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On the Rigidity of Noncompact Quotients of Bounded Symmetric Domains

By

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Introduction

Let $\mathscr{X}\xrightarrow{\pi}B$ be a differentiable family of complex manifolds, let o be a point of B and let X be the special fibre $\pi^{-1}(o)$. For any tangent vector at o, the infinitesimal variation of the complex structure of X is defined by Kodaira and Spencer [18] as an elment of $H^1(X, \Theta)$, where Θ denotes the sheaf of the germs of holomorphic tangent vectors of X. In case X is compact and $H^1(X, \Theta)=0$, it follows that one can find a neghbourhood $U \ni o$ such that $\pi^{-1}(p)\cong X$ for any $p\in U$; in other words X is (locally) rigid (cf. [18]). In this spirit, Calabi and Vesentini [10] has shown that X is rigid if X is compact and its universal covering space is biholomorphic to an irreducible bounded symmetric domain of dimension ≥ 2, applying the harmonic theory developed by Bochner [6], Kodaira [17] and Nakano [20].

The purpose of the present article is to extend Calabi-Vesentini's theorem to noncompact manifolds which arise in the theory of generalized automorphic functions (cf Siegel[24] and Baily-Borel [4]). Our main result is as follows.

Theorem. *Let X be a complex manifold whose universal covering space is biholomorphic to an irreducible bounded symmetric domain. If the covering transformation group of that covering is arithmetic in the sense of Borel, then $H^1(X, \Theta)=0$ except for the cases where the universal covering is either of type $(I)_{m,m'}$, $m+m'<4$, $(I)_{2,3}$, $(I)_{3,2}$, $(II)_m$, $m<4$, $(III)_m$, $m<4$,*

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$(III)_m$, $m<3$, or $(IV)_m$, $m<4$.

Corollary. *Let X be as above (with the same exceptions). Then X is* ***analytically rigid*** (see § 3 for the definition).

For the proof of the theorem we observe first of all that $H^1_{(2)}(X, \Theta)=0$, i.e. the vanishing of the L^2 cohomology with respect to the metric induced by the invariant metric of the symmetric domain. Our next, of course main, task is to prove the bijectivity of the natural homomorphism from $H^1_{(2)}(X, \Theta)$ to $H^1(X, \Theta)$. By taking the duals, the "obstructions" to be killed turn out to be $\lim_{K \subset\subset X} H^{n,n-k}_{(2)}(X \backslash \bar{K}, \Theta^*)$ for $k=1, 2$, whose vanishing can be verified within the framework of Andreotti-Vesentini [2] and Hörmander[15], once we know that X is hyper $(n-2)$-concave with respect to the invariant metric. For the proof of the hyper $(n-2)$-concavity, we rely on the compactification theory of Pyatetskii-Shapiro [22], Baily-Borel [4] and Ash-Mumford-Rapoport-Tai [3]. The fact we need is the existence of an ideal sheaf supported on the boundary which has a nice Fourier-Jacobi series expansion at every point.

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§ 1. Notations and Preliminaries

Let (X, ds^2) be a Hermitian complex manifold of dimension n, and (E, h) a Hermitian holomorphic vector bundle over X. The following notations shall be used throughout this paper.

$C^{p,q}(X, E) := \{E\text{-valued } C^\infty \ (p, q)\text{-forms on } X\}$

$L^{p,q}(X, E) := \{E\text{-valued square integrable } (p, q)\text{-forms on } X\}$

$L^{p,q}_{loc}(X, E) := \{E\text{-valued, locally square integrable } (p, q)\text{-forms on } X\}$

$C^{p,q}_0(X, E) := \{f \in C^{p,q}(X, E)\ ;\ \text{supp } f \subset\subset X\}$

$L^{p,q}_0(X, E) := \{f \in L^{p,q}_{loc}(X, E)\ ;\ \text{supp} f \subset\subset X\}$

$H^{p,q}(X, E) := \{f \in C^{p,q}(X, E)\ ;\ \bar{\partial} f=0\}/\{g \in C^{p,q}(X, E)\}\ ;$

$\exists u \in C^{p,q-1}(X, E) \text{ such that } g=\bar{\partial} u\}$

$$H_{(2)}^{p,q}(X, E) := \{f \in L^{p,q}(X, E) ;\ \bar\partial f = 0\} / \{g \in L^{p,q}(X, E) ;$$
$$\exists u \in L^{p,q-1}(X, E) \text{ such that } g = \bar\partial u\}$$
$$H_0^{p,q}(X, E) := \{f \in C_0{}^{p,q}(X, E) ;\ \bar\partial f = 0\} / \{g \in C_0^{p,q}(X, E) ;$$
$$\exists u \in C_0^{p,q}(X, E) \text{ such that } g = \bar\partial u\} .$$

Let x be any point of X. Then, by a theorem of L. Hörmander (cf. [15], Theorem 4.2.2), $\lim_U H_{(2)}^{p,q}(U, E) = 0$ for $q > 0$, where U runs through the neighbourhoods of x.

