Polynomial Growth or Decay of Eigenfunctions of Second-order Elliptic Operators

By

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§ 0. Introduction

In this paper we shall study the asymptotic behaviors as $|x| \to \infty$ of the not identically vanishing solution $u(x) \in H^2_{loc}(\Omega)$ of second-order elliptic equation in an unbounded domain $\Omega = \{x \mid |x| > R_0\} \subset R^n$

$$-\sum_{i,j=1}^{n} \left(\frac{\partial}{\partial x_i} + \sqrt{-1}\, b_i(x) \right) a_{ij}(x) \left(\frac{\partial}{\partial x_j} + \sqrt{-1}\, b_j(x) \right) u(x)$$
$$+ (q_1(x) + q_2(x)) u(x) = 0,$$

where the matrix $(a_{ij}(x))$ is uniformly positive definite, $b_i(x)$ $(1 \le i \le n)$ and $q_1(x)$ are real-valued functions, and $q_2(x)$ is a complex-valued function. In case $a_{ij}(x) = \delta_{ij}$ (Kronecker's delta) we state the main parts of the assumptions: there exist some constants $\alpha, \beta, \gamma, \eta$ satisfying $\beta, \eta > 0$ and $2 - 2\beta < \gamma < \min\{2, 2\alpha\}$ such that

$$\lim_{r \to \infty} \sup r^{2-2\beta} [r \partial_r q_1(x) + \gamma q_1(x)$$
$$+ (1+\eta) \left\{ \frac{1}{2\alpha - \gamma} |r q_2(x)|^2 + \frac{1}{2-\gamma} |B(x)x|^2 \right\}] < 0,$$

where $B(x) = (\partial_i b_j(x) - \partial_j b_i(x))$ is an $n \times n$ matrix. Moreover there exist some constants $0 < a < 1$, $-\infty < \delta_1 < \infty$, $C > 0$, $\delta_2 \le \beta - 2$ such that

$$\int_{\Omega} (q_1)_- |w|^2 dx \le a \int_{\Omega} |\nabla w|^2 dx + C \int_{\Omega} r^{\delta_1} |w|^2 dx \quad \text{for any } w(x) \in C_0^\infty(\Omega),$$

$$(\mathrm{Re}[q_2])_-(x) \le C r^{\delta_2} \qquad \qquad \text{for } |x| > R_0,$$

where $(f)_-(x) = \max\{0, -f(x)\}$ for a real-valued function $f(x)$. More

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detailed conditions are stated in §1. Then by Theorem 1.3 in §1 we have for any $\varepsilon > 0$

$$\lim_{R \to \infty} \inf R^{\alpha + \max\{0, \delta_1, \delta_2\}} \int_{R < |x| < R + \varepsilon} |u(x)|^2 dx > 0.$$

Moreover if $\alpha + \max\{0, \delta_1, \delta_2\} \leq 1$, then $u(x) \notin L^2(\Omega)$.

This result can be recognized as a generalization of Kato [5], Agmon [1, Theorem 3], Agmon [2, Theorems 3, 4 and 5], Ikebe-Uchiyama [4], Uchiyama [7], Mochizuki [6], Eastham-Kalf [3, Theorems 6.3.3 and 6.4.2] and Yamada [12] from following points of view. [1] and [12] did not give their own complete proof. The term $a_{ij}(x)$ is lacking in [5], [1], [2], [7], [3] and [12]. The term $b_i(x)$ is lacking in [5], [1], [2] and [7]. In [5], [2], [4] and [12] it was assumed that $q_1(x) < 0$ for $r > R_0$. So they are not applicable to atomic type many body potential treated in Example 5.3 and Remark 5.4 in §5. In [5], [4], [7] and [6] δ_1 was restricted to satisfy $\delta_1 \leq 0$. Except for [1] and [3, Theorem 6.3.3], it was difficult to treat Examples 5.1~5.8 in one theorem. But in [3] only the case $\beta = 1$ was treated. So in order to show $u(x) \notin L^2(\Omega)$, they assumed $\delta_1 < 1$ and $\delta_2 \leq -1$, which was needed to satisfy $\alpha + \max\{0, \delta_1, \delta_2\} \leq 1$ (noting $2 - 2\beta = 0 < \gamma < 2\alpha$). And in case $\delta_1 \geq 1$ they needed another theorem [3, Theorem 6.4.2], which corresponds to Example 5.1 in §5 and [2, Theorem 5].

Here we note that [1], [2], [7] and [12] treated the critical case $2 - 2\beta \leq \gamma \leq 2\alpha$ and they gave better estimates than ours under more strict conditions on $q_2(x)$ and curl $b(x)$.

