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The Stable Cohomotopy Ring of G_2

Dedicated to Professor Hirosi Toda on his 60th birthday

By

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§ 1. Introduction

The fact that a Lie group (generally a finite H-space) has a stably trivial attaching map of its top cell makes a little bit easier to determine the cohomotopy groups, especially when the space has a few cells. Actually, for $Sp(2)$ and $SU(3)$, it is easy to obtain 0-th cohomotopy groups, and moreover ring structure can also be calculated. These are carried out by G. Walker in [9]. But the more cells the space has, the more difficult the determination becomes.

In this paper we shall give the 0-th stable cohomotopy group of G_2, the exceptional Lie group, by means of G. Walker's method in the above mentioned paper and S. Oka's accurate study of the stable homotopy type of G_2 in [6]. We shall also determine the ring structure by the results of P. Eccles and G. Walker [3]. Then we shall be able to recover that $[G_2, L]=\kappa$ (see §4).

We denote the q-th reduced stable cohomotopy of X by $\pi^q(X)$ ($=\varinjlim[S^m X, S^{m+q}]$). We state our main results.

Theorem 1.1. $\pi^0(G_2)=Z_2\oplus Z_2\oplus Z_2\oplus Z_4\oplus Z_8\oplus Z_8\oplus Z_{27}\oplus Z_7$. *Generators are* $q^*\sigma^2$, $q^*\kappa$, ν^2pj', $\widetilde{\mathrm{Ext}\,\varepsilon}-\sigma\,\widetilde{\mathrm{Ext}\,\eta}$, $\widetilde{\mathrm{Ext}\,\varepsilon}$, $\tilde{\nu}$, $\tilde{\alpha}_1$, $\tilde{\alpha}_{1,7}$, *respectively* (see §3).

Theorem 1.2. 1). $\tilde{\nu}^2\equiv\nu^2pj'+\tilde{\tilde{\nu}}$ mod $\widetilde{4\mathrm{Ext}\,\varepsilon}(=4\sigma\,\widetilde{\mathrm{Ext}\,\eta})$. 2). $(\nu^2pj')\tilde{\nu}=2\tilde{\tilde{\nu}}$. 3). $\tilde{\nu}^3=2\tilde{\tilde{\nu}}+q^*\kappa$, *where* $\tilde{\tilde{\nu}}=\sigma\widetilde{\mathrm{Ext}\,\eta}+\widetilde{\mathrm{Ext}\,\varepsilon}$. *Other products are trivial.*

This paper is organized as follows. In Section 2, we recall the result of [6]. In Section 3, we shall prove our main Theorem 1.1.

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In Section 4, we shall give above results on the ring structure and prove our application to $[G_2, L]$.

§ 2. $\pi^0(X^3)$ and $\pi^0(Y^{11})$

First we recall that G_2 is stably equivalent to the space $Q\vee S^{14}$. For the space Q, there exists a cofibration $X^3\to Q\to Y^{11}$, where X^3 and Y^{11} are following cofibers. ([6]).

(2. 1) $$M^4\xrightarrow{\bar\eta}S^3\xrightarrow{i'}X^3\xrightarrow{j'}M^5.$$

(2. 2) $$S^{10}\xrightarrow{\bar\eta}M^8\xrightarrow{i''}Y^{11}\xrightarrow{j''}S^{11}.$$

Here M^n denotes the Moore space $S^n\underset{2}{\cup}e^{n+1}$.

From above cofibrations we obtain exact sequences as follows.

(2. 3) $$0\longleftarrow\pi^0(S^3)\xleftarrow{i'^*}\pi^0(X^3)\xleftarrow{j'^*}\pi^0(M^5)\longleftarrow 0.$$

(2. 4) $$\pi^0(S^{10})\xleftarrow{\bar\eta^*}\pi^0(M^8)\xleftarrow{i''^*}\pi^0(Y^{11})\xleftarrow{j''^*}\pi^0(S^{11})\longleftarrow\pi^0(M^9).$$

Lemma A. *In the exact sequence* (2. 4),
a). ker $\bar\eta^*=Z_4\langle\sigma\text{Ext }\eta\rangle+Z_4\langle\text{Ext }\varepsilon\rangle$. b). ker $j''^*=Z_4\langle 2\zeta\rangle$.