Since a locally square integrable function f is holomorphic if and only if $\bar\partial f = 0$, vanishing of the local L^2 cohomology as above implies that there exist canonical isomorphisms ;

$$H^{p,q}(X, E) \cong \{f \in L_{loc}^{p,q}(X, E) ;\ \bar\partial f = 0\} / \{g \in L_{loc}^{p,q}(X, E) ;$$
$$\exists u \in L_{loc}^{p,q-1}(X, E) \text{ such that } g = \bar\partial u\}$$
$$H_0^{p,q}(X, E) \cong \{f \in L_0^{p,q}(X, E) ;\ \bar\partial f = 0\} / \{g \in L_0^{p,q}(X, E) ;$$
$$\exists u \in L_0^{p,q-1}(X, E) \text{ such that } g = \bar\partial u\} .$$

Hence we have the following exact sequence :

(1) $$\lim_K H_{(2)}^{p,q-1}(X \backslash K, E) \longrightarrow H_0^{p,q}(X, E) \longrightarrow H_{(2)}^{p,q}(X, E)$$
$$\longrightarrow \lim_K H_{(2)}^{p,q}(X \backslash K, E) ,$$

where K runs through the compact subsets of X.

Therefore we have

Lemma 1.1. *If* $\lim_K H_{(2)}^{p,q-1}(X \backslash K, E) = 0$ *and* $\lim_K H_{(2)}^{p,q}(X \backslash K, E) = 0$, *then* $H_0{}^{p,q}(X, E) \cong H_{(2)}^{p,q}(X, E)$.

We are going to state a sufficient condition for $\lim_K H_{(2)}^{n,q}(X \backslash K, E)$ to vanish, which is to be verified for the quotients of bounded symmetric domains.

For that purpose we need to fix several notations used in differential geometry and present a fundamental inequality which includes a well known inequality due to Nakano [20] and Calabi-Vesentini [10] as a special case.

Let $* : C^{p,q}(X, E) \longrightarrow C^{n-q,n-p}(X, E)$ be Hodge's star operator. Then, the (formal) adjoints of the exterior differentiations d, ∂ and $\bar\partial$

acting on the scalar forms are expressed as $-*d*$, $-\bar{*}\partial\bar{*}$ and $-\bar{*}\bar{\partial}\bar{*}$, respectively, where we put $\bar{*}u=\overline{*u}$. We denote $d^*=-*d*$, $\partial^*=-\bar{*}\partial\bar{*}$ and $\bar{\partial}^*=-\bar{*}\bar{\partial}\bar{*}$. Note that ∂^* operates on $\bigoplus_{p,q} C^{p,q}(X, E)$ as well as $\bar{\partial}$, since $\partial^*(fu)=f\partial^*u$ for any holomorphic function f and C^∞ differential form u. We shall identify ∂^* with its maximal closed extension to $\bigoplus_{p,q} L^{p,q}(X, E)$. The adjoints of the operators $\bar{\partial}$ and ∂^* on $\bigoplus_{p,q} L^{p,q}(X, E)$ will be denoted by $\bar{\partial}_h^*$ and ∂_h, respectively. The operator $(\bar{\partial}+\partial_h)^2$ is known to be function-linear (cf. Wells [25]). For any differential form θ with values in $Hom(E, E)$, or values in the trivial bundle, let $e(\theta)$ denote the wedge multiplication by θ from the left hand side. The adjoint of $e(\theta)$ will be denoted by $e(\theta)^*$. Let ω be the fundamental form of the metric ds^2. We put $\Lambda:=e(\omega)^*$. Since $(\partial+\partial_h)^2$ contains no differentiation, it is expressed as $e(\Theta_h)$, where $\Theta_h \in C^{1,1}(X, Hom(E, E))$. Θ_h is called the curvature form of (E, h).

Let S and T be two homogeneous linear operators on the space $\bigoplus_{p,q} C^{p,q}(X, E)$. We put $[S, T]:=ST-(-1)^{st}TS$, where $s=\deg S$ and $t=\deg T$. The complex Laplacian $\Box_h$ is defined as $[\bar{\partial}, \bar{\partial}_h^*]$. We put $\overline{\Box}_h:=[\partial^*, \partial_h]$.

Proposition 1.2 (Jacobi's identity).

$$[[S, T], U]-[S, [T, U]]=(-1)^{tu}[[S, U], T],$$

where $t=\deg T$ and $u=\deg U$.

The proof is left to the reader.

From now on, we assume that (X, ds^2) is a Kähler manifold.

