

\mathcal{D}_n -Modules with Support on a Curve

By

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§ 0. Introduction

0.1. In [9] (see also [2], [4]) Kashiwara proves the following

Theorem. *Let X be a complex analytic manifold and Y a closed submanifold. Let $i: Y \hookrightarrow X$ be the inclusion. Then the direct image functor i_+ establishes an equivalence between the category of \mathcal{D}_Y -modules and the category of \mathcal{D}_X -modules with support contained in Y .*

What happens in case Y is no longer smooth, but only a closed subvariety? Following [8], [5], [12], [11] one defines the ring $\mathcal{D}(Y)$ of differential operators on Y . In case Y is non singular this definition coincides with the usual one, i. e. the subalgebra of $\text{End}_C(\mathcal{O}_Y)$ generated by \mathcal{O}_Y and $\text{Der}_C(\mathcal{O}_Y)$.

Bloom [5], [6], Vigué [12], Bernstein a. o. [1], and recently Smith and Stafford [11] in the algebraic case, investigated these kind of rings and showed that in general they fail to have some nice properties such as being left or right noetherian. However, as already Bloom and Vigué noticed, in case Y is a curve the situation is more pleasant. Investigations have culminated in a nice THEOREM of Smith and Stafford ([11], Th. B.). Let X be an affine curve and $\pi: \tilde{X} \rightarrow X$ the normalization. Assume π is injective. Then $\mathcal{D}(X)$ is Morita equivalent to $\mathcal{D}(\tilde{X})$. It goes without saying that \tilde{X} is non-singular, hence $\mathcal{D}(\tilde{X})$ is well known. (See e. g. [3]). Using this we are able to modify Kashiwara's theorem as follows.

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Theorem. *Let $(X, 0)$ be an irreducible germ of a curve in $(\mathbf{C}^n, 0)$. Then the category of $\mathcal{D}_{X,0}$ -modules is equivalent to the category of $\mathcal{D}_{n,0}$ -modules with support contained in X .*

0.2. In this paper we take a ringtheoretic point of view. \mathcal{O}_n is the formal (resp. convergent) power series ring in n indeterminates over k , an algebraically closed field of characteristic zero (resp. \mathbf{C}). $\mathfrak{p} \subset \mathcal{O}_n$ is a prime ideal of height $n-1$. Our aim is to prove that the category of \mathcal{D}_n -modules with support $V(\mathfrak{p})$ is equivalent to the category of $\mathcal{D}(\mathcal{O}_n/\mathfrak{p})$ -modules and thus to the category of \mathcal{D}_1 -modules.

In §1 we collect some facts concerning differential operators over a commutative k -algebra. In §2 we introduce the functors which are going to establish the required equivalence. We derive a necessary and sufficient condition for the equivalence to hold. In §3 we investigate “distribution” modules, i. e. $\mathcal{D}(\mathcal{O}_n/\mathfrak{p})$ -modules with support at the origin. We exhibit the equivalence for these modules. In §4 we use Kashiwara’s theorem for the regular case and the result of §3 to obtain that the afore mentioned condition in §2 is fulfilled. §5 contains an application. We show that for irreducible $f \in \mathcal{O}_2 = \mathcal{O}$ the left \mathcal{D} -module $\mathcal{O}_f/\mathcal{O}$ is simple. (Cf [13], [15]).

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§1. Generalities on Differential Operators

1.1. Let A be a commutative k -algebra. Throughout this paper k will denote an algebraically closed field of characteristic zero. Let M and N be A -Modules. One defines $\mathcal{D}_A^n(M, N)$, the space of k -linear differential operators from M to N of order $\leq n$, inductively by $\mathcal{D}_A^{-1}(M, N) := 0$ and for $n \geq 0$

$$\mathcal{D}_A^n(M, N) := \{\theta \in \text{Hom}_k(M, N) \mid [\theta, a] \in \mathcal{D}_A^{n-1}(M, N) \text{ for all } a \in A\}.$$

Put $\mathcal{D}_A(M, N) := \bigcup_{n=0}^{\infty} \mathcal{D}_A^n(M, N)$.

$$\mathcal{D}_A(M) := \mathcal{D}_A(M, M) \text{ is a } k\text{-subalgebra of } \text{End}_k(M).$$

$\mathcal{D}_A(M, N)$ is a $\mathcal{D}_A(N) - \mathcal{D}_A(M)$ bimodule. The module action is given by composition of maps. We refer the reader to the paper of Smith and Stafford [11], §1, where a nice survey of results on differential operators is given. The reader may also consult [8] or [10].

1.2. We would like to add the following observation :

*Let M be an A -module of finite presentation.
Then $\mathcal{D}_A(M, N) \cong \text{Hom}_A(M, \mathcal{D}_A(A, N))$ as A -modules,
where the A -module structure on $\mathcal{D}_A(A, N)$ is the one coming
from the right $\mathcal{D}_A(A)$ -structure.*

The short proof runs as follows :

$$\mathcal{D}_A^n(M, N) \cong \text{Hom}_A(P_A^n \otimes_A M, N) \cong \text{Hom}_A(M, \text{Hom}_A(P_A^n, N)),$$

where $\text{Hom}_A(P_A^n, N)$ is formed by viewing P_A^n as an A -module through the left action of A . (See note below.) $\text{Hom}_A(P_A^n, N)$ is considered as an A -module through the right action of A on P_A^n . As M is finitely presented we may apply [7], §1, Prop. 8a and conclude

$$\lim_n \mathcal{D}_A^n(M, N) \cong \text{Hom}_A(M, \lim_n \text{Hom}_A(P_A^n, N)).$$

Note. Let $A \otimes_k A \xrightarrow{\mu} A$ be the multiplication map $a \otimes b \mapsto ab$. Set $J_A = \text{Ker } \mu$, $P_A^n = A \otimes_k A / J_A^{n+1}$. P_A^n has two structures of an A -module. Namely multiplication on the left, giving the “left” structure and multiplication on the right, giving the “right” structure. (See e.g. [8]). Observe that in case the P_A^n are projective A -modules of finite type, then $\mathcal{D}_A(A, N) = N \otimes_A \mathcal{D}_A(A)$ and $\mathcal{D}_A(A)$ is a flat A -module. This occurs for example if A is a regular k -algebra of finite type or when $A = \mathcal{O}_n$.