Proof. By J. Mukai [4], $\pi^0(M^8)=Z_4\langle\text{Ext }\varepsilon\rangle\oplus Z_4\langle\sigma\text{Ext }\eta\rangle\oplus Z_2\langle\mu p\rangle$. Now $\bar\eta^*(\mu p)=\mu\eta$ is the generator of π_{10}^s(Toda [8]), where p is a projection map.

Consider elements $\bar\eta^*(\sigma\text{Ext }\eta)$, $\bar\eta^*(\text{Ext }\varepsilon)$, these are nothing but Toda brackets $\{\sigma\eta, 2, \eta\}$ and $\{\varepsilon, 2, \eta\}$. We see easily these contain zero. Thus a) is obvious. For b), this time we need to investigate $\{\mu, 2, \eta\}$, $\{\eta\sigma\eta, 2, \eta\}$ and $\{\eta\varepsilon, 2, \eta\}$. $\{\eta\sigma\eta, 2, \eta\}\supset\eta\sigma\{\eta, 2, \eta\}$, $\{\eta\varepsilon, 2, \eta\}\supset\varepsilon\{\eta, 2, \eta\}$ and $\{\eta, 2, \eta\}=\{\nu', -\nu'\}$ (5. 4 [8]). These contain zero since $\nu'=2\nu$. We see easily that $\{\mu, 2, \eta\}$ contains 2ζ, for example by e-invariant of Adams (Theorem 11. 1 in [1]). q. e. d.

We shall determine group extension in (2. 3). Since $\pi_3^s=Z_8\langle\nu\rangle\oplus Z_3\langle\alpha_1\rangle$, $\pi^0(M^5)=Z_2\langle\nu^2p\rangle$, we only consider the two primary component. We obtain an equality as follows.

$$8\iota i'^{*-1}(\nu)=j'^*\{8\iota, \nu, \bar\eta\} \bmod j'(8\iota[M^5, S^0]).$$

This equality is due to Toda (Proposition 1.9 [8], also refer Walker [9]). By the natural property of Toda brackets, $i_0\{8\iota, \nu, \bar{\eta}\} = \{i_0, 8\iota, \nu\}(-S\bar{\eta})$, where i_0: $S^0 \to M^0$ is the inclusion. We obtain $\{i_0, 8\iota, \nu\} = (\text{Coext } \eta)\eta^2$ since $p\{i_0, 8\iota, \nu\}$ can be easily seen to be $4\nu = p(\text{Coext } \eta)\eta^2$ and by Theorem 3.2. [4] $(\text{Coext } \eta)\eta^2$ is a generator of $[S^4, M^0] = Z_2$. On the other hand, as $(\text{Coext } \eta)\eta^2(-\text{Ext } S\bar{\eta}) = \eta_2^2\eta_3 = 0$ by vi) of Proposition 2.1 [5], $\{i_0, 8\iota, \nu\}(-\bar{\eta}) = 0$. Finally, i_0 induces a monomorphism i_{0*}: $[M^5, S^0] \to [M^5, M^0]$ again by [4. Theorem 3.1 and Theorem 3.3]. So $\{8\iota, \nu, \bar{\eta}\} = 0$. Thus (2.3) is split. We summarize our result as follows.

Proposition 2.5. $\pi^0(X^3) = Z_8\langle\tilde{\nu}\rangle \oplus Z_3\langle\tilde{\alpha}_1\rangle \oplus Z_2\langle\nu^2 pj'\rangle$, *where* $i'^*(\tilde{\nu}) = \nu$, $i'^*(\tilde{\alpha}_1) = \alpha_1$.

Analogously, we obtain the following.

