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Asymptotic Expansions of Distribution Solutions of Some Fuchsian Hyperbolic Equations

By

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Introduction

In [3] a class of Fuchsian hyperbolic operators has been considered and a general result was given concerning the structure of distribution solutions defined in a full neighborhood of a point of the characteristic hypersurface $t=0$. The operators treated in [3] are strictly hyperbolic for $t\neq 0$.

Our aim is to consider in this paper the more general case where the operators are strictly hyperbolic only for $t>0$. For some results in this direction, see Bernardi [1]. In this Introduction we state our main result in a particular, but typical, example.

Consider the following operator:

$$(0.1) \qquad P=(t\partial_t)^2-t\sum_{j=1}^{n}\partial_{x_j}^2+\alpha(t,x)t\partial_t+\sum_{j=1}^{n}\beta_j(t,x)t\partial_{x_j}+\gamma(t,x)$$

defined on some neighborhood Ω of the origin in $\mathbf{R}_t\times\mathbf{R}_x^n$ (the coefficients are supposed to be in $C^\infty(\Omega)$).

We explicitly remark that the results contained in [3] cannot be applied directly to the operator (0.1), since P is hyperbolic only for $t>0$. Denote by $\rho_1(x)$, $\rho_2(x)$ the roots of the indicial equation

$$(0.2) \qquad \rho^2+\alpha(0,x)\rho+\gamma(0,x)=0$$

and denote by $\mathscr{D}'_+$ the set of all germs of distributions $u(t,x)$ defined on some $\Omega'\cap\{t>0\}$, with Ω' an open neighborhood of $(t=0, x=0)$. Then we have the following result.

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Theorem. *Suppose that*:

$$\rho_1(0),\ \rho_2(0) \notin \frac{1}{2}\boldsymbol{Z};\ \rho_1(0)-\rho_2(0) \notin \frac{1}{2}\boldsymbol{Z}.$$

Then:

i) *Every* $u \in \mathscr{D}'_+$ *for which* $Pu=0$ *has the asymptotic expansion as* $t \to 0+$:

$$u(t, x) \sim \sum_{j=1,2} [\varphi_j(x) t^{\rho_j(x)} + \sum_{k=1}^{\infty} \sum_{h=0}^{k} (L_{j,k,h}(x, \partial_x)\varphi_j(x)) \times t^{\rho_j(x)+k/2} (\log t)^h] \tag{0.3}$$

for some unique germs of distributions $\varphi_1(x)$, $\varphi_2(x)$ *defined near* $x=0$, *where the* $L_{j,k,h}(x, \partial_x)$ *are linear differential operators* (*with smooth coefficients*) *depending only on* P.

ii) *Conversely, for every germs* $\varphi_1(x)$, $\varphi_2(x)$ *there exists a unique germ* $u \in \mathscr{D}'_+$ *satisfying* $Pu=0$ *and having the asymptotic expansion* (0.3) *as* $t \to 0+$.

The precise meaning of the expansion (0.3) will be defined in Sect. 3.

The example (0.1) is a particular case of the class of operators we will consider. Actually we shall prove that asymptotic expansions as (0.3) hold for solutions of equations of the following form:

$$Pu = \sum_{j+|\alpha| \le m} a_{j,\alpha}(t, x) (t\partial_t)^j (t^{1/k}\partial_x)^\alpha u = 0, \qquad t>0,$$

where k is a positive integer (in the case (0.1) $k=2$). For precise definitions, see Sect. 1.

It is worth to mention that asymptotic expansions like (0.3) were established first in [5] for C^∞-solutions on $t>0$. The possibility of passing from C^∞ to distribution solutions relies on two essential results: an extendability theorem, which is proved in Sect. 2, and the local representation formula proved in [3].

§ 1. Class of Operators

We consider operators P of the form:

$$P = \sum_{j+|\alpha| \le m} a_{j,\alpha}(t, x) (t\partial_t)^j (t^{1/k}\partial_x)^\alpha \tag{1.1}$$

defined on some box $[0, T[\times U \subset \overline{\boldsymbol{R}_t^+} \times \boldsymbol{R}_x^n$, where U is a neighborhood of $x=0$ and $0<T\leq+\infty$; m and k are positive integers.

We shall make the following assumptions on P:

i) $a_{m,0}\neq 0$ on $[0, T[\times U$.

ii) For every j, α, $a_{j,\alpha}\in C^\infty(]0, T[\times U)\cap C^0([0, T[\times U)$

and has the following expansion

$$a_{j,\alpha}\sim\sum_{l=0}^{\infty} a_{j,\alpha,l}(x)t^{l/k}, \quad \text{as } t\longrightarrow 0+, \tag{1.2}$$

for some $a_{j,\alpha,l}\in C^\infty(U)$, $l\geq 0$. The expansion (1.2) means that for every $N\in \boldsymbol{Z}_+$ and for every $h\in \boldsymbol{Z}_+$ we have:

$$t^{h-N/k}\partial_t^h[a_{j,\alpha}(t, x)-\sum_{l=0}^{N} a_{j,\alpha,l}(x)t^{l/k}]\longrightarrow 0 \tag{1.3}$$

in $\mathscr{E}(U)$ as $t\to 0+$.