Proposition 1.3 (Kähler identity).

$$[\bar{\partial}, \Lambda]=\sqrt{-1}\partial^*$$

$$[\partial_h, \Lambda]=-\sqrt{-1}\bar{\partial}_h^*.$$

For the proof, the reader is referred to [25].

Substituting $S=\bar{\partial}$, $T=\Lambda$ and $U=\partial_h$ into Jacobi's identity, we obtain

$$\sqrt{-1}([\partial^*, \partial_h]-[\bar{\partial}, \bar{\partial}_h^*])=[[\bar{\partial}, \partial_h], \Lambda].$$

Since $[\bar{\partial}, \partial_h]=(\bar{\partial}+\partial_h)^2$, the above equality is interpreted as

(2) $$\Box_h - \overline{\Box}_h = [\sqrt{-1}e(\Theta_h), \Lambda] \qquad \text{(Nakano's equality[20])}.$$

If we put $U = e(\theta)$, $\theta \in C^{p,q}(X, Hom(E, E))$, instead of putting $U = \partial_h$, we obtain

(3) $$\sqrt{-1}[\partial^*, e(\theta)] - [\bar\partial, [\Lambda, e(\theta)]] = [e(\bar\partial\theta), \Lambda].$$

If θ is of type (1, 0), $[e(\theta), \Lambda] = \sqrt{-1}e(\bar\theta)^*$. Hence the equality (3) becomes

(4) $$[\partial^*, e(\theta)] + [\bar\partial, e(\bar\theta)^*] = [-\sqrt{-1}e(\bar\partial\theta), \Lambda].$$

Let $f, g \in L^{p,q}_{loc}(X, E)$. We denote by $\langle f, g\rangle$ the pointwise inner product of f and g. For any open set $D \subset X$ we put $(f, g)_D = \int_D \langle f, g\rangle dv$, dv : the volume element, whenever the right hand side converges, and $\|f\|_D^2 := (f, f)_D$.

Proposition 1.4. Let D be a domain in X with C^∞ boundary and let ψ be a C^∞ function on X with $D = \{x \in X\,;\, \psi(x) > 0\}$. Then,

(5) $$\|\sqrt{\psi + |\bar\partial\psi|^2}\,\bar\partial_h{}^* f\|_D^2 + \|\sqrt{\psi}\,\bar\partial f\|_D^2$$
$$\geq ([\sqrt{-1}e(\psi\Theta_h - \partial\bar\partial\psi), \Lambda]f, f)_D - \|f\|_D^2,$$

for any $f \in C_0^{n,q}(X, E)$ satisfying $\bar{*} f \wedge \bar\partial\psi = 0$ on ∂D.

Proof. We have

$$\bar\partial e(\psi)\bar\partial_h^* + \bar\partial_h^* e(\psi)\bar\partial - \partial_h e(\psi)\partial^* - \partial^* e(\psi)\partial_h$$
$$= \psi(\Box_h - \overline{\Box}_h) + e(\bar\partial\psi)\bar\partial_h^* - e(\bar\partial\psi)^*\bar\partial - e(\partial\psi)\partial^* + e(\partial\psi)^*\partial_h$$
$$= [\sqrt{-1}e(\psi\Theta_h), \Lambda] + e(\bar\partial\psi)\bar\partial_h^* - e(\bar\partial\psi)^*\bar\phi - [\sqrt{-1}e(\partial\bar\partial\psi), \Lambda]$$
$$+ \partial^* e(\partial\psi) + [\bar\partial, e(\bar\partial\psi)^*] + e(\partial\psi)^*\partial_h,$$

by(2) and (4).

Since $\psi = 0$ on ∂D, by Stokes' theorem

$$((\bar\partial e(\psi)\bar\partial_h^* + \bar\partial_h^* e(\psi)\bar\partial - \partial_h e(\psi)\partial^* - \partial^* e(\psi)\partial_h)f, f)_D$$
$$= \|\sqrt{\psi}\,\partial_h^* f\|_D^2 + \|\sqrt{\psi}\,\bar\partial f\|_D^2 - \|\sqrt{\psi}\,\partial^* f\|_D^2 - \|\sqrt{\psi}\,\partial_h f\|_D^2,$$

for any $f \in C_0^{p,q}(X, E)$. If $p = n$, then $\partial\psi \wedge f = 0$ and $\partial_h f = 0$. Hence, for any $f \in C_0^{n,q}(X, E)$

$$\|\sqrt{\psi}\,\partial_h^* f\|_D{}^2 + \|\sqrt{\psi}\,\bar\partial f\|_D{}^2 - \|\sqrt{\psi}\,\partial^* f\|_D{}^2 - \|\sqrt{\psi}\,\partial_h f\|_D{}^2$$
$$= ([\sqrt{-1}e(\psi\Theta_h), \Lambda]f, f)_D + (e(\bar\partial\psi)\bar\partial_h^* f, f)_D - (e(\bar\partial\psi)^*\bar\partial f, f)_D$$

$$-([\sqrt{-1}e(\partial\bar{\partial}\psi), \Lambda]f, f)_D+([\bar{\partial}, e(\bar{\partial}\psi)^*]f, f)_D .$$