Proposition 2.6. $\pi^0(Y^{11}) = Z_4\langle\widetilde{\text{Ext } \varepsilon - \sigma\text{Ext } \eta}\rangle \oplus Z_8\langle\widetilde{\text{Ext } \varepsilon}\rangle \oplus Z_9\langle\alpha_3'\rangle \oplus Z_7\langle\alpha_{1,7}\rangle$.

Proof. $\{4\iota, \text{Ext } \varepsilon, \bar{\eta}\}4\iota = \{2\iota, 2\text{Ext } \varepsilon, \bar{\eta}\}4\iota = \{2\iota, \varepsilon\eta p, \bar{\eta}\}4\iota = \{2\iota, \varepsilon\eta, \eta\}4\iota = 4\zeta$ since $\{2\iota, \varepsilon\eta, \eta\} = \zeta + 2\pi_{11}^s$[8, (9.4)]. Therefore $\{4\iota, \text{Ext } \varepsilon, \bar{\eta}\}$ contains the element ζ. Thus the extension of Ext ε, we denote it by $\widetilde{\text{Ext } \varepsilon}$, is the element of order 8. Similarly $\widetilde{\sigma\text{Ext } \eta}$ has the order 8. Finaly, by (2.4) and Lemma A we obtain our proposition.

§ 3. The Determination of $\pi^0(G_2)$

Let ϕ be a map given in [6]. Then there exists the cofibration as follows.

(3.1) $$X^3 \xrightarrow{i} Q \xrightarrow{j} Y^{11} \xrightarrow{\phi} \sum X^3 (= X^4).$$

Because first we see that $\pi^1(Y^{11})$ is easily seen to be zero and $\pi^{-1}(X^3)$ contains only elements of order 2, on the other hand ϕ is equal to $2(\Sigma i')\sigma j''$ by [6. Theorem 4.12]. Then it is not hard to show that the following is exact.

(3.2) $$0 \longleftarrow \pi^0(X^3) \longleftarrow \pi^0(Q) \longleftarrow \pi^0(Y^{11}) \longleftarrow 0.$$

We have to determine this group extension. First we consider the

2-component. As in Section 2, we need to know Toda brackets $\{2\iota, \nu^2 pj', \Sigma^{-1}\phi\}$ and $\{8\iota, \tilde{\nu}, \Sigma^{-1}\phi\}$. $\{2\iota, \nu^2 pj', \Sigma^{-1}\phi\} \supset \{2\iota, \nu^2, pj' \Sigma^{-1}\phi\}$ contains zero since $\phi = 2(\Sigma i')\sigma j''$ and pj' is order 2. Thus the Z_2-summand splits. We claim that $\{8\iota, \tilde{\nu}, \Sigma^{-1}\phi\} = 0$, since without indeterminacy we obtain the equality: $\{8\iota, \tilde{\nu}, \Sigma^{-1}\phi\} = \{8\iota, \tilde{\nu}, 2i'\sigma \Sigma^{-1}j''\} = \{8\iota, \tilde{\nu}i'\sigma, 2\Sigma^{-1}j''\} = 0$ since $\tilde{\nu}i'\sigma = \nu\sigma = 0$. Therefore Z_8-summand also splits. As at the prime 3 Q is stably equivalent to $(S^3 \underset{2\alpha_2}{\cup} e^{11})$, we only have to consider the Toda bracket $\{3\iota, \alpha_1, 2\alpha_2\}$. By Theorem 11. 4 [1], we see that its e_C-invariant, $e_C\{3\iota, \alpha_1, \alpha_2\} = -\delta(4, 6)/3 \bmod Z$ and $(1/3)Z$. As we may take $\delta(4, 6) = 2\cdot 5\cdot 23/3\cdot 7$, our invariant is nontrivial. Thus we obtain a nontrivial extension on the 3-primary part. Now we complete the proof.