When $k=1$, the expansion (1.2) is equivalent to say that $a_{j,\alpha}\in C^\infty([0, T[\times U)$.

iii) For every $(t, x, \xi)\in[0, T[\times U\times(\boldsymbol{R}^n\backslash 0)$, the polynomial

$$\lambda\longrightarrow\sum_{j+|\alpha|=m} a_{j,\alpha}(t, x)\lambda^j\xi^\alpha$$

has m real distinct roots $\lambda_1(t, x, \xi), \dots, \lambda_m(t, x, \xi)$.

We denote by $\Phi^m_{1/k}([0, T[\times U)$ the class of all operators as P. For any $P\in\Phi^m_{1/k}([0, T[\times U)$ we define the indicial polynomial by

$$I_P(x;\rho)=\sum_{j=0}^{m} a_{j,0}(0, x)\rho^j. \tag{1.4}$$

The roots of $I_P(x;\rho)$ will be denoted by $\rho_1(x), \dots, \rho_m(x)$. We observe that the change of variables $(t, x)\to(t^{1/k}, x)$ transforms operators P in the class $\Phi^m_{1/k}$ into operators $\tilde{P}\in\Phi^m_1$ and we have the identity

$$I_{\tilde{P}}(x;\rho)=I_P(x;\rho/k). \tag{1.5}$$

In Sect. 2 we shall consider only the case $k=1$ and we need operators defined in a full box neighborhood $]-T, T[\times U$ of $(t=0, x=0)$ and satisfying conditions i)$\sim$iii) on $]-T, T[\times U$. We denote by $\Phi^m_1(]-T, T[\times U)$ the class of such operators and remark that every $P\in\Phi^m_1([0, T[\times U)$ has an extension $\hat{P}\in\Phi^m_1(]-T, T[\times U)$.

§ 2. Extendability Results

In this Section we shall prove some preliminary results to our main theorem.

Theorem 1. *Let* $P\in\Phi_1^m(]-T,T[\times U)$ *and suppose that the roots* $\rho_j(x)$, $j=1,\dots,m$, *of the indicial polynomial satisfy the condition*:

$$\rho_j(0)\notin\{-1,-2,\dots,-n,\dots\},\quad j=1,\dots,m. \tag{2.1}$$

Then, for every distribution $u\in\mathcal{D}'(]0,\varepsilon[\times\omega)$, *with* ω *a neighborhood of* $x=0$ *and* $]0,\varepsilon[\times\omega\subset]0,T[\times U$, *for which* $Pu=0$ *on* $]0,\varepsilon[\times\omega$, *there exists a distribution* v *such that*:

1) v *is defined on some neighborhood* $]-\varepsilon',\varepsilon'[\times\omega'\subset]-T,T[\times U$ *of the origin and* $Pv=0$ *on* $]-\varepsilon',\varepsilon'[\times\omega'$.

2) $v|_{]0,\varepsilon'[\times\omega'}=u|_{]0,\varepsilon'[\times\omega'}$.

The proof will follow from some lemmas.

Lemma 1. *Let* $P\in\Phi_1^m(]-T,T[\times U)$ *and let* $u\in\mathcal{D}'(]0,\varepsilon[\times\omega)$ *be as in the statement of Theorem 1. Then there is a distribution* $w\in\mathcal{D}'(\boldsymbol{R}_t\times\boldsymbol{R}^n)$ *such that*:

i) $\operatorname{supp}(w)\subset\overline{\boldsymbol{R}_t^+}\times\boldsymbol{R}_x^n$.

ii) $w|_{]0,\varepsilon'[\times\omega'}=u|_{]0,\varepsilon'[\times\omega'}$ *for some neighborhood* $]0,\varepsilon'[\times\omega'\subset]0,\varepsilon[\times\omega$.

Lemma 2. *Let* $P\in\Phi_1^m(]-T,T[\times U)$ *and suppose that the roots* $\rho_j(x)$, $j=1,\dots,m$, *of the indicial polynomial satisfy the condition*:

$$\rho_j(x)\notin\{-1,-2,\dots,-n,\dots\},\quad j=1,\dots,m,\quad x\in U. \tag{2.2}$$

Then, for every $f\in\mathcal{D}'(]-T,T[\times U)$ *with* $\operatorname{supp}(f)\subset\{t=0\}$, *there exists a unique* $g\in\mathcal{D}'(]-T,T[\times U)$ *with* $\operatorname{supp}(g)\subset\{t=0\}$ *such that* $Pg=f$ *on* $]-T,T[\times U$.

We now show how the two lemmas imply Theorem 1.

Proof of Theorem 1. Given u as in the statement, let $w\in\mathcal{D}'(\boldsymbol{R}_t\times\boldsymbol{R}^n)$ be as in Lemma 1. Put $\tilde{u}=w|_{]-\varepsilon',\varepsilon'[\times\omega'}$ and let $f=P\tilde{u}$. Then $f\in\mathcal{D}'(]-\varepsilon',\varepsilon'[\times\omega')$ with $\operatorname{supp}(f)\subset\{t=0\}$. By shrinking ω' and taking into account (2.1), we can suppose that condition (2.2) holds for every $x\in\omega'$. Application of Lemma 2 yields a distribution $g\in\mathcal{D}'(]-\varepsilon',\varepsilon'[\times\omega')$ with $\operatorname{supp}(g)\subset\{t=0\}$ and $Pg=f$. By defining

$v=\tilde{u}-g$, the theorem is proved.

