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Hodge Modules, Equivariant K-Theory and Hecke Algebras

By

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§ 0. Introduction

0.1. The Hecke algebra $H(W)$ of a Coxeter system (W, S) is an algebra over the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$ which has a free basis $\{T_w | w \in W\}$ and satisfies the following relations:

$$(T_s+1)(T_s-q)=0 \qquad (s \in S)$$

$$T_{w_1}T_{w_2}=T_{w_1}T_{w_2} \qquad (l(w_1)+l(w_2)=l(w_1w_2)),$$

where l is the length function.

When W is a Weyl group, this algebra appeared in connection with finite Chevalley groups ([I]) as we formulate in the following. Let G be a connected reductive algebraic group with Weyl group W defined and split over a finite field $\mathbb{F}_{q_0}$ and X the flag variety of G. We denote by H the $\mathbb{C}$-vector space consisting of $\mathbb{C}$-valued functions on $X(\mathbb{F}_{q_0}) \times X(\mathbb{F}_{q_0})$ which are invariant under the action of $G(\mathbb{F}_{q_0})$. H is endowed with an algebra structure via the convolution product:

$$(h_1 \cdot h_2)(x, y) = \sum_{z \in X(\mathbb{F}_{q_0})} h_1(x, z) h_2(z, y),$$

and it is isomorphic to the $\mathbb{C}$-algebra obtained by tensoring $\mathbb{C}$ to $H(W)$ over $\mathbb{Z}[q, q^{-1}]$ via the ring homomorphism $\mathbb{Z}[q, q^{-1}] \to \mathbb{C}$ $(q \to q_0)$.

Replacing functions on $X(\mathbb{F}_{q_0}) \times X(\mathbb{F}_{q_0})$ by $\mathbb{Q}_l$-sheaves on $X \times X$, we have a more sophisticated realization of the Hecke algebra (due to Beilin-

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son, Brylinski and Lusztig-Vogan [LV]). Let $\mathcal{C}$ be the category consisting of constructible $\mathbb{Q}_l$-sheaves on $X\times X$, whose restrictions to each G-orbit Y have composition factors of the type $\mathbb{Q}_{l,Y}(n)$ $(n\in\mathbb{Z})$, where (n) is the Tate twist. The Grothendieck group $K(\mathcal{C})$ is a $\mathbb{Z}[q, q^{-1}]$-module via $q^n[K]=[K(-n)]$. We define $p_{13}: X\times X\times X\to X\times X$ and $r: X\times X\times X\to X\times X\times X\times X$ by $p_{13}(a, b, c)=(a, c)$ and $r(a, b, c)=(a, b, b, c)$. By the product:

$$[K_1]\cdot[K_2]=\sum_j(-1)^j[R^j p_{13!}(r^*(K_1\boxtimes K_2))],$$

$K(\mathcal{C})$ is endowed with a $\mathbb{Z}[q, q^{-1}]$-algebra structure and it is isomorphic to $H(W)$.

0.2. It has been conjectured that there exists a theory in char$=0$, which corresponds to the theory of the weights for $\mathbb{Q}_l$-sheaves in char>0 (Deligne's philosophy, [Br] etc.). This was realized by M. Saito as a theory of Hodge modules quite recently ([Sa1~5]). He has defined, for a non-singular algebraic variety Y over $\mathbb{C}$, a certain abelian category $MHM(Y)$, which is a full subcategory of the category consisting of quartets $(\mathcal{M}, F, K, W)$, where $\mathcal{M}$ is a regular holonomic D_Y-module, F is a good filtration of $\mathcal{M}$, K is a perverse sheaf over $\mathbb{Q}$ on Y such that $DR(\mathcal{M})=\mathbb{C}\otimes_{\mathbb{Q}}K$ and W is a filtration of $(\mathcal{M}, F, K)$. This category corresponds to the category of mixed perverse sheaves in char>0, philosophically.

Using this theory we can give a realization of $H(W)$ in char$=0$. Let G be a connected reductive algebraic group over $\mathbb{C}$ whose Weyl group is W and let X be the flag variety of G. Then there exists a certain abelian category $\mathcal{A}$, which is a subcategory of the category consisting of the objects of $MHM(X\times X)$ with G-actions, so that its Grothendieck group $K(\mathcal{A})$ has two free bases $\{[\mathcal{M}_w]\mid w\in W\}$ and $\{[\mathcal{L}_w]\mid w\in W\}$ over $\mathbb{Z}[q, q^{-1}]$ (see Section 3). Here $\mathcal{M}_w$ and $\mathcal{L}_w$ are certain specified objects of $\mathcal{A}$ and the $\mathbb{Z}[q, q^{-1}]$-module structure is given by $q^n[\mathcal{V}]=[\mathcal{V}(-n)]$, where (n) is the counterpart of the Tate twist (see Section 1). Define p_{13} and r similarly to the case of char>0. We can show that $(\mathcal{H}^j p_{13!})(\mathcal{H}^{-\dim X} r^*)(\mathcal{V}_1\boxtimes\mathcal{V}_2)\in\mathcal{A}$ for $\mathcal{V}_1, \mathcal{V}_2\in\mathcal{A}$ and a $\mathbb{Z}[q, q^{-1}]$-algebra structure on $K(\mathcal{A})$ is defined by:

$$[\mathcal{V}_1]\cdot[\mathcal{V}_2]=\sum_j(-1)^j[(\mathcal{H}^j p_{13!})(\mathcal{H}^{-\dim X} r^*)(\mathcal{V}_1\boxtimes\mathcal{V}_2)].$$

Theorem A. *$K(\mathcal{A})$ is isomorphic to $H(W)$ as a $\mathbb{Z}[q, q^{-1}]$-algebra.*

The isomorphism is given by:

$$[\mathcal{M}_w] \leftrightarrow (-1)^{l(w)} T_w \quad \text{and} \quad [\mathcal{L}_w] \leftrightarrow (-1)^{l(w)} \sum_{y \leq w} P_{y,w}(q) T_y ,$$

where $P_{y,w}(q)$ *are the Kazhdan-Lusztig polynomials* (*see* [KL1]).

0.3. Recently Kazhdan-Lusztig [KL4] and Ginsburg [G2] have given a classification of the irreducible representations of the Hecke algebra of the affine Weyl group W_a using equivariant K-theory (conjecture of Deligne-Langlands-Lusztig). The first step of their work was to define an $(H(W_a), H(W_a))$-bimodule structure on the equivariant K-homology group $K^{G\times C^*}(Z)$ of the variety

$$Z = \{(x, y, A) \in X \times X \times \mathrm{Lie}(G) | A \text{ is nilpotent}, \quad A \in \mathrm{Lie}(B_x) \cap \mathrm{Lie}(B_y)\}$$

and to show that this coincides with the two-sided regular representation of $H(W_a)$. Here B_x is the Borel subgroup of G corresponding to $x \in X$. We review this briefly following the formulation of Ginsburg. Let $p: T^*X \to X$ be the cotangent bundle. Regarding Z as a subvariety of $T^*(X\times X) = T^*X \times T^*X$, we can view $K^{G\times C^*}(Z)$ as the Grothendieck group of the abelian category consisting of coherent $O_{T^*X\times T^*X}$-modules with $G\times \mathbb{C}^*$-actions supported in Z. Note that $K^{G\times C^*}(Z)$ is a $\mathbb{Z}[q, q^{-1}]$-module since the representation ring of $\mathbb{C}^*$ is identified with $\mathbb{Z}[q, q^{-1}]$. Let $p_{ij}: T^*X\times T^*X\times T^*X \to T^*X\times T^*X$ and p_2: $T^*X\times T^*X\times T^*X \to T^*X$ be the obvious projections. It is easily seen that a $\mathbb{Z}[q, q^{-1}]$-algebra structure on $K^{G\times C^*}(Z)$ is defined by:

$$[M_1]\cdot[M_2] = \sum_j (-1)^j [\mathcal{H}^j(\mathbb{R}p_{13*}(p_{12}{}^* M_1 \overset{L}{\otimes} p_{23}{}^* M_2 \overset{L}{\otimes} p_2{}^* p^* \Omega_X))] ,$$

where Ω_X is the sheaf of the differential forms of the highest degree on X. The result is that this algebra is isomorphic to $H(W_a)$, especially isomorphic to the two-sided regular representation as an $(H(W_a), H(W_a))$-bimodule (see Section 4.2 for the explicit description of the isomorphism).

0.4. Let $\mathcal{V} = (\mathcal{M}, F, K, W)$ be an object of $\mathcal{A}$. Then $\mathrm{Gr}^F \mathcal{M}$ is a coherent module over the $O_{X\times X}$-algebra $\mathrm{Gr}^F D_{X\times X} = (p\times p)_*(O_{T^*X\times T^*X})$. Hence

$$\mathrm{gr}\, \mathcal{V} = O_{T^*X\times T^*X} \underset{(p\times p)^{-1}(\mathrm{Gr}^F D_{X\times X})}{\otimes} (p\times p)^{-1}(\mathrm{Gr}^F \mathcal{M})$$

is a coherent $O_{T^*X\times T^*X}$-module with $G\times \mathbb{C}^*$-action. It is easily seen that the support of $\mathrm{gr}\,\mathcal{V}$ is contained in the union Λ of the conormal bundles of the G-orbits on $X\times X$. We define an involution a on $T^*X\times T^*X$ by $a(x, y, \xi, \eta)=(x, y, \xi, -\eta)$, where (x, y) is a coordinate of $X\times X$ and (ξ, η) is a coordinate of fibers. Since $a(\Lambda)=Z$, we have a $\boldsymbol{Z}[q, q^{-1}]$-module homomorphism:

$$\gamma=q^{\dim X}(a^*\circ\mathrm{gr}): K(\mathcal{A})\to K^{G\times C^*}(Z).$$

Theorem B. *γ is a homomorphism of $\boldsymbol{Z}[q, q^{-1}]$-algebra, and when we identify $K(\mathcal{A})$ and $K^{G\times C^*}(Z)$ with $H(W)$ and $H(W_a)$ respectively, γ coincides with the natural inclusion.*

The main difficulty in proving $K^{G\times C^*}(Z)\simeq H(W_a)$ is to show that the action of $H(W)$ is well-defined. Ginsburg and Kazhdan-Lusztig used the localization theorem in equivariant K-theory and reduced the problem to the case of $K^{G\times C^*}(X\times X)$. Then the problem turned out to be a combinatorial one, which had been already solved in [Lu] (see also Kato's simpler solution given in [KL4]). In a sence Theorem B gives a different proof of this fact. Although our proof relies on the deep theory of Hodge modules, it seems that it gives a more natural explanation of the fact that the Hecke algebra appears in the context of equivariant K-theory.

0.5. The contents of this paper are as follows. In Section 1 we give a brief summary of the theory of Hodge modules and state some facts concerning the Hodge modules with group actions. In Section 2 we review the definition of the equivariant K-homology groups and give some relation between Hodge modules with group actions and equivariant K-theory. In Sections 3 and 4 Theorem A and Theorem B are proved, respectively. In Section 5 we treat some problems concerning good filtrations of the modules over the enveloping algebra of the Lie algebra of G associated to Hodge modules.

In Sections 1 and 2 the letters G and X will be used for a general algebraic group and a general algebraic variety, respectively, while in Sections 3 to 5 they will be used for a connected reductive algebraic group and its flag variety, respectively. The letter W is used for both of the Weyl group and the weight filtration. We hope that readers will distinguish them from the context.

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§ 1. Hodge Modules

1.1. Hodge structures (see [D2])

We recall basic notions concerning Hodge structures.

Let H be a finite dimensional vector space over $\mathbf{Q}$ and F a decreasing filtration of $H_C=\mathbf{C}\underset{\mathbf{Q}}{\otimes}H$. Hence $F^p(H_C)$ is a $\mathbf{C}$-subspace of H_C for each $p\in\mathbf{Z}$, $F^p(H_C)\cap F^{p+1}(H_C)$, $F^p(H_C)=0$ for a sufficiently large p and $F^p(H_C)=H_C$ for a sufficiently small p. (H, F) is called a Hodge structure of weight n if $H_C=F^p\oplus\bar{F}^{n-p+1}$ for any p. Here barring denotes the complex conjugate. Setting $H^{p,q}=F^p\cap\bar{F}^q$ we have the Hodge decomposition $H_C=\underset{p}{\oplus}H^{p,n-p}$. When (H, F) and (H', F') are Hodge structures of weight n, a linear map $f: H\to H'$ is called a morphism (of Hodge structures of weight n) if $f(F^p)\subset F'^p$ for any p. We denote the category of Hodge structures of weight n by $SH(n)$.

A polarization of $(H, F)\in SH(n)$ is a bilinear form S on H, which is symmetric (resp. skew symmetric) if n is even (resp. odd) and satisfies the following condition:

$$S(H^{p,n-p}, H^{p',n-p'})=0 \quad \text{unless} \quad p+p'=n\,,$$

$$(\sqrt{-1})^{n-2p}S(v, \bar{v})>0 \quad \text{for} \quad v\in H^{p,n-p}, v\neq 0\,.$$

$(H, F)\in SH(n)$ is said to be polarizable if there exists a polarization of (H, F). We denote the full subcategory of $SH(n)$ consisting of polarizable Hodge structures by $SH(n)^p$. It is a semisimple abelian category.

Let H be a finite dimensional $\mathbf{Q}$-vector space, F a decreasing filtration of $\mathbf{C}\underset{\mathbf{Q}}{\otimes}H$ and $W=\{W_n\}$ an increasing filtration of (H, F). Then (H, F, W) is called a mixed Hodge structure if $\mathrm{Gr}_n^W(H, F)(=W_n(H, F)/W_{n-1}(H, F))\in SH(n)$ for any n. The category SHM of mixed Hodge structures is defined similarly. We denote by SHM^p the full subcategory of SHM consisting of (H, F, W) with $\mathrm{Gr}_n^W(H, F) \in SH(n)^p$ for any n.

Let X be a non-singular algebraic variety over $\mathbf{C}$. We denote the sheaf of algebraic differential operators on X by D_X. Let H be a $\mathbf{Q}$-local system on X. Hence H is a sheaf of $\mathbf{Q}$-vector spaces on the associated complex manifold X_{an} (in the classical topology) which is locally constant and has finite dimensional stalks. By the Riemann-Hilbert correspondence for local systems due to Deligne [D1] there exists a unique regular holonomic D_X-module $\mathcal{M}(H)$ which is locally free as an O_X-module and satisfies $\mathcal{M}(H)_{an} \simeq O_{X_{an}} \otimes_{\mathbf{Q}} H$. Here O_X is the structure sheaf of X, $O_{X_{an}}$ is the sheaf of holomorphic functions on X_{an}, $D_{X_{an}} = O_{X_{an}} \otimes_{O_X} D_X$ and $\mathcal{M}(H)_{an} = O_{X_{an}} \otimes_{O_X} \mathcal{M}(H) = D_{X_{an}} \otimes_{D_X} \mathcal{M}(H)$. Let F be a decreasing filtration of $\mathcal{M}(H)$ by O_X-submodules such that $F^p(\mathcal{M}(H))/F^{p+1}(\mathcal{M}(H))$ is locally free for any p. Then (H, F) is called a variation of Hodge structures of weight n if $(H_x, F(x)) \in SH(n)$ for any $x \in X$ and $\partial \cdot F^p(\mathcal{M}(H)) \subset F^{p-1}(\mathcal{M}(H))$ for any vector field ∂ and any p. Note that the fiber of $\mathcal{M}(H)$ at $x \in X$ is $\mathbf{C} \otimes_{\mathbf{Q}} H_x$ and F induces a filtration $F(x)$ of $\mathbf{C} \otimes_{\mathbf{Q}} H_x$. The category of variations of Hodge structures of weight n is denoted by $VSH(X, n)$. A polarization of $(H, F) \in VSH(X, n)$ is a $\mathbf{Q}_X$-linear map $H \otimes_{\mathbf{Q}_X} H \to \mathbf{Q}_X$ which gives a polarization of $(H_x, F(x))$ for any $x \in X$. The full subcategory of $VSH(X, n)$ consisting $(H, F) \in VSH(X, n)$ which are polarizable is denoted by $VSH(X, n)^p$. Categories $VSHM(X)$ and $VSHM(X)^p$ are defined similarly to SHM and SHM^p, respectively.

If $f: X \to Y$ is a morphism of non-singular varieties, we have natural functors $VSH(Y, n) \to VSH(X, n)$, $VSH(Y, n)^p \to VSH(X, n)^p$, $VSHM(Y) \to VSHM(X)$ and $VSHM(Y)^p \to VSHM(X)^p$. All of them are denoted by f^*.