§4. The Ring Structure (Proof of Theorem 1.2)

To prove Theorem 1. 2, we use the results of [3] and the spectral sequence of Atiyah-Hirzebruch associated to the filtration $F^q(X)$, $F^q(X) = \ker[\pi^0(X) \to \pi^0(X^{q-1})]$, X^{q-1} is a $(q-1)$-skeleton of X. Thus $\tilde{\nu}$, $\tilde{\alpha}_1 \in F^3$, $\nu^2 pj' \in F^6$, $\widetilde{\text{Ext}\,\varepsilon}$, $\widetilde{\sigma\text{Ext}\,\eta} \in F^8$, $\widetilde{4\text{Ext}\,\varepsilon} = \widetilde{4\sigma\text{Ext}\,\eta} = j''(\zeta)$, $\tilde{\alpha}_{1,7} \in F^{11}$, $q^*(\sigma^2)$, $q^*(\kappa) \in F^{14}$, where $F^m = F^m(G_2)$. It is easy to see that all products except $\tilde{\nu}^2$, $\tilde{\alpha}_1^2$, $(\nu^2 pj')^2$, $(\nu^2 pj')\tilde{\nu}$, $(\nu^2 pj')\tilde{\nu}^2$, $\tilde{\nu}x$ and $(\nu^2 pj')x$ $(x = \widetilde{\text{Ext}\,\varepsilon}$ or $\widetilde{\sigma\text{Ext}\,\eta})$, $\tilde{\nu}^3$, $\tilde{\nu}^4$, $\tilde{\nu}\cdot j''(\zeta)$ are zero for filtration reasons.

In the Atiyah-Hirzebruch spectral sequence,

$$E_2^{i,j} = H^i(G_2 : \pi_j^S) \Longrightarrow \pi^{i-j}(G_2).$$

$\nu \in E_2^{3,3}$ converges to $\tilde{\nu}$. By the multiplicative properties, $\nu^2 \in E_2^{6,6}$ converges to $\nu^2 pj'$, $\nu^3 \in E_2^{9,9}$ converges to $\tilde{\nu}^3$. Since $\tilde{\nu}j''(\zeta)$ has the filtration 14 and corresponds to $\nu\zeta = 0$, it is trivial. Also relations $\nu\sigma = \nu\varepsilon = 0$ give the results $(\nu^2 pj')x = 0$, $(x = \widetilde{\text{Ext}\,\varepsilon}$ or $\widetilde{\sigma\text{Ext}\,\eta})$. On the other hand, the element $\tilde{\nu}^2$ is equal to $\nu^2 pj'$ at filtration 6, $\tilde{\nu}^3$ and $(\nu^2 pj')\tilde{\nu}$ corresponds to $2\tilde{\nu}$ at F^9 since $\nu^3 = \eta^2\sigma + \eta\varepsilon$ which is $2(\sigma\text{Ext}\,\eta + \text{Ext}\,\varepsilon)$ in $\pi^0(M^8)$. In $\pi^0(SU(3))$ it has been proved that $\tilde{\nu}^2 = \tilde{\nu}$, thus by the natural inclusion we obtain that $\tilde{\nu}^2 = \nu^2 pj' + \tilde{\tilde{\nu}} + t$, where t is an element of higher filtration. As G_2 is stably self dual, we can apply Proposition 3. 1 of [3]. Using this proposition, a composition $S^{14} \xrightarrow{d} G_2 \wedge G_2 \xrightarrow{\tilde{\nu}\wedge\tilde{\nu}} S^0 \wedge S^0 = S^0$ is the Toda bracket $\{\tilde{\nu}, \phi, \tilde{\nu}^*\}$, where d is a duality map and $\tilde{\nu}^*$ means the dual of $\tilde{\nu}$. The bracket $\{\tilde{\nu}, \phi, \tilde{\nu}^*\}$

contains zero since $2\{\tilde{\nu}, i'\sigma\Sigma^{-1}j'', \tilde{\nu}^*\}=0$ $(\pi_{14}^S(S^0)=(2)^2)$. Thus the restriction of t to the top cell $(=S^{14})$ is trivial. This is 1). Similarly, $(\nu^2 pj')\tilde{\nu}\equiv 2\tilde{\tilde{\nu}} \bmod j''(\zeta)$ since $\{\tilde{\nu}, \phi, (\nu^2 pj')^*\}$ also cotains zero. Moreover the element $(\nu^2 pj')\tilde{\nu}$ can not involve $j''(\zeta)$ by the e_C-invariant argument. Namely, we define e_C-invariant on $[Q, S^0]$ and $[Y^{11}, S^0]$ in terms of the Chern charactor as in [6], so that we obtain the following commutative diagram.

