The Uncertainty Principle and Hypoelliptic Operators

By

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Introduction

Let P be a differential operator of second order with coefficients in $C^\infty(R^n)$, that is,

$$(1) \qquad P = \sum_{j,k} a_{jk}(x) D_j D_k + \sum_j ib_j(x) D_j + c(x), \quad D_j = -i\partial_{x_j}.$$

We assume that

$$(2) \qquad \begin{cases} a_{jk} \text{ and } b_j \text{ are real valued,} \\ \sum a_{jk}(x)\xi_j\xi_k \geq 0 \quad \text{for any } (x, \xi) \in R^{2n}. \end{cases}$$

Let $\log \Lambda$ be a pseudodifferential operator with symbol $\log\langle\xi\rangle$. As a criterion for P to be hypoelliptic in R^n the author has given the following in [8] (see also [6]):

Theorem A. (Corollary 2 in [8]) *If for any $\varepsilon > 0$ and any compact set K of R^n there exists a constant $C_{\varepsilon,K}$ such that*

$$(3) \qquad \|(\log \Lambda)u\|^2 \leq \varepsilon \operatorname{Re}(Pu, u) + C_{\varepsilon,K}\|u\|^2, \quad u \in C_0^\infty(K),$$

then P is hypoelliptic in R^n. Furthermore, we have

$$(4) \qquad\qquad \operatorname{WF} Pv = \operatorname{WF} v \quad \text{for } v \in \mathscr{D}'(R^n).$$

Though more general criteria than Theorem A were given in [8] and [9], in the present paper we shall restrict our consideration to the application of this theorem to some class of degenerate elliptic operators of second order. To verify the estimate (3), it is convenient

to employ "the uncertainty principle" mathematically formulated by Fefferman-Phong [2] and Fefferman [1].

This paper consists of two sections. In Section 1, we slightly modify the result in [2, Section 2] (, see Theorem B in Section 1). In Section 2 we study the hypoellipticity of infinitely degenerate elliptic operators by using Theorems A and B.

We mention that only the simplest parts of arguments in [1] and [2] are used here. One might expect to characterize the operators satisfying (3) by means of much deeper arguments in [1] and [2].

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§1. The Uncertainty Principle

Let λ be $0<\lambda\leq1$. We consider a symbol of the form $a(x,\xi)=|\xi|^{2\lambda}+V(x)$, $x\in R^n$, where $V(x)$ is a non-negative measurable function and depends on a large parameter $M>0$, that is, $V(x)=V(x, M)$. (In the next section we shall employ the case when $V(x)=g(x)M^2$, $g\in C^\infty$.)

Let $R\equiv R(M)$ be a non-decreasing function of M such that $R\to\infty$ as $M\to\infty$. Let $\mathscr{C}$ denote a set of boxes

(5) $B\equiv\{(x,\xi)\in R^{2n}\,;\,|x_j-x_{0j}|\leq\delta/2,\ |\xi_j-\xi_{0j}|\leq\delta^{-1}/2\}$

for all $(x_0,\xi_0)\in R^{2n}$ and all $\delta>0$. Clearly, the volume of $B\in\mathscr{C}$ is equal to 1.

Theorem B. *Assume that there exists a* $c>0$ *such that for any* $B\in\mathscr{C}$

(6) $m(\{(x,\xi)\in B\,;\,a(x,\xi)\geq R\})\geq c,$

where $m(.)$ *denotes Lebesgue measure. Then we have*

(7) $|||D_x|^\lambda u||^2+(V(x)u,u)\geq c'R||u||^2,\ u\in C_0^\infty(R^n),$

where $c'>0$ *depends only on* c *and* n.

For the comparison of Theorem B with the original version in [2, Section 2] we set the following corollary though we shall not use it throughout the present paper.

Corollary. *Let* $V(x)$ *belong to* $C^\infty(R_x^n)$ *and satisfy*

(8) $$|D^\alpha V(x)| \leq C_\alpha M^2, \quad x \in R^n.$$

Assume that for some $\varepsilon > 0$ and $c > 0$ we have $R \geq cM^\varepsilon$. Then the estimate (7) still holds when the hypothesis (6) is replaced by

(9) $$\min_{B \in \mathscr{C}} \max_{(x, \xi) \in B} a(x, \xi) \geq R.$$

For the proof of Theorem B in the case $\lambda = 1$ we use the following lemma given by Fefferman [1]:

Lemma 1. *Assume that $V(x) \geq 0$, measurable on a cube Q in R^n. Suppose that there exists a $c > 0$ such that*

(10) $$m(\{x \in Q \, ; \, V(x) \geq c(\mathrm{diam}\ Q)^{-2}\}) \geq c|Q|.$$

Then for $u \in C^1$ we have

(11) $$\int_Q \{|\nabla u(x)|^2 + V(x)|u(x)|^2\}\, dx \geq c'(\mathrm{diam}\ Q)^{-2} \int_Q |u(x)|^2 dx.$$

The constant $c' > 0$ depends only on n and c.

