=\omega_{\xi_2}$ for $\xi_1, \xi_2\in\mathscr{P}_{\xi_0}^\natural\cap\mathscr{D}$. Since $[\mathscr{M}]''_{w\sigma}=\overline{[\mathscr{M}]}^{t_s^*}$ and (5.1), we have $\xi_1=\xi_2$. This completes the proof.

Theorem 5.2. *Suppose $\phi\in\mathscr{M}_*^+$. Then the following statements hold.*

(1) *ϕ is strongly ϕ_0-absolutely continuous if and only if ϕ is represented as*

$$\phi(X)=(XH'\xi_0|H'\xi_0),\ X\in\mathscr{M}$$

for some positive self-adjoint operator H' affiliated with $\mathscr{M}'$ such that $\mathscr{M}\xi_0$ is a core for H'. In this case, such an operator H' for ϕ is unique, which is denoted by H'_ϕ.

(2) *ϕ is ϕ_0-singular if and only if $P(\mathscr{M}, \phi)\cap P(\mathscr{M}, \phi_0)=\{0\}$.*

(3) *ϕ is decomposed into the sum:*

$$\phi=\phi_c+\phi_s,$$

where ϕ_c is maximal in $P_c^{\phi_0}(\mathscr{M}, \phi)$ and $\phi_s\in P_s^{\phi_0}(\mathscr{M}, \phi)$.

Proof. By Theorem 5.1, $\phi_0+\phi$ is represented as

$$(\phi_0+\phi)(X)=(X\xi_{\phi_0+\phi}|\xi_{\phi_0+\phi}),\ X\in\mathscr{M} \tag{5.2}$$

for a unique vector $\xi_{\phi_0+\phi}\in\mathscr{P}^\natural_{\xi_0}\cap\mathscr{D}$, which implies by $[\mathscr{M}]''_{w\sigma}=\overline{[\mathscr{M}]}^{t_s^*}$ that $\xi_{\phi_0+\phi}$ is a separating vector for $\mathscr{M}''$. Since $\xi_{\phi_0+\phi}\in\mathscr{P}^\natural_{\xi_0}$, it follows that $\xi_{\phi_0+\phi}$ is also cyclic for $\mathscr{M}''$. We put

$$U\lambda_{\phi_0+\phi}(X)=X\xi_{\phi_0+\phi},\ X\in\mathscr{M}.$$

By (5.2) U is extended to a unitary operator of $\mathfrak{H}_{\phi_0+\phi}$ onto $\mathfrak{H}(\mathscr{D})$, which is also denoted by U. Using $[\mathscr{M}]''_{w\sigma}=\overline{[\mathscr{M}]}^{t_s^*}$ and $\xi_{\phi_0+\phi}$ is a cyclic vector for $\mathscr{M}''$, we can prove that $\pi_{\phi_0+\phi}(\mathscr{M})'=U^*\mathscr{M}'U$, so that $\pi_{\phi_0+\phi}(\mathscr{M})'$ is a von Neumann algebra. Hence, the statements (2) and (3) follow from Corollary 4.7 and Theorem 4.6, respectively.

We show the statement (1). Suppose ϕ is strongly ϕ_0-absolutely continuous. We denote by $T_\phi^{\phi_0}$ the closure of a closable map:

$$X\xi_0\in\mathscr{M}\xi_0\rightarrow X\xi_\phi\in\mathscr{M}\xi_\phi.$$

Then, it follows from $[\mathscr{M}]''_{w\sigma}=\overline{[\mathscr{M}]}^{t_s^*}$ that $\mathscr{M}''\xi_0\subset\mathscr{D}(T_\phi^{\phi_0})$ and $T_\phi^{\phi_0}A\xi_0=A\xi_\phi$ for all $A\in\mathscr{M}''$, which implies $\mathscr{M}''\xi_0$ is a core of $T_\phi^{\phi_0}$ and $T_\phi^{\phi_0}$ is affiliated with $\mathscr{M}'$. Put

$$H'_\phi=((T_\phi^{\phi_0})^*T_\phi^{\phi_0})^{1/2}.$$

Then it is easily shown that H'_ϕ is a positive self-adjoint operator affiliated with $\mathscr{M}'$ such that $\mathscr{M}\xi_0$ is a core for H'_ϕ and $\phi=\omega_{H'_\phi\xi_0}$. The uniqueness of H'_ϕ follows from that of polar decomposition. The converse follows from Theorem 4.5. This completes the proof.

Remark 5.3. Representing operators H' for ϕ in Theorem 4.6 satisfy $\mathscr{M}\xi_0\subset\mathscr{D}(H')$ but without the condition $[\mathscr{M}]''_{w\sigma}=\overline{[\mathscr{M}]}^{t_s^*}$, there does not necessarily exist a representing operator H' for ϕ such that $\mathcal{M}\xi_0$ is core for H'.

Remark 5.4. Suppose $\mathcal{M}''$ is finite. Then every $\phi\in\mathcal{M}_*^+$ is strongly ϕ_0-absolutely continuous. This is proved in similar to ([19] Corollary 2.3).

Theorem 5.5. *Suppose $(\mathcal{M}, \mathcal{D}, \xi_0)$ is a standard system and $\phi\in\mathcal{M}_*^+$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$. Then the following statements hold.*

(1) *ϕ is represented as*

$$\phi(X)=(XH\xi_0|H\xi_0),\ X\in\mathcal{M}$$

for some positive self-adjoint operator H affiliated with $\mathcal{M}'\cap\mathcal{M}''$ such that $\xi_0\in\mathcal{D}(H)$ and $H\xi_0\in\mathcal{D}$. Further, if ϕ is faithful; that is, $\phi(X^\dagger X)=0$ implies $X=0$, then ϕ is a standard positive linear functional on $\mathcal{M}$ with $\mathcal{D}_\phi=\mathcal{D}(\pi_\phi)$.

(2) *Suppose $(\mathcal{M}, \mathcal{D}, \xi_0)$ is full. Then ϕ is a standard positive linear functional on $\mathcal{M}$ with $\mathcal{D}_\phi=\mathcal{D}(\pi_\phi)$. Further, if ϕ is faithful, then $(\pi_\phi(\mathcal{M}), \mathcal{D}(\pi_\phi), \lambda_\phi(I))$ is a full standard system.*

Proof. (1) It follows from Theorem 5.1 that $\phi=\omega_{\xi_\phi}$ for $\xi_\phi\in\mathcal{P}_{\xi_0}^\natural\cap\mathcal{D}$. Since $[\mathcal{M}]''_{w\sigma}=\overline{[\mathcal{M}]}^{t_s^*}$, it follows that $\omega''_{\xi_\phi}\in(\mathcal{M}'')_*^+$ satisfies the KMS-condition with respect to $\{\sigma_t^{\xi_0}\}$, so that by ([32] Theorem 15.4) there exists a positive self-adjoint operator H affiliated with $\mathcal{M}'\cap\mathcal{M}''$ such that

$$(A\xi_\phi|\xi_\phi)=(AH\xi_0|H\xi_0) \tag{5.3}$$

for all $A\in\mathcal{M}''$. We denote by U' the partial isometry on $\mathfrak{H}(\mathcal{D})$ defined by:

$$A\xi_\phi\rightarrow AH\xi_0,\ A\in\mathcal{M}''.$$

Using $[\mathcal{M}]''_{w\sigma}=\overline{[\mathcal{M}]}^{t_s^*}$, we can prove $U'\in\mathcal{M}'$, and hence $H\xi_0=U'\xi_\phi\in\mathcal{D}$, which implies $\phi=\omega_{H\xi_0}$ by (5.3).

Suppose ϕ is faithful. Since the projection E of $\mathfrak{H}(\mathcal{D})$ onto Ker H is contained in $\mathcal{M}'\cap\mathcal{M}''$ and $(\mathcal{M}, \mathcal{D})$ is a generalized von Neumann algebra, it follows that $E_0\equiv E/\mathcal{D}\in\mathcal{M}$, and hence

$$\phi(E_0)=(EH\xi_0|H\xi_0)=0.$$

Since ϕ is faithful, we have $E_0=0$, and hence H is nonsingular. It follows from Lemma 3.4 and Lemma 3.5 that ϕ is a standard positive linear functional on $\mathscr{M}$ with $\mathscr{D}_\phi=\mathscr{D}(\pi_\phi)$.