$$\begin{array}{ccc} e_C\colon [Q, S^0] & \longrightarrow & Q/2Z \oplus Q/\frac{1}{2}Z \\ \uparrow & & \uparrow \\ e_C\colon [Y^{11}, S^0] & \longrightarrow & Q/\frac{1}{2}Z, \end{array}$$

in which vertical arrows are monic. On $[Y^{11}, S^0]$, $e_C(j''(\zeta))=1/4 \bmod (1/2)Z$, thus e_C of $j''(\zeta)$ on $[Q, S^0]$ is also nontrivial. Since we can easily see that $e_C((\nu^2 pj')\tilde{\nu})=e_C(2\tilde{\tilde{\nu}})=0$, we obtain our result.

Part 3). As $\tilde{\nu}^3=\tilde{\nu}(\nu^2 pj'+\tilde{\nu})=2\tilde{\tilde{\nu}}+\tilde{\nu}\tilde{\tilde{\nu}}$ by 1) and 2), we have to determine $\tilde{\nu}\tilde{\tilde{\nu}}$. Since this element has the filtration 14, we can use the similar method as above to obtain that at the top cell $\tilde{\nu}\tilde{\tilde{\nu}}$ is equal to the bracket $\{\tilde{\nu}, \text{Ext}\ \eta, \text{Coext}\ \bar{\nu}\}$ which is κ by [8] p. 96. Samely $\tilde{\nu}^4$ and $(\nu^2 pj')^2$ are also seen to be trivial.

(Odd prime case). It is well known that at the prime 3, G_2 is equivalent to $(S^3 \cup_{2\alpha_2} e^{11}) \cup e^{14}$. We obtain the following homotopy commutative diagram.

$$\begin{array}{ccccc} C & \xrightarrow{\Delta} & C\wedge C & \xrightarrow{g\wedge g} & S^0\wedge S^0 \\ \downarrow & & \| & & \uparrow{\scriptstyle g\wedge 1} \\ S^{11} & \xrightarrow{\bar{\Delta}} & C\wedge C & \xrightarrow{1\wedge g} & C\wedge S^0, \end{array}$$

where Δ is the diagonal map, $C=S^3 \cup_{2\alpha_2} e^{11}$, g is a representative of the restriction of $\tilde{\alpha}_1$ to C. Obviously, there exists $\bar{\Delta}$ which makes this diagram commutative. We observe that $\pi_{11}^S(C\wedge C)=0$, thus the top rows of the diagram are trivial. Therefore $\tilde{\alpha}_1^2$ is contained in $F^{12}(G_2)$. Since $\pi_{14}^S(S^0)_{(3)}=0$ we can conclude that $\tilde{\alpha}_1^2=0$.

Let $[G_2, L]$ be a stable homotopy element obtained by applying the Pontryagin-Thom construction to the left invariant framing L of

G_2. By [7], [10], it has been shown that $[G_2, L]=\kappa$. Also in [2], this fact is stated without the full proof. Combining our theorem above with the method in [2], we can easily obtain the result.

Corollary 4.1. ([7], [10] and [2]). $[G_2, L]=\kappa$.

Proof. $q^*[G_2, L]=J_R^2(J_R-2)$ by [2. (5.4) Theorem (a)], where J_R is the Hopf construction of 7-dimensional representation of G_2. As it is seen by the natural inclusion $SU(3)\to G_2$ that $J_R=\pm\tilde{\nu}+t$, t an element of higher filtration. Thus $q^*[G_2, L]=2\tilde{\nu}^2\pm\tilde{\nu}^3=q^*\kappa$ by our theorem above.

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