In the Main Lemma of [1, p. 146], it is assumed that $V(x)$ is polynomial and $(\mathrm{Av}_Q V) \geq (\mathrm{diam}\ Q)^{-2}$. The proof in [1] is still valid under the above hypotheses.

For the case $0 < \lambda < 1$ we need the following two lemmas: The first is a modification of Lemma 1.

Lemma 2. *Assume that $V(x) \geq 0$, measurable on a cube Q in R^n. For a $0 < \lambda < 1$, suppose that there exists a $c > 0$ such that*

(10)′ $$m(\{x \in Q \, ; V(x) \geq c(\mathrm{diam}\ Q)^{-2\lambda}\}) \geq c|Q|.$$

Then for $u \in C^1$ we have

(11)′ $$\int_{Q \times Q} \{|u(x) - u(y)|^2 / |x - y|^{n+2\lambda}\}\, dxdy$$
$$+ \int_Q V(x)|u(x)|^2 dx \geq c'(\mathrm{diam}\ Q)^{-2\lambda} \int_Q |u(x)|^2 dx.$$

Proof of Lemma 2. Note that

$$\int_{Q \times Q} \{|u(x) - u(y)|^2 / |x - y|^{n+2\lambda}\}\, dxdy$$
$$\geq c \frac{(\mathrm{diam}\ Q)^{-2\lambda}}{|Q|} \int_{Q \times Q} |u(x) - u(y)|^2\, dxdy,$$

where c depends only on n. Using this formula instead of the similar one in the proof of the Main Lemma of [1], we can obtain the desired estimate. The detail is omitted.   Q. E. D

Lemma 3. *For $0 < \lambda < 1$ we have*

$$\| |D_x|^\lambda u \|^2$$

$$= c \int_{R^n \times R^n} \{ |u(x) - u(y)|^2 / |x-y|^{n+2\lambda} \} dx dy, \quad u \in C_0^\infty(R^n),$$

where c depends only on λ and n. (See [11]).

Proof of Theorem B. Let Q_{x_0} be a cube in R_x^n, that is,

(12) $Q_{x_0} = \{ x \in R^n ; |x_j - x_{0j}| \le \delta/2 \}$ for some $x_0 \in R^n, \delta > 0$.

If $\delta = 2^{-2} n^{1/2} (2/R)^{1/2\lambda}$ it follows from (6) that

(13) $m(\{ x \in Q_{x_0} ; V(x) \ge R/2 \}) \ge c |Q_{x_0}|$.

In fact, this is obvious if we consider a box $Q_{x_0} \times \{ \xi ; |\xi_j| \le n^{-1/2} (R/2)^{1/2\lambda} \}$ belonging to $\mathscr{C}$. Let $\{Q_j\}$ be a partition of R_x^n such that each Q_j is a translation of Q_{x_0} with $\delta = 2^{-2} n^{1/2} (2/R)^{1/2\lambda}$. Note diam $Q_j = 2^{-1} n^{1/2} (2/R)^{1/2\lambda}$. When $\lambda = 1$, in view of (13) we have (7) by Lemma 1 because $\| |D_x| u \|^2 = \| \nabla u \|^2$. Note that

$$\int_{R^n \times R^n} dx dy \ge \sum_j \int_{Q_j \times Q_j} dx dy.$$

Then, the estimate (7) for the case $0 < \lambda < 1$ also follows from Lemmas 2 and 3.   Q. E. D.