We now prove the lemmas.

Proof of Lemma 1. By a modification of P outside a neighborhood of the origin we can suppose that $P\in\Phi_T^m(]-T,T[\times\mathbb{R}^n)$ with constant coefficients $a_{j,\alpha}$ for $|t|+|x|$ large enough. By using a bounded domain of dependence argument and partial hypoellipticity of P for $t\neq 0$ we can suppose that $u\in C^\infty(]0,\varepsilon[;\mathscr{D}'(\mathbb{R}^n))$ and $Pu=0$ on $]0,\varepsilon[\times\mathbb{R}^n$.

We now prove that for every $K\Subset\mathbb{R}^n$ there exists a positive number a such that:

$$t^a\langle u(t,\cdot),\ \varphi(\cdot)\rangle_{\mathscr{D}'(\mathbb{R}^n)\times\mathscr{D}(\mathbb{R}^n)}=O(1),\ \text{as } t\longrightarrow 0+, \tag{2.3}$$

uniformly with respect to bounded sets of $\varphi\in\mathscr{D}(K)$.

It is easy to show that property (2.3) implies the extendability of u and hence the lemma.

Now let us fix $K\Subset\mathbb{R}^n$. By a cut-off argument we can find a distribution $v\in C^\infty(]0,\varepsilon[;H^{-\infty}(\mathbb{R}^n))$ such that $Pv=0$ on $]0,\varepsilon[\times\mathbb{R}^n$ and $u=v$ on $]0,\varepsilon[\times K$.

Now, following [2; p. 185], define

$$v_j^{(h)}=(t\Lambda)^{m-h-j+1}(t\partial_t)^{j-1}v,\ h=1,\dots,m,\quad j=1,\dots,m-h+1, \tag{2.4}$$

where $\Lambda=(1+|D_x|^2)^{1/2}$.

The vector $\vec{v}=(v_1^{(1)},\dots,v_m^{(1)},v_1^{(2)},\dots,v_{m-1}^{(2)},\dots,v_1^{(m)})\in C^\infty(]0,\varepsilon[;H^{-\infty}(\mathbb{R}^n)^N)$, $N=m(m+1)/2$, satisfies a first order system on $]0,\varepsilon[\times\mathbb{R}^n$:

$$I_N t\partial_t\vec{v}=tA(t,x,D_x)\vec{v}+B(t,x,D_x)\vec{v}, \tag{2.5}$$

where:

i) $A(t,x,D_x)$ is an $N\times N$ matrix of classical pseudodifferential operators of order 1 (depending smoothly on $t\in[-T,T]$ and satisfying uniform estimates on (x,ξ)). The principal symbol of A has the following form

$$\sigma_1(A)(t,x,\xi)=\begin{matrix} & \overbrace{}^{m} & \\ m\Bigg\{ & \left[\begin{array}{c:c} A'(t,x,\xi) & \square \\ \hdashline \square & \square \end{array}\right] & \Big\}N-m \\ & & \underbrace{}_{N-m} \end{matrix}, \tag{2.6}$$

where the matrix $A'(t, x, \xi)$ has the roots $\lambda_j(t, x, \xi)$, $j=1, \dots, m$, as eigenvalues.

ii) $B(t, x, D_x)$ is an $N\times N$ matrix of classical pseudo differential operators of order 0 (depending smoothly on $t\in[-T, T]$ and satisfying uniform estimates on (x, ξ)).

For any $a>0$, define $\vec{v}_a=t^a\vec{v}$; then $\vec{v}_a$ satisfies the system:

$$(2.7)\qquad \begin{cases} \mathscr{P}_a\vec{v}_a=0 & \text{on }]0, \varepsilon[\times \boldsymbol{R}^n, \\ \mathscr{P}_a=I_N t\partial_t-tA-(B+aI_N). \end{cases}$$

To prove (2.3) it will be enough to show that there exists $a_0>0$ such that:

$$(2.8)\qquad t<\vec{v}_a(t, \cdot), \vec{\varphi}(\cdot)>_{H^{-\infty}\times H^{\infty}}=O(1), \text{ as } t\longrightarrow 0+,$$

uniformly with respect to bounded sets of $\vec{\varphi}\in H^{\infty}(\boldsymbol{R}^n)^N$ and $a\geq a_0$.

For the adjoint system $\mathscr{P}_a^*=-I_N t\partial_t-tA^*-(B_*+(a+1)I_N)$ and for every $s\in]0, \varepsilon[$, consider the following Cauchy problem:

$$(2.9)\qquad \begin{cases} \mathscr{P}_a^*\vec{\phi}_s=0 & \text{on }]0, \varepsilon[\times \boldsymbol{R}^n, \\ \vec{\phi}_s|_{t=s}=\vec{\varphi}\in H^{\infty}(\boldsymbol{R}^n)^N. \end{cases}$$

Since for $t>0$ $\mathscr{P}_a^*$ is a symmetrizable hyperbolic system (see e.g. [6]), we know that (2.9) has a unique solution $\vec{\phi}_s\in C^{\infty}(]0, \varepsilon[; H^{\infty}(\boldsymbol{R}^n)^N)$.