(2) We denote by $E'_{H\xi_0}$ the projection of $\mathfrak{H}(\mathscr{D})$ onto $\overline{\mathscr{M}H\xi_0}$. It follows from $[\mathscr{M}]''_{w\sigma}=\overline{[\mathscr{M}]}^{t_s^*}$ that $\overline{\mathscr{M}H\xi_0}$ is a closed subspace which is invariant for $\mathscr{M}''$, and hence $E'_{H\xi_0}\in\mathscr{M}'$. It is easily shown that the restriction $\mathscr{M}/E'_{H\xi_0}\mathscr{D}$ of the O_p^*-algebra $\mathscr{M}$ to $E'_{H\xi_0}\mathscr{D}$ is a closed O_p^*-algebra such that $(\mathscr{M}/E'_{H\xi_0}\mathscr{D})'=E'_{H\xi_0}\mathscr{M}'/E'_{H\xi_0}\mathfrak{H}(\mathscr{D})$ and $(\mathscr{M}/E'_{H\xi_0}\mathscr{D})''=\mathscr{M}''/E'_{H\xi_0}\mathfrak{H}(\mathscr{D})$. Let $H=\int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of H. Put

$$K_n=\int_{1/n}^n \frac{1}{\lambda}dE(\lambda),\quad E_n=\int_{1/n}^n dE(\lambda)$$

for $n\in\mathbb{N}$. Then we have $K_n/\mathscr{D}$, $E_n/\mathscr{D}\in\mathscr{M}$ and

$$\begin{aligned}\lim_{n\to\infty} E_nE'_{H\xi_0}X\xi_0&=\lim_{n\to\infty} K_nXH\xi_0\\&=E'_{H\xi_0}X\xi_0-E(0)E'_{H\xi_0}X\xi_0\\&=E'_{H\xi_0}X\xi_0,\\ \lim_{n\to\infty} YE_nE'_{H\xi_0}X\xi_0&=E'_{H\xi_0}YX\xi_0\end{aligned}$$

for each X, $Y\in\mathscr{M}$, which implies $E'_{H\xi_0}X\xi_0\in\overline{\mathscr{M}H\xi_0}^{t_\mathscr{M}}$ for each $X\in\mathscr{M}$. On the other hand, it is easily shown that $E'_{H\xi_0}\mathscr{D}\subset\overline{E'_{H\xi_0}\mathscr{M}\xi_0}^{t_\mathscr{M}}$. Hence, we have

$$\overline{E'_{H\xi_0}\mathscr{M}\xi_0}^{t_\mathscr{M}}=\overline{\mathscr{M}H\xi_0}^{t_\mathscr{M}}=E'_{H\xi_0}\mathscr{D}\,; \tag{5.4}$$

that is, $H\xi_0$ is a strongly cyclic vector for $(\mathscr{M}/E'_{H\xi_0}\mathscr{D},\ E'_{H\xi_0}\mathscr{D})$. It is clear that $H\xi_0$ is a separating vector for $(\mathscr{M}/E'_{H\xi_0}\mathscr{D})''=\mathscr{M}''/E'_{H\xi_0}\mathfrak{H}(\mathscr{D})$. Since $\Delta_{\xi_0}^{it}\mathscr{M}H\xi_0=\mathscr{M}H\xi_0$ for all $t\in\mathbb{R}$, we have $\Delta_{\xi_0}^{it}E'_{H\xi_0}=E'_{H\xi_0}\Delta_{\xi_0}^{it}$ for all $t\in\mathbb{R}$, which implies that $\omega''_{H\xi_0}$ satisfies the KMS-condition with respect to a strongly continuous one-parameter group of $*$-automorphisms:

$$A/E'_{H\xi_0}\mathfrak{H}(\mathscr{D})\to(\Delta_{\xi_0}^{it}E'_{H\xi_0})A(E'_{H\xi_0}\Delta_{\xi_0}^{-it})$$

of the von Neumann algebra $\mathscr{M}''/E'_{H\xi_0}\mathfrak{H}(\mathscr{D})$. By ([32] Theorem 13.2) we have

$$\Delta''^{it}_{H\xi_0}A\Delta''^{-it}_{H\xi_0}E'_{H\xi_0}\xi=(\Delta_{\xi_0}^{it}E'_{H\xi_0})A(\Delta_{\xi_0}^{-it}E'_{H\xi_0})E'_{H\xi_0}\xi$$

for all $\xi\in\mathfrak{H}(\mathscr{D})$ and $t\in\mathbb{R}$, which implies $\Delta''^{it}_{H\xi_0}=\Delta_{\xi_0}^{it}E'_{H\xi_0}$ for all $t\in\mathbb{R}$. Hence $H\xi_0$ is a modular vector for $(\mathscr{M}/E'_{H\xi_0}\mathscr{D},\ E'_{H\xi_0}\mathscr{D})$ with $\mathscr{D}_{H\xi_0}=$

$E'_{H\xi_0}\mathscr{D}$. Since $[\mathscr{M}]''_{w\sigma}=\overline{[\mathscr{M}]}^{t_s^*}$, it follows that $H\xi_0$ is standard, which implies by (5.4) that ϕ is a standard positive linear functional on $\mathscr{M}$ with $\mathscr{D}_\phi=\mathscr{D}(\pi_\phi)$.

Suppose ϕ is faithful. By (1), H is non-singular, and so $E'_{H\xi_0}=I$. By (5.4) $H\xi_0$ is a strongly cyclic vector for $\mathscr{M}$, and hence $(\mathscr{M}, \mathscr{D}, H\xi_0)$ is a full standard system, which implies that so is $(\pi_\phi(\mathscr{M}), \mathscr{D}(\pi_\phi), \lambda_\phi(I))$. This completes the proof.

We can similarly prove the following result using ([32] Theorem 15.2).

Theorem 5.6. *Suppose $(\mathscr{M}, \mathscr{D}, \xi_0)$ is a standard system and $\phi\in\mathscr{M}_*^+$. Then ϕ is $\{\sigma_t^{\xi_0}\}$-invariant if and only if ϕ is represented as*

$$\phi(X)=(XH\xi_0|H\xi_0),\ X\in\mathscr{M}$$

for some positive self-adjoint operator H affiliated with $\mathscr{M}''^{\sigma^{\xi_0}}$ such that $\xi_0\in\mathscr{D}(H)$ and $H\xi_0\in\mathscr{D}$.

We apply Radon-Nikodym theorems obtained the above to the spatial theory for O_p^*-algebras. The spatial theory for O_p^*-algebras was investigated in [13, 16, 33, 35]. In particular, it was obtained that every $*$-automorphism of the maximal O_p^*-algebra is unitarily implemented [33, 35] and each $*$-automorphism α of the O_p^*-algebra $\pi_0(\mathscr{A})$ of the Schrödinger representation π_0 of the canonical algebra $\mathscr{A}$ for one degree of freedom satisfying $\alpha(\pi_0(\mathscr{A})^+)\subset\pi_0(\mathscr{A})^+$ is unitarily implemented [33]. In the case of von Neumann algebras $\mathscr{M}_0$ with a cyclic and separating vector, each $*$-automorphism of $\mathscr{M}_0$ is always unitarily implemented, but in [33] Takesue gave an example of the self-adjoint O_p^*-algebra (the polynomial algebra $(\mathscr{P}\left(-i\frac{d}{dt}/\mathscr{D}\right), \mathscr{D})$, where $\mathscr{D}=\{f\in C^\infty[0,1];\ f^{(n)}(0)=f^{(n)}(1),\ n=0,1,2,\dots\}$) with a strongly cyclic and separating vector for which the above fact does not necessarily hold, and so we need consider the spatial theory for a self-adjoint O_p^*-algebra with a strongly cyclic and separating vector.

In a previous paper [16], we obtained the following Propositions 5.7, 5.8.

Proposition 5.7. *Let $(\mathscr{M}, \mathscr{D})$ be a self-adjoint O_p^*-algebra, a vector ξ_0 in $\mathscr{D}$ be strongly cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$ and α be a*

$*$-automorphism of $\mathscr{M}$. Then the following statements hold.

(1) *Suppose both the map $X\xi_0 \to \alpha(X)\xi_0$ and $X\xi_0 \to \alpha^{-1}(X)\xi_0$ are continuous. Then α is represented as*

$$\alpha(X) = U^\dagger X U, \quad X \in \mathscr{M}$$

for some $U \in \mathscr{L}^\dagger(\mathscr{D})_u \equiv \{U \in \mathscr{L}^\dagger(\mathscr{D});\ \bar{U} \text{ is unitary}\}$.