Proof of Corollary. It follows from (9) that we have

(14) $\max_{Q_{x_0}} V(x) \ge R/2$

for any Q_{x_0} defined by (12) with $\delta = 2^{-2} n^{1/2} (2/R)^{1/2\lambda} (\equiv \delta_0)$. Setting $F(y) = V(\delta_0 y + x_0)$, by (8) we have

$$|D_y^\alpha F(y)| \le C_\alpha' M^2 R^{-|\alpha|/2\lambda} \le C_\alpha'' M^{2-\varepsilon|\alpha|/2\lambda}, \quad y \in R^n.$$

Fix a large integer N such that $\varepsilon N \ge 4\lambda$ and set

$$P(y) = \sum_{|\alpha| < N} \partial_y^\alpha F(0) y^\alpha / \alpha!.$$

Then for some C_0 independent of M we have

(15) $|F(y) - P(y)| \le C_0$

on a unit cube $Q_0 \equiv \{y; |y_j| \le 1/2\}$.

By (14) and (15) we have $\max_{Q_0} P \ge R/4$ when M is large enough. Since P is polynomial of order N, we have $\max_{Q_0} |\nabla P| \le C \max_{Q_0} |P|$, where C depends only on N and n. In view of (15) and $F \ge 0$ we have $\max_{Q_0} P = \max_{Q_0} |P|$ if M is large enough. If $P(y_0) = \max_{Q_0} P$ and if Q_0' is a subcube of Q_0 such that $y_0 \in Q_0'$, then for $y \in Q_0'$ we have

$$P(y) \ge P(y_0) - \max_{Q_0'} |\nabla P| \, |y - y_0|$$
$$\ge P(y_0) - \max_{Q_0} |\nabla P| \, |y - y_0|$$
$$\ge P(y_0)(1 - C|y - y_0|).$$

Hence, there exists a subcube Q_0' of Q_0 such that $P(y) \ge R/8$ on Q_0' and $|Q_0'| \ge c_0$ for a constant c_0 depending only on n and N. Using (15) again, we have $V(x) \ge R/16$ on some subcube Q_x' of Q_x with $|Q_x'|/|Q_x| \ge c_0$ if M is large enough. So, for any Q_x defined by (12) with $\delta = \delta_0$ we have the similar formula as (13).   Q. E. D.

§2. Hypoelliptic Operators

Let $P_0 = p_0(x, y, D_x)$ be a differential operator of second order with C^∞ coefficients of the form

(16) $P_0 = a(x, y, D_x) + g(x') b(x, y, D_y)$ in $R^n = R_x^{n_1} \times R_y^{n_2}$,

where $x = (x', x'') \in R_{x'}^{n_1'} \times R_{x''}^{n_1''}$. We assume that $a(x, y, D_x)$ and $b(x, y, D_y)$ are strongly elliptic with respect to x and y, respectively, that is,

(A-1) $\begin{cases} \text{Re } a(x, y, \xi) \ge c_1 |\xi|^2 & \text{for } (x, y) \in R^n \text{ and large } |\xi|, \\ \text{Re } b(x, y, \eta) \ge c_2 |\eta|^2 & \text{for } (x, y) \in R^n \text{ and large } |\eta|, \end{cases}$

where c_1 and c_2 are positive constants. Furthermore we assume

(A-2) $\begin{cases} g^{(j)}(0) = 0 \text{ for any integer } j \ge 0 \\ \text{and } g(x') > 0 \text{ for } x' \ne 0. \end{cases}$

The conjunction of Theorems A and B gives the following:

Proposition 1. *Let P_0 be the above operator and assume $(A-1)$ and*

(A-2). *Furthermore, assume that* $g(x')$ *satisfies*

(17) $$\lim_{x'\to 0} |x'| \,|\log g(x')| = 0.$$

Then P_0 *is hypoelliptic in* R^n.

Remark 1. The theorem is nothing but Theorem 1.1 of [7], where we employed a primitive method in place of Theorems A and B.

Remark 2. As a typical example of P_0 we have $D_{x_1}^2 + D_{x_2}^2 + g(x_1) D_y^2$ in R^3. The theorem for this example under a little stringent assumption on g was proved by Kusuoka-Strook by using the Malliavin calculus (see Theorem 8.41 of [4]). For this example, it was proved in [4] and [5] that if $g(x_1) = g(-x_1)$ and g is nondecreasing in R_+ then (17) is necessary for the hypoellipticity.

Remark 3. If $g = \exp(-1/|x'|^\sigma)$, $\sigma > 0$, then (17) means $\sigma < 1$.