1.2. Filtered $\mathcal{D}$-modules and functors (see [Be], [Sa2; Section 2])

For a non-singular algebraic variety X over $\mathbf{C}$ let $M_{rh}(D_X)$ be the category of regular holonomic D_X-modules. Since we are working in the algebraic category, the regularity here includes the regularity at infinity (see [Be]).

For $\mathcal{M} \in M_{rh}(D_X)$ we set $\mathbb{D}(\mathcal{M}) = \mathcal{E}xt_{D_X}^{\dim X}(\mathcal{M}, D_X) \otimes_{O_X} \Omega_X^{-1}$, where Ω_X is the sheaf of differential forms of degree dim X. It is known that $\mathcal{E}xt^i_{D_X}(\mathcal{M}, D_X)=0$ for $i\neq\dim X$ and $\mathbb{D}(\mathcal{M})$ is a regular holonomic D_X-module. More generally, for a bounded complex $\mathcal{M}$ of D_X-modules such that $\mathcal{H}^i(\mathcal{M})\in M_{rh}(D_X)$ for each i, we set:

$$\mathbb{D}(\mathcal{M})=\mathbb{R}\mathcal{H}om_{D_X}(\mathcal{M}, D_X)\underset{O_X}{\otimes}\Omega_X{}^{-1}[\dim X].$$

Let $f: X\to Y$ be a morphism of non-singular varieties. An $(f^{-1}D_Y, D_X)$-bimodule $D_{Y\leftarrow X}$ and a $(D_X, f^{-1}D_Y)$-bimodule $D_{X\to Y}$ are defined by:

$$D_{Y\leftarrow X}=f^{-1}(D_Y\underset{O_Y}{\otimes}\Omega_Y{}^{-1})\underset{f^{-1}O_Y}{\otimes}\Omega_X\,,\quad D_{X\to Y}=O_X\underset{f^{-1}O_Y}{\otimes}f^{-1}D_Y\,.$$

Then for each $j\in\mathbb{Z}$ additive functors:

$$\mathcal{H}^jf_*\quad\text{and}\quad\mathcal{H}^jf_!: M_{rh}(D_X)\to M_{rh}(D_Y)\,,$$

$$\mathcal{H}^jf^!\quad\text{and}\quad\mathcal{H}^jf^*: M_{rh}(D_Y)\to M_{rh}(D_X)$$

are defined as the j-th cohomologies of the functors f_*, $f_!$, $f^!$, f^* between derived categories given by:

$$f_*(\mathcal{M})=\mathbb{R}f_*(D_{Y\leftarrow X}\overset{L}{\underset{D_X}{\otimes}}\mathcal{M})\,,\quad f_!(\mathcal{M})=\mathbb{D}(f_*(\mathbb{D}(\mathcal{M})))\,,$$

$$f^!(\mathcal{M})=(D_{X\to Y}\overset{L}{\underset{f^{-1}D_Y}{\otimes}}f^{-1}\mathcal{M})[\dim X-\dim Y]\,,\quad f^*(\mathcal{M})=\mathbb{D}(f^!(\mathbb{D}(\mathcal{M})))\,.$$

We have a natural increasing filtration F of D_X given by the orders of differential operators. If an increasing filtration F of a D_X-module $\mathcal{M}$ by O_X-submodules satisfies the conditions:

$$F_p(D_X)F_q(\mathcal{M})\subset F_{p+q}(\mathcal{M})\quad\text{for any } p,\ q\in\mathbb{Z}\,,$$

$$\mathcal{M}=\bigcup_p F_p(\mathcal{M})\,,$$

$$F_p(\mathcal{M})=0\quad\text{for a sufficiently small } p\,,$$

then $(\mathcal{M}, F)$ is called a filtered D_X-module. When $\mathrm{Gr}^F\mathcal{M}$ is a coherent $\mathrm{Gr}^F D_X$-module, F is called a good filtration. Let $MF_{rh}(D_X)$ be the category consisting of filtered D_X-module $(\mathcal{M}, F)$ such that $\mathcal{M}$ is regular holonomic and F is a good filtration. This is not an abelian category but an exact category.

For a projective morphism $f: X\to Y$ and $(\mathcal{M}, F)\in MF_{rh}(D_X)$, an object $f_*(\mathcal{M}, F)$ of the derived category consisting of complexes of filtered D_Y-modules is defined (see [Sa2: Section 2]). Forgetting the filtration this coincides with $\boldsymbol{R}f_*(D_{Y\leftarrow X}\overset{L}{\underset{D_X}{\otimes}}\mathcal{M})$. When $f_*(\mathcal{M}, F)$ is strict, that is,

$$\mathcal{H}^j(F_p(f_*(\mathcal{M}, F)))\to\mathcal{H}^j(\boldsymbol{R}f_*(D_{Y\leftarrow X}\overset{L}{\otimes}\mathcal{M}))\ (=\mathcal{H}^jf_*(\mathcal{M}))$$

is injective for any j and p, a good filtration of $\mathcal{H}^jf_*(\mathcal{M})$ is given by $F_p(\mathcal{H}^jf_*(\mathcal{M}))=\mathcal{H}^j(F_p(f_*(\mathcal{M}, F)))$. This object of $MF_{rh}(D_Y)$ is denoted by $\mathcal{H}^jf_*(M, F)$.

When f is a closed immersion, $\mathcal{H}^jf_*(\mathcal{M})=0$ for $j\neq 0$ and $f_*(\mathcal{M}, F)$ is always strict. $f_*(\mathcal{M}, F)\ (=\mathcal{H}^0f_*(\mathcal{M}, F)=(f_*(D_{Y\leftarrow X}\otimes\mathcal{M}), F))$ is given by:

$$F_p(f_*(D_{Y\leftarrow X}\otimes\mathcal{M}))=f_*(\sum_q F_q(D_{Y\leftarrow X})\otimes F_{p-q+\dim X-\dim Y}(\mathcal{M}))\,,$$

where the filtration of $D_{Y\leftarrow X}$ is induced from that of D_Y.

When $X=Y\times Z$ and $f: X\to Y$ is the projection (Z is a projective non-singular variety with dimension m), $D_{Y\leftarrow X}\overset{L}{\underset{D_X}{\otimes}}\mathcal{M}$ is quasi-isomorphic to the relative de Rham complex:

$$DR_{X/Y}(\mathcal{M})=[\Omega^0_{X/Y}\underset{O_X}{\otimes}\mathcal{M}\to\Omega^1_{X/Y}\underset{O_X}{\otimes}\mathcal{M}\to\cdots\to\Omega^m_{X/Y}\underset{O_X}{\otimes}\mathcal{M}]\,,$$

where $\Omega^i_{X/Y}$ is the sheaf of relative differential forms of degree i and the last term $\Omega^m_{X/Y}\otimes\mathcal{M}$ has the complex degree 0. With the filtration:

$$F_p(DR_{X/Y}(\mathcal{M}))=[\Omega^0_{X/Y}\underset{O_X}{\otimes}F_p(\mathcal{M})\to\cdots\to\Omega^m_{X/Y}\underset{O_X}{\otimes}F_{p+m}(\mathcal{M})]\,,$$

$DR_{X/Y}(\mathcal{M}, F)=(DR_{X/Y}(\mathcal{M}), F)$ is a complex of filtered $f^{-1}D_Y$-modules. Then $f_*(\mathcal{M}, F)$ is strict if and only if the homomorphism $\mathcal{H}^j(\boldsymbol{R}f_*(F_p(DR_{X/Y}(\mathcal{M}))))\to\mathcal{H}^j(\boldsymbol{R}f_*(DR_{X/Y}(\mathcal{M})))\ (=\mathcal{H}^jf_*(\mathcal{M}))$ is injective for any j and p, and in this case $\mathcal{H}^jf_*(\mathcal{M}, F)$ is given by $F_p(\mathcal{H}^jf_*(\mathcal{M}))=\mathcal{H}^j(\boldsymbol{R}f_*(F_p(DR_{X/Y}(\mathcal{M}))))$.

Example. Let $f: X\to Y$ be a $\boldsymbol{P}^1$-bundle. We define a good filtration of O_X and O_Y by $\mathrm{Gr}_j^F O_X=0$ and $\mathrm{Gr}_j^F O_Y=0$ for $j\neq 0$. Then it is easily seen that $f_*(O_X, F)$ is strict and $\mathcal{H}^jf_*(O_X, F)=0$ for $j\neq\pm 1$, $\mathcal{H}^{-1}f_*(O_X, F)=(O_Y, F)$ and $\mathcal{H}^1f_*(O_X, F)=(O_Y, F[-1])$. For an increasing filtration F and $n\in\boldsymbol{Z}$, $F[n]$ is a new filtration given by $F[n]_p=F_{p-n}$.

1.3. Pure Hodge modules

Let X be a non-singular algebraic variety over $\mathbb{C}$. We denote by $\mathrm{Perv}(\mathbb{C}_X)$ (resp. $\mathrm{Perv}(\mathbb{Q}_X)$) the abelian category of perverse sheaves over $\mathbb{C}$ (resp. $\mathbb{Q}$) on X([BBD]). For a regular holonomic D_X-module $\mathcal{M}$

$$DR_X(\mathcal{M})=\mathbf{R}\mathcal{H}om_{D_{X^{an}}}(O_{X^{an}}, D_{X^{an}}\underset{D_X}{\otimes}\mathcal{M})[\dim X]$$

belongs to $\mathrm{Perv}(\mathbf{C}_X)$ and the functor:

$$DR_X : M_{rh}(D_X)\to \mathrm{Perv}(\mathbf{C}_X)$$

gives an equivalence of abelian categories (the Riemann-Hilbert correspondence, [K], [Me1, 2], see also [Be] for the algebraic version stated above). It is known that the functor DR_X is compatible with direct images and inverse images, that is, we have:

$$DR_Y\circ(\mathcal{H}^j f_*)=({}^p\mathcal{H}^j f_*)\circ DR_X\,,\quad DR_Y\circ(\mathcal{H}^j f_!)=({}^p\mathcal{H}^j f_!)\circ DR_X\,,$$

$$DR_X\circ(\mathcal{H}^j f^!)=({}^p\mathcal{H}^j f^!)\circ DR_Y\,,\quad DR_X\circ(\mathcal{H}^j f^*)=({}^p\mathcal{H}^j f^*)\circ DR_Y\,,$$

where ${}^p\mathcal{H}^j$ is the perverse cohomology.

Let $MF_{rh}(D_X, \mathbf{Q})$ be the fiber product of the categories $MF_{rh}(D_X)$ and $\mathrm{Perv}(\mathbf{Q}_X)$ over $\mathrm{Perv}(\mathbf{C}_X)$. An object of $MF_{rh}(D_X, \mathbf{Q})$ is a triple $(\mathcal{M}, F, K)$, where $\mathcal{M}$ is a regular holonomic D_X-module, F is a good filtration of $\mathcal{M}$ and K is a perverse sheaf over $\mathbf{Q}$ with a given isomorphism $DR(\mathcal{M})\simeq \mathbf{C}\underset{\mathbf{Q}}{\otimes}K$. A fully faithful functor:

$$\phi_X{}^n: VSH(X, n)\to MF_{rh}(D_X, \mathbf{Q})$$

is defined by:

$$\phi_X{}^n(H, F)=(\mathcal{M}(H), F, H[\dim X])\quad \text{with}\quad F_p=F^{-p}$$

(see Section 1.1).

In [Sa1, 2] certain full subcategories $MH(X, k)^p$ and $MH_Z(X, k)^p$ of $MF_{rh}(D_X, \mathbf{Q})$ are defined. Here k is an integer and Z is an irreducible closed subvariety of X. We do not reproduce their definitions but list some properties which will be used later.

(p1) $MH(X, k)^p$ and $MH_Z(X, k)^p$ are abelian categories whose morphisms are always strict with respect to F.

(p2) $MH(X, k)^p=\bigoplus_Z MH_Z(X, k)^p$. That is, any object of $MH(X, k)^p$ is decomposed uniquely into the direct sum of the objects of $MH_Z(X, k)^p$, and if $\mathcal{V}_i\in MH_{Z_i}(X, k)^p$ $(i=1, 2)$ with $Z_1\neq Z_2$, then $Hom(\mathcal{V}_1, \mathcal{V}_2)=0$.

(p3) Let X be the union of open subsets U_λ and let $\mathcal{V}\in MF_{rh}(D_X, \mathbf{Q})$. Then $\mathcal{V}\in MH(X, k)^p$ if and only if $\mathcal{V}|U_\lambda\in MH(U_\lambda, k)^p$ for any λ.

(p4) If $\mathcal{V}=(\mathcal{M}, F, K)\in MH(X, k)^p$, then

$$\mathcal{V}(n)=(\mathcal{M}\underset{Q}{\otimes}\boldsymbol{Q}(n), F[n], K\underset{Q}{\otimes}\boldsymbol{Q}(n))\in MH(X, k-2n)^p ,$$

where $\boldsymbol{Q}(n)=(2\pi\sqrt{-1})^n\boldsymbol{Q}\subset\boldsymbol{C}$ and $F[n]_p=F_{p-n}$.

(p5) $\phi_X{}^n(H, F)$ belongs to $MH_X(X, n+\dim X)^p$ for $(H, F)\in VSH(X, n)^p$. Especially

$$\mathcal{L}_X=(O_X, F, \boldsymbol{Q}_X[\dim X]) \quad \text{with} \quad \mathrm{Gr}_i{}^F O_X=0 \quad \text{for} \quad i\neq 0$$

is an object of $MH(X, \dim X)^p$.

(p6) For an object $(\mathcal{M}, F, K)$ of $MH_Z(X, k)^p$, there exist a non-singular open subset U of Z $(Y=Z-U, i: Z\hookrightarrow X, i_0: U\hookrightarrow X-Y)$ and $(H, F)\in VSH(U, k-\dim Z)^p$ such that $K=i_*\mathcal{IC}(H)$ and $(\mathcal{M}, F, K)|X-Y=(i_{0*}(\mathcal{M}(H), F), i_{0*}H[\dim Z])$. Here $\mathcal{IC}(H)$ is the DGM-extension of H(see [GM], [BBD]).

(p7) Let $(\mathcal{M}, F, K)$ and $(\mathcal{M}', F', K')$ be objects of $MH_Z(X, k)^p$. Choose a non-singular open subset U of Z and (H, F), $(H', F')\in VSH(U, k-\dim Z)^p$ so that U and (H, F) (resp. (H', F')) satisfy the conclusion of (p6) for $(\mathcal{M}, F, K)$ (resp. $(\mathcal{M}', F', K')$). Then any morphism from (H, F) to (H', F') in $VSH(U, k-\dim Z)^p$ extends uniquely to a morphism from $(\mathcal{M}, F, K)$ to $(\mathcal{M}', F', K')$ in $MH_Z(X, k)^p$. Especially $(\mathcal{M}, F, K)$ in (p6) is uniquely determined by U and (H, F).

(p8) For a projective morphism $f: X\to Y$ of non-singular varieties $f_*(\mathcal{M}, F)$ is strict for $\mathcal{V}=(\mathcal{M}, F, K)\in MH(X, k)^p$ and

$$\mathcal{H}^j f_*(\mathcal{V}):=(\mathcal{H}^j f_*(\mathcal{M}, F), {}^p\mathcal{H}^j f_*(K))\in MH(Y, j+k)^p .$$

(p9) Let $f: X\to Y$ and $g: Y\to Z$ be projective morphisms of non-singular varieties. Then $\mathcal{H}^j(g\circ f)_*(\mathcal{V})=\bigoplus_k(\mathcal{H}^k g_*)(\mathcal{H}^{j-k}f_*)(\mathcal{V})$ for $\mathcal{V}\in MH(X, n)^p$.

Definition. Let Z be an irreducible closed subvariety of a non-singular variety X with singular locus Z_{sing} and natural inclusion $i: Z-Z_{\mathrm{sing}}\to X-Z_{\mathrm{sing}}$. It follows from (p2), (p6), (p7) and the desingularization theorem of Hironaka that there exists a unique object $\mathcal{V}$ of $MH_Z(X, \dim Z)^p$ such that $\mathcal{V}|X-Z_{\mathrm{sing}}=i_*\mathcal{L}_{Z-Z_{\mathrm{sing}}}$. We denote this $\mathcal{V}$ by $\mathcal{L}(Z, X)$.