1.3. Due to the absence of an appropriate reference we mention the following : (See also [11], end of §4)

Let $I \subset A$ be an ideal, then (by induction on the order of an operator)

$$\mathcal{D}_A(A, A/I) \subset \bigcup_{n=1}^{\infty} \{\theta \in \text{Hom}_k(A, A/I) \mid \theta(I^n) = 0\}.$$

Hence $\mathcal{D}_A(A, A/I) \cong \varinjlim_n \text{Hom}_A(A/I^n, \mathcal{D}_A(A, A/I))$

i. e. $\text{supp}(\mathcal{D}_A(A, A/I)) \subset V(I)$.

In particular, if $\mathfrak{m} \subset A$ is a maximal ideal such that $k \cong A/\mathfrak{m}$, then as one easily verifies by induction on n :

$$\begin{aligned} \mathcal{D}_A(A, k) &\cong \bigcup_{n=1}^{\infty} \{\theta \in \text{Hom}_k(A, k) \mid \theta(\mathfrak{m}^n) = 0\} \\ &\cong \varinjlim_n \text{Hom}_k(A/\mathfrak{m}^n, k). \end{aligned}$$

Hence in case A is a noetherian, local k -algebra with maximal ideal \mathfrak{m} such that $A/\mathfrak{m} \cong k$, then, according to [7], exercise 32 of §1, $\mathcal{D}_A(A, k)$ is a dualizing module for A . So in particular $\mathcal{D}_A(A, k)$ is the injective hull of the A -module k . (This fact was kindly pointed out to us by S. P. Smith).

Note that we are considering $\mathcal{D}_A(A, A/I)$ as an A -module through it's right $\mathcal{D}(A)$ -structure.

1.4. To finish this section we fix the setting for the rest of the paper. Let $n \in \mathbb{N}$, $n \neq 0$. $\mathcal{O} := \mathcal{O}_{n+1}$ denotes the formal (resp. convergent) power series ring in the indeterminates x, x_1, \dots, x_n over k (resp. \mathbb{C}). \mathcal{O}_1 denotes the formal (resp. convergent) power series ring in the indeterminate t over k (resp. \mathbb{C}). \mathcal{O}_0 denotes the formal (resp. convergent) power series ring in the indeterminate x over k (resp. \mathbb{C}).

$$\mathcal{D} := \mathcal{D}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) = \mathcal{O}[\partial, \partial_1, \dots, \partial_n]. \quad \mathcal{D}_1 := \mathcal{D}_{\mathcal{O}_1}(\mathcal{O}_1, \mathcal{O}_1) = \mathcal{O}_1[\partial_t].$$

$\mathfrak{m} := (x, x_1, \dots, x_n)$ denotes the maximal ideal in \mathcal{O} . Let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal of height n such that $x \notin \mathfrak{p}$. $A := \mathcal{O}/\mathfrak{p}$ is a local ring of dimension 1 with maximal ideal $\bar{\mathfrak{m}} = \mathfrak{m}/\mathfrak{p}$. The normalization of A , i. e. the integral closure of A in its field of fractions, is \mathcal{O}_1 .

In the sequel we will identify $\mathcal{O}/\mathfrak{m} = k = A/\bar{\mathfrak{m}}$, $k = \mathcal{O}_1/t\mathcal{O}_1$. We fix once and for all some canonical maps

$$\pi: \mathcal{O} \longrightarrow A \quad \tau: \mathcal{O} \longrightarrow k \quad \bar{\tau}: A \longrightarrow k$$

with $\bar{\tau}\pi = \tau$.

We have

$$\mathcal{D}(A) := \mathcal{D}_A(A, A) \cong I(\mathfrak{p}\mathcal{D})/\mathfrak{p}\mathcal{D} \cong \text{End}_{\mathcal{D}}(\mathcal{D}/\mathfrak{p}\mathcal{D})$$

where $I(\mathfrak{p}\mathcal{D}) =$ the idealizer of $\mathfrak{p}\mathcal{D}$ in \mathcal{D}

$$:= \{D \in \mathcal{D} \mid D(\mathfrak{p}) \subset \mathfrak{p}\}.$$

See [11], 1.6 or [5], [10], [12]. The identification arises as follows. If $D \in \mathcal{D}$ such that $D(\mathfrak{p}) \subset \mathfrak{p}$, then $\pi D(\mathfrak{p}) = 0$. Hence it induces a k -linear map $\bar{D}: A \rightarrow A$ such that $\pi D = \bar{D}\pi$. In fact $\bar{D} \in \mathcal{D}(A)$. Note that 1.2 implies that, at least as left $\mathcal{D}(A)$ -modules, $\mathcal{D}(A) = \mathcal{D}_\theta(A, A) \cong \text{Hom}_\theta(\mathcal{O}/\mathfrak{p}, \mathcal{D}_\theta(\mathcal{O}, A)) = \text{End}_{\mathcal{D}}(\mathcal{D}/\mathfrak{p}\mathcal{D})$.

1.5. Morita equivalence. One has the $\mathcal{D}(A) - \mathcal{D}_1$ bimodule $P := \mathcal{D}_A(\mathcal{O}_1, A)$, which is isomorphic to a right ideal in \mathcal{D}_1 . Hence P is projective and a generator, because $\text{gl. dim } \mathcal{D}_1 = 1$. The rings $\mathcal{D}(A)$ and \mathcal{D}_1 are Morita equivalent if the natural map $P \otimes_{\mathcal{D}_1} \mathcal{O}_1 \rightarrow A, \mathfrak{p} \otimes f \mapsto \mathfrak{p}(f)$ is surjective. (See [11], prop. 3.3).

As P is a left $\mathcal{D}(A)$ -module we only need verify that 1 is in the image. That is the case; arguing as in [11] we get $\text{Ann}_A(\mathcal{O}_1/A) \supset t^N \mathcal{O}_1$, for some $N \in \mathbb{N}$. Put $\mathfrak{p} = \prod_{j=1}^{N-1} (t\delta - j)$ then $\mathfrak{p}(t^j) = 0$, all $j \in \{1, \dots, N-1\}$, $\mathfrak{p}(t^N) = (N-1)! t^N$ and $\mathfrak{p}(1) = (-1)(-2)\dots(-N+1)$, thus $\mathfrak{p} \in P$.