The introduction of $\beta > 0$ in the above assumptions can be found in [1]. Other articles treated only the case $\beta = 1$ (i.e. $2 - 2\beta = 0$), and are not applicable to Example 5.2 in §5. Such conditions depending on β are also found in Uchiyama [8, 9, 10], which deal the exponential decay of $u(x)$ at infinity.

In §1 the Assumptions and main results are explained. §2 is the preliminary stage of the proofs of the theorems. The identity given in Lemma 2.8 is fully used in §3 and §4. In §5 we give several applications of Theorems 1.3 and 1.4. Some of Examples in §5 were also considered in [8, 9, 10] which showed

$$\lim_{R \to \infty} \exp\{R^\varepsilon\} \int_{R < |x| < R + 1} |u(x)|^2 dx = +\infty \quad \text{for any } \varepsilon > 0.$$

But here for such solutions $u(x)$ we shall show that

$$\liminf_{R\to\infty} R^\mu \int_{R<|x|<R+1} |u(x)|^2 dx > 0$$

holds, and we shall calculate μ.

§ 1. Assumptions and Main Results

At first we shall list the notations which will be freely used in the sequel, and are the same as given in Uchiyama [8].

Notations:

$\langle \xi, \eta \rangle = \xi_1 \eta_1 + \cdots + \xi_n \eta_n$ for $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n$;

$|\xi| = (\langle \xi, \bar{\xi} \rangle)^{1/2}$ for $\xi \in \mathbb{C}^n$;

$\hat{x} = x/|x|$ and $r = |x|$ for $x = (x_1, \ldots, x_n) \in R^n$;

$\partial_j = \partial/\partial x_j$ and $\partial_r = \partial/\partial r$;

$D_j = \partial_j + \sqrt{-1}\, b_j(x)$ and $D = (D_1, \ldots, D_n)$;

$\nabla f = (\partial_1 f, \ldots, \partial_n f)$ for a scalar valued function $f(x)$;

div $g = \partial_1 g_1 + \cdots + \partial_n g_n$ for a vector valued function $g(x) = (g_1(x), \ldots, g_n(x))$;

$A = A(x) = (a_{ij}(x))$ is an $n \times n$ matrix;

$B = B(x) = \text{curl } b(x) = (\partial_i b_j(x) - \partial_j b_i(x))$ is an $n \times n$ matrix;

$(f)_{\pm}(x) = \max\{0, \pm f(x)\} \geq 0$ for a real-valued function $f(x)$;

supp$[f]$ denotes the closure of $\{x \,|\, f(x) \neq 0\}$;

$C^j(\Omega)$ denotes the class of j-times continuously differentiable functions;

$C_0^\infty(\Omega) = \{f(x) \,|\,$ for any $j = 0, 1, 2, \ldots, f \in C^j(\Omega)$ and supp$[f]$ is a compact set in $\Omega\}$;

$L^p(\Omega) = \{f(x) \,|\, \int_\Omega |f(x)|^p dx < \infty\}$ for $p \geq 1$;

$L_{\text{loc}}^p(\Omega) = \{f(x) \,|\,$ for any compact set $K \subset \Omega, \int_k |f(x)|^p dx < \infty\}$ for $p \geq 1$;

$H^m(\Omega)$ denotes the class of L^2-functions in Ω such that all distribution derivatives up to m belong to $L^2(\Omega)$;

$H_{\text{loc}}^m(\Omega)$ denotes the class of L_{loc}^2-functions in Ω such that all distribution derivatives up to m belong to $L_{\text{loc}}^2(\Omega)$;

$$\left(\int_{|x|=t} - \int_{|x|=s}\right) f(x)\, dS = \int_{|x|=t} f(x)\, dS - \int_{|x|=s} f(x)\, dS.$$

Next we shall state the conditions required in the theorems.

Assumptions:
(A1)   each $a_{ij}(x) \in C^2(\Omega)$ is a real-valued function;
(A2) $a_{ij}(x) = a_{ji}(x)$;
(A3) $a_{ij}(x) \to \delta_{ij}$ as $|x| \to \infty$;
(A4)   there exists some constant $C_1 \geq 1$ such that for any $x \in \Omega$ and any $\xi \in C^n$ we have