Moreover, if $\bar{*}f\wedge\bar{\partial}\psi=0$ on ∂D, then $(\bar{\partial}e(\bar{\partial}\psi)^*f, f)_D=(e(\bar{\partial}\psi)^*f, \bar{\partial}_h{}^*f)_D$. Thus we obtain

$$\|\sqrt{\psi}\,\bar{\partial}_h^*f\|_D{}^2+\|\sqrt{\psi}\,\bar{\partial}f\|_D{}^2$$
$$\geq([\sqrt{-1}e(\psi\Theta_h-\partial\bar{\partial}\psi), \Lambda]f, f)_D+2\mathrm{Re}e(\bar{\partial}\psi)\,\bar{\partial}_h^*f, f)_D ,$$

whence

$$\|\sqrt{\psi+|\partial\psi|^2}\,\bar{\partial}_h^*f\|_D{}^2+\|\sqrt{\psi}\,\bar{\partial}f\|_D{}^2$$
$$\geq([\sqrt{-1}e(\psi\Theta_h-\partial\bar{\partial}\psi), \Lambda]f, f)_D-\|f\|_D{}^2 ,$$

for any $f\in C_0^{n,q}(X, E)$ satisfying $\bar{*}f\wedge\bar{\partial}\psi=0$ on ∂D.

Remark. In case $D=X$, (5) was obtained in [21]. If we put $\psi=$const. and let $\psi\to\infty$, then we obtain Nakano-Calabi-Vesentini's inequality on X.

By a theorem of Gaffney [12], Proposition 1.4 immediately implies the following.

Theorem 1.5. *Let (X, ds^2) be a complete Kähler manifold of dimension n, (E, h) a holomorphic Hermitian vector bundle over X, and D an open subset of X with C^∞ smooth boundary. For a fixed integer q, suppose that there exists a C^∞ function ψ: $X\longrightarrow\mathbf{R}$ with $D=\{x\in X\,;\ \psi(x)>0\}$, satisfying the following properties :*

a) *There exist $A>1$ such that*

$$\langle\sqrt{-1}e(\psi\Theta_h-\partial\bar{\partial}\psi)\Lambda f, f\rangle\geq A\langle f, f\rangle \quad \textit{on} \quad D,$$
$$\textit{for any} \quad f\in C^{n,q}(X, E) .$$

b) $\psi+|\partial\psi|^2<B$ *on D, for some $B>0$.*

Then $H_{(2)}^{n,q}(X, E)=0$. More presicely, for any $f\in L_{(2)}^{n,q}(X, E)$ with $\bar{\partial}f=0$, one can find $u\in L_{(2)}^{n,q-1}(X, E)$ such that

$$\bar{\partial}u=f \quad \textit{and} \quad \|u\|_D^2\leq\frac{B}{A-1}\|f\|_D^2 .$$

Definition. (E, h) is called q-positive if there exists $A_1>0$ such that $\langle\sqrt{-1}e(\Theta_h)\Lambda f, f\rangle\geq A_1\langle f, f\rangle$ for any $f\in C^{n,q}(X, E)$.

The notion of 1-positivity of (E, h) was first esiablished by S. Nakano [20]. He showed that 1-positivity is equivalent to that Θ_h naturally defines a positive definite quadratic form along the fibers of $T_X\otimes E$ (the so called

Nakano positivity).

Definition. A smoothly bounded open subset $D\subset X$ is said to be hyper q-convex at a boundary point $p\in\partial D$ if there exist a neighbourhood $U\ni p$ in X and a C^∞ function $\varphi: U\longrightarrow \mathbf{R}$ with $U\cap D=\{x\in U\,;\ \varphi(x)<0\}$ such that, for any C^∞ (n, q) form f on U,

$$\langle\sqrt{-1}e(\partial\bar{\partial}\varphi)\Lambda f, f\rangle\geq c\langle f, f\rangle \quad \text{for some} \quad c>0\,.$$

D is said to be hyper q-convex if it is hyper q-convex at every boundary point.

The notion of hyper q-convexity is first due to H. Grauert and O. Riemenschneider [13]. It is easy to verify that the above definition is equivalent to their definition except for the regularity of the boundary, which is not so important for our purpose.

Theorem 1.5 is now paraphrased by using the above terminology.