So $\mathcal{D}(A)$ and \mathcal{D}_1 are Morita equivalent. The functor $N \mapsto N \otimes_{\mathcal{D}(A)} P$ from $\text{Mod-}\mathcal{D}(A)$, the category of right $\mathcal{D}(A)$ -modules, to $\text{Mod-}\mathcal{D}_1$, the category of right \mathcal{D}_1 -modules, is an equivalence of categories. The inverse functor is $M \mapsto \text{Hom}_{\mathcal{D}_1}(P, M)$. Similarly $N \mapsto \text{Hom}_{\mathcal{D}(A)}(P, N) = P^* \otimes_{\mathcal{D}(A)} N$ gives an equivalence between $\mathcal{D}(A)\text{-Mod}$, the category of left $\mathcal{D}(A)$ -modules and $\mathcal{D}_1\text{-Mod}$, the category of left \mathcal{D}_1 -modules. One has $P^* := \text{Hom}_{\mathcal{D}(A)}(P, \mathcal{D}(A)) \cong \mathcal{D}_A(A, \mathcal{O}_1)$. The reader is referred to [11], §2, 3 for the details.

Remark. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a curve. Let $\pi: \tilde{X} \rightarrow X$ be the normalization. According to [11], Th. 3.13: If $\#\pi^{-1}(0) = 1$, then $\mathcal{D}_{\tilde{X}, \pi^{-1}(0)}$ is Morita equivalent to $\mathcal{D}_{X,0}$. Now $\#\pi^{-1}(0) =$ number of the irreducible components of the germ $(X, 0)$. Hence in case $(X, 0)$ is an irreducible germ of a curve, $\mathcal{D}_{X,0}$ is Morita equivalent to $\mathcal{D}_{\mathbb{C},0}$. (See also [12], II).

§ 2. The Main Theorem

2.1. As we mentioned in the introduction we want to compare

\mathcal{D} -modules with support on $V(\mathfrak{p})$ and $\mathcal{D}(A)$ -modules. Let $\text{Mod}_-, \mathcal{D}$ denote the category of right \mathcal{D} -modules M such that, considered as \mathcal{O} -module $\text{supp}(M) \subset V(\mathfrak{p})$. It is a full abelian subcategory of $\text{Mod}-\mathcal{D}$, which is closed under extensions. In case A is regular, i.e. $V(\mathfrak{p})$ is non-singular, the A -module $A \otimes_{\mathcal{O}} \mathcal{D}$ can be given the structure of a left $\mathcal{D}(A)$ -module. (See e.g. [2], [3]). This enables one to define inverse images of \mathcal{D} -modules. Now $A \otimes_{\mathcal{O}} \mathcal{D} = \mathcal{D}_{\mathcal{O}}(\mathcal{O}, A)$ and it is not difficult to show that the above mentioned left $\mathcal{D}(A)$ -module structure on $A \otimes_{\mathcal{O}} \mathcal{D}$, in case A is regular, coincides with the usual left $\mathcal{D}(A)$ -module structure on $\mathcal{D}_{\mathcal{O}}(\mathcal{O}, A)$. This motivates the following

Definition. $B := \mathcal{D}_{\mathcal{O}}(\mathcal{O}, A)$.

B is a $\mathcal{D}(A)$ - \mathcal{D} bimodule and as we already saw, $\text{supp}_{\mathcal{O}}(B) \subset V(\mathfrak{p})$, where B is considered as an \mathcal{O} -module via the action of \mathcal{D} . Moreover the natural inclusion

$$\lim_{\substack{\longrightarrow \\ n}} \text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{p}^n, N \otimes_{\mathcal{D}(A)} B) \xrightarrow{\sim} N \otimes_{\mathcal{D}(A)} B$$

is an isomorphism for all $N \in \text{Mod}-\mathcal{D}(A)$.

So $N \otimes_{\mathcal{D}(A)} B$ is a right \mathcal{D} -module with $\text{supp}(N \otimes_{\mathcal{D}(A)} B) \subset V(\mathfrak{p})$.

This justifies the following

Definition. $i_+ : \text{Mod}-\mathcal{D}(A) \rightarrow \text{Mod}_-, \mathcal{D}, \quad N \mapsto N \otimes_{\mathcal{D}(A)} B$
 $i^+ : \text{Mod}-\mathcal{D} \rightarrow \text{Mod}-\mathcal{D}(A), \quad M \mapsto \text{Hom}_{\mathcal{D}}(B, M)$.

We make the following observations:

- i_+ is a left adjoint of i^+ .
- i^+ is left exact; i_+ is right exact.
- If $M \in \text{Mod}_-, \mathcal{D}, M \neq 0$, then $i^+(M) \neq 0$, because $i^+M = \text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{p}, M)$.

2.2 Theorem. i_+ defines an equivalence between the category of right $\mathcal{D}(A)$ -modules and the category of right \mathcal{D} -modules with support on $V(\mathfrak{p})$.

In the remainder of the paper we shall be mainly concerned with

the proof of this theorem. In fact the theorem follows directly from proposition 2 and the corollary to proposition 5. As a first step we have

Proposition 1. i_+ is an exact, faithful functor.

Proof. We want to prove that i_+ is an exact functor. Or what amounts to the same B is a flat $\mathcal{D}(A)$ -module. Now as we already saw $B = \varinjlim B_n$, where $B_n := \text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{p}^n, B)$. By induction on n we show that each B_n is a projective left $\mathcal{D}(A)$ -module. This certainly implies the flatness of B .

$B_1 = \text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{p}, B) = \text{Hom}_{\mathcal{D}}(\mathcal{D}/\mathfrak{p}\mathcal{D}, \mathcal{D}/\mathfrak{p}\mathcal{D}) = \mathcal{D}(A)$, hence projective. For each $n \in \mathbb{N}$ we have an exact sequence of \mathcal{O} -modules

$$\mathfrak{p}^n/\mathfrak{p}^{n+1} \hookrightarrow \mathcal{O}/\mathfrak{p}^{n+1} \twoheadrightarrow \mathcal{O}/\mathfrak{p}^n,$$

which gives rise to an exact sequence of left $\mathcal{D}(A)$ -modules

$$B_n \hookrightarrow B_{n+1} \twoheadrightarrow \text{Hom}_{\mathcal{O}}(\mathfrak{p}^n/\mathfrak{p}^{n+1}, B).$$