Proof of Proposition 1. By Theorem A it suffices to derive (3) for P_0. Since our consideration is local on a fixed compact set K, by modifying g outside of K we may assume $g(x') \geq c_0 > 0$ for $|x'| \geq 1$. It follows from (A-1) that for some constant C we have

(18) $C(\mathrm{Re}\ (P_0 u, u) + \|u\|^2)$
$$\leq \|\nabla_{x'} \check{u}\|^2 + \|\nabla_{x''} \check{u}\|^2 + (g(x')\,|\eta|^2 \check{u}, \check{u}), u \in C_0^\infty,$$

where $\check{u} = \check{u}(x, \eta)$ is the Fourier transform of u with respect to y. Set $V(x') = g(x') |\eta|^2$. We shall show that for any integer $k > 0$ there exists a $M_k > 0$ such that

(19) $$\int \{|\nabla_{x'} u(x')|^2 + V(x')\,|u(x')|^2\} dx'$$
$$\geq c(k \log |\eta|)^2 \int |u(x')|^2 dx'$$
$$for\ u \in C_0^\infty(R_{x'}^{n_1})\quad if\ |\eta| \geq M_k,$$

where $c > 0$ is a constant independent of k and $|\eta|$. The estimate (3) easily follows from (19). Instead of x' and $|\eta|$, write x and M. Use Theorem B with $\lambda = 1$. We shall check (6). In view of (17), for any k there exists a $\tau_k > 0$ such that

(20) $-\log\ g(x) = |\log\ g(x)\ | \leq 1/(10k(\ |x\ |)$ $for\ |x\ | \leq \tau_k.$

Take a large $M_k > 0$ such that

$$M_k^{-1} \leq \min_{|x| \geq \tau_k} g(x)\ \ and\ \ \tau_k \geq (10k\ \log\ M_k)^{-1}.$$

When $M \geq M_k$ we have

(21) $V(x) \geq M$ $for\ \ |x\ | \geq \tau_k.$

Set $R \equiv R(M) = (k\ \log\ M)^2$ and note that

(22) $V(x) = M\ \exp\{\log\ M - |\log\ g(x)\ |\}$
                  $\geq M\ \exp\{\log\ M - 1/(10k\ |x\ |)\} \geq M$
                  $if\ \ (10R)^{-1/2} \leq |x\ | \leq \tau_k\ and\ M \geq M_k.$

Inequalities (21) and (22) show that for Q_{x_0} defined by (12) with $\delta \geq (8R)^{-1/2}$ we have

(23) $m(\{x \in Q_{x_0}; V(x) \geq M\}) \geq |Q_{x_0}|/10.$

From this we see that (6) holds for $B \in \mathscr{C}$ with $\delta \geq (8R)^{-1/2}$. On account of the term $|\xi|^2$, (6) is trivial for others $B \in \mathscr{C}$.

                                                      Q. E. D.

The above discussion can apply to more degenerate elliptic operators than P_0. For example:

Corollary. *Proposition 1 is still true when $g(x')$ is replaced by $g(x')$* $\sin^2(1/\ |x'\ |)$.

Proof. As in the proof of Proposition 1, we write x and M instead of x' and $|\eta|$. Let Q_{x_0} be a cube defined by (12). For any Q_{x_0} where $\sin(1/\ |x\ |)$ vanishes at least with two different values of $|x\ |$, we have

(24) $m(\{x \in Q_{x_0}; \sin^2(1/\ |x\ |) \geq 1/2\}) \geq |Q_{x_0}|/4.$

For others Q_{x_0} in $\{\ |x\ | \leq T\}$ with $\delta \geq (8R)^{-1/2}$ we have

(25) $m(\{x \in Q_{x_0}; \sin^2(1/\ |x\ |) \geq c_n T^{-4} R^{-1}\}) \geq |Q_{x_0}|/2$

because $\sin t \geq 2t/\pi$ for $0 \leq t \leq \pi/2$. Here $c_n > 0$ depends only on n. Noting (23), from (24) or (25) we see that (6) holds even when $V(x) = g(x)\sin^2(1/\ |x\ |)M^2.$ Q. E. D.