Example. If $f: X\to Y$ is a $\boldsymbol{P}^1$-bundle of non-singular varieties, we have $\mathcal{H}^j f_*(\mathcal{L}_X)=0$ for $j\neq\pm 1$, $\mathcal{H}^{-1}f_*(\mathcal{L}_X)=\mathcal{L}_Y$ and $\mathcal{H}^1 f_*(\mathcal{L}_X)=\mathcal{L}_Y(-1)$.

1.4. Mixed Hodge modules

For a non-singular variety X let $MHW(X)^p$ be the category consisting of quartets $(\mathcal{M}, F, K, W)$ where $(\mathcal{M}, F, K)$ is an object of $MF_{rh}(D_X, \mathbb{Q})$ and W is a finite increasing filtration of $(\mathcal{M}, F, K)$ in $MF_{rh}(D_X, \mathbb{Q})$ such that $\mathrm{Gr}_k{}^W(\mathcal{M}, F, K)$ is an object of $MH(X, k)^p$ for any k. In view of (p5) we have a natural functor:

$$\phi_X : VSHM(X)^p \to MHW(X)^p .$$

Saito has defined a certain full subcategory $MHM(X)$ of $MHW(X)^p$ and additive functors:

$$\mathcal{H}^j f_! : MHM(X) \to MHM(Y), \quad \mathcal{H}^j f^* : MHM(Y) \to MHM(X)$$

for a morphism $f : X \to Y$ of non-singular varieties ($\mathcal{H}^j f_*$ and $\mathcal{H}^j f^!$ are also defined. But we do not use them.). We list some of their properties in the following ([Sa3~5]).

(m1) $MHM(X)$ is an abelian category whose morphisms are always strict for both F and W.

(m2) $MHM(X)$ is closed under subquotients in $MHW(X)^p$.

(m3) If $\mathcal{H}^j f_!(\mathcal{M}, F, K, W) = (\mathcal{M}', F', K', W')$, then $\mathcal{M}' = \mathcal{H}^j f_!(\mathcal{M})$ and $K' = {}^p\mathcal{H}^j f_!(K)$.

(m4) If $\mathcal{H}^j f^*(\mathcal{M}, F, K, W) = (\mathcal{M}', F', K', W')$, then $\mathcal{M}' = \mathcal{H}^j f^*(\mathcal{M})$ and $K' = {}^p\mathcal{H}^j f^*(K)$.

(m5) If $\mathcal{V} = (\mathcal{M}, F, K, W) \in MHM(X)$, then we have:

$$\mathcal{V}(n) : = (\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}(n), F[n], K \otimes_{\mathbb{Q}} \mathbb{Q}(n), W[-2n]) \in MHM(X) .$$

(m6) For a short exact sequence $0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \mathcal{V}_3 \to 0$ in $MHM(X)$ we have a long exact sequence:

$$\cdots \to \mathcal{H}^j f_!(\mathcal{V}_1) \to \mathcal{H}^j f_!(\mathcal{V}_2) \to \mathcal{H}^j f_!(\mathcal{V}_3) \to \mathcal{H}^{j+1} f_!(\mathcal{V}_1) \to \cdots$$

in $MHM(Y)$ which coincides with the usual long exact sequence caused by $\mathcal{H}^j f_!$ (resp. ${}^p\mathcal{H}^j f_!$) on the level of $M_{rh}(D_Y)$ (resp. $\mathrm{Perv}(\mathbb{Q}_Y)$).

(m7) For a short exact sequence $0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \mathcal{V}_3 \to 0$ in $MHM(Y)$ we have a long exact sequence:

$$\cdots \to \mathcal{H}^j f^*(\mathcal{V}_1) \to \mathcal{H}^j f^*(\mathcal{V}_2) \to \mathcal{H}^j f^*(\mathcal{V}_3) \to \mathcal{H}^{j+1} f^*(\mathcal{V}_1) \to \cdots$$

in $MHM(X)$ which coincides with the usual long exact sequence caused by $\mathcal{H}^j f^*$ (resp. ${}^p\mathcal{H}^j f^*$) on the level of $M_{rh}(D_X)$ (resp. $\mathrm{Perv}(\mathbb{Q}_X)$).

(m8) Let $f : X \to Y$ be a smooth morphism with relative dimension m. (Hence $\mathcal{H}^j f^* = 0$ for $j \neq m$.) Set $(\mathcal{H}^m f^*)(\mathcal{M}, F, K, W) = (\mathcal{M}', F', K', W')$

for $(\mathcal{M}, F, K, W) \in MHM(Y)$. Then we have:

$$\mathcal{M}' = O_X \underset{f^{-1}O_Y}{\otimes} f^{-1}\mathcal{M}, \quad K' = {}^p\mathcal{H}^m f^*(K), \quad F_p'(\mathcal{M}') = O_X \otimes f^{-1}(F_p(\mathcal{M})),$$

$$W_q(\mathcal{M}', F', K') = (\mathcal{M}'', F'', K'') \quad \text{with}$$

$$\mathcal{M}'' = O_X \otimes f^{-1}(W_{q-m}(\mathcal{M})), \quad K'' = {}^p\mathcal{H}^m f^*(W_{q-m}(K)),$$

$$F_p''(\mathcal{M}'') = O_X \otimes f^{-1}(W_{q-m}(\mathcal{M}) \cap F_p(\mathcal{M})).$$

(m9) Let $f: X \to Y$ be a projective morphism. For an object $\mathcal{V} = (\mathcal{M}, F, K, W)$ of $MHM(X)$ with $\mathrm{Gr}_i^W \mathcal{V} = 0$ for $i \neq k$, we have $\mathrm{Gr}_i^W (\mathcal{H}^j f_!(\mathcal{V})) = 0$ for $i \neq j+k$ and $\mathrm{Gr}_{j+k}^W(\mathcal{H}^j f_!(\mathcal{V}))$ coincides with $\mathcal{H}^j f_*(\mathcal{M}, F, K)$ in the sence of Section 1.3.

(m10) For $\mathcal{V} = (\mathcal{M}, F, K, W) \in MHM(X)$ and $\mathcal{V}' = (\mathcal{M}', F', K', W') \in MHM(X')$ we have:

$$\mathcal{V} \boxtimes \mathcal{V}' = (\mathcal{M} \boxtimes \mathcal{M}', F'', K \boxtimes K', W'') \in MHM(X \times X'),$$

with $F_p'' = \sum_q F_q \boxtimes F_{p-q}'$ and $W_p'' = \sum_q W_q \boxtimes W_{p-q}'$.

(m11) Let $f: \mathrm{X} \to \mathrm{Y}$ be a morphism of non-singular varieties and T a non-singular variety. For a natural morphism $f \times 1: X \times T \to Y \times T$ we have:

$$\mathcal{H}^j (f \times 1)_! (\mathcal{V}_1 \boxtimes \mathcal{V}_2) = (\mathcal{H}^j f_! \mathcal{V}_1) \boxtimes \mathcal{V}_2,$$

$$\mathcal{H}^j (f \times 1)^* (\mathcal{V}_1 \boxtimes \mathcal{V}_2) = (\mathcal{H}^j f^* \mathcal{V}_1) \boxtimes \mathcal{V}_2.$$

(m12) For a closed immersion $f: X \to Y$, $f_! (= \mathcal{H}^0 f_!)$ gives a category equivalence between $MHM(X)$ and the full subcategory of $MHM(Y)$ whose objects are supported in X. Its quasi-inverse is $\mathcal{H}^0 f^*$.

(m13) Let $i: Y \to X$ be a closed immersion of non-singular varieties with $\operatorname{codim} Y = 1$. Set $j: U = X - Y \hookrightarrow X$. For $\mathcal{V} \in MHM(X)$ we have an exact sequence:

$$0 \to i_!(\mathcal{H}^{-1} i^* \mathcal{V}) \to (\mathcal{H}^0 j_!)(j^* \mathcal{V}) \to \mathcal{V} \to i_!(\mathcal{H}^0 i^* \mathcal{V}) \to 0.$$

(Note that $\mathcal{H}^k i^* = 0$ for $k \neq 0, -1$ and $\mathcal{H}^k j_! = 0$ for $k \neq 0$.)

(m14) $(\mathcal{L}_X, W)$ with $\mathrm{Gr}_k^W(\mathcal{L}_X) = 0$ ($k \neq \dim X$) belongs to $MHM(X)$. Hence $(\mathcal{L}(Z, X), W)$ with $\mathrm{Gr}_k^W(\mathcal{L}(Z, X)) = 0$ ($k \neq \dim Z$) belongs to $MHM(X)$. $(\mathcal{L}_X, W)$ and $(\mathcal{L}(Z, X), W)$ will be denoted by $\mathcal{L}_X$ and $\mathcal{L}(Z, X)$ in the following.

(m15) If $f: X \to Y$ is a morphism of non-singular varieties and $\phi_Y(H) \in MHM(Y)$ for $H \in VSHM(Y)^p$, then $\mathcal{H}^j f^*(\phi_Y(H)) = 0$ for $j \neq \dim X$

$-\dim Y$ and $\mathcal{H}^{\dim X-\dim Y} f^*(\phi_Y(H))=\phi_X(f^*(H))$. Especially we have $\mathcal{H}^{\dim X-\dim Y} f^*(\mathcal{L}_Y)=\mathcal{L}_X$.

(m16) Let $X' \xrightarrow{g'} X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y' \xrightarrow{g} Y$ be morphisms of non-singular varieties which form a cartesian diagram.

(a) Assume that f is projective and $\mathcal{V} \in MHM(X)$. If $\mathcal{H}^i g'^*(\mathcal{V})=0$ $(i \neq k)$ and $(\mathcal{H}^i g^*)(\mathcal{H}^j f_!)(\mathcal{V})=0$ $(i \neq k)$ for any j, then $(\mathcal{H}^k g^*)(\mathcal{H}^j f_!)(\mathcal{V})=(\mathcal{H}^j f'_!)(\mathcal{H}^k g'^*)(\mathcal{V})$ for any j.

(b) If g is smooth with relative dimension m, then $(\mathcal{H}^m g^*)(\mathcal{H}^j f_!)=(\mathcal{H}^j f'_!)(\mathcal{H}^m g'^*)$.

(m17) Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of non-singular varieties.

(a) If $\mathcal{V} \in MHM(X)$ satisfies $(\mathcal{H}^i f_!)(\mathcal{V})=0$ for $i \neq k$, then $\mathcal{H}^j(g \circ f)_!(\mathcal{V})=(\mathcal{H}^{j-k} g_!)(\mathcal{H}^k f_!)(\mathcal{V})$.

(b) If g is a closed immersion, $\mathcal{H}^j(g \circ f)_!=g_! \circ (\mathcal{H}^j f_!)$.

(m18) Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of non-singular varieties.

(a) If $\mathcal{V} \in MHM(Z)$ satisfies $(\mathcal{H}^i g^*)(\mathcal{V})=0$ for $i \neq k$, then $\mathcal{H}^j(g \circ f)^*(\mathcal{V})=(\mathcal{H}^{j-k} f^*)(\mathcal{H}^k g^*)(\mathcal{V})$.

(b) If f is smooth with relative dimension m, $\mathcal{H}^j(g \circ f)^*=(\mathcal{H}^m f^*)(\mathcal{H}^{j-m} g^*)$.

Using the terminology of the derived category the properties (m16~18) above can be formulated without assuming vanishing of cohomologies ([Sa 4]). Here we formulate them in a weaker form.

We denote the Grothendieck group of $MHM(X)$ by $KH(X)$. For a morphism $f: X \to Y$ of non-singular varieties, $\mathbb{Z}$-linear maps $f_!: KH(X) \to KH(Y)$ and $f^*: KH(Y) \to KH(X)$ are defined by $f_!([\mathcal{V}])=\sum_j (-1)^j [\mathcal{H}^j f_!(\mathcal{V})]$ and $f^*([\mathcal{V}])=\sum_j (-1)^j [\mathcal{H}^j f^*(\mathcal{V})]$.

(m16′) If g is smooth or f is projective in the cartesian diagram of (m16), then the two maps $g^* \circ f_!$ and $f'_! \circ g'^*$ from $KH(X)$ to $KH(Y')$ coincide.

(m17′) Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of non-singular varieties. Then the two maps $g_! \circ f_!$ and $(g \circ f)_!$ from $KH(X)$ to $KH(Z)$ coincide.

(m18′) Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of non-singular varieties. Then the two maps $f^* \circ g^*$ and $(g \circ f)^*$ from $KH(Z)$ to $KH(X)$

coincide.

Let pt be the algebraic variety consisting of a single point. Set $R=KH(pt)$. R is endowed with a ring structure via the tensor product $\boxtimes$ (commutative with unit $[\mathcal{L}_{pt}]$) and the Laurent polynomial ring $\boldsymbol{Z}[q, q^{-1}]$ is a subring of R $(q^i \leftrightarrow [\mathcal{L}_{pt}(-i)])$. We have an R-module structure on $KH(X)$ via the tensor product $\boxtimes$ and f^* and $f_!$ are R-homomorphisms.

1.5. Hodge modules with group actions

Let G be an algebraic group over $\boldsymbol{C}$ acting on a non-singular algebraic variety X. Let $m: G\times G\to G$ and $\sigma: G\times X\to X$ be the product in G and the action of G on X, respectively.

Definition. A Hodge module on X with G-action is a pair $(\mathcal{V}, \varphi)$, where $\mathcal{V}$ is an object of $MHM(X)$ and $\varphi: (\mathcal{H}^{\dim G}\sigma^*)(\mathcal{V})\to(\mathcal{H}^{\dim G}p_2^*)(\mathcal{V})$ is an isomorphism in $MHM(G\times X)$ such that the two morphisms $((\mathcal{H}^{\dim G}p_{23}^*)\varphi)\circ(\mathcal{H}^{\dim G}(1_G\times\sigma)^*\varphi)$ and $(\mathcal{H}^{\dim G}(m\times 1_X)^*\varphi)$ from $\mathcal{H}^{2\dim G}(\sigma\circ(1_G\times\sigma))^*(\mathcal{V})=\mathcal{H}^{2\dim G}(\sigma\circ(m\times 1_X))^*(\mathcal{V})$ to $\mathcal{H}^{2\dim G}(p_2\circ p_{23})^*(\mathcal{V})=\mathcal{H}^{2\dim G}(p_2\circ(m\times 1_X))^*(\mathcal{V})$ in $MHM(G\times G\times X)$ coincide. Here $p_2: G\times X\to X$ and $p_{23}: G\times G\times X\to G\times X$ are projections.

The above formalism is due to Mumford. We define a category $MHM(X, G)$ as follows. An object is a Hodge module with G-action. A morphism from $(\mathcal{V}, \varphi)$ to $(\mathcal{V}', \varphi')$ is a morphism $u: \mathcal{V}\to\mathcal{V}'$ in $MHM(X)$ satisfying $((\mathcal{H}^{\dim G}p_2^*)u)\circ\varphi=\varphi'\circ((\mathcal{H}^{\dim G}\sigma^*)u)$. It is easily seen that $MHM(X, G)$ is an abelian category. We denote the Grothendieck group of $MHM(X, G)$ by $KH^G(X)$.

For a G-equivariant morphism $f: X\to Y$ of non-singular varieties we have additive functors:

$$\mathcal{H}^j f_!: MHM(X, G)\to MHM(Y, G), \quad \mathcal{H}^j f^*: MHM(Y, G)\to MHM(X, G),$$

which induce $\boldsymbol{Z}$-linear maps:

$$f_!: KH^G(X)\to KH^G(Y), \quad f^*: KH^G(Y)\to KH^G(X).$$

When X_i $(i=1, 2)$ are non-singular G_i-varieties, we have a bi-exact functor:

$$\boxtimes: MHM(X_1, G_1)\times MHM(X_2, G_2)\to MHM(X_1\times X_2, G_1\times G_2),$$

which induces a $\boldsymbol{Z}$-linear map:

$$\boxtimes : KH^{G_1}(X_1)\underset{\boldsymbol{Z}}{\otimes} KH^{G_2}(X_2) \to KH^{G_1\times G_2}(X_1\times X_2)\,.$$

Especially $KH^G(X)$ is an R $(=KH(pt))$-module. It is seen that $f_!$ and f^* are R-homomorphisms and $\boxtimes$ is R-bilinear.

For a $\boldsymbol{Q}$-local system S on a non-singular variety X we define $H_S=(S, F, W)\in VSHM(X)$ by:

$$F^p=\begin{cases}\mathcal{M}(S) & (p\leqq 0)\\ 0 & (p>0)\end{cases} \quad\text{and}\quad W_p=\begin{cases}(S, F) & (p\geqq 0)\\ 0 & (p<0)\,.\end{cases}$$

If the monodromy representation of S factors through a finite group, then $H_S\in VSHM(X)^p$.