For every $(s, S)\in\Delta=\{(s, S)\,|0<s<S<\varepsilon\}$ from the identity

$$0=\int_s^S<\mathscr{P}_a\vec{v}_a(t), \vec{\phi}_s(t)>dt-\int_s^S<\vec{v}_a(t), \mathscr{P}_a^*\vec{\phi}_s(t)>dt$$

we get the relation:

$$(2.10)\qquad s<\vec{v}_a(s), \vec{\varphi}>=S<\vec{v}_a(S), \vec{\phi}_s(S)>.$$

To prove (2.8) it is enough to show that for any $k\in\boldsymbol{Z}_+$ and any bounded subset $\mathscr{B}\subset H^{\infty}(\boldsymbol{R}^n)^N$ we have

$$(2.11)\qquad \sup_{\substack{(s,S)\in\Delta \\ \vec{\varphi}\in\mathscr{B}}}\|\Lambda^k\vec{\phi}_s(S)\|<\infty$$

($\|\cdot\|$ means here the L^2-norm).

We prove (2.11) by induction on k. Denote by $R(t, x, D_x)$ an $N\times N$ matrix of classical pseudodifferential operators of order 0 (depending smoothly on t and satisfying uniform estimates on (x, ξ)) such that:

i) $\sigma_0(R)(t, x, \xi)$ is a symmetrizer for $\sigma_1(A^*)(t, x, \xi)$.

ii) $R=R^*$.

iii) There exists a $\gamma>0$ for which $(R\vec{\varphi}, \vec{\varphi}) \geq \gamma \|\vec{\varphi}\|^2$ for every $\vec{\varphi} \in L^2(\mathbf{R}^n)^N$ and any $t \in [0, \varepsilon]$.

(For the existence of R see e.g. [6]).

To simplify notation we write $\vec{\varphi}$ instead of $\vec{\varphi}_s$. We have:

$$(2.12) \quad t\frac{d}{dt}(R\vec{\varphi}, \vec{\varphi}) = -t((RA^*+AR)\vec{\varphi}, \vec{\varphi}) - ((RB^*+BR)\vec{\varphi}, \vec{\varphi}) -2(a+1)(R\vec{\varphi}, \vec{\varphi}) + t\left(\left(\frac{d}{dt}R\right)\vec{\varphi}, \vec{\varphi}\right)$$
$$\leq Ct\|\vec{\varphi}\|^2 + 2\|RB^*\|\,\|\vec{\varphi}\|^2 - 2(a+1)(R\vec{\varphi}, \vec{\varphi})$$

for some $C>0$ independent of $\vec{\varphi}$ and $t \in]0, \varepsilon[$. Taking into account property iii) of R, from (2.12) we obtain

$$(2.13) \quad t\frac{d}{dt}(R\vec{\varphi}, \vec{\varphi}) \leq \frac{C}{\gamma}t(R\vec{\varphi}, \vec{\varphi}) + \frac{2}{\gamma}\|RB^*\|(R\vec{\varphi}, \vec{\varphi}) -2(a+1)(R\vec{\varphi}, \vec{\varphi}).$$

Choose $a_0>0$ such that

$$(2.14) \quad a_0+1-\frac{1}{\gamma}\sup_{t\in[0,\varepsilon]}\|RB^*(t)\|>0.$$

Then for every $a \geq a_0$ we obtain from (2.13)

$$(2.15) \quad \left(t\frac{d}{dt}-\alpha t+\beta\right)(R\vec{\varphi}, \vec{\varphi}) \leq 0$$

with $\alpha=C/\gamma$, $\beta=2(a+1)-\frac{2}{\gamma}\sup_{t\in[0,\varepsilon]}\|RB^*(t)\|>0$. Inequality (2.15) is equivalent to

$$(2.16) \quad \frac{d}{dt}(e^{-\alpha t}t^{\beta}(R\vec{\varphi}, \vec{\varphi})) \leq 0.$$

Integrating from s to S we get

$$(R(S)\vec{\varphi}_s(S),\ \vec{\varphi}_s(S)) \leq e^{-\alpha(s-S)}\left(\frac{s}{S}\right)^{\beta}(R(s)\vec{\varphi}, \vec{\varphi})$$

and finally, by iii),

$$\|\vec{\varphi}_s(S)\|^2 \leq \frac{1}{\gamma}e^{\alpha\varepsilon}\sup_{t\in[0,\varepsilon]}\|R(t)\|\,\|\vec{\varphi}\|^2,$$

which proves (2.11) for $k=0$.