We can give a more pathological hypoelliptic operator. Let P_1 be a second order differential operator of the form (16) with $g(x') = f(|x'|)$ for $f \in C^\infty(\bar{R}_+)$. We assume that $f(t)$ vanishes on a Cantor set E with measure 0 defined as follows: Set $E = I_0 \backslash \overset{\infty}{\underset{j=1}{\cup}} I_j$, where $I_0 = [0, 1]$, I_1 is an open interval with length $1/3$ whose center is $1/2$. Here I_2 and I_3 are open intervals with length $(1/3)^2$ whose centers coincide with the centers of each connected component of $I_0 \backslash I_1$, respectively. Furthermore, $I_4, \ldots, I_7$ are open intervals with length $(1/3)^3$ whose centers coincide with the centers of each connected component of $I_0 \backslash \overset{3}{\underset{j=1}{\cup}} I_j$, respectively. We define open intervals I_j in this manner, recursively. On each $I_j = (-a+b, a+b)$, $j \geq 1$, we set $f(t) = \exp\{(-1/|t+a-b|^\sigma) + (-1/|t-a-b|^\sigma)\}$ and we set $\exp(-1/(|t-1|^\sigma))$ for $(1, \infty)$, where $\sigma > 0$.

Proposition 2. *Let P_1 be a second order differential operator of the form (16) with $g(x') = f(|x'|)$ for $f \in C^\infty(\bar{R}_+)$ defined above. We assume (A-1). If $0 < \sigma < 1$ then P_1 is hypoelliptic in R^n.*

Proof. Recall Remark 3 of Proposition 1, that is, that the condition (17) is equivalent to $\sigma < 1$ when $g(x') = \exp(-1/|x'|^\sigma)$. Noting the definition of the Cantor set E we can easily check the condition (6) of Theorem B, as in the proof of Proposition 1. The detail is omitted.   Q.E.D.

The assumption (A-1) can be also weakened. Assume that

$$a(x, y, D_x) = -\sum_{j=1}^{r} X_j^2 \quad and \quad b(x, y, D_y) = -\sum_{j=1}^{s} Y_j^2,$$

where X_j (resp. Y_j) are real vector fields in $R_x^{n_1}$ (resp. $R_y^{n_2}$) for every fixed $y \in R_y^{n_2}$ (resp. $x \in R_x^{n_1}$). In place of (A-1) we assume that

(A-1)′ $\begin{cases} \textit{For any fixed } y \in R_y^{n_2} \textit{ (resp. } x \in R_x^{n_1}\textit{) the vector fields } X_1, \ldots, \\ X_r \textit{ (resp. } Y_1, \ldots, Y_s\textit{) and their repeated commutators span the} \\ \textit{tangent space at each point in } R_x^{n_1} \textit{ (resp. } R_y^{n_2}\textit{).} \end{cases}$

and

(B) $\left\{ \begin{array}{l} \textit{For any fixed } y \in R_y^{n_2} \textit{ the vector fields } X_1, \ldots, X_r \textit{ and their} \\ \textit{repeated commutators of order up to } l \textit{ span the subtangent} \\ \textit{space } T(R_{x'}^{n'_1}) \textit{ at each point in } R_x^{n_1}. \end{array} \right.$

Proposition 3. *Let P_2 be a differential operator of the form* (16). *Assume that* $(A\text{-}1)'$, $(A\text{-}2)$ *and* (B). *If $g(x')$ satisfies*

$$(26) \qquad \lim_{x' \to 0} |x'|^{1/(l+1)} |\log g(x')| = 0.$$

then P_2 is hypoelliptic in R^n.

Proof. It follows from $(A\text{-}1)'$ and (B) that for a $k > 0$ we have

$$(27) \qquad C(\mathrm{Re}(P_2 u, u) + \|u\|^2)$$
$$\geq \| <D_x>^k \check{u} \|^2 + \| |D_{x'}|^{1(l+1)} \check{u} \|^2 + (g(x') |\eta|^{2k} \check{u}, \check{u}), \; u \in C_0^\infty,$$

instead of (18). Indeed, this is a direct consequence of Hörmander's classical theorem in [3] and its sharp version given in [10] and [2]. In place of x' and $|\eta|^k$, write x and M. Set $V(x) = g(x)M^2$. Then the proof of the proposition is reduced to show that for any integer $k > 0$ there exists a $M_k > 0$ such that

$$(28) \qquad \| |D_x|^{1/(l+1)} u \|^2 + (V(x)u, u)$$
$$\geq c(k \log M)^2 \|u\|^2, \quad u \in C_0^\infty,$$
$$\textit{if } M \geq M_k.$$

For the proof of (28) it suffices to apply Theorem B with $\lambda = 1/(l+1)$. Similarly as in the proof of Proposition 1, the condition (6) easily follows from (26). Q. E. D.

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