(2) *Suppose $\pi_{\phi_0+\phi_0\circ\alpha}(\mathscr{M})'$ is a von Neumann algebra, and the map $X\xi_0 \to \alpha(X)\xi_0$ and $X\xi_0 \to \alpha^{-1}(X)\xi_0$ are closable. Then α is represented as*

$$\alpha(X) = U^\dagger X U, \quad X \in \mathscr{M}$$

for some $U \in \mathscr{L}^\dagger(\mathscr{D})_u$.

Throughout the rest of this section, let $(\mathscr{M}, \mathscr{D})$ be a selfadjoint O_p^*-algebra such that $\overline{[\mathscr{M}]}^{t_s^*} = [\mathscr{M}]''_{w\sigma}$, a vector ξ_0 in $\mathscr{D}$ be strongly cyclic for $\mathscr{M}$ and separating for $\mathscr{M}''$ and α be a $*$-automorphism of $\mathscr{M}$.

Proposition 5.8. *Suppose $\phi_0\circ\alpha$ and $\phi_0\circ\alpha^{-1}$ in $\mathscr{M}_*^+$; in particular, α and α^{-1} are continuous relative to the σ-weak topology for $\mathscr{M}$. Then α is represented as*

$$\alpha(X) = U^\dagger X U, \quad X \in \mathscr{M}$$

for some $U \in \mathscr{L}^\dagger(\mathscr{D})_u$.

We here weaken the condition of the continuity of α and α^{-1} in Proposition 5.8.

Theorem 5.9. *Suppose α is continuous relative to the σ-strong topology for $\mathscr{M}$. Then α is represented as*

$$\alpha(X) = U^\dagger X U, \quad X \in \mathscr{M}$$

for some $U \in \mathscr{L}^\dagger(\mathscr{D})_i \equiv \{U \in \mathscr{L}^\dagger(\mathscr{D});\ U^*\bar{U} = I\}$. Further, suppose α^{-1} is closable relative to the σ-strong topology for $\mathscr{M}$. Then α is represented as*

$$\alpha(X) = U^\dagger X U, \quad X \in \mathscr{M}$$

for some $U \in \mathscr{L}^\dagger(\mathscr{D})_u$.

Proof. Since α is continuous relative to the σ-strong topology for

$\mathscr{M}$, it follows that

$$\|\alpha(X)\xi_0\| \leqq \|[X]\{\xi_n\}\|, \quad X \in \mathscr{M}$$

for some $\{\xi_n\} \in \mathscr{D}^\infty(\mathscr{M})$. In similar to the proof of ([16] Lemma 5.2), we can prove that

$$(\alpha(X)\xi_0 | \xi_0) = (X\zeta_0 | \zeta_0), \quad X \in \mathscr{M}$$

for some $\zeta_0 \in \mathscr{D}$. By Theorem 5.1 there exists a vector η_0 in $\mathscr{P}^\dagger_{\xi_0} \cap \mathscr{D}$ such that

$$(5.5) \qquad (\alpha(X)\xi_0 | \xi_0) = (X\eta_0 | \eta_0)$$

for all $X \in \mathscr{M}$. Put

$$U_0\alpha(X)\xi_0 = X\eta_0, \quad X \in \mathscr{M}.$$

Then, by (5.5) the closure $\overline{U_0}$ of U_0 is an isometry on $\mathfrak{H}(\mathscr{D})$. We now put

$$U = \overline{U_0}/\mathscr{D}.$$

Then it is easily shown that $U \in \mathscr{L}^\dagger(\mathscr{D})_i$, $\bar{U}U^* \in \mathscr{M}'$ and $\alpha(X) = U^\dagger XU$ for all $X \in \mathscr{M}$.

Suppose α^{-1} is closable relative to the σ-strong* topology. Then we show $U \in \mathscr{L}^\dagger(\mathscr{D})_u$. Suppose $A\eta_0 = 0$, $A \in \mathscr{M}''$. Since $[\mathscr{M}''] \subset \overline{[\mathscr{M}]}^{t_s^*}$, there exists a net $\{X_\lambda\}$ in $\mathscr{M}$ such that

$$(5.6) \qquad \lim_\lambda [X_\lambda]\{\xi_n\} = [A]\{\xi_n\}, \ \lim_\lambda [X_\lambda^\dagger]\{\xi_n\} = [A^*]\{\xi_n\}$$

for each $\{\xi_n\} \in \mathscr{D}^\infty(\mathscr{M})$. Since α is continuous relative to the σ-strong topology for $\mathscr{M}$, there exist elements $\alpha''(A)$ and $\alpha''(A^*)$ of $\mathscr{L}(\mathscr{D}, \mathfrak{H}(\mathscr{D}))$ such that

$$(5.7) \qquad \begin{aligned} &\lim_\lambda \ [\alpha(X_\lambda)]\{\xi_n\} = [\alpha''(A)]\{\xi_n\}, \\ &\lim_\lambda \ [\alpha(X_\lambda^\dagger)]\{\xi_n\} = [\alpha''(A^*)]\{\xi_n\} \end{aligned}$$

for each $\{\xi_n\} \in \mathscr{D}^\infty(\mathscr{M})$. Then we have

$$\begin{aligned} ([\alpha''(A)]\{\xi_n\} \mid \{\eta_n\}) &= \lim_\lambda ([\alpha(X_\lambda)]\{\xi_n\} \mid \{\eta_n\}) \\ &= \lim_\lambda (\{\xi_n\} \mid [\alpha(X_\lambda^\dagger)]\{\eta_n\}) \\ &= (\{\xi_n\} \mid [\alpha''(A^*)]\{\eta_n\}) \end{aligned}$$

for each $\{\xi_n\}, \{\eta_n\} \in \mathscr{D}^\infty(\mathscr{M})$. Hence, we have $\alpha''(A) \in \mathscr{C}^\dagger(\mathscr{D}, \mathfrak{H}(\mathscr{D}))$ and $\alpha''(A)^\dagger = \alpha''(A^*)$. By (5.5), (5.6) and (5.7) we have

$$\begin{aligned}||\alpha''(A)\xi_0|| &= \lim_\lambda ||\alpha(X_\lambda)\xi_0|| \\ &= \lim_\lambda ||X_\lambda\eta_0|| \\ &= ||A\eta_0|| = 0,\end{aligned}$$

and hence $\alpha''(A)\xi_0=0$, and further we have

$$\begin{aligned}(\alpha''(A^*)\xi_0|C\xi_0) &= \lim_\lambda(\alpha(X_\lambda^\dagger)\xi_0|C\xi_0) \\ &= \lim_\lambda(C^*\xi_0|\alpha(X_\lambda)\xi_0) \\ &= (C^*\xi_0|\alpha''(A)\xi_0) \\ &= 0\end{aligned}$$

for each $C\in\mathcal{M}'$, and hence $\alpha''(A^*)\xi_0=0$, which implies

$$(\alpha''(A)\xi|C\xi_0) = (C^*\xi_0|\alpha''(A^*)\xi_0) = 0$$

for each $\xi\in\mathscr{D}$ and $C\in\mathcal{M}'$. Hence, $\alpha''(A)=0$. By (5.6) and (5.7), a net $\{\alpha(X_\lambda)\}$ in $\mathcal{M}$ converges to 0 and $\{\alpha^{-1}(\alpha(X_\lambda))\}$ is a Cauchy net in $\mathcal{M}$ relative to the σ-strong* topology for $\mathcal{M}$. Since α^{-1} is closable relative to the σ-strong* topology for $\mathcal{M}$, it follows that $\lim_\lambda X_\lambda=0$, and hence $A=0$. Hence, η_0 is a separating vector for $\mathcal{M}''$. It follows from $\eta_0\in\mathscr{P}_{\xi_0}^\natural$ that η_0 is a cyclic vector for $\mathcal{M}''$, which implies that $\bar{U}$ is a unitary operator on $\mathfrak{H}(\mathscr{D})$. This completes the proof.

Theorem 5.10. *Suppose $\phi_0\circ\alpha\in\mathcal{M}_*^+$ and the map $X\xi_0\to\alpha^{-1}(X)\xi_0$ is closable. Then α is represented as*

$$\alpha(X) = U^\dagger XU, \quad X\in\mathcal{M}$$

for some $U\in\mathscr{L}^\dagger(\mathscr{D})_u$.