For a homogeneous space X of G we denote by $\mathrm{Loc}(X, G)$ the category of $\boldsymbol{Q}$-local systems on X with G-actions. This category is equivalent to the category of finite dimensional representations over $\boldsymbol{Q}$ of the finite group $G^x/(G^x)^\circ$, where x is a point of X, G^x is its stabilizer in G and $(G^x)^\circ$ is the connected component of G^x containing the identity. For $S\in\mathrm{Loc}(X, G)$, H_S belongs to $VSHM(X)^p$ and is naturally endowed with a G-action.

Lemma 1.1. *Let X be a homogeneous space of G. We assume that irreducible representations of $G^x/(G^x)^\circ$ over $\boldsymbol{Q}$ are absolutely irreducible for some (and hence for any) point x of X.*

(i) *For $S\in\mathrm{Loc}(X, G)$ $\phi_X(H_S)$ belongs to $MHM(X)$ and is naturally endowed with an action of G.*

(ii) *If S and H are simple objects of $\mathrm{Loc}(X, G)$ and $MHM(pt)$ respectively, then $\phi_X(H_S)\boxtimes H$ is a simple object of $MHM(X, G)$.*

(iii) *If $\mathcal{V}$ is an object of $MHM(X, G)$ such that $\mathrm{Gr}_i^W\mathcal{V}=0$ for $i\neq k$, then $\mathcal{V}$ is a direct sum of the simple objects of the type given in* (ii) *with* $\mathrm{Gr}_i^W H=0$ $(i\neq k-\dim X)$.

(iv) *$KH^G(X)$ is a free R-module with basis $\{\phi_X(H_S)|S$ is a simple object of* $\mathrm{Loc}(X, G)\}$.

We prepare a lemma in order to prove Lemma 1.1.

Lemma 1.2. *Let X be a homogeneous space of G. Choose a point x of X and set $pt=\{x\}$, $i: pt\hookrightarrow X$ and $q: X\to pt$. We assume that G^x is connected. Then $\mathcal{H}^{-\dim X}i^*$ gives an equivalence of abelian categories $MHM(X, G)$ and $MHM(pt)$. Its quasi inverse is given by $\mathcal{H}^{\dim X}q^*$.*

Proof. It follows from (p6) that any object of $MHM(X, G)$ lies in the image of Φ_X. Let $VSHM(X, G)^p$ be the category of objects of $VSHM(X)^p$ with G-actions. We have exact fully faithful functors $F_X : MHM(X, G) \to VSHM(X, G)^p$ and $F_{pt} : MHM(pt) \to SHM^p$. Since G^x is connected, it is easily seen that $i^* : VSHM(X, G)^p \to SHM^p$ gives an equivalence of categories with quasi-inverse q^*. By (m15) we see that $i^* \circ F_X = F_{pt} \circ (\mathcal{H}^{-\dim X} i^*)$ and $q^* \circ F_{pt} = F_X \circ (\mathcal{H}^{\dim X} q^*)$. Hence the lemma.

Proof of Lemma 1.1.

Let $f : X_0 = G/(G^x)^\circ \to X = G/G^x$ be the natural map and S_0 the $\boldsymbol{Q}$-local system on X with G-action corresponding to the regular representation of $G^x/(G^x)^\circ$.

(i) Since $\mathcal{L}_{X_0} \in MHM(X_0, G)$, we have $\Phi_X(H_{S_0}) = (\mathcal{H}^0 f_!)(\mathcal{L}_{X_0}) \in MHM(X, G)$. Since any representation (over $\boldsymbol{Q}$) of a finite group is a direct sum of irreducible representations and since any irreducible representation is a direct summand of the regular representation, $\Phi_X(H_S) \in MHM(X, G)$ for any $S \in \mathrm{Loc}(X, G)$ by (m2), and (i) is proved.

(ii) Let $SHM(G^x/(G^x)^\circ)^p$ be the category of polarizable mixed Hodge structures with $G^x/(G^x)^\circ$-actions. As in the proof of Lemma 1.2 we have fully faithful functors:

$$MHM(X, G) \to VSHM(X, G)^p \to SHM(G^x/(G^x)^\circ)^p .$$

Hence it is enough to show that $H \otimes V$ is a simple object of $SHM(G^x/(G^x)^\circ)^p$ if H is a simple object of SHM^p and V is an irreducible $G^x/(G^x)^\circ$-module over $\boldsymbol{Q}$. This follows from our assumption on $G^x/(G^x)^\circ$.

(iii) By Lemma 1.2 there exists an object H of $MHM(pt)$ such that $(\mathcal{H}^0 f^*)(\mathcal{V}) = \mathcal{L}_{X_0} \boxtimes H$ with $\mathrm{Gr}_i^W H = 0$ $(i \neq k - \dim X)$. Since $SH(n)^p$ is a semisimple category, H is a direct sum of simple objects by (m2). Therefore the assertion follows from the fact that $\mathcal{V}$ is a direct summand of $(\mathcal{H}^0 f_!)(\mathcal{H}^0 f^*)(\mathcal{V}) = \Phi_X(H_{S_0}) \boxtimes H$.

(iv) This follows from (ii) and (iii).

Proposition 1.3. *Let X be a non-singular G-variety and Y a G-orbit containing $x \in X$. Set $\partial Y = \bar{Y} - Y$ and $i : Y \hookrightarrow X - \partial Y$. We assume that any irreducible representation of $G^x/(G^x)^\circ$ over $\boldsymbol{Q}$ is absolutely irreducible. For simple objects H and S of $MHM(pt)$ and $Loc(Y, G)$ respectively, there exists a unique simple object $\mathcal{V}$ of $MHM(X, G)$ such that $\mathcal{V} | X - \partial Y = i_!(\phi_Y(H_S) \boxtimes H)$.*

Proof. Since H is simple, we have $\mathrm{Gr}_j^W H = 0$ $(j \neq k)$ for some k. If such $\mathcal{V}$ exists, the underlying object $\mathcal{V}_1$ of $MHM(X)$ satisfies the following condition:

(P) $\mathcal{V}_1|X-\partial Y=i_!(\Phi_Y(H_S)\boxtimes H)$, $\mathrm{Gr}_j^W(\mathcal{V}_1)=0$ $(j\neq n=k+\dim Y)$ and $\mathrm{Gr}_n^W(\mathcal{V}_1)$ is an object of $MH_{\bar{Y}}(X, n)^p$.

If there exists $\mathcal{V}_1\in MHM(X)$ satisfying (P), the action of G on $\Phi_Y(H_S)\boxtimes H$ uniquely extends to that of G on $\mathcal{V}_1$ by (p7) and the resulting object of $MHM(X, G)$ is simple by (m12). Hence it is enough to prove the existence of $\mathcal{V}_1\in MHM(X)$ satisfying (P). This follows from the desingularization theorem of Hironaka and the arguments as in the proof of Lemma 1.1.

Notation. We denote $\mathcal{V}$ in Proposition 1.3 by $\mathcal{L}(\bar{Y}, X, S, H)$. Set $\mathcal{L}(\bar{Y}, X, S)=\mathcal{L}(\bar{Y}, X, S, \mathcal{L}_{pt})$.

Lemma 1.4. *We have* $\mathcal{L}(\bar{Y}, X, S, H)=\mathcal{L}(\bar{Y}, X, S)\boxtimes H$.

Proof. It is easy to see that $\mathcal{L}(\bar{Y}, X, S)\boxtimes H$ satisfies the condition (P) in the proof of Proposition 1.3.

Proposition 1.5. *Let X be a non-singular G-variety with finitely many orbits. We assume that irreducible representations of $G^x/(G^x)^\circ$ over $\mathbb{Q}$ are absolutely irreducible for any point x of X.*

(i) *If $\mathcal{V}$ is an object of $MHM(X, G)$ such that $\mathrm{Gr}_i^W\mathcal{V}=0$ for $i\neq k$, then $\mathcal{V}$ is a direct sum of the simple objects of the type $\mathcal{L}(\bar{Y}, X, S, H)$, where Y is a G-orbit, S and H are simple objects of $Loc(Y, G)$ and $MHM(pt)$ respectively with $\mathrm{Gr}_i^W H=0$ $(i\neq k-\dim Y)$.*

(ii) *$KH^G(X)$ is a free R-module with basis $\{[\mathcal{L}(\bar{Y}, X, S)]|(Y, S)\}$, where (Y, S) is running through pairs of a G-orbit Y and a simple object S of* $\mathrm{Loc}(Y, G)$.

Proof. (i) Since $\mathrm{Gr}_k^W(\mathcal{V})$ is an object of $MH(X, k)^p$, we have a direct sum decomposition $\mathrm{Gr}_k^W(\mathcal{V})=\bigoplus_Y \mathcal{V}_Y$ (Y is a G-orbit and $\mathcal{V}_Y$ is an object of $MH_{\bar{Y}}(X, k)^p$) in $MH(X, k)^p$. Then each $\mathcal{V}_Y$ (with $\mathrm{Gr}_j^W\mathcal{V}_Y=0$ for $j\neq k$) is an object of $MHM(X, G)$. Hence we may assume that $\mathrm{Gr}_k^W(\mathcal{V})$ belongs to $MH_{\bar{Y}}(X, k)^p$. Set $\partial Y=\bar{Y}-Y$ and $i: Y\hookrightarrow X-\partial Y$. By (m12) there exists an object $\mathcal{V}_1$ of $MHM(Y, G)$ such that $\mathcal{V}|X-\partial Y=i_!(\mathcal{V}_1)$ and $\mathrm{Gr}_i^W(\mathcal{V}_1)=0$ $(i\neq k)$. Hence the assertion follows from Lemma 1.1, Proposition 1.3 and (p7).

(ii) This follows from (i) and Lemma 1.4.

§ 2. Equivariant K-theory and Hodge Modules with Group Actions

2.1. Equivariant K-theory (see [Th])

Let G be an algebraic group and $i: X \hookrightarrow Y$ a G-equivariant closed immersion of G-varieties (not necessarily irreducible nor non-singular). We denote the abelian category consisting of coherent O_Y-modules with G-actions supported in X by $C^G(X, Y)$. Let $K^G(X, Y)$ be its Grothendieck group. When M is a bounded complex of O_Y-modules with G-action so that each $\mathscr{H}^i(M)$ belongs to $C^G(X, Y)$, we set $[M]=\sum_i(-1)^i[\mathscr{H}^i(M)] \in K^G(X, Y)$. Since the exact functor $i_*: C^G(X, X) \to C^G(X, Y)$ induces an isomorphism $i_*: K^G(X, X) \to K^G(X, Y)$, $K^G(X, Y)$ does not depend on the choice of the ambient space Y. When we do not have to specify Y we denote it by $K^G(X)$. The abelian group $R_G=K^G(pt)$ is endowed with a ring structure and $K^G(X)$ is an R_G-module via the tensor product (R_G is called the representation ring of G.).

Let Y_i $(i=1, 2)$ be G-varieties, X_i G-stable closed subvarieties of Y_i and $f: Y_1 \to Y_2$ be a G-equivariant morphism. When $f(X_1)$ is contained in X_2 and $X_1 \to X_2$ is proper, an R_G-linear map:

$$f_*: K^G(X_1, Y_1) \to K^G(X_2, Y_2)$$

is defined by $f_*([M])=[Rf_*(M)]$. When $f^{-1}(X_2)$ is contained in X_1 and Y_2 is non-singular, an R_G-linear map:

$$f^*: K^G(X_2, Y_2) \to K^G(X_1, Y_1)$$

is defined by $f^*([M])=[Lf^*(M)]$. Let X_i $(i=1, 2, 3)$ be G-stable closed subvarieties of a non-singular G-variety Y so that $X_1 \cap X_2 \subset X_3$. Then

$$\otimes: K^G(X_1, Y) \underset{R_G}{\otimes} K^G(X_2, Y) \to K^G(X_3, Y)$$

is defined by $[M_1] \otimes [M_2]=[M_1 \overset{L}{\underset{O_Y}{\otimes}} M_2]$. Note that f_* does not depend on the choice of the ambient space while f^* and $\otimes$ do.

The following well-known facts will be used frequently later.

Lemma 2.1. (projection formula). *Let $f: Y_1 \to Y_2$ be a G-equivariant morphism of non-singular G-varieties. When M_i $(i=1, 2)$ are coherent O_{Y_i}-modules with G-actions so that* $\mathrm{Supp}(M_1) \to Y_2$ *is proper, we have :*

$$\mathbb{R}f_*(M_1 \overset{L}{\underset{O_{Y_1}}{\otimes}} \mathbb{L}f^*(M_2)) = \mathbb{R}f_*(M_1) \overset{L}{\underset{O_{Y_2}}{\otimes}} M_2 .$$

Lemma 2.2. (smooth base change theorem). *Let $f: Y_1 \to Y_2$ and $g_2: Y_2' \to Y_2$ be G-equivariant morphisms of non-singular G-varieties. Set $Y_1' = Y_1 \underset{Y_2}{\times} Y_2'$ and let $g_1: Y_1' \to Y_1$ and $f': Y_1' \to Y_2'$ be natural maps. We assume that g_2 is smooth. When M is a coherent C_{Y_1}-module with G-action so that* $\mathrm{Supp}M \to Y_2$ *is proper, we have :*

$$\mathbb{L}g_2^* \circ \mathbb{R}f_*(M) = \mathbb{R}f_*' \circ \mathbb{L}g_1^*(M) .$$

2.2. Coherent sheaves on the cotangent bundles associated to filtered D-modules

For a non-singular variety X over $\mathbb{C}$, we denote the cotangent bundle by $p: T^*X \to X$. The O_X-algebra $\mathrm{Gr}^F D_X$ is naturally identified with $p_* O_{T^*X}$. For an object $(\mathcal{M}, F)$ of $MF_{rh}(D_X)$ we have a coherent O_{T^*X}-module :

$$\mathrm{gr}(\mathcal{M}, F) := O_{T^*X} \underset{p^{-1}\mathrm{Gr}^F D_X}{\otimes} p^{-1}(\mathrm{Gr}^F \mathcal{M}) .$$

The group $\mathbb{C}^*$ acts on T^*X by $z \cdot (x, \xi) = (x, z\xi)$ (x is a coordinate of X and ξ is a coordinate of fibres.). We have a natural $\mathbb{C}^*$-action on $\mathrm{gr}(\mathcal{M}, F)$ by :

$$z \cdot (f(x, \xi) \otimes m_i) = f(x, z^{-1}\xi) \otimes z^{-i} m_i$$
$$(z \in \mathbb{C}^*, f(x, \xi) \in O_{T^*X} , \quad m_i \in \mathrm{Gr}_i^F \mathcal{M}) .$$

For a morphism $f: X \to Y$ of non-singular varieties, set $\Omega_{X/Y} = \Omega_X \underset{O_X}{\otimes} f^*(\Omega_Y^{-1})$. Consider the following commutative diagram :

$$\begin{array}{ccccc} T^*Y & \xleftarrow{\varpi_f} & T^*Y \underset{Y}{\times} X & \xrightarrow{\rho_f} & T^*X \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_{X/Y}} & \swarrow{\scriptstyle p_X} & \\ Y & \xleftarrow{f} & X & & \end{array}$$

Here morphisms are the natural ones.

Lemma 2.3. *Let $f: X \to Y$ be a projective morphism of non-singular varieties and $(\mathcal{M}, F)$ an object of $MF_{rh}(D_X)$. If $f_*(\mathcal{M}, F)$ is strict, we have :*

$$\mathrm{gr}(\mathcal{H}^i f_*(\mathcal{M}, F)) = \mathcal{H}^i(\boldsymbol{R}\bar{\omega}_{f*}(\boldsymbol{L}\rho_f{}^*(\mathrm{gr}(\mathcal{M}, F) \underset{O_{T^*X}}{\otimes} p_X{}^*\Omega_{X/Y}))) \underset{\boldsymbol{C}}{\otimes} V_{\dim X - \dim Y},$$

as a coherent O_{T^*X}-module with $\boldsymbol{C}^*$-action, where V_i denotes the one-dimensional $\boldsymbol{C}^*$-module such that the action of $z \in \boldsymbol{C}^*$ is given by the multiplication of z^i.