Theorem 1.6. *Let (X, ds^2) be a complete Kähler manifold of dimension n, (E, h) a q-positive vector bundle over X, and $D\subset X$ a hyper q-convex open subset whose boundary is compact. Then $H^{n,q}_{(2)}(D, E)=0$.*

Definition. A complete Kähler manifold (X, ds^2) is called hyper q-concave if X is exhausted by an increasing family of compact subsets $\{K_j\}_{j=1}^\infty$ such that $X\backslash K_j$ is hyper q-convex for every j.

Combining Theorem 1.6 with Lemma 1.1, we obtain the following.

Theorem 1.7. *Let (X, ds^2) be a complete Kähler manifold of dimension n and (E, h) a Hermitian holomorphic vector bundle over X. Let k be an integer such that (X, ds^2) is hyper q-concave for $q\geq k$ and (E, h) is q-positive for $q\geq k$. Then $H_0^{n,p}(X, E)=0$ for $q\geq k+1$.*

§ 2. Hyper q-concavity of the Quotients of Bounded Symmetric Domains

Let M be a complex Hermitian manifold. M is called a Hermitian symmetric space if for every point $x\in M$ there exists an involutive automorphism (i.e. isometric, as well as holomorphic) s_x which has x as an isolated fixed point. If M is a Hermitian symmetric space, then the Hermitian manifold M decomposes as

$$M=M_0\times M_1\times\cdots\times M_n\,,$$

where M_0 is the quotient of a complex vector space with a translation invariant metric by a discrete group of translation (such a space is called of euclidean type), and M_i $(i\neq 0)$ is an irreducible and non-euclidean Hermitian symmetric space (cf. Helgason [14] and Wolf [26]). A non-compact factor M_i $(i\neq 0)$ is called of noncompact type. The classification of the irreducible Hermitian symmetric spaces D is found in [11]. They are given as follows according to the notation of [5], and are simply connected bounded domains in C^n equipped with the Bergman metric. (cf. [7] or [23]). In what follows we shall call them irreducible bounded symmetric domains.

Type $I_{m,m'}$: $D=U^m(m+m')/U(m)\times U(m')$, the space of complex $m\times m'$ matrices Z such that $I_{(m')}-{}^tZ\bar{Z}$ is positive definite.

Type II_m: $D=SO^*(2m)/U(m)$, $m\geq 2$, the space of complex, skew-symmetric $m\times m$ matrices Z such that $I_{(m)}-{}^tZ\bar{Z}>0$.

Type III_m: $D=Sp(m, \boldsymbol{R})/U(m)$, the space of complex, symmetric $m\times m$ matrices Z such that $I_{(m)}-Z\bar{Z}>0$.

Type IV_m: $D=SO^m(m+2)/SO(m)\times SO(2)$, $m\geq 3$, the space of $m\times 1$ matrices Z, satisfying $1+|{}^tZZ|^2-2{}^t\bar{Z}Z>0$, ${}^t\bar{Z}Z<1$.

Type V: $D=E_6^3/\mathrm{Spin}(10)\times SO(2)$, $\dim D=16$.

Type VI: $D=E_7^3/E_6\times SO(2)$, $\dim D=27$.

The equalities $D=U^m(m+m')/U(m)\times U(m')$, etc. should be read; $U^m(m+m')$ operates transitively on D as a group of automorphisms and the stabilizer of some point $x\in D$ is $U(m)\times U(m')$, etc.

Let D be an irreducible bounded symmetric domain. From the above, $D=G(\boldsymbol{R})/K$, where $G(\boldsymbol{R})$ is the group of real points of a connected algebraic matric group G defined over $\boldsymbol{Q}$, simple over $\boldsymbol{Q}$, such that the topological identity component $G(\boldsymbol{R})^\circ$ is isomorphic to, and shall be identified with the connected component of the group of holomorphic automorphisms of D, and K is a maximal compact subgroup of $G(\boldsymbol{R})$. A subgroup $\Gamma\subset G(\boldsymbol{R})^\circ\cap G(\boldsymbol{Q})$ is called *arithmetic* if Γ satisfies $[\Gamma:\Gamma\cap G(\boldsymbol{Z})]<\infty$ and $[G(\boldsymbol{R})^\circ\cap G(\boldsymbol{Z}):\Gamma\cap G(\boldsymbol{Z})]<\infty$. An arithmetic subgroup Γ is said to be *neat* if the group $W^*\subset \boldsymbol{C}\backslash\{0\}$ generated by the eigenvalues of the elements of Γ is torsion free. Note that such a Γ is â fortiori torsion free.