Proof. By Theorem 5.1 there exists an element η_0 of $\mathscr{P}_{\xi_0}^\natural\cap\mathscr{D}$ such that

$$(\alpha(X)\xi_0|\xi_0) = (X\eta_0|\eta_0) \tag{5.8}$$

for all $X\in\mathcal{M}$. Suppose $A\eta_0=0$, $A\in\mathcal{M}''$. Since $[\mathcal{M}'']\subset\overline{[\mathcal{M}]}^{t_s^*}$ and (5.8), there exists a net $\{X_\lambda\}$ in $\mathcal{M}$ such that

$$\lim_\lambda\ \alpha(X_\lambda)\xi_0=0 \ \textit{and}\ \lim_\lambda\ \alpha^{-1}(\alpha(X_\lambda))\xi_0=A\xi_0.$$

Since $X\xi_0\to\alpha^{-1}(X)\xi_0$ is closable, we have $A\xi_0=0$, and hence $A=0$. Hence, η_0 is a separating vector for $\mathcal{M}''$. It follows from $\eta_0\in\mathscr{P}_{\xi_0}^\natural$ that η_0 is a cyclic vector for $\mathcal{M}''$, which implies by (5.8) that α is represented as

$$\alpha(X)=U^{\dagger}XU, \quad X\in\mathscr{M}$$

for some $U\in\mathscr{L}^{\dagger}(\mathscr{D})_u$.

§6. Examples

In this section we investigate the absolute continuity and singularity of positive linear functionals on the O_p^*-algebra generated by the differential operator, the O_p^*-algebra defined by the Schrödinger representation and the maximal O_p^*-algebra $\mathscr{L}^{\dagger}(\mathscr{S}(\mathbf{R}))$ on the Schwartz space $\mathscr{S}(\mathbf{R})$.

Example 6.1. Put

$$\mathscr{D}=\{\xi\in C^{\infty}[0,1];\ \xi^{(n)}(0)=\xi^{(n)}(1),\ n=0,1,2,\dots\},$$

$$X_0=-i\frac{d}{dt}\,|\,\mathscr{D},$$

$$\xi_0(t)=\left[\exp\left\{-\exp\left(-\frac{d^2}{dt^2}\right)\right\}\right]5-4\cos 2\pi t)^{-1}, \quad t\in[0,1].$$

Then the polynomial algebra $\mathscr{P}(X_0)$ generated by X_0 is a self-adjoint O_p^*-algebra on $\mathscr{D}$ and a vector ξ_0 in $\mathscr{D}$ is strongly cyclic for $\mathscr{P}(X_0)$ and separating for $\mathscr{P}(X_0)''$. We consider positive linear functionals on $\mathscr{P}(X_0)$ defined by

$$\phi_a^b(p(X_0))=(p(aX_0+b)\xi_0\,|\,\xi_0), \quad a\neq 0,\ b\in\mathbf{R}.$$

Then the following statements hold.

(1) $\phi_n^{2\pi m}$ $(n\neq 0,\ m\in\mathbf{Z})$ *are strongly* ω_{ξ_0}*-absolutely continuous.*

In fact, by ([33] Example) $\phi_n^{2\pi m}$ is represented as

$$\phi_n^{2\pi m}(p(X_0))=(p(X_0)U\xi_0\,|\,U\xi_0)$$

for some $U\in\mathscr{L}^{\dagger}(\mathscr{D})_i\equiv\{U\in\mathscr{L}^{\dagger}(\mathscr{D});\ U^*\overline{U}=I\}$. We put

$$(\phi_n^{2\pi m})''(A)=(AU\xi_0\,|\,U\xi_0), \quad A\in\mathscr{P}(X_0)''.$$

Since $\mathscr{P}(X_0)''$ is a commutative von Neumann algebra ([15] Theorem 2.1 and [25] Theorem 7.1) and ξ_0 is a cyclic vector for $\mathscr{P}(X_0)''$, it follows that $\mathscr{P}(X_0)''$ is finite, so that by ([19] Corollary 2.3) $(\phi_n^{2\pi m})''$ is strongly ω_{ξ_0}-absolutely continuous. Hence we have

$$(AU\xi_0\,|\,U\xi_0)=(\phi_n^{2\pi m})''(A)=(AH'\xi_0\,|\,H'\xi_0), \quad A\in\mathscr{P}(X_0)''$$

for some positive self-adjoint operator H' in $L^2[0,1]$ affiliated with $\mathscr{P}(X_0)'$, which implies

$$H'\xi_0 \in \mathscr{D} \text{ and } (\phi_n^{2\pi m})(p(X_0)) = (p(X_0)H'\xi_0 | H'\xi_0).$$

Hence, $\phi_n^{2\pi m}$ is strongly ω_{ξ_0}-absolutely continuous.

(2) *For each bounded subset B of $\mathbb{R}$ we define positive linear functionals on $\mathscr{P}(X_0)$ by*

$$(\omega_{\xi_0}\circ\chi_B)(p(X_0)) = (\chi_B(X_0)p(X_0)\xi_0 | \xi_0),$$

$$(\phi_a^b\circ\chi_B)(p(X_0)) = (\chi_B(X_0)p(aX_0+b)\xi_0 | \xi_0).$$

Then $\phi_a^b\circ\chi_B$ $(a\in\mathbb{Z}$ or $b\notin 2\pi\mathbb{Z})$ are $(\omega_{\xi_0}\circ\chi_B)$-singular.

In fact, for each polynomial p and $n\in\mathbb{N}$ we define a polynomial p_n by

$$p_n(t) = \sum_{k=1}^{2n+1} \alpha_k\{(t+2n\pi)(t+2(n-1)\pi)\dots(t+2\pi)t(t-2\pi)\dots(t-2n\pi)\}^k,$$

where $\{\alpha_1, \alpha_2, \dots, \alpha_{2n+1}\}$ is a unique solution of the equation:

$$p_n(2m\pi a+b) = p(2m\pi a+b),\ m=-n, \dots, -1, 0, 1, \dots, n$$

(the existence of the unique solution dues to $a\neq 0$). Since B is a bounded subset of $\mathbb{R}$, it follows that

$$(\omega_{\xi_0}\circ\chi_B)(p_n(X_0)^\dagger p_n(X_0)) = 0,$$
$$(\phi_a^b\circ\chi_B)((p_n(X_0)-p(X_0))^\dagger(p_n(X_0)-p(X_0)) = 0$$

for sufficient large all $n\in\mathbb{N}$. Hence, $\phi_a^b\circ\chi_B$ is $\omega_{\xi_0}\circ\chi_B$-singular.

Let $\mathscr{S}=\mathscr{S}(\mathbb{R})$ be the Schwartz space of infinitely differentiable rapidly decreasing functions and $\{f_n\}_{n=0,1,2,\dots}$ be an orthonormal basis in the Hilbert space $L^2=L^2(\mathbb{R})$ contained in $\mathscr{S}$ consisting of the normalized Hermite functions. We denote by $L^2\otimes\overline{L^2}$ the Hilbert space with inner product $\langle\ |\ \rangle$ of Hilbert-Schmidt operators on L^2, by $\mathscr{S}\otimes\overline{L^2}$ the subspace $\{T\in L^2\otimes\overline{L^2};\ TL^2\subset\mathscr{S}\}$ of $L^2\otimes\overline{L^2}$ and by $(\mathscr{S}\otimes L_2)_+$ the set of all positive operators of $\mathscr{S}\otimes\overline{L^2}$. Let K be a densely defined closed operator in L^2. We define densely defined closed operators $\pi''(K)$ and $\pi'(K)$ as follows:

$$\begin{cases}\mathscr{D}(\pi''(K)) = \{T\in L^2\otimes\overline{L^2};\ KT\in L^2\otimes\overline{L^2}\},\\ \pi''(K)T = KT,\quad T\in\mathscr{D}(\pi''(K));\end{cases}$$

$$\begin{cases}\mathscr{D}(\pi'(K)) = \{T\in L^2\otimes\overline{L^2};\ \overline{TK}\in L^2\otimes\overline{L^2}\},\\ \pi'(K) = \overline{TK},\quad T\in\mathscr{D}(\pi'(K)).\end{cases}$$

Then $\pi''(K)$ (resp. $\pi'(K)$) is a densely defined closed operator in $L^2\otimes\overline{L^2}$ affiliated with the von Neumann algebra $\pi''(\mathscr{B}(L^2))$ (resp. $\pi''(\mathscr{B}(L^2))'=\pi'(\mathscr{B}(L^2))$). In particular, if K is a positive self-adjoint operator in L^2, then $\pi''(K)$ and $\pi'(K)$ are positive self-adjoint operators in $L^2\otimes\overline{L^2}$ ([14] Lemma 5.1).