Proof. We first consider the case when f is a projection. By the assumption $\mathcal{H}^i(\boldsymbol{R}f_*(F_p(DR_{X/Y}(\mathcal{M})))) \to \mathcal{H}^i(\boldsymbol{R}f_*(DR_{X/Y}(\mathcal{M})))$ is injective for each i and p and we have $\mathcal{H}^i f_*(\mathcal{M}, F) = (\mathcal{H}^i(\boldsymbol{R}f_*(DR_{X/Y}(\mathcal{M}))), F)$ with $F_p(\mathcal{H}^i(\boldsymbol{R}f_*(DR_{X/Y}(\mathcal{M})))) = \mathcal{H}^i(\boldsymbol{R}f_*(F_p(DR_{X/Y}(\mathcal{M}))))$ (see Section 1.2). Apply $\boldsymbol{R}f_*$ to the distinguished triangle:

$$F_{p-1}(DR_{X/Y}(\mathcal{M})) \to F_p(DR_{X/Y}(\mathcal{M})) \to \mathrm{Gr}_p{}^F(DR_{X/Y}(\mathcal{M})) \to F_{p-1}(DR_{X/Y}(\mathcal{M}))[1]$$

and consider the long exact sequence of cohomologies. Then we have a short exact sequence:

$$0 \to \mathcal{H}^i(\boldsymbol{R}f_*(F_{p-1}DR_{X/Y}(\mathcal{M}))) \to \mathcal{H}^i(\boldsymbol{R}f_*(F_pDR_{X/Y}(\mathcal{M}))) \to$$
$$\mathcal{H}^i(\boldsymbol{R}f_*(\mathrm{Gr}_p{}^F DR_{X/Y}(\mathcal{M})) \to 0$$

for each i and p. Hence $\mathrm{Gr}^F(\mathcal{H}^i f_*(\mathcal{M}, F)) = \mathcal{H}^i(\boldsymbol{R}f_*(\mathrm{Gr}^F DR_{X/Y}(\mathcal{M})))$.

It is easily seen that the natural actions of O_X and $f^{-1}\,\mathrm{Gr}\,D_Y$ on $\mathrm{Gr}^F(DR_{X/Y}(\mathcal{M}))$ induce an $f^*\mathrm{Gr}D_Y$-module structure on $\mathrm{Gr}^F(DR_{X/Y}(\mathcal{M}))$. By definition we have:

$$\mathrm{Gr}^F(DR_{X/Y}(\mathcal{M})) = \mathrm{Gr}^F D_{Y \leftarrow X} \overset{\boldsymbol{L}}{\underset{\mathrm{Gr}D_X}{\otimes}} \mathrm{Gr}^{F'}\mathcal{M} = f^*\mathrm{Gr}D_Y \overset{\boldsymbol{L}}{\underset{\mathrm{Gr}D_X}{\otimes}} \mathrm{Gr}^{F'}\mathcal{M} \overset{\boldsymbol{L}}{\underset{O_X}{\otimes}} \Omega_{X/Y},$$

where $F' = F[\dim Y - \dim X]$. Set $V = T^*Y \underset{Y}{\times} X$ and $p = p_{X/Y}$ for simplicity. Then we have:

$$\mathrm{gr}(\mathcal{H}^i f_*(\mathcal{M}, F))$$

$$= O_{T^*Y} \underset{p_Y{}^{-1}\mathrm{Gr}D_Y}{\otimes} p_Y{}^{-1}\mathcal{H}^i(\boldsymbol{R}f_*\mathrm{Gr}^F DR_{X/Y}\mathcal{M})$$

$$= \mathcal{H}^i(O_{T^*Y} \underset{p_Y{}^{-1}\mathrm{Gr}D_Y}{\otimes} p_Y{}^{-1}\boldsymbol{R}f_*\mathrm{Gr}^F DR_{X/Y}\mathcal{M})$$

$$= \mathcal{H}^i(O_{T^*Y} \underset{p_Y{}^{-1}\mathrm{Gr}D_Y}{\otimes} \boldsymbol{R}\bar{\omega}_{f*}p^{-1}\mathrm{Gr}^F DR_{X/Y}\mathcal{M})$$

$$= \mathcal{H}^i(\boldsymbol{R}\bar{\omega}_{f*}(O_V \underset{p^{-1}f^*\mathrm{Gr}D_Y}{\otimes} p^{-1}\mathrm{Gr}^F DR_{X/Y}\mathcal{M})$$

$$=\mathcal{H}^{i}(\mathbb{R}\bar{\omega}_{f*}(O_V \overset{L}{\underset{p^{-1}\mathrm{Gr}D_X}{\otimes}} p^{-1}(\mathrm{Gr}^{F'}\mathcal{M}\underset{O_X}{\otimes}\Omega_{X/Y})))$$

$$=\mathcal{H}^{i}(\mathbb{R}\bar{\omega}_{f*}(O_V \overset{L}{\underset{\zeta_f^{-1}O_{T^*X}}{\otimes}} \rho_f^{-1}(O_{T^*X} \underset{p_X^{-1}\mathrm{Gr}D_X}{\otimes} p_X^{-1}\mathrm{Gr}^{F'}\mathcal{M} \overset{L}{\underset{p_X^{-1}O_X}{\otimes}} p_X^{-1}\Omega_{X/Y})))$$

$$=\mathcal{H}^{i}(\mathbb{R}\bar{\omega}_{f*}\mathbb{L}\rho_f^{*}(\mathrm{gr}(\mathcal{M}, F')\underset{O_{T^*X}}{\otimes} p_X^{*}\Omega_{X/Y}))\,.$$

Since $\mathrm{gr}(\mathcal{M}, F[i])=\mathrm{gr}(\mathcal{M}, F)\underset{C}{\otimes} V_{-i}$, the assertion is proved when f is a projection. When f is a closed immersion, our claim is shown by the similar arguments as above. Since any projective morphism is a composit of morphisms of these two types (a closed immersion followed by a projection), our assertion follows from the above two cases.

Remark. Saito informed us that the above Lemma follows directly from [Sa2 ; Section 2.3].

2.3. For a non-singular G-variety X with finitely many G-orbits let Λ $(=\Lambda_{(X,G)})$ be the union of the conormal bundles T_O^*X of G-orbits O. It is a $G\times C^*$-stable closed subvariety of T^*X. For an object $\mathcal{V}=(\mathcal{M}, F, K, W)$ of $MHM(X, G)$ we have an object $\mathrm{gr}\mathcal{V} := \mathrm{gr}(\mathcal{M}, F)$ of $C^{G\times C^*}(\Lambda, T^*X)$. This induces a $\mathbb{Z}$-linear map:

$$\mathrm{gr} : KH^G(X)\to K^{G\times C^*}(\Lambda)=K^{G\times C^*}(\Lambda, T^*X)\,.$$

$K^{G\times C^*}(\Lambda)$ is an $R_{G\times C^*}$-module, hence an R_{C^*}-module. We identify R_{C^*} with $\mathbb{Z}[q, q^{-1}]$ via $[V_i]\leftrightarrow q^i$. On the other hand $KH^G(X)$ is an $R(=KH(pt))$-module, hence a $\mathbb{Z}[q, q^{-1}]$-module (see Section 1.3). It is easily seen from the definition that gr is a homomorphism of $\mathbb{Z}[q, q^{-1}]$-module. The following lemma is clear from Lemma 2.3 (compare with [La]).

Lemma 2.4. *Let $f : X\to Y$ be a projective G-equivariant morphism of non-singular G-varieties with finitely many G-orbits. Then for $u\in KH^G(X)$ we have :*

$$\mathrm{gr}(f_! u)=q^{\dim X-\dim Y}\bar{\omega}_{f*}(\rho_f^*(\mathrm{gr}(u)\otimes p_X^*[\Omega_{X/Y}]))\,.$$

The following is also clear from the definition.

Lemma 2.5. *Let $f : X\to Y$ be a smooth G-equivariant morphism of non-singular G-varieties with finitely many G-orbits. Then for $u\in KH^G(Y)$ we have :*

$$\mathrm{gr}(f^*u)=(-1)^{\dim X-\dim Y}\rho_{f*}(\bar{\omega}_f{}^*(\mathrm{gr}(u)))\,.$$

§ 3. A Realization of Hecke Algebras of Weyl Groups

In Sections 3 to 5 G is a connected reductive algebraic group over $\mathbb{C}$ and X is the flag variety of G.

3.1. It is well-known that the set of G-orbits on $X\times X$ is parametrized by the Weyl group W. In fact if we identify X with the quotient G/B for a fixed Borel subgroup B, $X\times X$ is the disjoint union of the G-orbits Y_w containing (B, wB), where w is running through the elements of W. Moreover we have $\dim Y_w=N+l(w)$ and $\bar{Y}_w\supset\bar{Y}_y$ if and only if $w\geqq y$, where $N=\dim X$, $l(w)$ is the length of w and $\geqq$ is the Bruhat ordering on W. These facts are direct consequences of the corresponding facts concerning B-orbits on X.

Let $i_w: Y_w\to X\times X$ be the natural inclusion. We set:

$$\mathcal{L}_w=\mathcal{L}(\bar{Y}_w, X\times X) \quad \text{and} \quad \mathcal{M}_w=\mathcal{H}^0 i_{w!}(\mathcal{L}_{Y_w})\,.$$

They are objects of $MHM(X\times X, G)$. Note that $\mathcal{H}^j i_{w!}(\mathcal{L}_{Y_w})=0$ for $j\neq 0$ since i_w is an affine morphism. By Proposition 1.5 the R-module $KH^G(X\times X)$ has a free basis $\{[\mathcal{L}_w] | w\in W\}$. Since $[\mathcal{M}_w]$ belongs to $[\mathcal{L}_w]+\sum_{y<w}R[\mathcal{L}_y]$, $\{[\mathcal{M}_w] | w\in W\}$ is also a free basis of $KH^G(X\times X)$.

We define $p_{13}: X\times X\times X\to X\times X$ and $r: X\times X\times X\to X\times X\times X\times X$ by $p_{13}(a, b, c)=(a, c)$ and $r(a, b, c)=(a, b, b, c)$. For $u, v\in KH^G(X\times X)$ set

$$u\cdot v=(-1)^N p_{13!}r^*(u\boxtimes v)\in KH^G(X\times X)\,.$$

It follows from (m16′) that this product satisfies the associativity. For $s\in S=\{\text{simple reflection of } W\}$ let X^s be the generalized flag variety consisting of parabolic subgroups with semisimple rank 1 corresponding to s and $\pi_s: X\to X^s$ the natural morphism.

Lemma 3.1. *For $u\in KH^G(X\times X)$, $s\in S$ and $w\in W$ we have:*

(i) $[\mathcal{L}_e]\cdot u=u\cdot[\mathcal{L}_e]=u$,

(ii) $[\mathcal{L}_s]\cdot u=-(\pi_s\times 1)^*(\pi_s\times 1)_!(u)$,

(ii′) $u\cdot[\mathcal{L}_s]=-(1\times\pi_s)^*(1\times\pi_s)_!(u)$,

(iii) $[\mathcal{M}_s]=[\mathcal{L}_s]+[\mathcal{L}_e]$ *and* $[\mathcal{L}_s]=[\mathcal{M}_s]-[\mathcal{M}_e]$,

(iv) $[\mathcal{L}_s]\cdot[\mathcal{L}_s]=-(q+1)[\mathcal{L}_s]$,

(v) $[\mathcal{M}_s]\cdot[\mathcal{M}_w]=[\mathcal{M}_{sw}]$ *if* $sw>w$,

(v′) $[\mathcal{M}_w]\cdot[\mathcal{M}_s]=[\mathcal{M}_{ws}]$ *if* $ws>w$.

Proof. First note that $\bar{Y}_s$ is non-singular and hence $\mathcal{L}_s=\bar{i}_{s!}(\mathcal{L}_{\bar{Y}^s})$ with $\bar{i}_s: \bar{Y}_s\hookrightarrow X\times X$. Thus (i), (ii), (ii′), (v) (v′) follow from (m16′), and (iii) follows from (m13). (iv) is a consequence of (ii), (m16′) and Example in 1.3.

We see from Lemma 1.1 that $KH^G(Y_w)$ is a free R-module of rank one generated by $[\mathcal{L}_{Y_w}]$. We define an R-linear map:

$$h: KH^G(X\times X)\to R \underset{Z[q,q^{-1}]}{\otimes} H(W)$$

by:

$$h(u)=\sum_{w\in W}(-1)^{l(w)}h_w(u)T_w \quad \text{with} \quad i_w{}^*(u)=h_w(u)[\mathcal{L}_{Y_w}].$$

Proposition 3.2. *h is an isomorphism of R-algebras.*

Proof. We see easily from (m3), (m4), (m12) that $h([\mathcal{M}_w])=(-1)^{l(w)}T_w$. Hence the assertion follows from Lemma 3.1.

Lemma 3.3. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be objects of $MHM(X\times X, G)$.*

(i) $\mathcal{H}^j r^*(\mathcal{V}_1\boxtimes\mathcal{V}_2)=0$ *for* $j\neq -N$.

Hence $[\mathcal{V}_1]\cdot[\mathcal{V}_2]=\sum_j(-1)^j[(\mathcal{H}^j p_{13!})(\mathcal{H}^{-N}r^*)(\mathcal{V}_1\boxtimes\mathcal{V}_2)]$.

(ii) $W_p(\mathcal{H}^{-N}r^*(\mathcal{V}_1\boxtimes\mathcal{V}_2))=(\mathcal{H}^{-N}r^*)(\sum_q W_q(\mathcal{V}_1)\boxtimes w_{p-q+N}(\mathcal{V}_2))$.

Proof. Fix $x_0\in X$ and let U be the unipotent radical of a Borel subgroup which is opposite to the Borel subgroup corresponding to x_0. We define

$$\varphi_1: X\times U\to X\times X,\quad \varphi_2: U\times X\to X\times X,\quad \psi: X\times U\times X\to X\times X\times X,$$

$$k_1: X\to X\times X,\quad k_2: X\to X\times X$$

by

$$\varphi_1(x, u)=(u\cdot x, u\cdot x_0),\quad \varphi_2(u, y)=(u\cdot x_0, u\cdot y),$$

$$\psi(x, u, y)=(u\cdot x, u\cdot x_0, u\cdot y),$$

$$k_1(x)=(x, x_0),\quad k_2(y)=(x_0, y).$$

φ_1, φ_2, ψ are open immersions and k_1, k_2 are closed immersions. Consider the commutative diagram:

$$\begin{array}{ccc}
 & X\times X & \\
 {}^{\varphi_1}\nearrow & & \nwarrow \\
X\times U & \xrightarrow{k} & G\times(X\times X) \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\
X & \xrightarrow{k_1} & X\times X
\end{array} ,$$

where $k(x, u)=(u, x, x_0)$ and $\sigma(g, x, y)=(g\cdot x, g\cdot y)$. For an object $\mathcal{V}$ of $MHM(X\times X, G)$ we have:

$$\begin{aligned}
\mathcal{H}^i\varphi_1{}^*(\mathcal{V}) &= (\mathcal{H}^{i-\dim G}k^*)(\mathcal{H}^{\dim G}\sigma^*)(\mathcal{V}) \\
&\simeq (\mathcal{H}^{i-\dim G}k^*)(\mathcal{H}^{\dim G}p_2{}^*)(\mathcal{V}) \\
&= \mathcal{H}^i(p_2\circ k)^*(\mathcal{V}) \\
&= (\mathcal{H}^N p_1{}^*)(\mathcal{H}^{i-N}k_1{}^*)(\mathcal{V}) \\
&= (\mathcal{H}^{i-N}k_1{}^*)(\mathcal{V})\boxtimes\mathcal{L}_U
\end{aligned}$$

in $MHM(X\times U)$. Since φ_1 is an open immersion, $\mathcal{H}^i\varphi_1{}^*(\mathcal{V})=0$ for $i\neq 0$. Hence $\mathcal{H}^i k_1{}^*(\mathcal{V})=0$ for $i\neq -N$, $\mathcal{H}^{-N}k_1{}^*$ is an exact functor from $MHM(X\times X, G)$ to $MHM(X)$ and $\varphi_1{}^*(\mathcal{V})\simeq(\mathcal{H}^{-N}k_1{}^*(\mathcal{V}))\boxtimes\mathcal{L}_U$. Under this identification we have:

$$\begin{aligned}
(\mathcal{H}^{-N}k_1{}^*)(W_p(\mathcal{V}))\boxtimes\mathcal{L}_U &\simeq \varphi_1{}^*(W_p(\mathcal{V})) \\
&= W_p(\varphi_1{}^*(\mathcal{V})) \\
&\simeq W_p((\mathcal{H}^{-N}k_1{}^*(\mathcal{V}))\boxtimes\mathcal{L}_U) \\
&= W_{p-N}(\mathcal{H}^{-N}k_1{}^*(\mathcal{V}))\boxtimes\mathcal{L}_U ,
\end{aligned}$$

and hence $W_p(\mathcal{H}^{-N}k_1{}^*(\mathcal{V}))=\mathcal{H}^{-N}k_1{}^*(W_{p+N}(\mathcal{V}))$. In consequence we have:

$$\begin{aligned}
&\varphi_1{}^*(\mathcal{V}_1)\simeq(\mathcal{H}^{-N}k_1{}^*(\mathcal{V}_1))\boxtimes\mathcal{L}_U , \\
&W_p(\mathcal{H}^{-N}k_1{}^*(\mathcal{V}_1))=\mathcal{H}^{-N}k_1{}^*(W_{p+N}(\mathcal{V}_1)) .
\end{aligned}$$

Similarly we have:

$$\begin{aligned}
&\varphi_2{}^*(\mathcal{V}_2)\simeq\mathcal{L}_U\boxtimes(\mathcal{H}^{-N}k_2{}^*(\mathcal{V}_2)) , \\
&W_p(\mathcal{H}^{-N}k_2{}^*(\mathcal{V}_2))=\mathcal{H}^{-N}k_2{}^*(W_{p+N}(\mathcal{V}_2)) .
\end{aligned}$$

Let $\Delta: U\to U\times U$ be the diagonal embedding. Since $r\circ\psi=(\varphi_1\times\varphi_2)\circ(1\times\Delta$

$\times 1)$, it is easily seen that:

$$\psi^*(\mathcal{H}^i r^*(\mathcal{V}_1 \boxtimes \mathcal{V}_2))=0 \quad \text{for} \quad i \neq -N,$$

$$\psi^*(\mathcal{H}^{-N} r^*(\mathcal{V}_1 \boxtimes \mathcal{V}_2)) \simeq (\mathcal{H}^{-N} k_1^*(\mathcal{V}_1)) \boxtimes \mathcal{L}_U \boxtimes (\mathcal{H}^{-N} k_2^*(\mathcal{V}_2)),$$

$$\psi^*(\mathcal{H}^{-N} r^*(W_p(\mathcal{V}_1 \boxtimes \mathcal{V}_2)))=\psi^*(\mathcal{H}^{-N} r^*)(\sum_q W_q(\mathcal{V}_1) \boxtimes W_{p-q+N}(\mathcal{V}_2)).$$

Hence the lemma.