Suppose now that (2.11) is proved up to $k-1$. Then it is easy to show that $\Lambda^k\vec{\phi}_s$ satisfies the following Cauchy problem:

$$(2.17)\qquad \begin{cases}(\mathscr{P}_a^* + t[A^*, \Lambda^k]\Lambda^{-k})\Lambda^k\vec{\phi}_s = -[B^*, \Lambda^k]\Lambda^{-k+1}\Lambda^{k-1}\vec{\phi}_s,\\ \Lambda^k\vec{\phi}_s\,|_{t=s} = \Lambda^k\vec{\varphi}.\end{cases}$$

Since $[A^*, \Lambda^k]\Lambda^{-k}$ and $[B^*, \Lambda^k]\Lambda^{-k+1}$ are of order 0, proceding as above we obtain

$$(2.18)\qquad \begin{aligned} t\frac{d}{dt}(R\Lambda^k\vec{\phi}, \Lambda^k\vec{\phi}) &\le C_k t\|\Lambda^k\vec{\phi}\|^2 + 2\|RB^*\|\,\|\Lambda^k\vec{\phi}\|^2\\ &\quad -2(a+1)(R\Lambda^k\vec{\phi}, \Lambda^k\vec{\phi}) + C'_k\|\Lambda^{k-1}\vec{\phi}\|\,\|\Lambda^k\vec{\phi}\|\\ &\le C_k t\|\Lambda^k\vec{\phi}\|^2 + (2\|RB^*\| + \delta C'_k)\|\Lambda^k\vec{\phi}\|^2\\ &\quad -2(a+1)(R\Lambda^k\vec{\phi}, \Lambda^k\vec{\phi}) + \frac{C'_k}{\delta}\|\Lambda^{k-1}\vec{\phi}\|^2\end{aligned}$$

for some $C_k, C'_k > 0$ (independent of $\vec{\phi}$ and $t\in]0,\varepsilon[$) and for every $\delta>0$. By choosing δ small enough and $a\ge a_0$ we get

$$(2.19)\qquad \left(t\frac{d}{dt} - \alpha_k t + \beta_k\right)(R\Lambda^k\vec{\phi}, \Lambda^k\vec{\phi}) \le \frac{C'_k}{\delta}\|\Lambda^{k-1}\vec{\phi}\|^2$$

for some $\alpha_k, \beta_k > 0$.

By multiplying both sides of (2.19) for $e^{-\alpha_k t}t^{\beta_k}$ and integrating from s to S we get

$$(2.20)\qquad \begin{aligned}(R(S)\Lambda^k\vec{\phi}_s(S), \Lambda^k\vec{\phi}_s(S)) &\le e^{-\alpha_k(s-S)}\left(\frac{s}{S}\right)^{\beta_k}(R(s)\Lambda^k\vec{\varphi}, \Lambda^k\vec{\varphi})\\ &\quad + \frac{C'_k}{\delta}\frac{1}{S^{\beta_k}}\int_s^S e^{-\alpha_k(\sigma-S)}\sigma^{\beta_k-1}\|\Lambda^{k-1}\vec{\phi}_s(\sigma)\|^2 d\sigma.\end{aligned}$$

Since

$$\frac{1}{S^{\beta_k}}\int_s^S \sigma^{\beta_k-1}d\sigma \le \frac{1}{\beta_k},$$

from (2.20) we obtain

$$\begin{aligned}&(R(S)\Lambda^k\vec{\phi}_s(S), \Lambda^k\vec{\phi}_s(S))\\ &\quad \le e^{\alpha_k\varepsilon}\Big(\sup_{t\in[0,\varepsilon]}\|R(t)\|\,\|\Lambda^k\vec{\varphi}\|^2 + \frac{C'_k}{\delta\beta_k}\sup_{\substack{(s,S)\in\Delta\\ \vec{\varphi}\in\mathscr{B}}}\|\Lambda^{k-1}\vec{\phi}_s(S)\|^2\Big).\end{aligned}$$

By induction the above inequality implies (2.11). The proof of Lemma 1 is completed.

Proof of Lemma 2. Let $f=\sum_{j=0}^{N} f_j(x)\otimes\partial_t^j\delta_t$ and $g=\sum_{j=0}^{N} g_j(x)\otimes\partial_t^j\delta_t$ be two distributions with f_j, $g_j\in\mathscr{D}'(\omega)$, $j=0,\dots,N$, $N\in\mathbb{Z}_+$, $\omega\subset U$. We remark that the operator P can be decomposed as

$$P=I_P(x\,;t\partial_t)-tR(t,x,t\partial_t,\partial_x) \tag{2.21}$$

for some differential operator R with smooth coefficients.

Taking into account the identities

$$(t\partial_t)^j\partial_t^l\delta_t=(-1)^j(1+l)^j\partial_t^l\delta_t,\qquad j,l=0,1,\dots, \tag{2.22}$$

it is easy to see that the equation $Pg=f$ is equivalent to the following triangular system:

$$\begin{cases} I_P(x\,;-(N+1))g_N=f_N, \\ I_P(x\,;-N)g_{N-1}=f_{N-1}+L_{N-1}(g_N), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ I_P(x\,;-1)g_0=f_0+L_0(g_N,g_{N-1},\dots,g_1), \end{cases} \tag{2.23}$$

where L_j, $j\geq 0$, are linear differential operators depending only on P.

Under condition (2.2) system (2.23) is uniquely solvable in $\mathscr{D}'(\omega)$. The proof of the lemma is now a trivial consequence of this remark.

§ 3. Asymptotic Expansions

In this Section we prove the main result of this paper.