Now $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is a \mathcal{O}/\mathfrak{p} -module of finite type. Hence we have a surjection

$$\bigoplus_{i=1}^r \mathcal{O}/\mathfrak{p} \twoheadrightarrow \mathfrak{p}^n/\mathfrak{p}^{n+1}$$

and an injection

$$\text{Hom}_{\mathcal{O}}(\mathfrak{p}^n/\mathfrak{p}^{n+1}, B) \hookrightarrow \bigoplus_{i=1}^r \text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathfrak{p}, B) = \bigoplus_{i=1}^r \mathcal{D}(A).$$

So B_{n+1}/B_n may be identified with a submodule of $\bigoplus_{i=1}^r \mathcal{D}(A)$. Because $\mathcal{D}(A)$ is Morita equivalent to \mathcal{D}_1 , $\text{gl. dim } \mathcal{D}(A) = 1$. This implies that every submodule of a projective $\mathcal{D}(A)$ -module is itself projective. So B_{n+1}/B_n is a projective left $\mathcal{D}(A)$ -module and we have a split exact sequence

$$B_n \hookrightarrow B_{n+1} \twoheadrightarrow B_{n+1}/B_n.$$

By induction on n , B_{n+1} is a projective left $\mathcal{D}(A)$ -module. We can say even more, namely $B \cong B_1 \oplus B_2/B_1 \oplus B_3/B_2 \oplus \dots$. So $B_1 \cong \mathcal{D}(A)$ is a direct sum factor of B and this implies $N \otimes_{\mathcal{D}(A)} B = 0$ iff $N = 0$. Hence i_+ is faithful.

Our aim is to show that i_+ defines an equivalence of categories. Now $\text{Mod}_\nu\text{-}\mathcal{D}$ is closed under extensions in $\text{Mod}\text{-}\mathcal{D}$, hence we should have

$$\text{Ext}_{\mathcal{D}}^1(B, i_+N) \cong \text{Ext}_{\mathcal{D}(A)}^1(\mathcal{D}(A), N) = 0, \text{ all } N \in \text{Mod}\text{-}\mathcal{D}(A).$$

We claim that this is also sufficient. Let us first mention the existence of natural transformations $\eta: 1 \rightarrow i^+i_+$, $\varepsilon: i_+i^+ \rightarrow 1$, arising from the adjointness of i_+ and i^+ .

Proposition 2. *Assume $\text{Ext}_{\mathcal{D}}^1(B, i_+N) = 0$, for all $N \in \text{Mod}\text{-}\mathcal{D}(A)$. Then i_+ is an equivalence of categories.*

Proof. Let N be a right $\mathcal{D}(A)$ -module. Because $\text{gl. dim } \mathcal{D}(A) = 1$, N has a projective resolution of length 1

$$P_1 \longleftarrow P_0 \longrightarrow N.$$

Applying i^+i_+ we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} i^+i_+P_1 & \longleftarrow & i^+i_+P_0 & \longrightarrow & i^+i_+N & \longrightarrow & \text{Ext}_{\mathcal{D}}^1(B, i_+P_1) = 0 \\ \uparrow \eta(P_1) & & \uparrow \eta(P_0) & & \uparrow \eta(N) & & \\ P_1 & \longleftarrow & P_0 & \longrightarrow & N & & . \end{array}$$

Now $\eta(P_1)$ and $\eta(P_0)$ are isomorphisms, [7], §6, Prop. 7. Hence $\eta(N)$ is an isomorphism. Hence η is an equivalence. Furthermore we have for any $M \in \text{Mod}\text{-}\mathcal{D}$ a composition of maps

$$\begin{aligned} i^+M &\xrightarrow{\eta(i^+M)} i^+i_+i^+M \xrightarrow{i^+\varepsilon(M)} i^+M \\ i^+\varepsilon(M) \circ \eta(i^+M) &= 1 \end{aligned}$$

Since $\eta(i^+M)$ is bijective, $i^+\varepsilon(M)$ is bijective. Hence $i^+(\text{Ker } \varepsilon(M)) = 0$, implying $\text{Ker } \varepsilon(M) = 0$ because $\text{Ker } \varepsilon(M)$ is a submodule of i_+i^+M , hence $\text{Ker } \varepsilon(M) \in \text{Mod}_\nu\text{-}\mathcal{D}$.

Consider

$$i_+i^+M \xrightarrow{\varepsilon(M)} M \longrightarrow \text{Coker } \varepsilon(M)$$

Applying i^+ yields an exact sequence

$$i^+i_+i^+M \xrightarrow{i^+\varepsilon(M)} i^+M \longrightarrow i^+\text{Coker } \varepsilon(M) \longrightarrow \text{Ext}_{\mathcal{D}}^1(B, i_+i^+M) = 0.$$

Hence $i^+\text{Coker } \varepsilon(M) = 0$, because $i^+\varepsilon(M)$ is surjective. Now if $M \in \text{Mod}_\nu\text{-}\mathcal{D}$, then $\text{Coker } \varepsilon(M) \in \text{Mod}_\nu\text{-}\mathcal{D}$ and so it follows then $\text{Coker } \varepsilon(M) = 0$.

This proves that $\varepsilon(M)$ is an isomorphism for all $M \in \text{Mod}_\mathfrak{p}\text{-}\mathcal{D}$.

2.3. So we see that a necessary and sufficient condition for i_+ to be an equivalence is the vanishing of the $\text{Ext}_\mathcal{D}(B, i_+N)$. Later we will see that it suffices to show $\text{Ext}_\mathcal{D}^k(B, B) = 0$ for all $k \in \{1, \dots, n\}$. Even $k \in \{1, 2\}$ suffices.

§ 3. The Module $\mathcal{D}_A(A, k)$

3.1. Before we proceed, we focus our attention on a $\mathcal{D}(A)$ -module with support $\{\mathfrak{m}\}$, namely $\mathcal{D}_A(A, k)$.

We already mentioned in the introduction (1.2) that, for A -Modules of finite presentation, one has an isomorphism of A -modules $\mathcal{D}_A(M, N) \cong \text{Hom}_A(M, \mathcal{D}_A(A, N))$.

It follows immediately that

$$i^+(\mathcal{D}_\theta(\mathcal{O}, k)) \cong \text{Hom}_\theta(A, \mathcal{D}_\theta(\mathcal{O}, k)) \cong \mathcal{D}_\theta(A, k) = \mathcal{D}_A(A, k).$$

This means that we have a bijective map

$$\phi: \mathcal{D}_A(A, k) \longrightarrow i^+(\mathcal{D}_\theta(\mathcal{O}, k)),$$

which is A -linear. It is straightforward to check that for all $D \in \mathcal{D}_A(A, k)$, $\phi(D) \in i^+(\mathcal{D}_\theta(\mathcal{O}, k)) = \text{Hom}_\mathcal{D}(B, \mathcal{D}_\theta(\mathcal{O}, k))$ is the \mathcal{D} -linear map

$$E \longrightarrow DE, \text{ all } E \in B = \mathcal{D}_\theta(\mathcal{O}, A).$$

Hence ϕ is a right $\mathcal{D}(A)$ -linear isomorphism.