3.2. Kazhdan-Lusztig Polynomials and $\mathcal{L}_w$

For each $w \in W$ there exists a unique element C_w'' of $H(W)$ of the form:

$$C_w''=(-1)^{l(w)} \sum_{y \leq w} P_{y,w}(q) T_y=(-q)^{l(w)} \sum_{y \leq w} P_{y,w}(q^{-1}) T_{y^{-1}}^{-1}$$

such that $P_{w,w}(q)=1$ and $P_{y,w}(q)$ is a polynomial in q with degree$\leqq (l(w)-l(y)-1)/2$ for $y<w$([KL1]). For $y<w$ with $l(w)-l(y)$ odd, we denote the coefficient of $q^{(l(w)-l(y)-1)/2}$ in $P_{y,w}(q)$ by $\mu(y, w)$. The following two lemmas are known.

Lemma 3.4 ([KL1]). *Let $s \in S$ and $w \in W$.*

(i) *If $sw>w$, we have*:

$$C_s'' C_w''=C_{sw}''+\sum_z \mu(z, w) q^{(l(w)-l(z)+1)/2} C_z'',$$

where z is running through elements of W so that $z<w$, $sz<z$ and $l(w)-l(z)$ is odd.

(ii) *If $sw<w$, we have*:

$$C_s'' C_w''=-(q+1) C_w''.$$

Lemma 3.5 ([KL2], [Sp]). *Let $s \in S$ and $w, y \in W$.*

(i) *If $j+l(w)-l(y)$ is odd, then ${}^p\mathcal{H}^j i_y^*(\mathcal{IC}(\bar{Y}_w))=0$.*

(ii) *If $sw>w$, we have*:

$${}^p\mathcal{H}^1(\pi_s \times 1)^* {}^p\mathcal{H}^j(\pi_s \times 1)_!(\mathcal{IC}(\bar{Y}_w))$$

$$=\begin{cases} 0 & (j \neq 0) \\ \mathcal{IC}(\bar{Y}_{sw}) \oplus (\bigoplus_z \mathcal{IC}(\bar{Y}_z)^{\oplus \mu(z,w)}) & (j=0), \end{cases}$$

where z is running through elements of W so that $z<w$, $sz<z$ and $l(w)-l(z)$ is odd.

(iii) *If* $sw<w$, *we have*:

$$ {}^{p}\mathcal{H}^{1}(\pi_{s}\times 1)^{*p}\mathcal{H}^{j}(\pi_{s}\times 1)_{!}(\mathcal{IC}(\bar{Y}_{w}))=\begin{cases}0 & (j\neq\pm 1)\\ \mathcal{IC}(\bar{Y}_{w}) & (j=\pm 1)\,.\end{cases} $$

Lemma 3.6. *Assume* $sw>w$ *for* $s\in S$ *and* $w\in W$.
(i) $\mathcal{H}^{1}(\pi_{s}\times 1)^{*}\mathcal{H}^{j}(\pi_{s}\times 1)_{!}(\mathcal{L}_{w})=0$ *for* $j\neq 0$.
(ii) $\mathrm{Gr}_{k}^{W}(\mathcal{H}^{1}(\pi_{s}\times 1)^{*}\mathcal{H}^{0}(\pi_{s}\times 1)_{!}(\mathcal{L}_{w}))=0$ *for* $k\neq N+l(w)+1$.
(iii) $\mathcal{L}_{sw}$ *is a direct summand of* $\mathcal{H}^{1}(\pi_{s}\times 1)^{*}\mathcal{H}^{0}(\pi_{s}\times 1)_{!}(\mathcal{L}_{w})$.

Proof. (i) is clear from Lemma 3.5. (ii) follows from the fact that $\pi_{s}\times 1$ is projective and smooth with relative dimension 1.

(iii) It follows from (m16) that $\mathcal{L}_{sw}$ and $\mathcal{H}^{1}(\pi_{s}\times 1)^{*}\mathcal{H}^{0}(\pi_{s}\times 1)_{!}(\mathcal{L}_{w})$ coincide on $(X\times X)-(\bar{Y}_{sw}-Y_{sw})$. Hence the assertion follows from (ii).

For a non-singular G-variety V we denote by $KH^{G}(V)^{+}$ (resp. R^{+}) the set of the elements in $KH^{G}(V)$ (resp. R) represented by objects of $MHM(V, G)$ (resp. $MHM(pt)$). Let $\{H_{\gamma}(i)|\gamma\in\Gamma, i\in\mathbb{Z}\}$ be the set of isomorphism classes of simple objects of $MHM(pt)$. For each $\gamma\in\Gamma$ an integer n_{γ} is determined by $\mathrm{Gr}_{i}^{W}(H_{\gamma})=0$ for $i\neq n_{\gamma}$. We may assume that $H_{\gamma_{0}}=\mathcal{L}_{pt}$. Then we have:

$$ R^{+}=\bigoplus_{\substack{\gamma\in\Gamma\\ i\in\mathbb{Z}}}\mathbb{Z}_{\geq 0}[H_{\gamma}(i)]=\bigoplus_{\gamma\in\Gamma}\mathbb{Z}_{\geq 0}[q, q^{-1}][H_{\gamma}]\,, $$

$$ KH^{G}(X\times X)^{+}=\bigoplus_{w\in W}R^{+}[\mathcal{L}_{w}]\,, $$

$$ KH^{G}(Y_{y})^{+}=R^{+}[\mathcal{L}_{Y_{y}}]\,. $$

Proposition 3.7. *Let* $s\in S$ *and* $y, w\in W$.
(i) $h_{y}([\mathcal{L}_{w}])\in(-1)^{l(w)-l(y)}\mathbb{Z}_{\geq 0}[q, q^{-1}]$.
(ii) *If* $sw>w$, *we have*:

$$ \mathcal{H}^{1}(\pi_{s}\times 1)^{*}\mathcal{H}^{0}(\pi_{s}\times 1)_{!}(\mathcal{L}_{w}) $$

$$ =\mathcal{L}_{sw}\oplus(\bigoplus_{z}\mathcal{L}_{z}(-(l(w)-l(z)+1)/2)^{\oplus\mu(z,w)})\,, $$

where z *is running through elements of* W *so that* $z<w$, $sz<z$ *and* $l(w)-l(z)$ *is odd.*

(iii) *If* $sw<w$, *we have*:

$$\mathcal{H}^1(\pi_s\times 1)^*\mathcal{H}^j(\pi_s\times 1)_!(\mathcal{L}_w)=\begin{cases}\mathcal{L}_w(1) & (j=-1)\\ \mathcal{L}_w & (j=1)\\ 0 & (j\neq\pm 1)\,.\end{cases}$$

(iv) $h([\mathcal{L}_w])=C_w''$.

Proof. We first prove (i) and (ii). Assume $sw>w$ for $w\in W$ and $s\in S$. By induction we have only to show the statement:

(*) $h_y([\mathcal{L}_{sw}])\in(-1)^{l(w)-l(y)+1}\mathbb{Z}_{\geq 0}[q, q^{-1}]$ for any $y\in W$ and $\mathcal{H}^1(\pi_s\times 1)^*\mathcal{H}^0(\pi_s\times 1)_!(\mathcal{L}_w)=\mathcal{L}_{sw}\oplus(\bigoplus_z \mathcal{L}_z(-(l(w)-l(z)+1)/2)^{\oplus\mu(z,w)})$, where z is running through elements of W so that $z<w$, $sz<z$ and $l(w)-l(z)$ is odd,

assuming:

(**) $h_y([\mathcal{L}_z])\in(-1)^{l(z)-l(y)}\mathbb{Z}_{\geq 0}[q, q^{-1}]$ for any y, $z\in W$ with $l(z)\leqq l(w)$.

Set $\mathcal{V}=\mathcal{H}^1(\pi_s\times 1)^*\mathcal{H}^0(\pi_s\times 1)_!(\mathcal{L}_w)$. It follows from Lemma 3.6 (ii), (iii) that we have $\mathcal{V}=\mathcal{L}_{sw}\oplus(\bigoplus_{(z,\gamma,i)\in J}(H_\gamma(i)^{\oplus m_{z,\gamma,i}})\boxtimes\mathcal{L}_z)$ for some integers $m_{z,\gamma,i}$, where $J=\{(z,\gamma,i)\in W\times\Gamma\times\mathbb{Z}\,|\,z<w,\ sz<z,\ l(w)-l(z)\equiv 1(\mathrm{mod}\ 2),\ n_\gamma=l(w)-l(z)+2i+1\}$. Since $h_y([\mathcal{L}_{sw}])=(-1)^{l(w)-l(y)+1}\sum_{\gamma\in\Gamma}f_\gamma(q)[H_\gamma]$ for some $f_\gamma(q)\in\mathbb{Z}_{\geq 0}[q, q^{-1}]$ by Lemma 3.5(i), we have:

$$(-1)^{l(w)-l(y)+1}h_y([\mathcal{V}])=\sum_{\gamma\in\Gamma}k_\gamma(q)[H_\gamma]$$

with

$$k_\gamma(q)=f_\gamma(q)+\sum_{(z,\gamma,i)\in J}m_{z,\gamma,i}q^{-i}((-1)^{l(z)-l(y)}h_y([\mathcal{L}_z]))\,.$$

On the other hand we have $h([\mathcal{V}])=h([\mathcal{L}_s]\cdot[\mathcal{L}_w])=h([\mathcal{L}_s])h([\mathcal{L}_w])\in H(W)$ and hence $k_\gamma(q)=0$ for $\gamma\neq\gamma_0$. Since $f_\gamma(q)$, $(-1)^{l(z)-l(y)}h_y([\mathcal{L}_z])\in\mathbb{Z}_{\geq 0}[q, q^{-1}]$ and $m_{z,\gamma,i}\in\mathbb{Z}_{\geq 0}$, we have $f_\gamma(q)=0$ for $\gamma\neq\gamma_0$ and $m_{z,\gamma,i}=0$ for $(z,\gamma,i)\in J$ with $\gamma\neq\gamma_0$. Thus our assertion follows from Lemma 3.5(ii) and Lemma 3.6(ii).

(iv) This is easily proved by induction on $l(w)$ in view of (ii) and Lemma 3.4(i).

(iii) Since $\mathrm{Gr}_k^W(\mathcal{H}^1(\pi_s\times 1)^*\mathcal{H}^j(\pi_s\times 1)_!(\mathcal{L}_w))=0$ for $k\neq N+l(w)+j+1$, the assertion follows from (iv) and Lemma 3.5 (ii).

Definition. Let $\mathcal{A}$ be the full subcategory of $MHM(X\times X, G)$ consisting of $\mathcal{V}\in MHM(X\times X, G)$ so that $\mathrm{Gr}_k^W\mathcal{V}$ is a direct sum of the simple objects of the form $\mathcal{L}_w(n)$ with $l(w)+N-2n=k$.

It is easily seen that $\mathcal{A}$ is an abelian category.

Lemma 3.8. *For* $\mathcal{V}\in MHM(X\times X, G)$, $h([\mathcal{V}])\in H(W)$ *if and only if* $\mathcal{V}\in\mathcal{A}$. *Especially, we have* $\mathcal{M}_w\in\mathcal{A}$ *for any* $w\in W$.

Proof. If $[\mathcal{V}]=\sum_{\substack{\gamma\in\Gamma\\ w\in W}} f_{\gamma,w}(q)[H_\gamma\boxtimes\mathcal{L}_w]$ $(f_{\gamma,w}(q)\in\boldsymbol{Z}_{\geq 0}[q, q^{-1}])$ for an object $\mathcal{V}$ of $MHM(X\times X, G)$, then $h([\mathcal{V}])=\sum_{\substack{\gamma\in\Gamma\\ w\in W}} f_{\gamma,w}(q)[H_\gamma]C_w''$. Thus $h([\mathcal{V}])\in H(W)$ if and only if composition factors of $\mathcal{V}$ are of the form $\mathcal{L}_w(n)$ with $w\in W$ and $n\in\boldsymbol{Z}$. Hence the lemma.

Proposition 3.9. $(\mathcal{H}^j p_{13!})(\mathcal{H}^{-N}\gamma^*)(\mathcal{V}_1\boxtimes\mathcal{V}_2)\in\mathcal{A}$ *for any* j *and* $\mathcal{V}_1$, $\mathcal{V}_2\in\mathcal{A}$.

Proof. By Lemma 3.3(i) we may assume that $\mathrm{Gr}_k^W\mathcal{V}_i=0$ for $k\neq n_i$ $(i=1, 2)$. Then by Lemma 3.3 (ii) we have $\mathrm{Gr}_k^W((\mathcal{H}^j p_{13!})(\mathcal{H}^{-N}\gamma^*)(\mathcal{V}_1\boxtimes\mathcal{V}_2))=0$ for $k\neq n+j$ with $n=n_1+n_2-N$. Hence we have:

$$\begin{aligned}[(\mathcal{H}^j p_{13!})(\mathcal{H}^{-N}\gamma^*)(\mathcal{V}_1\boxtimes\mathcal{V}_2)]&=\sum_{(w,\gamma,i)\in J_j} m_{w,\gamma,i}^j q^i[\mathcal{L}_w\boxtimes H_\gamma]\\ &=\sum_{(w,\gamma,i)\in J_j} m_{w,\gamma,i}^j q^i[H_\gamma]C_w'',\end{aligned}$$

where $J_j=\{(w,\gamma,i)\in W\times\Gamma\times\boldsymbol{Z}\,|\,N+l(w)+n_\gamma+2i=n+j\}$ and $m_{w,\gamma,i}^j\in\boldsymbol{Z}_{\geq 0}$. On the other hand we have:

$$\sum_j(-1)^j[(\mathcal{H}^j p_{13!})(\mathcal{H}^{-N}\gamma^*)(\mathcal{V}_1\boxtimes\mathcal{V}_2)]=[\mathcal{V}_1]\cdot[\mathcal{V}_2]\in H(W)$$

by Lemma 3.8. Hence $m_{w,\gamma,i}^j=0$ for $\gamma\neq\gamma_0$ and the assertion is proved.

In consequence we have the following.