Definition. A Siegel domain of the third kind is a domain $S\subset \boldsymbol{C}^{m+\ell+k}$ of the form

$$\{(z, u, t)\,;\, \mathrm{Im}z-\mathrm{Re}L_t(u, u)\in V,\ t\in F\}$$

which is equivalent to a bounded domain. Here $z \in \mathbb{C}^m$, $u \in \mathbb{C}^\ell$, $t \in \mathbb{C}^k$, V is a nondegenerate cone in $\mathbb{R}^m$, F is a bounded domain in $\mathbb{C}^k$, and L_t is a vector-valued nondegenerate semihermitian form (i.e. $L_t = L_t^0 + L_t^1$, where L_t^0 is a Hermitian form and L_t^1 is a symmetric bilinear form) with domain $\mathbb{C}^m$ and range $\mathbb{C}^\ell$ which depends differentiably on t.

By $\pi_F : \mathcal{S} \longrightarrow F$ we denote the natural projection. A one-to-one transformation of a Siegel domain of the third kind $\mathcal{S}$ having the form $z \longrightarrow z + a(u, t)$, $u \longrightarrow u + b(t)$, $t \longrightarrow t$ is called a parallel translation of $\mathcal{S}$. The group of parallel translations of $\mathcal{S}$ will be denoted by Δ.

Let D be a bounded symmetric domain and Γ an arithmetic subgroup of the automorphism group D. Let $\mathcal{S}$ be a Siegel domain of the third kind biholomorphic to D, and let $\varphi : D \longrightarrow \mathcal{S}$ be a biholomorphic map. We say that the fibration $\pi_F \circ \varphi : D \longrightarrow F$ is Γ-rational if: (1) the factor space $\varphi^*(\Delta)/\Gamma \cap \varphi^*(\Delta)$ is compact and (2) the subgroup of Γ consisting of the fibration preserving automorphisms induces a discrete subgroup $\Gamma(F)$ of automorphisms of F. By an abuse of notation, we also regard Γ as a group of automorphisms of $\mathcal{S}$ via φ, and identify e.g. $\Delta/\Gamma \cap \Delta$ with $\varphi^*(\Delta)/\Gamma \cap \varphi^*(\Delta)$. Let Δ_0 be the set of parallel translations of the form $z \longrightarrow z + a$, $a \in \mathbb{R}$, $u \longrightarrow u$, $t \longrightarrow t$. Then $\Delta_0 \cap \Gamma$ is a commutative group with m generators, more accurately $\Delta_0 \cap \Gamma$ is a lattice for Δ_0, since Γ is arithmetic and the fibration is Γ-rational. Let D^* be the set-theoretic union of the domain D and the domains F that appear as the bases of Γ-rational fibrations (D is also regarded as a base). The action of the group Γ is naturally defined on the space D^* (see the following diagram).

$$\begin{array}{ccccc} D & \xrightarrow{\varphi} & \mathcal{S} & \longrightarrow & F \\ \downarrow{\scriptstyle \gamma \in \Gamma} & & \| & & \downarrow{\scriptstyle \gamma' \in \Gamma(F)} \\ D & \xrightarrow{\varphi'} & \mathcal{S} & \longrightarrow & F \end{array}$$

We denote by $p_F : F \longrightarrow F/\Gamma(F)$ the natural projections. Clearly, $F/\Gamma(F) \subset D^*/\Gamma$ for any base F. The following is due to I.I. Pyatetskii-Shapiro [22] and Baily-Borel [4].

Theorem 2.1. *D/Γ has a compactification $\overline{D/\Gamma}$ as an irreducible normal analytic space such that there exists a one-to-one map $\iota : D^*/\Gamma \longrightarrow \overline{D/\Gamma}$ such that, $\iota | F/\Gamma(F)$ is an isomorphic embedding onto a locally closed* *analytic subset of* $\overline{D/\Gamma}$. *The topology of* $\overline{D/\Gamma}$ *satisfies the following property: Let* $x\in\iota(F/\Gamma(F))$. *Then, for some compact subset* $W\subset \boldsymbol{C}^{\ell}$, *the closures of the images of the sets* $\{(z, u, t)\in \mathcal{S}\ ;\ \mathrm{Im}z-\mathrm{Re}L_t(u, u)\in V+v_0,\ u\in W,\ t\in T\}$ *under* ι *constitute a system of neighbourhoods of* x. *Here* v_0 *and* T *run through* V *and the neighbourhoods of* $p_F^{-1}(x)$ *in* F, *respectively.*

In what follows we shall identify D^*/Γ with $\overline{D/\Gamma}$ by the above theorem whose proof we omitted because of its length. Let $x\in F/\Gamma(F)\subset D^*/\Gamma$ be any point. Then holomorphic function f around x are described by the Fourier-Jacobi series

$$f=\sum\theta_\chi(u, t)\exp 2\pi\sqrt{-1}\langle\chi, z\rangle$$

$$\langle\chi, z\rangle=\sum_{j=1}^{m}\chi_j z_j\,,$$

where χ runs through the lattice $\Lambda\subset Hom(\Delta_0, \boldsymbol{R})$ which is the dual lattice of $\Delta_0\cap\Gamma\subset\Delta_0$ over $\boldsymbol{Z}$. We note that θ_χ are nothing but the sections of a line bundle $\mathcal{L}_\chi$ over a family of complex tori (cf. [3] p.318). A description of $\mathcal{L}_\chi$ is as follows.