3.2. Let $I \subset \mathcal{O}$ be an ideal containing \mathfrak{p} and $\bar{I} = I/\mathfrak{p}$ the corresponding ideal in A . Then

$$i_+(\mathcal{D}(A)/\bar{I}\mathcal{D}(A)) = A/\bar{I} \otimes_A \mathcal{D}_\theta(\mathcal{O}, A) = \mathcal{D}_\theta(\mathcal{O}, \mathcal{O}/I).$$

Applied to $I = \mathfrak{m}$ this gives

$$i_+(\mathcal{D}(A)/\mathfrak{m}\mathcal{D}(A)) = \mathcal{D}_\theta(\mathcal{O}, k).$$

The faithfulness of i_+ and the fact that $\mathcal{D}_\theta(\mathcal{O}, k)$ is a simple right \mathcal{D} -module imply then that also $\mathcal{D}(A)/\mathfrak{m}\mathcal{D}(A)$ is a simple right $\mathcal{D}(A)$ -module. Hence the natural map

$$\mathcal{D}(A)/\mathfrak{m}\mathcal{D}(A) \longrightarrow \mathcal{D}_A(A, k)$$

is injective. The surjectivity is established by the following

Lemma 1. $\mathcal{D}_A(A, k)$ is a simple right $\mathcal{D}(A)$ -module.

Proof. Consider the canonical map

$$(\mathcal{D}_1/t\mathcal{D}_1) \otimes_{\mathcal{D}_1} P^* = \mathcal{D}_{\mathcal{O}_1}(\mathcal{O}_1, k) \otimes_{\mathcal{D}_1} \mathcal{D}_A(A, \mathcal{O}_1) \longrightarrow \mathcal{D}_A(A, k).$$

Clearly this map is non zero and hence injective because $\mathcal{D}_1/t\mathcal{D}_1$ is a simple right \mathcal{D}_1 -module. It remains to show the surjectivity. According to 1.5 $P \otimes_{\mathcal{D}_1} P^* \cong \mathcal{D}(A)$.

Hence $1 = \sum_{\alpha \in I} p_\alpha q_\alpha$ (for some finite set I), with $p_\alpha \in P = \mathcal{D}_A(\mathcal{O}_1, A)$, $q_\alpha \in P^* = \mathcal{D}_A(A, \mathcal{O}_1)$, for all $\alpha \in I$.

Now let $\theta \in \mathcal{D}_A(A, k)$. Then $\theta p_\alpha \in \mathcal{D}_{\mathcal{O}_1}(\mathcal{O}_1, k)$, for all $\alpha \in I$, and

$$\sum_{\alpha \in I} (\theta p_\alpha) \otimes q_\alpha \longmapsto \theta \left(\sum_{\alpha \in I} p_\alpha q_\alpha \right) = \theta.$$

Corollary. $i^+ i_+(\mathcal{D}_A(A, k)) \cong \mathcal{D}_A(A, k)$.

3.3. Remark. In case of a right ideal $I \subset \mathcal{D}(A)$, $I = (A_1, \dots, A_m) \mathcal{D}(A)$ one finds a right ideal $J \subset \mathcal{D}$ such that

$$i_+(\mathcal{D}(A)/I) = \mathcal{D}/J.$$

One may argue as follows. Choose a finite presentation of $\mathcal{D}(A)/I$, i. e. an exact sequence

$$\mathcal{D}(A) \xrightarrow{m-\alpha} \mathcal{D}(A) \longrightarrow \mathcal{D}(A)/I.$$

Apply i_+ and recall that $B = \mathcal{D}/\mathfrak{p}\mathcal{D}$; the map $i_+(\alpha)$ lifts to a map $\bar{\alpha} : \mathcal{D}^m \rightarrow \mathcal{D}$ to give a commutative diagram with exact rows

$$\begin{array}{ccc} B \xrightarrow{i_+(\alpha)} B & \longrightarrow & i_+(\mathcal{D}(A)/I) \\ \hat{\uparrow} & & \hat{\uparrow} \\ \mathcal{D}^m \xrightarrow{\bar{\alpha}} \mathcal{D} & \longrightarrow & \mathcal{D}/\text{Im } \bar{\alpha} \end{array}$$

and $\text{Ker } \beta = \mathfrak{p}\mathcal{D}/\text{Im } \bar{\alpha} \cap \mathfrak{p}\mathcal{D} = (\mathfrak{p}\mathcal{D} + \text{Im } \bar{\alpha})/\text{Im } \bar{\alpha}$.

Furthermore $\text{Im } \bar{\alpha} = (D_1, \dots, D_m) \mathcal{D}$ with $\pi D_i = A_i \pi$, all i . So $D_i(\mathfrak{p}) \subset \mathfrak{p}$ and D_i induces $A_i \in \mathcal{D}(A)$. One concludes

$$i_+(\mathcal{D}(A)/I) = \mathcal{D}/J$$

where $J = \mathfrak{p}\mathcal{D} + \text{Im } \bar{\alpha} = \mathfrak{p}\mathcal{D} + (D_1, \dots, D_m) \mathcal{D}$.

§ 4. The Modules $\text{Ext}_\theta^i(A, B)$

4.1. Let us now attack the problem of proving $\text{Ext}_\mathcal{D}^1(B, i_+N) = 0$ for all right $\mathcal{D}(A)$ -modules N .