Theorem A. (i) *The Grothendieck group* $K(\mathcal{A})$ *of the abelian category* $\mathcal{A}$ *is endowed with a* $\boldsymbol{Z}[q, q^{-1}]$*-algebra structure by*:

$$[\mathcal{V}_1]\cdot[\mathcal{V}_2]=\sum_j(-1)^j[(\mathcal{H}^j p_{13!})(\mathcal{H}^{-N}\gamma^*)(\mathcal{V}_1\boxtimes\mathcal{V}_2)]$$

and

$$q^n[\mathcal{V}]=[\mathcal{V}(-n)].$$

(ii) *$K(\mathcal{A})$ is isomorphic to $H(W)$ as a $\boldsymbol{Z}[q, q^{-1}]$-algebra via the correspondence :*

$$[\mathcal{M}_w] \leftrightarrow (-1)^{l(w)} T_w \quad and \quad [\mathcal{L}_w] \leftrightarrow (-1)^{l(w)} \sum_{y \leqq w} P_{y,w}(q) T_y .$$

3.3. Let K be a closed subgroup of G which is either

(a) a Borel subgroup of G,

or

(b) a subgroup of G^ϑ containing $(G^\vartheta)^0$, where ϑ is an involutive automorphism of G.

Then it is known that the number of K-orbits on X is finite and for any $x \in X$ the component gvoup $K^x/(K^x)^0$ of the stabilizer K^x is isomovphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^N$ for some $N \geqq 0$ (see [Ma],[LV]).

For a K-orbit O on X and a simple object S of $\mathrm{Loc}(O, K)$, we set :

$$\mathcal{L}(O, S) = \mathcal{L}(\bar{O}, X, S) \quad \text{and} \quad \mathcal{M}(O, S) = \mathcal{H}^0 i_!(\Phi_O(H_S)) ,$$

where $i: O \to X$ is the natural inclusion (see Section 1.5). Let $\mathcal{A}^K$ be the full subcategory of $MHM(X, K)$ consisting of $\mathcal{V} \in MHM(X, K)$ such that for any $k \in \boldsymbol{Z}$ $\mathrm{Gr}_k^W(\mathcal{V})$ is a direct sum of the objects of the form $\mathcal{L}(O, S)(n)$ with $k = \dim O - 2n$. We define $p_1 : X \times X \to X$ and $q : X \times X \to X \times X \times X$ by $p_1(a, b) = a$ and $q(a, b) = (a, b, b)$.

Theorem A′. (i) $\mathcal{M}(O, S) \in \mathcal{A}^K$.

(ii) *Both of $\{[\mathcal{L}(O, S)] | (O, S)\}$ and $\{[\mathcal{M}(O, S)] | (O, S)\}$ are bases of $K(\mathcal{A}^K)$ over $\boldsymbol{Z}[q, q^{-1}]$.*

(iii) *For $\mathcal{V} \in \mathcal{A}$ and $\mathcal{N} \in \mathcal{A}^K$ we have $(\mathcal{H}^j q^*)(\mathcal{V} \boxtimes \mathcal{N}) = 0$ for $j \neq -N$ and $(\mathcal{H}^i p_{1!})(\mathcal{H}^{-N} q^*)(\mathcal{V} \boxtimes \mathcal{N}) \in \mathcal{A}^K$ for any i.*

(iv) *An action of the $\boldsymbol{Z}[q, q^{-1}]$-algebra $K(\mathcal{A})$ on $K(\mathcal{A}^K)$ is defined by :*

$$[\mathcal{V}] \cdot [\mathcal{N}] = \sum_j (-1)^j [(\mathcal{H}^j p_{1!})(\mathcal{H}^{-N} q^*)(\mathcal{V} \boxtimes \mathcal{N})] .$$

Hence $K(\mathcal{A}^K)$ is an $H(W)$-module.

(v) *When K is of type* (a) *(hence a Borel subgroup B), $K(\mathcal{A}^B)$ is isomorphic to $K(\mathcal{A})$ as a left $H(W)$-module via the correspondence :*

$$[\mathcal{L}(X_w, \boldsymbol{Q})] \leftrightarrow [\mathcal{L}_{w^{-1}}] \quad and \quad [\mathcal{M}(X_w, \boldsymbol{Q})] \leftrightarrow [\mathcal{M}_{w^{-1}}] .$$

Here X_w is the Schubert cell BwB/B.

(vi) *When K is of type* (b), *let M be the $H(W)$-module constructed in* [LV]. *It has two free bases $\{\delta|\delta\in\mathcal{D}\}$ and $\{C_\delta|\delta\in\mathcal{D}\}$ over $\mathbb{Z}[q, q^{-1}]$, where $\mathcal{D}$ is the set of the pairs (O, S) of K-orbits O and simple objects S of* Loc(O, K). *Then $K(\mathcal{A}^K)$ is isomorphic to M as an $H(W)$-module via the correspondence :*

$$[\mathcal{M}(O, S)]\leftrightarrow(-1)^{\dim O}\delta \quad \textit{and} \quad [\mathcal{L}(O, S)]\leftrightarrow(-1)^{\dim O}C_\delta$$

with $\delta=(O, S)$.

The proof is similar to that of Theorem A.

§ 4. Equivariant K-Theory and Hecke Algebras of Affine Weyl Groups

4.1. Hecke algebras of affine Weyl groups

Let B be a Borel subgroup of G and T a maximal torus of G contained in B. We choose an ordering on the root system so that the weights of Lie(G)/Lie(B) are positive roots. The Weyl group W $(=N_G(T)/T)$ acts naturally on the weight lattice P $(=Hom(B, \mathbb{C}^*)=Hom(T, \mathbb{C}^*))$. We denote the semidirect product $W\propto P$ by W_a and call it the affine Weyl group of G.

When G is an adjoint group, W_a is a Coxeter group and the Hecke algebra $H(W_a)$ is defined. Besides the usual Iwahori-Matsumoto relation ([IM]), there is another presentation of $H(W_a)$ due to Bernstein. Let us recall Bernstein's description of $H(W_a)$. (It is also defined for general G.) Let α_s be the simple root corresponding to $s\in S$. The Hecke algebra $H(W_a)$ is a $\mathbb{Z}[q, q^{-1}]$-algebra which satisfies the following conditions $(h1)$ $\sim(h3)$.

$(h1)$ $H(W_a)=H(W)\underset{\mathbb{Z}[q,q^{-1}]}{\otimes}\mathbb{Z}[q,q^{-1}][P]$ as a $\mathbb{Z}[q, q^{-1}]$-module.

$(h2)$ $H(W)\to H(W_a)$ $(h\to h\otimes 1)$ and $\mathbb{Z}[q, q^{-1}][P]\to H(W_a)(u\to 1\otimes u)$ are algebra homomorphisms.

We view $H(W)$ and $\mathbb{Z}[q, q^{-1}][P]$ as subalgebras of $H(W_a)$. The element of $\mathbb{Z}[q, q^{-1}][P]$ corresponding to $\lambda\in P$ is denoted by ϑ_λ when it is regarded as an element of $H(W_a)$.

$(h3)$ $T_s\vartheta_\lambda=\vartheta_{s(\lambda)}T_s+(q-1)\dfrac{\vartheta_{\alpha_s}(\vartheta_\lambda-\vartheta_{s(\lambda)})}{\vartheta_{\alpha_s}-1}$ for $s\in S$ and $\lambda\in P$.

4.2. A result of Ginsburg and Kazhdan-Lusztig

For $x\in X$ let B_x be the corresponding Borel subgroup and $\mathfrak{n}_x$ the Lie algebra of the unipotent radical of B_x. We consider the equivariant K-homology group $K^{G\times C^*}(Z)$ of the variety

$$Z=\{(x, y, A)\in X\times X\times \text{Lie } G|A\in\mathfrak{n}_x\cap\mathfrak{n}_y\},$$

where the action of $G\times \mathbb{C}^*$ is given by:

$$(g, z)\cdot(x, y, A)=(g\cdot x, g\cdot y, zAd(g)A).$$

Since the representation ring R_{C^*} of $\mathbb{C}^*$ is identified with $\mathbb{Z}[q, q^{-1}]$ via $[z\to z^i]\leftrightarrow q^i$, $K^{G\times C^*}(Z)$ is a $\mathbb{Z}[q, q^{-1}]$-module. Ginsburg and Kazhdan-Lusztig have defined an $(H(W_a), H(W_a))$-bimodule structure on $K^{G\times C^*}(Z)$ and shown that it is isomorphic to the two-sided regular representation ([KL4], [Gi2]). We explain this slightly modifying the formulation of Ginsburg.

Since the dual space of $\text{Lie}(G)/\text{Lie}(B_x)$ is naturally identified with $\mathfrak{n}_x$ via the Killing form, the cotangent bundle T^*X of X is identified with the variety $\{(x, A)\in X\times\text{Lie}(G)|A\in\mathfrak{n}_x\}$. Hence we can view Z as a $G\times\mathbb{C}^*$-stable closed subvariety of $T^*X\times T^*X=\{(x, y, A, A')|A\in\mathfrak{n}_x, A'\in n_y\}$ by $(x, y, A)\leftrightarrow(x, y, A, A)$. Let $p_{ij}: T^*X\times T^*X\times T^*X\to T^*X\times T^*X$ and $p_i: T^*X\times T^*X\times T^*X\to T^*X$ be the projections and $p: T^*X\to X$ the cotangent bundle. It is easily seen from Lemma 2.1 and 2.2 that a $\mathbb{Z}[q, q^{-1}]$-module structure on $K^{G\times C^*}(Z)$ $(=K^{G\times C^*}(Z, T^*X\times T^*X))$ is given by:

$$m_1\cdot m_2=p_{13*}(p_{12}{}^*m_1\otimes p_{23}{}^*m_2\otimes p_2{}^*p^*[\Omega_X]).$$

For $\lambda\in p$ let $O(\lambda)$ be the invertible O_X-module consisting of sections of the line bundle on X with G-action such that the action of B_x on the fiber at $x\in X$ is given by λ. For $s\in S$ we denote the closure of $\{(x, y, A)\in Z|(x, y)\in Y_s\}$ by Z_s. It is a G-equivariant vector bundle over $\bar{Y}_s$ via the natural projection $p_s: Z_s\to\bar{Y}_s$. Let $j: T^*X\to T^*X\times T^*X$ be the diagonal embedding and $j_s: Z_s\to T^*X\times T^*X$ the natural inclusion. We set:

$$e(\lambda)=j_*p^*([O(\lambda)\otimes\Omega_X{}^{-1}]) \quad\text{and}\quad a_s=j_{s*}p_s{}^*([\Omega_{\bar{Y}^s/X\times X}])$$

for $\lambda\in p$ and $s\in S$. They are elements of $K^{G\times C^*}(Z)=K^{G\times C^*}(Z, T^*X\times T^*X)$.

Theorem 4.1.([Gi, 2]). *$K^{G\times C^*}(Z)$ is isomorphic to $H(W_a)$ as a $\mathbf{Z}[q, q^{-1}]$-algebra via the correspondence :*

$$qa_s \leftrightarrow -(T_s+1)(s\in S) \quad \text{and} \quad e(-\lambda)\leftrightarrow \vartheta_\lambda\ (\lambda\in P)\,.$$

4.3. Let a be the involution on $T^*X\times T^*X$ given by $a(x, y, A, A')=(x, y, A, -A')$. Since $\Lambda_{(X\times X,G)}=\{(x, y, A, -A)\in T^*X\times T^*X|A\in \mathfrak{n}_x\cap \mathfrak{n}_y\}$, we have $a(Z)=\Lambda_{(X\times X,G)}$. We set :

$$\gamma=q^N(a^*\circ \mathrm{gr}): K(\mathcal{A})\to K^{G\times C^*}(Z)\,.$$

Theorem B. *γ is a homomorphism of $\mathbf{Z}[q, q^{-1}]$-algebras. If we identify $K(\mathcal{A})$ and $K^{G\times C^*}(Z)$ with $H(W)$ and $H(W_a)$, respectively, then γ coincides with the natural inclusion.*

Let K be a closed subgroup of G which is either of type (a) or (b) in Section 3.3. Let $q_i: T^*X\times T^*X\to T^*X$ be the obvious projections ($i=1$ 2). It is easily seen that an action of $K^{G\times C^*}(Z)$ on $K^{K\times C^*}(\Lambda_{(X,K)})$ is defined by :

$$m\cdot n=q_{1*}(m\otimes q_2{}^*n\otimes q_2{}^*p^*[\Omega_X])\ (m\in K^{G\times C^*}(Z)\,,\ n\in K^{K\times C^*}(\Lambda_{(X,K)}))\,.$$

Especially, $K^{K\times C^*}(\Lambda_{(X,K)})$ is an $H(W)$-module.

Theorem B′. gr *: $K(\mathcal{A}^K)\to K^{K\times C^*}(\Lambda_{(X,K)})$ is a homomorphism of $H(W)$-modules.*

We give the proof of Theorem B. Theorem B′ is proved similarly.

Proof of Theorem B. Let $\sigma: Y_e\to T^*Y_e$ be the zero section. Since $\mathcal{L}_e=i_{e*}(\mathcal{L}_{Y_e})$ and $\mathrm{gr}(\mathcal{L}_{Y_e})=\sigma_*(O_{Y_e})$, we see from Lemma 2.5 that $\gamma([\mathcal{L}_e])=e(0)$. Similarly we have $\gamma([\mathcal{L}_s])=qa_s$ for $s\in S$. Hence it is sufficient to show $\gamma([\mathcal{L}_s]\cdot m)=qa_s\cdot\gamma(m)$ for $m\in K(\mathcal{A})$. We set $u_i=q_i\circ j_s$ for $i=1$, 2. It is easily seen from Lemma 2.1 and Lemma 2.2 that :

$$a_s\cdot n=(u_1\times 1)_*(u_2\times 1)^*(n\otimes(p\times p)^*([\Omega_{X/X^s}\boxtimes O_X]))$$

for $n\in K^{G\times C^*}(Z)$. On the other hand we have $[\mathcal{L}_s]\cdot m=-(\pi_s\times 1)^*(\pi_s\times 1)_*(m)$. Thus we can see easily from Lemma 2.4 and 2.5 that :

$$\gamma([\mathcal{L}_s]\cdot m)=q(u_1\times 1)_*(u_2\times 1)^*(\gamma(m)\otimes(p\times p)^*([\Omega_{X/X^s}\boxtimes O_X]))\,.$$

Hence the assertion is proved.

Remark. Theorem B and Theorem B′ are generalization of the results in [KT] and [Ta].

§ 5. Good Filtrations of $U(\mathfrak{g})$-Modules Associated to Hodge Modules

5.1. We denote the enveloping algebra of $\mathfrak{g}=\mathrm{Lie}(G)$ by $U(\mathfrak{g})$.

By [BeB] the category of coherent D_X-modules is equivalent to the category of finitely generated $U(\mathfrak{g})$-modules with trivial central character. The $U(\mathfrak{g})$-module corresponding to a coherent D_X-module $\mathcal{M}$ is $\Gamma(X, \mathcal{M})$, the space of its global sections. Note that $H^i(X, \mathcal{M})=0$ for $i>0$([BeB]). When a good filtration F of a coherent D_X-module $\mathcal{M}$ is given (for example when $\mathcal{M}$ is an underlying D_X-module of a Hodge module), the corresponding $U(\mathfrak{g})$-module $M=\Gamma(X, \mathcal{M})$ is equipped with a good filtration via $F_p(M)=\Gamma(X, F_p\mathcal{M})$. A good filtration of a finitely generated $U(\mathfrak{g})$-module is defined similarly to the case of a coherent D-module using the order filtration of $U(\mathfrak{g})$. Let $MF(\mathfrak{g})$ be the category consisting of pairs (M, F) of finitely generated $U(\mathfrak{g})$-modules M with trivial central character and their good filtrations F. By the above arguments we have a functor:

$$\Gamma F : MHM(X) \to MF(\mathfrak{g}) .$$

The category $MF(\mathfrak{g})$ is not an abelian category but an exact category. A sequence:

$$[\cdots \to (M_{i-1}, F) \to (M_i, F) \to (M_{i+1}, F) \to \cdots]$$

in $MF(\mathfrak{g})$ is exact if and only if the associated graded sequence:

$$[\cdots \to \mathrm{Gr}^F M_{i-1} \to \mathrm{Gr}^F M_i \to \mathrm{Gr}^F M_{i+1} \to \cdots]$$

is exact in the abelian category of $\mathrm{Gr}\, U(\mathfrak{g})$ $(=S(\mathfrak{g}))$-modules.

It is natural to ask whether ΓF is an exact functor. Hence we are led to the following:

Question. Is it true that

$$(B) \quad H^i(X, F_p\mathcal{M})=0 \quad (i>0, p\in \mathbf{Z})$$

for $\mathcal{V}=(\mathcal{M}, F, K, W)\in MHM(X)$?