As stated in Section 3, a self-adjoint representation π of $\mathscr{L}^\dagger(\mathscr{S})$ in $L^2\otimes\overline{L^2}$ is defined by

$$\pi(X)T=XT, \quad X\in\mathscr{L}^\dagger(\mathscr{S}), \quad T\in\mathscr{S}\otimes\overline{L^2},$$

which satisfies

$$\pi(\mathscr{L}^\dagger(\mathscr{S}))'=\pi'(\mathscr{B}(L^2)) \text{ and } \pi(\mathscr{L}^\dagger(\mathscr{S}))''=\pi''(\mathscr{B}(L^2)).$$

We put

$$\boldsymbol{s}_+=\{\{\alpha_n\}\,;\ \alpha_n>0 \text{ for } n=0,1,2,\dots \text{ and } \sup_n n^k\alpha_n<\infty \text{ for each } k\in\mathbf{N}\},$$

$$\Omega_{\{\alpha_n\}}=\sum_{n=0}^{\infty}\alpha_n f_n\otimes\overline{f_n}, \quad \{\alpha_n\}\in\boldsymbol{s}_+.$$

Then, for each $\{\alpha_n\}\in\boldsymbol{s}_+$ $(\pi(\mathscr{L}^\dagger(\mathscr{S})), \mathscr{S}\otimes\overline{L^2}, \Omega_{\{\alpha_n\}})$ is a full standard system such that $J_{\Omega_{\{\alpha_n\}}}T=T^*$ for $T\in L^2\otimes\overline{L^2}$, $\Delta_{\Omega_{\{\alpha_n\}}}=\pi'(\Omega_{\{\alpha_n\}}^{-2})\pi''(\Omega_{\{\alpha_n\}}^{2})$ and $\{\sigma_t^{\Omega_{\{\alpha_n\}}}(\,\cdot\,)\equiv\Omega_{\{\alpha_n\}}^{2it}\cdot\Omega_{\{\alpha_n\}}^{-2it}\}_{t\in\mathbf{R}}$ is a one-parameter group of $*$-automorphisms of $\mathscr{L}^\dagger(\mathscr{S})$ satisfying $\Delta_{\Omega_{\{\alpha_n\}}}^{it}\pi(X)\Delta_{\Omega_{\{\alpha_n\}}}^{-it}=\pi(\sigma_t^{\Omega_{\{\alpha_n\}}}(X))$ for each $X\in\mathscr{L}^\dagger(\mathscr{S})$ and $t\in\mathbf{R}$ ([14] Theorem 5.4, Corollary 5.5). We define strongly positive linear functionals ϕ_ρ $(\rho\in\mathscr{S}\otimes\overline{L^2})$ on $\mathscr{L}^\dagger(\mathscr{L})$ by

$$\phi_\rho(X)=\operatorname{Tr}\rho\rho^*X=\langle\pi(X)\rho\,|\,\rho\rangle, \quad X\in\mathscr{L}^\dagger(\mathscr{S})$$

and in particular, when $\rho=\Omega_{\{\alpha_n\}}(\{\alpha_n\}\in\boldsymbol{s}_+)$ we simply write $\phi_{\Omega_{\{\alpha_n\}}}$ by $\phi_{\{\alpha_n\}}$.

Let π_1 be a self-adjoint representation of the canonical algebra $\mathscr{A}$ for one degree of freedom defined by

$$\pi_1(x)=\pi(\pi_0(x)), \quad x\in\mathscr{A},$$

where π_0 denotes the Schrödinger representation of $\mathscr{A}$. Then $\Omega_{\{e^{-n\beta}\}}$ is a standard vector for $(\pi_1(\mathscr{A}), \mathscr{S}\otimes\overline{L^2})$ with $(\mathscr{S}\otimes\overline{L^2})_{\Omega_{\{e^{-n\beta}\}}}=\mathscr{S}\otimes\overline{L^2}$ and there exists a one-parameter group $\{\Delta_{\{e^{-n\beta}\}}^{it}\}_{t\in\mathbf{R}}$ of $*$-automorphisms of $\mathscr{A}$ such that

$$\pi_1(\Delta_{\{e^{-n\beta}\}}^{it}x)=\Delta_{\Omega_{\{e^{-n\beta}\}}}^{it}\pi_1(x)\Delta_{\Omega_{\{e^{-n\beta}\}}}^{-it}$$

for each $x\in\mathscr{A}$ and $t\in\mathbb{R}$ ([10] Theorem 20 and [14] Corollary 5.6). For each $\rho\in(\mathscr{S}\otimes\overline{L^2})_+$ we simply denote by ϕ_ρ a positive linear functional $\phi_\rho\circ\pi_0$ on $\mathscr{A}$ and in particular, denote by $\phi_{\{\alpha_n\}}$ a positive linear functional $\phi_{\{\alpha_n\}}\circ\pi_0$ on $\mathscr{A}$.

In next Example 6.2 we consider the strongly $\phi_{\{e^{-n\beta}\}}$-absolute continuity, $\phi_{\{e^{-n\beta}\}}$-singularity and $\{\Delta^{it}_{\{e^{-n\beta}\}}\}_{t\in\mathbb{R}}$-invariance of positive linear functionals on $\mathscr{A}$, and in Example 6.3 we give concrete examples of $\phi_{\{e^{-n\beta}\}}$-singular positive linear functionals on $\mathscr{L}^\dagger(\mathscr{S})$ and strongly $\phi_{\{e^{-n\beta}\}}$-absolutely continuous positive linear functionals on $\mathscr{L}^\dagger(\mathscr{S})$, and characterize $\{\Delta^{it}_{\{e^{-n\beta}\}}\}_{t\in\mathbb{R}}$-invariant positive linear functionals on $\mathscr{L}^\dagger(\mathscr{S})$.

Example 6.2. Let ϕ be a positive linear functional on $\mathscr{A}$. It is well-known that $\phi\circ\pi_0^{-1}$ is strongly positive if and only if $\phi=\phi_\rho$ for some $\rho\in(\mathscr{S}\otimes\overline{L^2})_+$ [29]. Consider positive linear functionals ϕ_ρ.

(1) *Suppose* $\Omega^{-1}_{\{e^{-n\beta}\}}\rho$ *is densely defined. Then* ϕ_ρ *is strongly* $\phi_{\{e^{-n\beta}\}}$-*absolutely continuous.*

In fact, ϕ_ρ is represented as

$$\phi_\rho(x)=\langle\pi_1(x)\,|\pi'(\Omega^{-1}_{\{e^{-n\beta}\}}\rho)\,|\Omega_{\{e^{-n\beta}\}}\,|\,|\pi'(\Omega^{-1}_{\{e^{-n\beta}\}}\rho)\,|\Omega_{\{e^{-n\beta}\}}\rangle,\qquad x\in\mathscr{A}$$

for a positive self-adjoint operator $|\pi'(\Omega^{-1}_{\{e^{-n\beta}\}}\rho)|$ *affiliated* with $\pi'(\mathscr{B}(L^2))$ such that $|\pi'(\Omega^{-1}_{\{e^{-n\beta}\}}\rho)\,|\Omega_{\{e^{-n\beta}\}}\in\mathscr{S}\otimes\overline{L^2}$. Hence, ϕ_ρ is strongly $\phi_{\{e^{-n\beta}\}}$-absolutely continuous.

We next consider when ϕ_ρ is $\{\Delta^{it}_{\{e^{-n\beta}\}}\}_{t\in\mathbb{R}}$-invariant. It is clear that $\phi_{\{\alpha_n\}}(\{\alpha_n\}\in s_+)$ are $\{\Delta^{it}_{\{e^{-n\beta}\}}\}$-invariant. Hence, the following question arises: *If* ϕ_ρ *is* $\{\Delta^{it}_{\{e^{-n\beta}\}}\}$-*invariant, then* $\phi_\rho=\phi_{\{\alpha_n\}}$ *for* $\{\alpha_n\}\in s_+$? For this problem the following fact holds.