As noted before such a module N has a projective resolution of length 1

$$P_1 \longleftarrow P_0 \longrightarrow N.$$

Applying i^+i_+ one finds a commutative diagram with exact rows

$$(1) \quad \begin{array}{ccccc} i^+i_+P_1 & \longleftarrow & i^+i_+P_0 & \xrightarrow{\alpha} & i^+i_+N \\ \uparrow \eta^{(P_1)} & & \uparrow \eta^{(P_0)} & & \uparrow \eta^{(N)} \\ P_1 & \longleftarrow & P_0 & \longrightarrow & N \end{array}$$

and long exact sequences

$$(2) \quad \text{Ext}_\mathcal{D}^k(B, i_+P_0) \longrightarrow \text{Ext}_\mathcal{D}^k(B, i_+N) \longrightarrow \text{Ext}_\mathcal{D}^{k+1}(B, i_+P_1) \longrightarrow \text{Ext}_\mathcal{D}^{k+1}(B, i_+P_0).$$

For a projective right $\mathcal{D}(A)$ -module P the obvious map

$$P \otimes_{\mathcal{D}(A)} \text{Ext}_\mathcal{D}^k(B, B) \longrightarrow \text{Ext}_\mathcal{D}^k(B, P \otimes_{\mathcal{D}(A)} B)$$

is an isomorphism. ([7], §6, Prop. 7).

It follows that $\eta(N)$ is injective, establishing again that i_+ is faithful. Furthermore

$$(3) \quad \begin{aligned} \text{Coker } \eta(N) &\cong \text{Coker } \alpha \cong \text{Ker } (\text{Ext}_\mathcal{D}^1(B, i_+P_1) \longrightarrow \text{Ext}_\mathcal{D}^1(B, i_+P_0)) \\ &\cong \text{Ker } (P_1 \otimes_{\mathcal{D}(A)} \text{Ext}_\mathcal{D}^1(B, B) \longrightarrow P_0 \otimes_{\mathcal{D}(A)} \text{Ext}_\mathcal{D}^1(B, B)) \\ &\cong \text{Tor}_1^{\mathcal{D}(A)}(N, \text{Ext}_\mathcal{D}^1(B, B)). \end{aligned}$$

Observe that $\text{Ext}_\mathcal{D}^k(B, B)$, $k \in \{1, \dots, n\}$, has a left and a right $\mathcal{D}(A)$ -module structure. Because we are only interested in the left one, we prefer to write $\text{Ext}_\theta^k(A, B)$ instead of $\text{Ext}_\mathcal{D}^k(B, B)$. [Note that $\text{Ext}_\theta^k(A, B) \cong \text{Ext}_\mathcal{D}^k(B, B)$; this uses $A \otimes_\theta \mathcal{D} = \mathcal{D}_\theta(\theta, A) = B$ and the fact that \mathcal{D} is a flat θ -module. [7], §6, Prop. 8.]

For notational convenience we introduce the left $\mathcal{D}(A)$ -modules

$$C^k := \text{Ext}_\theta^k(A, B), \text{ all } k \in \{1, \dots, n\}.$$

The previous observations (2) and (3) can be reformulated as:

$$\text{Coker } \eta(N) = \text{Tor}_1^{\mathcal{D}(A)}(N, C^1)$$

$$(4) \quad P_0 \otimes_{\mathcal{D}(A)} C^k \longrightarrow \text{Ext}_{\mathcal{D}}^k(B, i_+N) \longrightarrow P_1 \otimes_{\mathcal{D}(A)} C^{k+1} \longrightarrow P_0 \otimes_{\mathcal{D}(A)} C^{k+1}$$

are exact sequences.

4.2. Our aim is to prove $C^k=0$ for all $k \in \{1, \dots, n\}$ and thereby establishing $\text{Ext}_{\mathcal{D}}^k(B, i_+N) = 0$ for all $k \in \{1, \dots, n\}$. The first result to this end is the following proposition, whose proof is postponed till the end of this section. We are still considering $C^k = \text{Ext}_{\mathcal{D}}^k(A, B)$ as a left $\mathcal{D}(A)$ -module. In particular C^k inherits the structure of an A -module. We discard the other A -module structure. Note that in forming $\text{Ext}_{\mathcal{D}}^k(A, B)$, B is viewed as an \mathcal{O} -module through it's right \mathcal{D} -module structure.

Proposition 3. $\text{supp}_A(C^l) \subset \{\bar{m}\}$, all $l \in \{1, \dots, n\}$.

The proposition emphasizes that the left $\mathcal{D}(A)$ -modules C^l are supported on the singular point $\{\bar{m}\}$.

Before we proceed we need a technical

Lemma 2. Let $p \in P^* = \mathcal{D}_A(A, \mathcal{O}_1)$, $m \in \mathbb{N}$, $t^m \in \mathcal{D}(A)$. There exist $q \in P^*$, $N \in \mathbb{N}$ such that $t^N p = q t^m$.

Proof. By induction on the order of p .

- $p \in \mathcal{D}_A^0(A, \mathcal{O}_1) = \text{Hom}_A(A, \mathcal{O}_1)$. Take $q = p$, $N = m$.
- $p \in \mathcal{D}_A^{d+1}(A, \mathcal{O}_1)$. Then $[p, t^m] \in \mathcal{D}_A^d(A, \mathcal{O}_1)$.

Hence there exist $q \in P^*$, $N \in \mathbb{N}$ such that $t^N [p, t^m] = q t^m$. It follows $t^N p t^m - t^{N+m} p = q t^m$, and thus $t^{N+m} p = (t^N p - q) t^m$.

Proposition 4. Let $l \in \{1, \dots, n\}$. Assume $\text{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C^l) = 0$. Then $C^l = 0$.

Proof. Let $l \in \{1, \dots, n\}$ and put $C = C^l$. Assume to the contrary that $C \neq 0$. It implies $P^* \otimes_{\mathcal{D}(A)} C \neq 0$. Hence there exist $p \in P^*$, $c \in C$, such that $p \otimes c \neq 0$. According to proposition 3 some power M of \bar{m} annihilates c . Choose $m \in \mathbb{N}$ big enough such that $t^m \in \bar{m}^M \subset \mathcal{D}(A)$. (This is possible because $\text{Ann}_A(\mathcal{O}_1/A) \neq 0$). By lemma 2 we can find $q \in P^*$, $N \in \mathbb{N}$ such that $t^N p = q t^m$.

It follows that $t^N(p \otimes c) = qt^m \otimes c = q \otimes t^m c = 0$.

We arrive at the conclusion that $P^* \otimes_{\mathcal{D}(A)} C$ contains a non-zero element which is annihilated by t . A contradiction, because

$$\text{Ker}(t \cdot, P^* \otimes_{\mathcal{D}(A)} C) = \text{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_1/t\mathcal{D}_1, P^* \otimes_{\mathcal{D}(A)} C) = \text{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C)$$

which by assumption vanishes.