Similar problems are treated in [BoB].

By the exact sequence $[0\to F_{p-1}\mathcal{M}\to F_p\mathcal{M}\to \mathrm{Gr}_p{}^F\mathcal{M}\to 0]$ we see easily that (B) is equivalent to:

$$(B') \quad H^i(X, \mathrm{Gr}^F \mathcal{M})=0 \quad (i>0),$$

and if this is true, then we have $\mathrm{Gr}^F M=\Gamma(X, \mathrm{Gr}^F \mathcal{M})$ for $(M, F) = \Gamma F(\mathcal{V})$ and the functor ΓF is exact.

Remark. Kashiwara has proved (B) for $X=\boldsymbol{P}^n$ using Saito's Kodaira vanishing theorem ([Sa3]).

5.2. Identifying the cotangent bundle T^*X with $\{(x, f)\in X\times\mathfrak{g}^* | f(\mathrm{Lie}(B_x))=0\}$ we define $\tau: T^*X\to\mathfrak{g}^*$ by $\tau(x, f)=f$ (the moment map). We fix a Borel subgroup B of G. It is easily seen that $\Lambda_{(X,B)}=\tau^{-1}(\mathfrak{b}^\perp)$ where $\mathfrak{b}$ is the Lie algebra of B and $\mathfrak{b}^\perp=\{f\in\mathfrak{g}^* | f(\mathfrak{b})=0\}$. $\mathfrak{b}^\perp$ can be identified with $[\mathfrak{b}, \mathfrak{b}]$ via the Killing form. Consider the maps:

$$K(\mathcal{A}^B)\xrightarrow{\mathrm{gr}} K^{B\times C^*}(\Lambda_{(X,B)}, T^*X)$$

$$\xrightarrow{\tau^*} K^{B\times C^*}(\mathfrak{b}^\perp)=K^{B\times C^*}(\mathfrak{b}^\perp, \mathfrak{g}^*).$$

$K^{B\times C^*}(\mathfrak{b}^\perp)$ can be identified with the representation ring $R_{B\times C^*}=\boldsymbol{Z}[q, q^{-1}][P] =\bigoplus_{\mu\in p}\boldsymbol{Z}[q, q^{-1}]e^\mu$ via the Thom isomorphism $\bar{p}^*: R_{B\times C^*}(=K^{B\times C^*}(pt))\to K^{B\times C^*}(\mathfrak{b}^\perp)$, where $\bar{p}: \mathfrak{b}^\perp\to pt$.

Lemma 5.1. ([Lu], see also Kato's proof given in [KL4]).
An action of the Hecke algebra $H(W_a)$ on $\boldsymbol{Z}[q, q^{-1}][P]$ is given by:

$$T_s\cdot x=\frac{x-s(x)e^{-2\alpha}}{e^\alpha-1}-q\frac{x-s(x)e^\alpha}{e^\alpha-1} \qquad (s\in S),$$

$$\vartheta_\lambda\cdot x=e^{-\lambda}x \qquad (\lambda\in p),$$

where α is the simple root corresponding to $s\in S$.

Proposition 5.2. *For $w\in W$ we have:*

$$\tau_*(\mathrm{gr}([\mathcal{L}(\bar{X}_w, X)]))=q^{-N}C_w''\cdot e^{2\rho}$$

in $K^{B\times C^*}(\mathfrak{b}^\perp)=\boldsymbol{Z}[q, q^{-1}][P]$, where ρ is the half of the sum of the positive roots.

Although ρ is not necessarily an element of P, 2ρ and $w\rho+\rho$ for $w\in W$ are elements of P.

The proof of Proposition 5.2 will be given in Section 5.3.

Let $(L_w, F)=\Gamma F(\mathcal{L}(\bar{X}_w, X))$. It is known that L_w is the irreducible lowest weight module with lowest weight $w\rho+\rho$ ([BK], [BeB]). Note that we have chosen the ordering on the root system so that the set of positive roots Δ^+ coincides with the weights in $\mathfrak{g}/\mathfrak{b}$.

Definition. For a finitely generated $U(\mathfrak{g})$-module M with B-action and a B-stable good filtration F of M we define the 'q-character' $\mathrm{ch}_q(M, F)$ of (M, F) by:

$$\mathrm{ch}_q(M, F)=\sum_{j\in\mathbf{Z}}\mathrm{ch}(\mathrm{Gr}_j^F M)q^{-j}\in\mathbf{Z}[P]((q^{-1})) .$$

Here $\mathrm{ch}(\mathrm{Gr}_j^F M)\in\mathbf{Z}[P]$ is the character of the B-module $\mathrm{Gr}_j^F M$.

Corollary 5.3. *If the condition* (B) *holds for* $\mathcal{V}=\mathcal{L}(\bar{X}_w, X)$, *then we have*:

$$\mathrm{ch}_q(L_w, F)=\frac{q^{-N}C_w''\cdot e^{2\rho}}{\prod_{\alpha\in\Delta^+}(1-q^{-1}e^{\alpha})} .$$

Proof. In general for $\mathcal{V}=(\mathcal{M}, F, K, W)\in MHM(X)$ we have

$$\begin{aligned}\mathbf{R}\Gamma(X, \mathrm{Gr}^F\mathcal{M})&=\mathbf{R}\Gamma(X, \mathbf{R}p_*(\mathrm{gr}(\mathcal{V})))\\&=\mathbf{R}\Gamma(T^*X, \mathrm{gr}(\mathcal{V}))\\&=\mathbf{R}\Gamma(\mathfrak{g}^*, \mathbf{R}\tau_*(\mathrm{gr}(\mathcal{V}))) .\end{aligned}$$

Hence when the condition (B) holds for V, we have:

$$\begin{aligned}\mathrm{Gr}^F M&=\Gamma(X, \mathrm{Gr}^F\mathcal{M})\ (=\mathbf{R}\Gamma(X, \mathrm{Gr}^F\mathcal{M}))\\&=\Gamma(\mathfrak{g}^*, \tau_*(\mathrm{gr}(\mathcal{V})))\ (=\mathbf{R}\Gamma(\mathfrak{g}^*, \mathbf{R}\tau_*(\mathrm{gr}(\mathcal{V}))))\end{aligned}$$

for $(M, F)=\Gamma F(\mathcal{V})$. Therefore

$$\mathrm{ch}_q(M, F)=\frac{\tau_*(\mathrm{gr}([\mathcal{V}]))}{\prod_{\alpha\in\Delta^+}(1-q^{-1}e^{\alpha})} ,$$

and the assertion follows from Proposition 5.2. Here $(\prod_{\alpha\in\Delta^+}(1-q^{-1}e^{\alpha}))^{-1}=\prod_{\alpha\in\Delta^+}(\sum_{k\geq 0}q^{-k}e^{k\alpha})$ appears as the character of the $B\times\mathbf{C}^*$-module $\Gamma(\mathfrak{b}^{\perp}, O_{\mathfrak{b}^{\perp}})$.

5.3. Proof of Proposition 5.2

The arguments below are inspired by [BoB].

We first give some relations of $\mathcal{A}$ and $\mathcal{A}^B$. Let $x_0\in X$ be the point corresponding to B. We define $k: X\to X\times X$ by $k(x)=(x, x_0)$.

Lemma 5.4. (i) $(\mathcal{H}^j k^*)(\mathcal{V})=0$ *for* $\mathcal{V}\in MHM(X\times X, G)$ *and* $j\neq -N$.

(ii) $(\mathcal{H}^{-N}k^*)(\mathcal{L}_w)=\mathcal{L}(\bar{X}_{w^{-1}}, X)$ *for* $w\in W$.

(iii) $\mathcal{H}^{-N}k^*$ *induces exact functors*:

$$MHM(X\times X, G)\to MHM(X, B) \quad \text{and} \quad \mathcal{A}\to\mathcal{A}^B .$$

Proof. (i) is shown in the proof of Lemma 3.3. Choose a Borel subgroup which is opposit to B and denote its unipotent radical by U. We define $\varphi: X\times U\to X\times X$ by $\varphi(x, u)=(u\cdot x, u\cdot x_0)$. It is an open immeresion. By the proof of Lemma 3.3 we have $\varphi^*\mathcal{V}\simeq(\mathcal{H}^{-N}k^*)(\mathcal{V})\boxtimes\mathcal{L}_U$ for $\mathcal{V}\in MHM(X\times X, G)$. Since $\varphi^{-1}(Y_w)=X_{w^{-1}}\times U$, we have:

$$\begin{aligned}\mathcal{L}(\bar{X}_{w^{-1}}, X)\boxtimes\mathcal{L}_U &\simeq\mathcal{L}(\bar{X}_{w^{-1}}\times U, X\times U)\\ &=\varphi^*\mathcal{L}_w\\ &=(\mathcal{H}^{-N}k^*)(\mathcal{L}_w)\boxtimes\mathcal{L}_U ,\end{aligned}$$

and (ii) is proved. (iii) is a consequence of (i) and (ii).

We identify $(T^*X)_{x_0}$, the fiber of T^*X at x_0, with $\mathfrak{n}=[\mathfrak{b}x_0, \mathfrak{b}x_0]$. Let ϖ: $T^*X\times\mathfrak{n}$ $(=T^*X\times(T^*X)_{x_0})\to T^*(X\times X)(=T^*X\times T^*X)$ be the inclusion and $\rho: T^*X\times\mathfrak{n}\to T^*X$ the projection. Identifying T^*X with $\{(x, A)\in X\times\mathfrak{g}|A\in\mathfrak{n}_x\}$ we have:

$$\Lambda_{(X\times X,G)}=\{(x, y, A, -A)\in X\times X\times\mathfrak{g}\times\mathfrak{g}|A\in\mathfrak{n}_x\cap\mathfrak{n}_y\} ,$$

$$\Lambda_{(X,B)}=\{(x, A)\in X\times\mathfrak{g}|A\in\mathfrak{n}_x\cap\mathfrak{n}\} .$$

We define subvarieties Λ^+ and Λ^- of $T^*X\times\mathfrak{n}$ by:

$$\Lambda^{\pm}=\{((x, A), \pm A)|A\in\mathfrak{n}_x\cap\mathfrak{n}\} .$$

Since ρ induces an isomorphism $\Lambda^-\xrightarrow{\sim}\Lambda_{(X,B)}$, and since $\varpi^{-1}(\Lambda_{(X\times X,G)})=\Lambda^-$, we have the natural maps:

$$\varpi^*: K^{G\times C^*}(\Lambda_{(X\times X,G)}, T^*(X\times X))\to K^{B\times C^*}(\Lambda^-, T^*X\times\mathfrak{n})$$

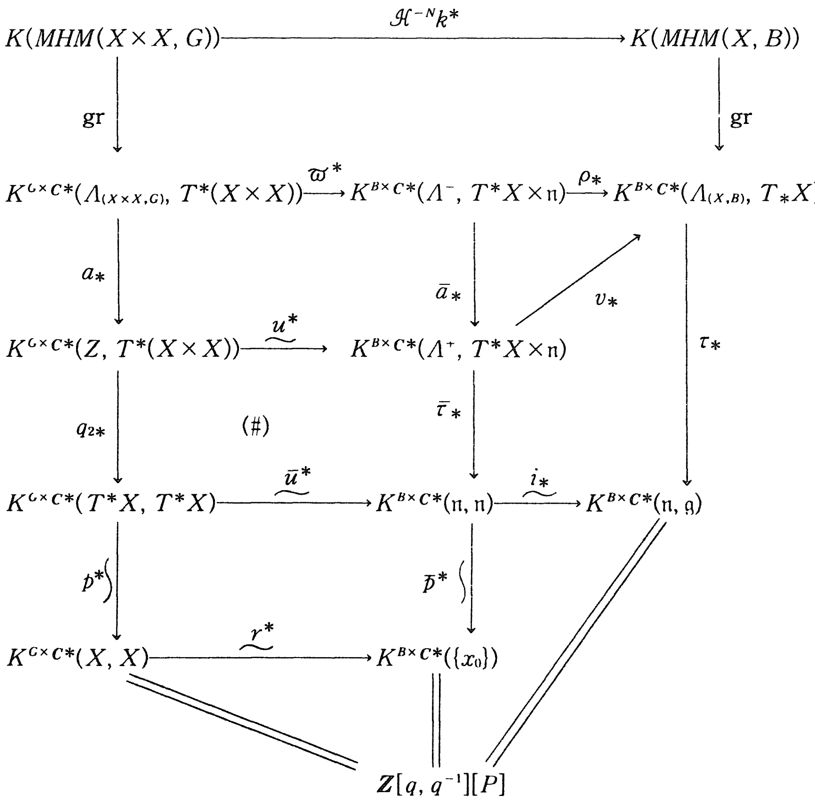
$$\rho_*: K^{B\times C^*}(\Lambda^-, T^*X\times\mathfrak{n})\to K^{B\times C^*}(\Lambda_{(X,B)}, T^*X) .$$

Lemma 5.5. *For* $\mathcal{V}\in MHM(X\times X, G)$ *we have*:

$$\rho_*\varpi^*(\mathrm{gr}([\mathcal{V}]))=\mathrm{gr}([(\mathcal{H}^{-N}k^*)(\mathcal{V})]) .$$

This follows from the fact that $\varphi^*\mathcal{V}=(\mathcal{H}^{-N}k^*)(\mathcal{V})\boxtimes\mathcal{L}_U$ for $\mathcal{V}\in MHM(X\times X, G)$ in the notation of the proof of Lemma 5.4. Details are left to the readers.

Consider the following commutative diagram.



Here u, $\bar{u}$, r, i are natural inclusions and

$$a(x, y, A, A')=(x, y, A, -A'),$$

$$\bar{a}((x, A), A')=((x, A), -A'),$$

$$v((x, A), A')=(x, A),$$

$$\bar{\tau}((x, A), A')=A'.$$

Note that $q_2|Z$ and $\bar{\tau}|\Lambda^+$ are projective morphisms. The commutativity of (#) follows easily from Lemma 2.1 and Lemma 2.2 since $\bar{u}$ is a closed immersion and q_2 is smooth.

By Lemma 5.4 we have $\tau_*(\mathrm{gr}([\mathcal{L}(\bar{X}_w, X)]))=q_{2*}(a_*(\mathrm{gr}(\mathcal{L}_{w^{-1}})))q_{1*}(a_*(\mathrm{gr}([\mathcal{L}_w])))$ in $\boldsymbol{Z}[q, q^{-1}][P]$. The last equality follows from an easy calculation involving the G-equivariant automorphism of $X\times X$ given by $(x, y)\to(y, x)$. By Theorem 4.1 $K^{G\times C^*}(Z)$ is identified with $H(W_a)$. Define $F: H(W_a)\to\boldsymbol{Z}[q, q^{-1}][P]$ by the commutativity of:

$$\begin{array}{ccc} K^{G\times C^*}(Z) & \xrightarrow{q_{1*}} & K^{G\times C^*}(T^*X) \\ \wr\| & & \wr\| \\ H(W_a) & \xrightarrow{F} & \mathbf{Z}[q, q^{-1}][P]. \end{array}$$

By Theorem A and Theorem B we have $\tau_*(\mathrm{gr}([\mathcal{L}(\bar{X}_w, X)]))=q^{-N}F(C_w'')$. Hence Proposition 5.2 is a consequence of the following:

Lemma 5.6. (i) *F is a homomorphism of $H(W_a)$-modules. Here the $H(W_a)$-module structure of $H(W_a)$ is given by the left multiplication and that of $\mathbf{Z}[q, q^{-1}][P]$ is the one given in Lemma* 5.1.

(ii) $F(1)=e^{2\rho}$.

Proof. It is easily seen that a $K^{G\times C^*}(Z)$-module structure on $K^{G\times C^*}(T^*X)$ is defined by:

$$h\cdot m=q_{1*}(h\otimes q_2^*(m\otimes p^*([\Omega_X])))$$
$$(h\in K^{G\times C^*}(Z, T^*(X\times X)), \quad m\in K^{G\times C^*}(T^*X, T^*X)).$$

By a standard argument we see that q_{1*} is a homomorphism of $K^{G\times C^*}(Z)$-modules and that the $K^{G\times C^*}(Z)$ $(=H(W_a))$-module structure on $K^{G\times C^*}(T^*X)$ $(=\mathbf{Z}[q, q^{-1}][P])$ coincides with the one given in Lemma 5.1. (i) is proved. (ii) is a consequence of $q_{1*}(e(0))=p^*([\Omega_X^{-1}])$.

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