(2) *If* ϕ_ρ *is* $\phi_{\{e^{-n\beta}\}}$-*dominated, then* $\phi_\rho=\phi_{\{\alpha_n\}}$ *for some* $\{\alpha_n\}\in s_+$. *In more general, if* $\Omega^{-1}_{\{e^{-n\beta}\}}\rho$ *is densely defined and* $\overline{\rho^2\Omega^{-1}_{\{e^{-n\beta}\}}}\in\mathscr{S}\otimes\overline{L^2}$, *then* $\phi_\rho=\phi_{\{\alpha_n\}}$ *for some* $\{\alpha_n\}\in s_+$.

In fact, we now suppose $\Omega^{-1}_{\{e^{-n\beta}\}}\rho$ is densely defined and $\overline{\rho^2\Omega^{-1}_{\{e^{-n\beta}\}}}\in\mathscr{S}\otimes\overline{L^2}$ and put

$$H_0=(\Omega^{-1}_{\{e^{-n\beta}\}}\rho)\,(\Omega^{-1}_{\{e^{-n\beta}\}}\rho)^*.$$

Then $\pi'(H_0)$ is a positive self-adjoint operator in L^2 affiliated with $\pi'(\mathscr{B}(L^2))$. Since $\overline{\rho^2\Omega^{-1}_{\{e^{-n\beta}\}}}\in\mathscr{S}\otimes\overline{L^2}$, it follows that

$$\Omega_{\{e^{-n\beta}\}} \in \mathscr{D}(\pi'(H_0)) \text{ and}$$
$$\pi'(H_0)\Omega_{\{e^{-n\beta}\}} = \overline{\rho^2 \Omega_{\{e^{-n\beta}\}}} \in \mathscr{S} \otimes \overline{L^2},$$

and hence

$$\pi_1(\mathscr{A})\Omega_{\{e^{-n\beta}\}} \subset \mathscr{D}(\pi'(H_0)),$$
$$\pi'(H_0)\pi_1(x)\Omega_{\{e^{-n\beta}\}} = \pi_1(x)\pi'(H_0)\Omega_{\{e^{-n\beta}\}}$$

and

$$\phi(x) = \langle \pi_1(x)\pi'(H_0)\Omega_{\{e^{-n\beta}\}} | \Omega_{\{e^{-n\beta}\}} \rangle$$

for all $x \in \mathscr{A}$. Since ϕ is $\{\Delta^{it}_{\{e^{-n\beta}\}}\}$-invariant, it follows that

$$\begin{aligned}
\phi(y^* \Delta^{it}_{\{e^{-n\beta}\}} x) &= \langle \pi_1(y^* \Delta^{it}_{\{e^{-n\beta}\}} x)\pi'(H_0)\Omega_{\{e^{-n\beta}\}} | \Omega_{\{e^{-n\beta}\}} \rangle \\
&= \langle \pi'(H_0)\pi_1(\Delta^{it}_{\{e^{-n\beta}\}} x)\Omega_{\{e^{-n\beta}\}} | \pi_1(y)\Omega_{\{e^{-n\beta}\}} \rangle \\
&= \langle \pi'(H_0)\Delta^{it}_{\Omega_{\{e^{-n\beta}\}}}\pi_1(x)\Omega_{\{e^{-n\beta}\}} | \pi_1(y)\Omega_{\{e^{-n\beta}\}} \rangle, \\
\phi(y^* \Delta^{it}_{\{e^{-n\beta}\}} x) &= \phi(\Delta^{-it}_{\{e^{-n\beta}\}}(y^* \Delta^{it}_{\{e^{-n\beta}\}} x)) \\
&= \phi((\Delta^{-it}_{\{e^{-n\beta}\}} y)^* x) \\
&= \langle \Delta^{it}_{\Omega_{\{e^{-n\beta}\}}}\pi_1(x)\pi'(H_0)\Omega_{\{e^{-n\beta}\}} | \pi_1(y)\Omega_{\{e^{-n\beta}\}} \rangle \\
&= \langle \Delta^{it}_{\Omega_{\{e^{-n\beta}\}}}\pi'(H_0)\pi_1(x)\Omega_{\{e^{-n\beta}\}} | \pi_1(y)\Omega_{\{e^{-n\beta}\}} \rangle
\end{aligned}$$

for all $x, y \in \mathscr{A}$, which implies since $\pi''(\mathscr{B}(L^2))\Omega_{\{e^{-n\beta}\}} \subset \mathscr{D}(\pi'(H_0))$ that

$$\begin{aligned}
&\langle \pi_1(x)\Omega_{\{e^{-n\beta}\}} | \Delta^{it}_{\Omega_{\{e^{-n\beta}\}}}\pi'(H_0)\pi''(A)\Omega_{\{e^{-n\beta}\}} \rangle \\
&= \langle \pi'(H_0)\Delta^{-it}_{\Omega_{\{e^{-n\beta}\}}}\pi_1(x)\Omega_{\{e^{-n\beta}\}} | \pi''(A)\Omega_{\{e^{-n\beta}\}} \rangle \\
&= \langle \Delta^{-it}_{\Omega_{\{e^{-n\beta}\}}}\pi'(H_0)\pi_1(x)\Omega_{\{e^{-n\beta}\}} | \pi''(A)\Omega_{\{e^{-n\beta}\}} \rangle \\
&= \langle \pi_1(x)\Omega_{\{e^{-n\beta}\}} | \pi'(H_0)\Delta^{it}_{\Omega_{\{e^{-n\beta}\}}}\pi''(A)\Omega_{\{e^{-n\beta}\}} \rangle
\end{aligned}$$

for all $A \in \mathscr{B}(L^2)$, $x \in \mathscr{A}$ and $t \in \mathbb{R}$. Hence we have

$$\begin{aligned}
&\pi''(\Omega^{2it}_{\{e^{-n\beta}\}})\pi'(\Omega^{-2it}_{\{e^{-n\beta}\}})\pi'(H_0)\pi''(A)\Omega_{\{e^{-n\beta}\}} \\
&= \pi'(H_0)\pi''(\Omega^{2it}_{\{e^{-n\beta}\}})\pi'(\Omega^{-2it}_{\{e^{-n\beta}\}})\pi''(A)\Omega_{\{e^{-n\beta}\}}
\end{aligned}$$

for all $A \in \mathscr{B}(L^2)$ and $t \in \mathbb{R}$. Since $f_k \in \mathscr{D}(H_0)$ for $k \in \mathbb{N} \cup \{0\}$, it follows that

$$\begin{aligned}
e^{-2k\beta it}(H_0 f_k | f_n) f_n &= (f_n \otimes \overline{f_n}) H_0 \Omega^{-2it}_{\{e^{-n\beta}\}} f_k \\
&= (f_n \otimes \overline{f_n}) \Omega^{-2it}_{\{e^{-n\beta}\}} H_0 f_k \\
&= e^{-2n\beta it}(H_0 f_k | f_n) f_n,
\end{aligned}$$

which implies that

$$H_0 f_n = (H_0 f_n | f_n) f_n, \quad n = 0, 1, 2, \ldots .$$

Hence we have

$$\{\alpha_n\}\equiv\{e^{-n\beta}(H_0 f_n\,|\,f_n)^{1/2}\}\in \mathfrak{s}_+ \text{ and } \phi_\rho=\phi_{\{\alpha_n\}}.$$

Suppose ϕ_ρ is $\phi_{\{e^{-n\beta}\}}$-dominated. Then ϕ is represented as

$$\phi_\rho(x)=\langle \pi_1(x)\pi'(H_0)\Omega_{\{e^{-n\beta}\}}\,|\,\pi'(H_0)\Omega_{\{e^{-n\beta}\}}\rangle,\qquad x\in\mathscr{A}$$

for some positive operator H_0 in $\mathscr{B}(L^2)$, and hence we can take $\pi'(H_0)\Omega_{\{e^{-n\beta}\}}\in\mathscr{S}\otimes\overline{L^2}$ as ρ. Since

$$\Omega^{-1}_{\{e^{-n\beta}\}}\pi'(H_0)\Omega_{\{e^{-n\beta}\}}=H_0\in\mathscr{B}(L^2),$$
$$\overline{(\pi'(H_0)\Omega_{\{e^{-n\beta}\}})^2\Omega^{-1}_{\{e^{-n\beta}\}}}=(\pi'(H_0)\Omega_{\{e^{-n\beta}\}})H_0\in\mathscr{S}\otimes\overline{L^2},$$

it follows from the above fact that $\phi_\rho=\phi_{\{\alpha_n\}}$ for some $\{\alpha_n\}\in\mathfrak{s}_+$.