Let $P\in\Phi_{1/k}^m([0,T[\times U)$ and denote by $\mathscr{D}'_+$ the set of all distributions defined on some open subset $]0,\varepsilon[\times\omega\subset]0,T[\times U$, ω being a neighborhood of $x=0$. Then we have the following theorem.

Theorem 2. *Suppose that the roots of the indicial polynomial of P satisfy the condition*:

$$\begin{cases} \rho_j(0)\notin\frac{1}{k}\mathbb{Z}, & i,j=1,\dots,m, \\ \rho_i(0)-\rho_j(0)\notin\frac{1}{k}\mathbb{Z}, & i\neq j. \end{cases} \tag{3.1}$$

Then:

i) *For every $u\in\mathscr{D}'_+$ with $Pu=0$ there exist uniquely determined germs* *of distributions* $\varphi_j(x)$, $j=1,\dots,m$, *defined near* $x=0$, *for which the following asymptotic expansion holds as* $t\to 0+$:

$$(3.2)\quad u(t,x)\sim\sum_{j=1}^{m}[\varphi_j(x)t^{\rho_j(x)}+\sum_{l=1}^{\infty}\sum_{h=0}^{l}(L_{j,l,h}(x,\partial_x)\varphi_j(x))t^{\rho_j(x)+l/k}(\log t)^h],$$

where the $L_{j,l,h}(x,\partial_x)$ *are linear differential operators* (*with smooth coefficients*) *depending only on* P.

ii) *Conversely, for every germs* $\varphi_j(x)$, $j=1,\dots,m$, *there exists a unique germ* $u\in\mathscr{D}'_+$ *satisfying* $Pu=0$ *and having the asymptotic expansion* (3.2) *as* $t\to 0+$.

Before proving the Theorem we make precise the meaning of the asymptotic expansion (3.2).

If $u\in\mathscr{D}'(]0,\varepsilon[\times\omega)$, we can suppose that $\varphi_j\in\mathscr{D}'(\omega')$, $j=1,\dots,m$, for some neighborhood of the origin $\omega'\subset\omega$. Since $Pu=0$ on $]0,\varepsilon[\times\omega$, by partial hypoellipticity we have $u\in C^\infty(]0,\varepsilon[;\mathscr{D}'(\omega))$. Moreover, by condition (3.1) and shrinking ω' if necessary we can suppose that the roots $\rho_j(x)$ are smooth functions of $x\in\omega'$.

Now the definition of (3.2) is the following one. *For every* $a>0$ *there exists* $N_0>0$ *such that: for every* $N>N_0$ *and for every* $p\in\mathbf{Z}_+$ *we have*

$$t^{-a}(t\partial_t)^p\{u(t,x)-\sum_{j=1}^{m}[\varphi_j(x)t^{\rho_j(x)}+\sum_{l=1}^{N}\sum_{h=0}^{l}(L_{j,l,h}(x,\partial_x)\varphi_j(x))t^{\rho_j(x)+l/k}(\log t)^h]\}\longrightarrow 0$$

in $\mathscr{D}'(\omega')$ *as* $t\to 0+$.

Proof of Theorem 2. Let $u\in\mathscr{D}'(]0,\varepsilon[\times\omega)$ satisfy $Pu=0$ in $]0,\varepsilon[\times\omega$. Consider the change of variables $\chi(s,x)=(t=s^k,x)$, $s>0$. By the remark in Sect. 1 the operator P is transformed to $\tilde{P}\in\Phi_1^m$ and we can actually suppose that $\tilde{P}\in\Phi_1^m(]-T,T[\times\omega)$, $T=\varepsilon^{1/k}$. Then the distribution $\tilde{u}(s,x)=\chi^*(u)$ satisfies $\tilde{P}\tilde{u}=0$ on $]0,T[\times\omega$. Application of Theorem 1 yields the existence of a distribution $v\in\mathscr{D}'(]-T',T'[\times\omega')$ such that $\tilde{P}v=0$ on $]-T',T'[\times\omega'$ and $v=\tilde{u}$ on $]0,T'[\times\omega'\subset]0,T[\times\omega$.

Now we use Theorem 2 in [3] and can represent v in the following form:

$$(3.3)\quad v(s,x)=\sum_{j=1}^{m}[\int s_+^{k\rho_j(y)}r_j(s,x,y)\varphi_j(y)dy+\int s_-^{k\rho_j(y)}r_j(s,x,y)\psi_j(y)dy]$$

for some germs φ_j, ψ_j, $j=1,\dots,m$, of distributions defined near $x=0$, uniquely determined by v. To obtain (3.3) from Theorem 2 in [3] we use the fact that $k\rho_j$, $j=1,\dots,m$, are the roots of the indicial polynomial of $\tilde{P}$ (see Sect. 1) and note that hypothesis (3.1) is equivalent to the hypothesis in Theorem 2 in [3]. The kernels $r_j(s,x,y)$ are suitable distributions defined near $s=0$, $x=y=0$, satisfying the following conditions:

$$(3.4)\quad \begin{aligned} &1)\ \operatorname{supp}(r_j)\subset\{(s,x,y)\mid |x-y|\le M|s|\},\\ &2)\ \mathrm{WF}(r_j)\subset\{((s,x,\sigma,\xi),(y,\eta))\mid \eta\neq 0,\\ &\qquad |x-y|\le M|s|,\ |\sigma|\le M|\eta|,\ |\xi+\eta|\le M|s|\,|\eta|\}\end{aligned}$$

for some $M>0$.