The invariance of f under $\Delta\cap\Gamma$ shows:

$$\theta_\chi(u+b(t), t)$$
$$=\theta_\chi(u, t)\exp 2\pi\sqrt{-1}<\chi,$$
$$-2\sqrt{-1}L_t(u, b(t))-\sqrt{-1}L_t(b(t), b(t))-a(u, t)\rangle\,,$$

if the transformation $z\longrightarrow z+a(u, t)$, $u\longrightarrow u+b(t)$, $t\longrightarrow t$ belongs to Γ. Note that $a(u, t)$ is uniquely determined by $b(t)$ modulo $\Delta_0\cap\Gamma$. Hence the right hand side of the above equality may well be expressed as $\theta_\chi(u, t)e_{\chi,b(t)}(u)$. The action of $\Delta\cap\Gamma$ on $\boldsymbol{C}^{\ell}\times F$ is well-defined and the line bundle $\mathcal{L}_\chi$ is defined on the family of complex tori $\boldsymbol{C}^{\ell}\times F/\Delta\cap\Gamma\longrightarrow F$ as $\boldsymbol{C}\times\boldsymbol{C}^{\ell}\times F$ modulo the following action of $\Delta\cap\Gamma$: $(s, u, t)\longrightarrow(e_{\chi,b(t)}(u)s, u+b(t), t)$.

Proposition 2.2. *If* Γ *is neat, then there exist an ideal sheaf* $\mathcal{J}$ *supported on* $\bigcup_{F\neq D}F/\Gamma(F)$ *and a convex set* $C_F\subset V^*=\{\rho\in Hom(\Delta_0, \boldsymbol{R})\ ;\ \rho(v)>0$ *on* $\bar{V}\backslash\{0\}\}$ *for each base* F, *such that*

(1) $$f\in\mathcal{J}_x\Longleftrightarrow f=\sum_{\chi\in C_F\cap\Lambda}\theta_\chi(u, t)\exp 2\pi\sqrt{-1}\langle\chi, z\rangle$$

around $x\in F/\Gamma(F)$.

(2) $\mathcal{L}_\chi|_{\pi_F^{-1}(t)}$ *is very ample for any* $\chi \in C_F \cap \Lambda$ *and* $t \in F$.

Proof. See [3] p.323 Proposition 5 (cf. also p.318).

Now we turn to the question of hyper q-concavity of the manifold D/Γ in case Γ is torsion free. We may assume that Γ is neat in virtue of Borel's theorem which states: every arithmetic subgroup of automorphisms of D contains a neat subgroup of finite index (cf. [9]). Let F be any base, $x \in F/\Gamma(F)$ and $U \subset \mathcal{S}$ be an open subset of the form

$$\{(z, u, t) \in \mathcal{S}\ ;\ \mathrm{Im}z - \mathrm{Re}L_t(u, u) \in V + v_0,\ u \in W,\ t \in T\},$$

where v_0, W and T are as above. It is easy to see that the Bergman metric $ds_{\mathcal{S}}^2$ of $\mathcal{S}$ satisfies an estimate;

$$\text{(3)}\quad A_1(t, u)\sum_{p=1}^{k} dt_p d\bar{t}_p \leq ds_{\mathcal{S}}^2$$

$$\leq A_2(T, u)\left\{\sum_{p=1}^{k} dt_p d\bar{t}_p + \frac{\sum_{q=1}^{\ell} du_q d\bar{u}_q}{\langle \mathrm{Im}z\rangle} + \frac{\sum_{r=1}^{m} dz_r d\bar{z}_r}{\langle \mathrm{Im}z\rangle^2}\right\}$$

on U. Here we put $\langle \mathrm{Im}z\rangle := \inf_r |\mathrm{Im}z_r| + 1$ and $A_1(t, u)$, $A_2(t, u)$ are positive continuous functions on $W \times T$.

Let $f = (f_1, \cdots, f_d)$ be a system of generators of $\mathcal{J}_x$ around x. Then, for a suitable choices of v_0 and T, f is defined on U. We put $\phi_x := |f|^2$. Then $\partial\bar{\partial}\phi_x = \partial f \wedge {}^t\bar{\partial}\bar{f}$. By (1) and (2), $\partial\bar{\partial}\phi_x$ dominates asymptotically $\phi_x ds_F{}^2$ near x. Thus from (3) it follows that the eigenvalues $\lambda_1(y) \geq \cdots \geq \lambda_{\ell+m+k}(y)$, $y \in U$, of the Levi form $\sqrt{-1}\partial\bar{\partial}\phi_x$ with respect to $ds_{\mathcal{S}}^2$ satisfy:

$$\text{(4)}\qquad \lambda_j(y) > 0 \quad \text{for any} \quad j$$

and

$$\text{(5)}\qquad \lim_{y \to x} \frac{\lambda_j(y)}{\phi_x(y)} = \infty \quad \text{for} \quad 1 \leq j \leq m + \ell\,,$$

since $\langle \mathrm{Im}z\rangle \longrightarrow \infty$ as U shrinks to x.