(3) *A positive linear functional ϕ on $\mathscr{A}$ which satisfies the KMS-condition with respect to $\{\Delta^{it}_{\{e^{-n\beta}\}}\}_{t\in\mathbf{R}}$ is represented as*

$$\phi=\gamma\phi_{\{e^{-n\beta}\}}$$

for some constant $\gamma>0$ ([10] *Theorem* 30).

Example 6.3. We consider $\phi_{\{e^{-n\beta}\}}$-absolute continuity, $\phi_{\{e^{-n\beta}\}}$-singularity and $\{\sigma_t^{\Omega_{\{e^{-n\beta}\}}}\}_{t\in\mathbf{R}}$-invariance of positive linear functionals on $\mathscr{L}^\dagger(\mathscr{S})$. The following examples (1)$\sim$(4) are modifications of examples constructed by Kosaki in [19].

(1) $\phi_{f_\infty\otimes\overline{f_\infty}}$ *is* $\phi_{\{e^{-n\beta}\}}$*-singular, where* $f_\infty=\sum_{n=0}^{\infty}e^{-n\beta}f_n\in\mathscr{S}$.

In fact, for each $X\in\mathscr{L}^\dagger(\mathscr{S})$ we put

$$X_m=\frac{1}{\log m}\sum_{k=1}^{m}\frac{1}{k}e^{\beta k}(Xf_\infty\otimes\overline{f_k}),\qquad m=2,3,\dots.$$

Then we have

$$X_m\in\mathscr{S}\otimes\overline{L^2},$$
$$\pi(X_m)\Omega_{\{e^{-n\beta}\}}=\frac{1}{\log m}\sum_{k=1}^{m}\frac{1}{k}\,(Xf_\infty\,|\,\overline{f_k})$$

and

$$\pi(X_m)(f_\infty\otimes\overline{f_\infty})=\Big(\frac{1}{\log m}\sum_{k=1}^{m}\frac{1}{k}\Big)\pi(X)(f_\infty\otimes\overline{f_\infty})$$

for $m=2,3,\dots$. It hence follows that

$$\lim_{m\to\infty}\pi(X_m)\Omega_{\{e^{-n\beta}\}}=0 \text{ and } \lim_{m\to\infty}\pi(X_m)(f_\infty\otimes\overline{f_\infty})=\pi(X)(f_\infty\otimes\overline{f_\infty})$$

for each $X \in \mathscr{L}^{\dagger}(\mathscr{S})$, which means that $\phi_{f_\infty \otimes \overline{f_\infty}}$ is $\phi_{\{e^{-n\beta}\}}$-singular.

(2) *$\phi_{f'_\infty \otimes \overline{f'_\infty}}$ is $\phi_{\{e^{-n\beta}\}}$-singular and $\phi_{f_\infty \otimes \overline{f_\infty}} + \phi_{f'_\infty \otimes \overline{f'_\infty}}$ is not $\phi_{\{e^{-n\beta}\}}$-singular, where $f'_\infty = 2f_0 - f_\infty$.*

In fact, it is shown in similar to (1) that $\phi_{f'_\infty \otimes \overline{f'_\infty}}$ is $\phi_{\{e^{-n\beta}\}}$-singular. We show that $\phi_{f_\infty \otimes \overline{f_\infty}} + \phi_{f'_\infty \otimes \overline{f'_\infty}}$ is not $\phi_{\{e^{-n\beta}\}}$-singular. Since

$$(f_\infty \otimes \overline{f_\infty})^2 + (f'_\infty \otimes \overline{f'_\infty})^2 = \frac{e^{2\beta}}{e^{2\beta}-1}(f_\infty \otimes \overline{f_\infty} + f'_\infty \otimes \overline{f'_\infty}),$$
$$((f_\infty \otimes \overline{f_\infty})^2 + (f'_\infty \otimes \overline{f'_\infty})^2)(f_\infty + f'_\infty) = \frac{2e^{2\beta}}{e^{2\beta}-1}(f_\infty + f'_\infty),$$
$$((f_\infty \otimes \overline{f_\infty})^2 + (f'_\infty \otimes \overline{f'_\infty})^2)(f_\infty - f'_\infty) = \frac{2e^{2\beta}}{(e^{2\beta}-1)^2}(f_\infty - f'_\infty),$$

it follows that $f_\infty + f'_\infty = 2f_0$ and $f_\infty - f'_\infty$ are eigenvectors for $((f_\infty \otimes \overline{f_\infty})^2 + (f'_\infty \otimes \overline{f'_\infty})^2)$ with eigenvalues $\frac{2e^{2\beta}}{e^{2\beta}-1}$ and $\frac{2e^{2\beta}}{(e^{2\beta}-1)^2}$, respectively, which implies

$$(f_\infty \otimes \overline{f_\infty})^2 + (f'_\infty \otimes \overline{f'_\infty})^2 \geqq \frac{2e^{2\beta}}{e^{2\beta}-1}(f_0 \otimes \overline{f_0}).$$

Hence we have

$$\begin{aligned}(\phi_{f_\infty \otimes \overline{f_\infty}} + \phi_{f'_\infty \otimes \overline{f'_\infty}})(X^{\dagger}X) &= \mathrm{Tr}((f_\infty \otimes \overline{f_\infty})^2 + (f'_\infty \otimes \overline{f'_\infty})^2)X^{\dagger}X \\ &\geqq \frac{2e^{2\beta}}{e^{2\beta}-1}\mathrm{Tr}(f_0 \otimes \overline{f_0})X^{\dagger}X \\ &= \frac{2e^{2\beta}}{e^{2\beta}-1}\phi_{f_0 \otimes \overline{f_0}}(X^{\dagger}X)\end{aligned}$$

for all $X \in \mathscr{L}^{\dagger}(\mathscr{S})$, and hence

$$\begin{aligned}&\phi_{f_0 \otimes \overline{f_0}} \neq 0 \\ &\in P\left(\mathscr{L}^{\dagger}(\mathscr{S}),\ \frac{2e^{2\beta}}{e^{2\beta}-1}(\phi_{f_\infty \otimes \overline{f_\infty}} + \phi_{f'_\infty \otimes \overline{f'_\infty}})\right) \cap P(\mathscr{L}^{\dagger}(\mathscr{S}),\ \phi_{\{e^{-n\beta}\}}).\end{aligned}$$

It hence follows from Theorem 5. 2, (2) that $(\phi_{f_\infty \otimes \overline{f_\infty}} + \phi_{f'_\infty \otimes \overline{f'_\infty}})$ is not $\phi_{\{e^{-n\beta}\}}$-singular.

(3) *The strongly $\phi_{\{e^{-n\beta}\}}$-absolutely continuous positive linear functional $\phi_{\{e^{-n/2\beta}\}}$ on $\mathscr{L}^{\dagger}(\mathscr{S})$ dominates a positive linear functional ϕ on $\mathscr{L}^{\dagger}(\mathscr{S})$ which is not strongly $\phi_{\{e^{-n\beta}\}}$-absolutely continuous.*

Let $\mathfrak{H}_1$ be the closed subspace of L^2 generated by $\{f_1, f_3, \ldots, f_{2n+1}, \ldots\}$ and P be the projection of L^2 onto $\mathfrak{H}_1$. Since $\Omega_{\{e^{-n/2\beta}\}}P = P\Omega_{\{e^{-n/2\beta}\}}$ is a non-singular compact operator on $\mathfrak{H}_1$, it follows from

([19] Lemma 8. 8) that there exists a unitary operator $\tilde{U}$ on $\mathfrak{H}_1$ such that

$$\mathscr{R}(\Omega_{\{e^{-n/2\beta}\}}P)\cap\tilde{U}\mathscr{R}(\Omega_{\{e^{-n/2\beta}\}}P)=\{0\}.$$

We here put

$$\rho=\Omega_{\{e^{-n/2\beta}\}}U\Omega_{\{e^{-n/2\beta}\}},\text{ where } U=\tilde{U}P+(1-P),$$
$$\psi(X)=\mathrm{Tr}\rho\rho^*X,\qquad X\in\mathscr{L}^\dagger(\mathscr{S}).$$