So we are reduced to prove that all these Tor_1 's vanish. This is the content of

Proposition 5. *For all $l \in \{1, \dots, n\}$ $\text{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C^l) = 0$.*

Proof. By induction on l .

$$l=1 : \text{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C^1) = \text{Coker } \eta(\mathcal{D}_A(A, k)) = 0$$

by the corollary at the end of §3.

Assume the proposition has been proven for $1, \dots, l$.

By the previous proposition $C^l = 0$. Applying the long exact sequence (4) with $k=l$ we get

$$\begin{aligned} \text{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C^{l+1}) &\cong \text{Ext}_{\mathcal{D}}^1(B, i_+ \mathcal{D}_A(A, k)) \\ &\cong \text{Ext}_{\mathcal{O}}^1(A, \mathcal{D}_{\mathcal{O}}(\mathcal{O}, k)) = 0, \end{aligned}$$

because $\mathcal{D}_{\mathcal{O}}(\mathcal{O}, k)$ is an injective \mathcal{O} -module. (See 1.3).

Corollary. $\text{Ext}_{\mathcal{D}}^l(B, i_+ N) = 0$, all $l \in \{1, \dots, n\}$, all $N \in \text{Mod-}\mathcal{D}(A)$.

4.3. Proof of proposition 3. Let $l \in \{1, \dots, n\}$. We have to show that $\text{supp}_A(C^l) \subset \{\bar{m}\}$. Now A is a local ring with only two prime ideals, (0) and \bar{m} . We need only show that $(0) \notin \text{supp}(C^l)$. Now

$$A_{(0)} \otimes_A C^l = \mathcal{O}_{\mathfrak{p}} \otimes_{\mathcal{O}} C^l = \text{Ext}_{\mathcal{O}}^l(A, \mathcal{O}_{\mathfrak{p}} \otimes_{\mathcal{O}} B).$$

Furthermore

$$\mathcal{O}_{\mathfrak{p}} \otimes_{\mathcal{O}} B = \mathcal{O}_{\mathfrak{p}} \otimes_{\mathcal{O}} \mathcal{D}_{\mathcal{O}}(\mathcal{O}, A) = \mathcal{D}_{\mathcal{O}_{\mathfrak{p}}}(\mathcal{O}_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}),$$

so it is a right $\mathcal{D}(\mathcal{O}_{\mathfrak{p}})$ -module and we are done if we can show that

$$\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^l(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}, \mathcal{D}_{\mathcal{O}_{\mathfrak{p}}}(\mathcal{O}_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}})) = 0.$$

In fact we will prove that for any right $\mathcal{D}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}[\partial, \partial_1, \dots, \partial_n]$ -module

M with $\text{supp}(M) \subset V(\mathfrak{p} \mathcal{O}_{\mathfrak{p}})$

$$\text{Ext}_{\mathcal{O}_{\mathfrak{p}}}^l(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p} \mathcal{O}_{\mathfrak{p}}, M) = 0, \text{ all } l \in \{1, \dots, n\}.$$

By a suitable change of coordinates we can manoeuvre ourselves into the following situation (Normalization theorem. See [3], Ch. 3, 3.22) :

(i) $x \notin \mathfrak{p}$ (this we already assumed from start).

(ii) \mathfrak{p} contains an element f_1 such that $f_1 \in \mathcal{O}_0[x_1]$ is an irreducible Weierstrass polynomial in x_1 .

(iii) Let $\Delta \in \mathcal{O}_0$ be the discriminant of f_1 . \mathfrak{p} contains elements f_2, \dots, f_n such that for any $i \in \{2, \dots, n\}$ $f_i = \Delta x_i - T_i$, for some $T_i \in \mathcal{O}_0[x_1]$.

(iv) $\mathfrak{p} \mathcal{O}_{\Delta} = (f_1, \dots, f_n)$.

(v) Notice that $\Delta \notin \mathfrak{p}$; hence $\mathfrak{p} \mathcal{O}_{\mathfrak{p}} = (f_1, \dots, f_n)$ and (f_1, \dots, f_n) is a regular sequence in $\mathfrak{p} \mathcal{O}_{\mathfrak{p}}$.

(vi) $\partial_i(f_i) \in \mathcal{O}_{\mathfrak{p}} - \mathfrak{p} \mathcal{O}_{\mathfrak{p}}$, i. e. $\partial_i(f_i)$ is a unit in $\mathcal{O}_{\mathfrak{p}}$ for all $i \in \{1, \dots, n\}$. $\partial_i(f_j) = 0$, if $i > j$ $i, j \in \{1, \dots, n\}$.

By induction on d one proves

Sublemma. *Let M be a right $\mathcal{O}_{\mathfrak{p}}[\partial_1, \dots, \partial_d]$ -module with $\text{supp}(M) \subset V(f_1, \dots, f_d)$.*

Then (f_1, \dots, f_d) is an M -coregular sequence.

[The reader is referred to [7], §9, No.6 for the definition of coregular sequence.]

Proof of sublemma. By induction on d (f_1, \dots, f_{d-1}) is an M -coregular sequence. We need to verify that right multiplication by f_d is surjective on $M' := \text{Ker}(f_1)_M \cap \dots \cap \text{Ker}(f_{d-1})_M$. Put $f := f_d$, $\delta := \partial_d$. The right $\mathcal{O}_{\mathfrak{p}}[\delta]$ -module M' has $\text{supp}(M') \subset V(f)$.

Let $m \in M'$. Some power N of f annihilates m , i. e. $mf^N = 0$. Hence $0 = mf^N \delta = (m\delta f - mN\delta(f)) f^{N-1}$.

By induction on N we may assume that $m\delta f - mN\delta(f) = m_0 f$, for some $m_0 \in M'$. Hence $m = (m\delta - m_0)(N\delta(f))^{-1} f$, because $\delta(f)$ is a unit in $\mathcal{O}_{\mathfrak{p}}$.

It follows that (f_1, \dots, f_n) is M -coregular for any right $\mathcal{O}_{\mathfrak{p}}$ -module M with $\text{supp}(M) \subset V(\mathfrak{p} \mathcal{O}_{\mathfrak{p}})$.

Hence for any such module

$$\text{Ext}_{\mathcal{O}_p}^l(\mathcal{O}_p/\mathfrak{p}\mathcal{O}_p, M) = H_{n-l}((f_1, \dots, f_n), M) = 0,$$

for all $l \in \{1, \dots, n\}$, according to [7], §9, No. 7.