Furthermore, by the construction performed in [3] it follows that every r_j has an asymptotic expansion of the following form as $s\to 0$:

$$(3.5)\qquad r_j(s,x,y)\sim\delta(x-y)+\sum_{l=1}^{\infty}\sum_{|\alpha|=0}^{l}(c_{j,l,\alpha}(x)\,\partial_x^\alpha\delta(x-y))\,s^l$$

for some C^∞-functions $c_{j,l,\alpha}(x)$ defined in a common neighborhood $\tilde{\omega}\subset\omega$ of $x=0$. The meaning of the expansion (3.5) is the following (noting that $r_j\in C^\infty(]-T',T'[\,;\mathscr{D}'(\tilde{\omega}\times\tilde{\omega}))$ as a consequence of (3.4), 2)): *For every* N, $h\in\boldsymbol{Z}_+$ *we have*

$$s^{-N}(s\partial_s)^h[r_j(s,x,y)-\delta(x-y)-\sum_{l=1}^{N}\sum_{|\alpha|=0}^{l}(c_{j,l,\alpha}(x)\,\partial_x^\alpha\delta(x-y))\,s^l]\longrightarrow 0$$

in $\mathscr{D}'(\tilde{\omega}\times\tilde{\omega})$ *as* $|s|\to 0$.

By restriction to $s>0$ we obtain from (3.3)

$$(3.6)\qquad \tilde{u}(s,x)=\sum_{j=1}^{m}\int s^{k\rho_j(y)}r_j(s,x,y)\,\varphi_j(y)\,dy.$$

By using χ^{-1} and the expansions (3.5) we get (3.2) for $u(t,x)$.

The uniqueness of the φ_j in (3.2) is proved as follows.

Suppose that for some distributions $\psi_j(x)$, $j=1,\dots,m$, defined on some neighborhood $\omega'\subset\omega$ of $x=0$ we have

$$(3.7)\quad 0\sim\sum_{j=1}^{m}[\psi_j(x)\,t^{\rho_j(x)}+\sum_{l=1}^{\infty}\sum_{h=0}^{l}(L_{j,l,h}(x,\partial_x)\psi_j(x))\,t^{\rho_j(x)+l/k}(\log t)^h].$$

We have to show that $\psi_j=0$ near $x=0$ for every j.

We can obviously rearrange the $\rho_j(x)$ in such a way that

$$(3.8)\qquad \begin{aligned}\operatorname{Re}\rho_1(0)&=\cdots=\operatorname{Re}\rho_{k_1}(0)<\operatorname{Re}\rho_{k_1+1}(0)=\cdots\\ &=\operatorname{Re}\rho_{k_1+k_2}(0)<\cdots<\operatorname{Re}\rho_{k_1+\cdots+k_{\nu-1}}(0)\\ &=\cdots=\operatorname{Re}\rho_m(0)\end{aligned}$$

and decompose accordingly $\{1,\dots,m\}=I_1\cup I_2\cup\dots\cup I_\nu$ (disjoint union). We may assume that there exist real numbers $m_1<m_2<\cdots<m_{\nu-1}$ such that

$$(3.9)\qquad \begin{cases}\sup_{j\in I_h} \mathrm{Re}\,\rho_j(x)<m_h<\inf_{j\in I_{h+1}} \mathrm{Re}\,\rho_j(x) \\ \inf_{j\in I_h} \mathrm{Re}\,\rho_j(x)+1/k>m_h\end{cases},\quad x\in\omega',\ h=1,\dots,\nu-1.$$

From (3.7) we obtain

$$t^{-m_1}(t\partial_t)^p[\phi_1(x)t^{\rho_1(x)}+\cdots+\phi_{k_1}(x)t^{\rho_{k_1}(x)}]\longrightarrow 0$$

in $\mathscr{D}'(\omega')$ as $t\to 0+$ for any $p\in\boldsymbol{Z}_+$.

By taking $p=0,1,\dots,k_1-1$ we get

$$(3.10)\qquad t^{-m_1}\begin{bmatrix}1 & \cdots\cdots & 1\\ \rho_1 & \cdots\cdots & \rho_{k_1}\\ \vdots & & \vdots\\ \rho_1^{k_1-1} & \cdots\cdots & \rho_{k_1}^{k_1-1}\end{bmatrix}\begin{bmatrix}\phi_1 t^{\rho_1}\\ \\ \\ \phi_{k_1}t^{\rho_{k_1}}\end{bmatrix}\longrightarrow 0$$

in $\mathscr{D}'(\omega')$ as $t\to 0+$. Since the matrix in (3.10) is invertible (by (3.1)) we get $\phi_j t^{\rho_j-m_1}\to 0$ in $\mathscr{D}'(\omega')$ as $t\to 0+$ for $j=1,\dots,k_1$. By condition (3.9) we conclude that $\phi_j=0$ on ω' for $j=1,\dots,k_1$. As a consequence, $L_{j,l,h}(x,\partial_x)\phi_j=0$ on ω' for every l and h and for $j=1,\dots,k_1$. Hence (3.7) is reduced to

$$0\sim\sum_{j=k_1+1}^{m}[\phi_j(x)t^{\rho_j(x)}+\sum_{l=1}^{\infty}\sum_{h=0}^{l}(L_{j,l,h}(x,\partial_x)\phi_j(x))t^{\rho_j(x)+l/k}(\log t)^h].$$

Using the same procedure as above we conclude that $\phi_j=0$ on ω' for $j=k_1+1,\dots,k_1+k_2$, and so on. Thus, part i) in Theorem 2 is proved.