Since

$$\begin{aligned}\psi(X^\dagger X)&=\|\pi(X)\Omega_{\{e^{-n/2\beta}\}}U\Omega_{\{e^{-n/2\beta}\}}\|_2^2\\&=\|\pi'(U\Omega_{\{e^{-n/2\beta}\}})\pi(X)\Omega_{\{e^{-n/2\beta}\}}\|_2^2\\&\leqq\|\pi'(U\Omega_{\{e^{-n/2\beta}\}})\|^2\|\pi(X)\Omega_{\{e^{-n/2\beta}\}}\|_2^2\\&\leqq\phi_{\{e^{-n/2\beta}\}}(X^\dagger X)\end{aligned}$$

for all $X\in\mathscr{L}^\dagger(\mathscr{S})$, it follows that ψ is $\phi_{\{e^{-n/2\beta}\}}$-dominated. Suppose ψ is strongly $\phi_{\{e^{-n\beta}\}}$-absolutely continuous. By Theorem 5. 2, ψ is represented as

$$\psi(X)=\langle\pi(X)H'_\psi\Omega_{\{e^{-n\beta}\}}|H'_\psi\Omega_{\{e^{-n\beta}\}}\rangle,\qquad X\in\mathscr{L}^\dagger(\mathscr{S}).$$

Hence, a positive linear functional ψ'' on $\mathscr{B}(L^2)$ defined by

$$\psi''(A)=\langle\pi''(A)H'_\psi\Omega_{\{e^{-n\beta}\}}|H'_\psi\Omega_{\{e^{-n\beta}\}}\rangle$$

is faithful and strongly $\phi''_{\{e^{-n\beta}\}}$-absolutely continuous, and so by ([19] Corollary 7. 3) $\pi'(\mathscr{B}(L^2))\Omega_{\{e^{-n\beta}\}}\cap\pi'(\mathscr{B}(L^2))\rho$ is dense in $L^2\otimes\overline{L^2}$. Take an arbitrary $H\in\pi'(\mathscr{B}(L^2))\Omega_{\{e^{-n\beta}\}}\cap\pi'(\mathscr{B}(L^2))\rho$. Then, since

$$H=\pi'(A)\rho=\pi'(B)\Omega_{\{e^{-n\beta}\}},\qquad A,\ B\in\mathscr{B}(L^2),$$

we have

$$U\Omega_{\{e^{-n/2\beta}\}}A\xi=\Omega_{\{e^{-n/2\beta}\}}B\xi$$

for each $\xi\in L^2$, which implies

$$P\Omega_{\{e^{-n/2\beta}\}}B\xi=\tilde{U}P\Omega_{\{e^{-n/2\beta}\}}A\xi\in\mathscr{R}(P\Omega_{\{e^{-n/2\beta}\}})\cap\tilde{U}\mathscr{R}(P\Omega_{\{e^{-n/2\beta}\}})=\{0\}.$$

Hence, we have

$$PH\xi=P\Omega_{\{e^{-n\beta}\}}B\xi=\Omega_{\{e^{-n/2\beta}\}}P\Omega_{\{e^{-n/2\beta}\}}B\xi=0$$

for each $\xi\in L^2$, and so $\mathscr{R}(H)\subset(1-P)\mathfrak{H}$, which contradicts $\pi'(\mathscr{B}(L^2))\ \Omega_{\{e^{-n\beta}\}}\cap\pi'(\mathscr{B}(L^2))\rho$ is dense in $L^2\otimes\overline{L^2}$. Hence, ψ is not strongly $\phi_{\{e^{-n\beta}\}}$-absolutely continuous.

(4) *The Lebesgue decomposition of* $\phi_{\{e^{-n/2\beta}\}}$ *is not unique.*

In fact, the strongly $\phi_{\{e^{-n\beta}\}}$-absolutely continuous positive linear functional $\phi_{\{e^{-n/2\beta}\}}$ on $\mathscr{L}^{\dagger}(\mathscr{S})$ is decomposed into

$$\begin{aligned}\phi_{\{e^{-n/2\beta}\}} &= \phi_{\{e^{-n/2\beta}\}} + 0 \\ &= \{(\phi_{\{e^{-n/2\beta}\}} - \psi) + \psi_c\} + \psi_s,\end{aligned}$$

where ψ is in (3). Since $\phi_{\{e^{-n/2\beta}\}} - \psi \leqq \phi_{\{e^{-n\beta}\}}$ and $\psi_s \neq 0$, it follows that $((\phi_{\{e^{-n/2\beta}\}} - \psi) + \psi_c)$ is $\phi_{\{e^{-n/2\beta}\}}$-dominated and strongly $\phi_{\{e^{-n\beta}\}}$-absolutely continuous and $\psi_s \neq 0$ is $\phi_{\{e^{-n\beta}\}}$-singular, which shows that the Lebesgue decomposition of $\phi_{\{e^{-n/2\beta}\}}$ is not unique.

(5) *Every strongly* $\phi_{\{e^{-n\beta}\}}$*-absolutely continuous and* $\{\sigma_t^{\Omega_{\{e^{-n\beta}\}}}\}_{t\in\mathbf{R}}$*-invariant, strongly positive linear functional* ϕ *on* $\mathscr{L}^{\dagger}(\mathscr{S})$ *is represented as*

$$\phi = \phi_{\{\alpha_n\}}$$

for some $\{\alpha_n\} \in \boldsymbol{s}_+$.

In fact, by Theorem 5.6 ϕ is represented as

$$\phi(X) = \langle \pi(X) H\Omega_{\{e^{-n\beta}\}} | H\Omega_{\{e^{-n\beta}\}} \rangle, \quad X \in \mathscr{L}^{\dagger}(\mathscr{S})$$

for some positive self-adjoint operator H in $L^2 \otimes \overline{L^2}$ affiliated with $\pi''(\mathscr{B}(L^2))^{\sigma^{\Omega_{\{e^{-n\beta}\}}}}$ such that $H\Omega_{\{e^{-n\beta}\}} \in \mathscr{S} \otimes \overline{L^2}$. It is easily shown that

$$\pi''(\mathscr{B}(L^2))^{\sigma^{\Omega_{\{e^{-n\beta}\}}}} = \{\pi''(A)\,;\ A = \sum_{n=0}^{\infty} \alpha_n f_n \otimes \overline{f_n} \in \mathscr{B}(L^2)\}.$$

Hence, we have

$$H_n = \sum_{k=0}^{\infty} \beta_k^{(n)} f_k \otimes \overline{f_k} \in \mathscr{B}(L^2), \quad n \in \mathbf{N}$$

and

$$\lim_{n\to\infty} \pi''(H_n)\Omega_{\{e^{-n\beta}\}} = H\Omega_{\{e^{-n\beta}\}},$$

which implies

$$\lim_{n\to\infty} \beta_k^{(n)} e^{-k\beta} = \alpha_k, \quad k = 0, 1, 2, \ldots$$

and

$$H\Omega_{\{e^{-n\beta}\}} = \sum_{k=0}^{\infty} \alpha_k f_k \otimes \overline{f_k} \in \mathscr{S} \otimes \overline{L^2},$$

and hence $\{\alpha_k\} \in \boldsymbol{s}_+$ and $\phi = \phi_{\{\alpha_n\}}$.

(6) *Every strongly positive linear functional* ϕ *on* $\mathscr{L}^{\dagger}(\mathscr{S})$ *which*

satisfies the KMS-condition with respect to $\{\sigma_t^{\Omega_{\{e^{-n\beta}\}}}\}_{t\in\mathbf{R}}$ *is represented as*

$$\phi=\gamma\phi_{\{e^{-n\beta}\}}$$

for some constant $\gamma>0$.

In fact, by Theorem 5.5 ϕ is represented as

$$\phi(X)=\langle\pi(X)H\Omega_{\{e^{-n\beta}\}}|H\Omega_{\{e^{-n\beta}\}}\rangle,\qquad X\in\mathscr{L}^{\dagger}(\mathscr{S})$$

for some positive self-adjoint operator H affiliated with $\pi''(\mathscr{B}(L^2))\cap\pi'(\mathscr{B}(L^2))$ such that $H\Omega_{\{e^{-n\beta}\}}\in\mathscr{S}\otimes\overline{L^2}$. It is easily shown that $\pi''(\mathscr{B}(L^2))\cap\pi'(\mathscr{B}(L^2))=\mathbf{C}I$, which implies $\phi=\gamma\phi_{\{e^{-n\beta}\}}$ for some constant $\gamma>0$.

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