$$C_1^{-1} |\xi|^2 \leq \langle A(x)\xi, \bar{\xi} \rangle \leq C_1 |\xi|^2;$$

(A5) $\partial_l a_{ij}(x) = o(r^{-1})$ as $|x| \to \infty$;
(B1)   each $b_i(x)$ is a real-valued function;
(B2)   for any $w \in H^1_{\mathrm{loc}}(\Omega)$ we have $|b_i|^2 |w|^2$, $|\partial_j b_j|^2 |w|^2 \in L^1_{\mathrm{loc}}(\Omega)$;
(C1) $q_1(x)$ is a real-valued function;
(C2)   for any $w \in H^1_{\mathrm{loc}}(\Omega)$ we have $|q_1| |w|^2 \in L^1_{\mathrm{loc}}(\Omega)$;
(C3)   for any $w \in H^1_{\mathrm{loc}}(\Omega)$ we have $|\nabla q_1| |w|^2 \in L^1_{\mathrm{loc}}(\Omega)$;
(D1) $q_2(x)$ is a complex-valued function;
(D2)   for any $w \in H^1_{\mathrm{loc}}(\Omega)$ we have $|q_2| |w|^2 \in L^1_{\mathrm{loc}}(\Omega)$;
(E) Ω is a domain in $\mathbf{R}^n$ and there exists some constant $R_0 > 1$ such that $\Omega \supset \{x \mid |x| > R_0\}$;
(F)   there exist some constants $\alpha, \beta, \delta, \eta$ satisfying $0 < \delta < \beta$, $\eta > 0$ and some real-valued function $\gamma(x) \in C^1(\Omega)$ such that (F1) $\sim$ (F5) hold.
(F1) $\alpha \geq \dfrac{1}{2}\gamma(x) + \delta$ for $|x| > R_0$;
(F2) $2 - 2\beta + \delta \leq \gamma(x) \leq 2 - \delta$ for $|x| > R_0$;
(F3)

$$\limsup_{r \to \infty} r^{2-2\beta} [r \langle A(x) \nabla q_1(x), \hat{x} \rangle + \gamma(x) \langle A(x)\hat{x}, \hat{x} \rangle q_1(x)$$

$$+ (1+\eta) \left\{ \frac{1}{2\alpha - \gamma(x)} |r q_2(x)|^2 \right.$$

$$\left. + \frac{1}{2 - \gamma(x)} \langle A(x)\hat{x}, \hat{x} \rangle^{-1} \langle A(x) B(x) A(x) x, B(x) A(x) x \rangle \right\}] < 0;$$

(F4) $\partial_l \partial_m a_{ij}(x) = o(r^{\beta-2})$ as $|x| \to \infty$;
(F5) $\nabla \gamma(x) = o(r^{\beta-1})$ as $|x| \to \infty$;
(G1)   there exist some constants $0 < a < 1$, $-\infty < \delta_1 < \infty$ and $C_2 > 0$ such that for any $w \in C_0^\infty(\Omega)$ we have

$$\int_{\Omega} (q_1)_- |w|^2 dx \leq a \int_{\Omega} |\nabla w|^2 dx + C_2 \int_{\Omega} r^{\delta_1} |w|^2 dx \ ;$$

(G2)   there exist some constants $\delta_2 \leq \beta - 2$ and $C_3 > 0$ such that for any $|x| > R_0$ we have

$$(\mathrm{Re}[q_2])_- (x) \leq C_3 r^{\delta_2},$$

where $\mathrm{Re}[z]$ means the real part of $z \in \mathbb{C}$.

Now we have the

Theorem 1.1. *Let $u(x)$ satisfy*

$$(*) : \begin{cases} -\langle D, ADu \rangle + (q_1(x) + q_2(x)) u(x) = 0 & \text{in } \Omega, \\ u(x) \in H^2_{loc}(\Omega), \\ \mathrm{supp}[u] \text{ is not a compact set in } \bar{\Omega} \text{ (closure of } \Omega). \end{cases}$$

Let conditions (A) $\sim$ (G) *hold. Then we have*

$$\liminf_{R \to \infty} R^\alpha \int_{|x|=R} [|\langle ADu, \hat{x} \rangle|^2 + (r^{-2} + q_1)_-] |u|^2] dS > 0,$$

and for any $\varepsilon > 0$

$$\liminf_{R \to \infty} R^{\alpha + \max\{0, \delta_1, \delta_2\}} \int_{R < |x| < R + \varepsilon} |u|^2 dx > 0.$$

Corollary 1.2. *Let $u(x)$ satisfy* (*), *and let conditions* (A) $\sim$ (G) *hold. If*

$$\alpha + \max\{0, \delta_1, \delta_2\} \leq 1,$$

then we have $u(x) \notin L^2(\Omega)$.

Now we shall consider the more special case under the weaker conditions.