From (3) one can find a C^∞ partition of unity $\{\rho_\alpha\}$ of the space D^*/Γ such that $|d\rho_\alpha|$ are bounded on D/Γ.

Therefore, patching the functions ϕ_x and the constants, say 1, we obtain a C^∞ defining function ϕ of the boundary $\bigcup_{F \neq D} F/\Gamma(F)$ on D^*/Γ such that the domain $\{y \in D/\Gamma\ ;\ \phi(y) < \varepsilon\}$ is hyper $(q+1)$-convex for sufficiently small ε. Here $q=\max_{F\neq D} k$.

According to the classification table of the bases in [23], p.114~p.118, the maximal dimensions of $F(\neq D)$ are as follows.

$I_{m,m'}(m\leq m')$: $(m-1)^2$ if $m=m'$ or $m=1$, m^2 otherwise.

II_m : $\frac{1}{2}(m-1)(m-2)$.

III_m : $\frac{1}{2}m(m-1)$.

IV_m : 1.

V : 8.

VI : 10.

Thus we obtain the following.

Proposition 2.3 *Let D be an irreducible bounded symmetric domain of dimension n. If D is one of the followings, then D/Γ is hyper $(n-2)$-concave for any torsion free arithmetic subgroup Γ of automorphisms of D.*

$I_{m,m'}(m\leq m')$: $m=1$ *and* $m'\geq 3$, $m=2$ *and* $m'\geq 4$, *or* $m\geq 3$.

II_m : $m\geq 4$.

III_m : $m\geq 3$.

IV_m : $m\geq 4$.

V, VI.

§ 3. Proof of Theorem

Let X be a complex manifold of dimension n whose universal covering D is biholomorphic to a bounded domain which appears in the list of Proposition 2.3. Let ds^2 be the metric on X that is induced by the Bergman metric of D. We know already that X is hyper $(n-2)$-concave. In order to be able to apply Theorem 1.7 we need to know the $(n-2)$-positivity of the tangent bundle of X which we denote by T_X. The curvaure of T_X has been calculated in [8] and [10] to which we owe the following.

Proposition 3.1. *Let (X, ds^2) and D be as above. If we assume* *moreover* $m+m'\geq 4$ *in case* D *is of type* $I_{m,m'}$, *then the dual of the bundle* (T_X, ds^2) *is* $(n-2)$*-positive.*

Proof. See [10], Table 1 in p.499.

By Andreotti- Grauert's theorem (cf. [1]), $H^k(X, \mathcal{F})$ is finite dimensional for any locally free analytic sheaf $\mathcal{F}$ over X if $k=0, 1$. Hence, $\dim H^1(X, \Theta)=\dim H_0^{n,n-1}(X, T_X{}^*)=0$ if X satisfies the condition of Proposition 3.1.

Thus we have accomplished the proof of Theorem.

A consequence of Theorem is the rigidity of X.

Definition. A complex manifold X is said to be analytically rigid if, for any complex analytic family $\pi : \mathcal{X}\longrightarrow B$ with $\pi^{-1}(o)\cong X$ for some $o\in B$ and a compact subset $K\subset\pi^{-1}(o)$, there exists a neighbourhood $U\supset K$ in $\mathcal{X}$ and a biholomophic map $U\xrightarrow{\eta}\pi(U)\times(U\cap\pi^{-1}(o))$ such that $\pi\circ\eta^{-1}$ is the projection to the first factor.

Similarly as in the compact case we have the following criterion for the local rigidity.

Theorem 3.2. *Let* M *be a complex manifold of dimension* m. *Suppose that* M *is* $(m-1)$*-concave, i.e. there exists a* C^∞ *function* $\varphi : M\longrightarrow\mathbb{R}$ *such that the subsets* $\{x\in M\,;\ \varphi(x)>c\}$ *are relatively compact for any* $c>\inf\varphi$ *and* $\sqrt{-1}\partial\bar{\partial}\varphi$ *has at least* 2 *positive eigenvalues outside a compact subset of* M. *Then* M *is analytically rigid in the above sense if* $H^1(M, \Theta)=0$.

Proof. Similar as in [19], Theorem 3.2.

Clearly every hyper $(n-2)$-concave manifold is $(n-1)$-concave. Thus we obtain our corollary stated in the introduction.

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