Remark. No doubt the reader familiar with the theory of \mathcal{D} -modules will have recognized this proof as one for a special case of Kashiwara's theorem. (See e. g. [4]).

4.4. *Remark.* Let Y be an affine non-singular variety over k , an algebraically closed field of characteristic zero.

Let $X \xrightarrow{i} Y$ be a closed subvariety of $\dim X = 1$.

Assume that the normalization map $\pi: \tilde{X} \rightarrow X$ is injective. Then $\text{Mod}_X\text{-}\mathcal{D}(Y)$, the category of right $\mathcal{D}(Y)$ -modules with support contained in X , is equivalent to $\text{Mod-}\mathcal{D}(X)$.

Of course the "same" proof as above applies: Put $B := \mathcal{D}_{\mathcal{O}(Y)}(\mathcal{O}(Y), \mathcal{O}(X))$ as in §2. Note that in case X is non-singular, B corresponds to the sheaf $\mathcal{D}_{X \rightarrow Y}$.

Put $i_+ := - \otimes_{\mathcal{D}(X)} B$, $i^+ := \text{Hom}_{\mathcal{D}(Y)}(B, -)$, a pair of adjoint functors. $\mathcal{D}(X)$ is Morita equivalent to $\mathcal{D}(\tilde{X})$ ([9], Th. B), which achieves that i_+ is faithful and exact.

By Kashiwara's theorem (See [2], [4]) $\text{Ext}_{\mathcal{D}(Y)}^l(B, B) = \text{Ext}_{\mathcal{O}(Y)}^l(\mathcal{O}(X), B) =: C^l$, to be viewed as a left $\mathcal{D}(X)$ -module, is, for $l \neq 0$, supported at the singular points of X .

Write $A := \mathcal{O}(X)$, $\bar{A} := \mathcal{O}(\tilde{X})$. Let $x \in X$ be a singular point, corresponding to a maximal ideal \mathfrak{m} in A . As π is injective, let $\bar{\mathfrak{m}}$ be the unique maximal ideal of \bar{A} , which is above \mathfrak{m} . Identify $k = A/\mathfrak{m} = \bar{A}/\bar{\mathfrak{m}}$. The $\mathcal{D}(X)$ -module $\mathcal{D}_A(A, k)$ has $\text{support} = \{\mathfrak{m}\}$. As in lemma 1 one obtains $\mathcal{D}_A(A, k) = \text{Hom}_{\mathcal{D}(A)}(P, \mathcal{D}_A(\bar{A}, k))$, where $P = \mathcal{D}(\tilde{X}, X)$ is the bimodule establishing the Morita equivalence. One derives that $0 = \text{Coker } \eta(\mathcal{D}_A(A, k)) = \text{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(A, k), C^1) = \text{Tor}_1^{\mathcal{D}(A)}(\mathcal{D}_A(\bar{A}, k), P^* \otimes_{\mathcal{D}(A)} C^1)$.

Now $R := \bar{A}_{\bar{\mathfrak{m}}}$ is a regular local ring, whose maximal ideal $\bar{\mathfrak{m}}_{\bar{\mathfrak{m}}}$ is a principal ideal; say $t\bar{A}_{\bar{\mathfrak{m}}} = \bar{\mathfrak{m}}_{\bar{\mathfrak{m}}}$.

Then $0 = \text{Tor}_1^{\mathcal{D}(R)}(\mathcal{D}(R)/t\mathcal{D}(R), P_{\bar{\mathfrak{m}}}^* \otimes_{\mathcal{D}(R)} C_{\bar{\mathfrak{m}}}^1)$, which means that $C_{\bar{\mathfrak{m}}}^1 = (C^1)_x$, the stalk at x , has no t -torsion. But then $C_{\bar{\mathfrak{m}}}^1 = 0$, because

support $(C_m^1) \subset \{mA_m\}$. By induction on $l: C_m^l = 0$.

4.5. Let $f \in \mathcal{O}_2$ be irreducible and let M be a right \mathcal{D}_2 -module with $\text{supp}(M) \subset V(f)$, i. e. $M_f = 0$.

Then the corollary to proposition 5 implies that $\text{Ext}_{\mathcal{D}_2}^1(B, M) = 0$. Hence the right multiplication by f on M is surjective.

§ 5. An Application

The application we have in mind is to show the following proposition. We will not dwell on its meaning but refer the reader to [13] or [15].

Proposition 6. *Let $f \in \mathcal{O}_2 =: \mathcal{O}$ be irreducible. Then $\mathcal{O}_f / \mathcal{O}$ is a simple left \mathcal{D} -module.*

Proof. There exists a k -linear involution on \mathcal{D} , transposition of differential operators, determined by (i) $a^t = a$, all $a \in \mathcal{O}$; (ii) $\partial_i^t = -\partial_i$, all i ; (iii) $(PQ)^t = Q^tP^t$, all $P, Q \in \mathcal{D}$. Clearly this involution turns every left \mathcal{D} -module M into a right one, denoted by M^t , and vice versa.

This involution induces a k -algebra anti-isomorphism

$$I(\mathcal{D}f) / \mathcal{D}f \cong I(f\mathcal{D}) / f\mathcal{D} = \mathcal{D}(A).$$

Furthermore, there exists a k -algebra isomorphism

$$\phi : I(f\mathcal{D}) / f\mathcal{D} \cong I(\mathcal{D}f) / \mathcal{D}f,$$

induced by the map: for all $D \in I(f\mathcal{D})$, $D \mapsto D'$, where $D' \in I(\mathcal{D}f)$ is the unique element such that $Df = fD'$.

Composing both maps gives a k -linear involution on $\mathcal{D}(A)$, which turns A into a right $\mathcal{D}(A)$ -module, provisionally denoted by A^t . It is straightforward to check that

$$i^+((\mathcal{O}_f / \mathcal{O})^t) \cong A^t.$$

Because A is a simple left $\mathcal{D}(A)$ -module (as $\mathcal{D}(A)$ is Morita equivalent to \mathcal{D}_1), it follows that A^t is a simple right $\mathcal{D}(A)$ -module. Hence $(\mathcal{O}_f / \mathcal{O})^t$ is a simple right \mathcal{D} -module, implying that $\mathcal{O}_f / \mathcal{O}$ is a simple left \mathcal{D} -module.

Remark. The above proposition is obtained independently by S. P. Smith [14] by a quite different method.

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