To prove ii), let $\varphi_j(x)\in\mathscr{D}'(\omega)$, $j=1,\dots,m$, and define

$$(3.11)\qquad v(s,x)=\sum_{j=1}^{m}\int s_+^{k\rho_j(y)}r_j(s,x,y)\varphi_j(y)dy.$$

By Theorem 2 in [3] it follows that $\tilde{P}v=0$ on some box $]-T',T'[\times\omega'$ $(\omega'\subset\omega)$. By defining $u=(\chi^{-1})^*(v|_{\{s>0\}})$ we obtain $u\in\mathscr{D}'_+$, with $Pu=0$, having the asymptotic expansion (3.2).

To prove uniqueness we observe that if two distributions u_1, $u_2\in\mathscr{D}'_+$ satisfy $Pu_1=Pu_2=0$ on some box $]0,\varepsilon[\times\omega$ and if they have the same asymptotic expansion (3.2), then u_1-u_2 satisfies $P(u_1-u_2)=0$ on $]0,\varepsilon[\times\omega$ and it is extendable as a C^∞ function in t up to $t=0$, i.e. $u_1-u_2\in C^\infty([0,\varepsilon'[;\mathscr{D}'(\omega'))$ for some box $[0,\varepsilon'[\times\omega'\subset[0,\varepsilon[\times\omega$. Furthermore, u_1-u_2 is flat at $t=0$. Hence, application of the local uniqueness results of [2, 4] yields that $u_1=u_2$ in a smaller box.

Thus, Theorem 2 is proved.

§ 4. Examples and Remarks

(1) The result stated in the Introduction is a consequence of Theorem 2.

(2) Let $P\in\Phi^m_{1/k}(\boldsymbol{R}^+_t\times\boldsymbol{R}^n_x)$. A consequence of our proof in Sect. 2 is that any distribution u defined in an open subset of $\boldsymbol{R}^+_t\times\boldsymbol{R}^n_x$ near a point $(0,x_0)$ of $\partial(\boldsymbol{R}^+_t\times\boldsymbol{R}^n_x)$ which satisfies $Pu=0$ is extendable as a distribution in a full neighborhood of $(0,x_0)$. Moreover, if the coefficients of P are smooth up to $t=0$ and if the roots $\rho_j(x)$ of the indicial polynomial satisfy the condition $\rho_j(x_0)\notin\{-1,-2,\dots\}$, $j=1,\dots,m$, then u can be extended as a distribution solution $\tilde{u}$ of $P\tilde{u}=0$.

(3) Under the hypotheses of Theorem 2 we can define "boundary values" of a solution u of $Pu=0$ by taking the leading coefficients $\varphi_1,\dots,\varphi_m$ of the asymptotic expansion (3.2) of u.

(4) Let us consider the Fuchsian hyperbolic operators of weight $m-h\geq 0$ considered in [3]:

$$P=t^hP_m+t^{h-1}P_{m-1}+\cdots+P_{m-h}.$$

By combining the results in [3] with the arguments in Theorem 2 one can prove that every local distribution solution u of $Pu=0$, defined in some box $]0,\varepsilon[\times\omega$, has an asymptotic expansion of the form:

$$u\sim\sum_{j=0}^{m-h-1}[\psi_j(x)t^j+\sum_{l=1}^{\infty}(\mathscr{L}_{j,l}(x,\partial_x)\psi_j(x))t^{j+l}]$$
$$+\sum_{j=1}^{h}[\varphi_j(x)t^{\rho_j(x)}+\sum_{l=1}^{\infty}\sum_{i=0}^{l}(L_{j,l,i}(x,\partial_x)\varphi_j(x))t^{\rho_j(x)+l}(\log t)^i],$$

as $t\to 0+$, for some germs of distributions $\psi_0,\dots,\psi_{m-h-1}$, $\varphi_1,\dots,\varphi_h$, provided the non trivial roots of the indicial polynomial $\rho_1(x),\dots,\rho_h(x)$ satisfy the conditions: $\rho_j(0)\notin\boldsymbol{Z}$, $j=1,\dots,h$ and $\rho_j(0)-\rho_{j'}(0)\notin\boldsymbol{Z}$ for every $j,j',j\neq j'$.

(5) The example $P=t\partial_t^2-\Delta_x+\alpha\partial_t+\sum_{j=1}^{n}\beta_j\partial_{x_j}+\gamma$ is not included in our classes and has been already treated by Bernardi [1].

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