Theorem 1.3. *Let $u(x)$ satisfy*

$$(**) : \begin{cases} -\langle D, Du \rangle + \{q_1(x) + q_2(x)\} u(x) = 0 & \text{in } \Omega, \\ u \in H^2_{loc}(\Omega), \\ \mathrm{supp}[u] \text{ is not a compact set in } \bar{\Omega}. \end{cases}$$

We assume (B) $\sim$ (G) *with $a_{ij}(x) = \delta_{ij}$ except for* (C3). *Instead of* (C3) *we assume*

(C3)′ *for any* $w \in H^1_{loc}(\Omega)$ *we have* $|\partial_r q_1| \, |w|^2 \in L^1_{loc}(\Omega)$.
Then we have

$$\liminf_{R \to \infty} R^\alpha \int_{|x|=R} [\, |\langle Du, \hat{x} \rangle|^2 + \{r^{-2} + (q_1)_-\} \, |u|^2] dS > 0,$$

and for any $\varepsilon > 0$

$$\liminf_{R \to \infty} R^{\alpha + \max\{0, \delta_1, \delta_2\}} \int_{R < |x| < R+\varepsilon} |u|^2 dx > 0.$$

Moreover if

$$\alpha + \max\{0, \delta_1, \delta_2\} \le 1,$$

then we have $u(x) \notin L^2(\Omega)$.

Lastly we shall consider the most special case under the weakest conditions.

Theorem 1.4. *Let* $u(x)$ *satisfy*

$$(***): \begin{cases} -\Delta u(x) + \{q_1(x) + q_2(x)\} u(x) = 0 & \text{in } \Omega, \\ u \in H^2_{loc}(\Omega), \\ \text{supp}[u] \text{ is not a compact set in } \bar{\Omega}, \end{cases}$$

where Δ *is a Laplacian in* $\mathbf{R}^n$. *We assume* (C)$\sim$(G) *with* $a_{ij}(x) = \delta_{ij}$ *and* $b_i(x) = 0$ *except for* (C3), (F2) *and* (F5). *Instead of* (C3), (F2) *and* (F5) *we assume* (C3)′, *and*
    (F2)′ $2 - 2\beta + \delta \le \gamma(x) \le 2$;
    (F5)′ $\partial_r \gamma(x) = o(r^{\beta-1})$ *as* $|x| \to \infty$ *and*
        $|(\nabla - \hat{x}\partial_r)\gamma(x)| \le \{2 - \gamma(x)\}^{1/2} p(x)$, *where* $p(x) = o(r^{\beta-1})$ *as* $|x| \to \infty$.

Then we have

$$\liminf_{R \to \infty} R^\alpha \int_{|x|=R} [\, |\partial_r u|^2 + \{r^{-2} + (q_1)_-\} \, |u|^2] dS > 0,$$

and for any $\varepsilon > 0$

$$\liminf_{R \to \infty} R^{\alpha + \max\{0, \delta_1, \delta_2\}} \int_{R < |x| < R+\varepsilon} |u|^2 dx > 0.$$

Moreover if

$$\alpha + \max\{0, \delta_1, \delta_2\} \le 1,$$

then we have $u(x) \notin L^2(\Omega)$.

Remark 1.5.   (F2)′ and (F5)′ are weaker conditions than (F2) and (F5).   In fact $\gamma(x) = 2$ (constant) satisfies (F2)′ and (F5)′, and does not satisfy (F2) and (F5).

§ 2. Preliminary Lemmas

In this section all the conditions (A)∼(G) are assumed. And let $u(x)$ satisfy (*).

Lemma 2.1. *We have*

(1) $\langle A(x)\hat{x}, \hat{x}\rangle = 1 + o(1)$ *as* $|x| \to \infty$,

(2) $\partial_i(\langle A(x)\hat{x}, \hat{x}\rangle) = o(r^{-1})$ *as* $|x| \to \infty$, *and*
   $\partial_i(\langle A(x)\hat{x}, \hat{x}\rangle^{-1}) = o(r^{-1})$ *as* $|x| \to \infty$,

(3) $\partial_i\partial_j(\langle A(x)\hat{x}, \hat{x}\rangle) = o(r^{\beta-2})$ *as* $|x| \to \infty$, *and*
   $\partial_i\partial_j(\langle A(x)\hat{x}, \hat{x}\rangle^{-1}) = o(r^{\beta-2})$ *as* $|x| \to \infty$,

(4) $\operatorname{div}(A(x)\hat{x}) = (n-1)r^{-1} + o(r^{-1})$ *as* $|x| \to \infty$,

(5) $\partial_i(\operatorname{div}(A(x)\hat{x})) = o(r^{\beta-2})$ *as* $|x| \to \infty$.

Proof. Noting (A3), (A5), (F4) and $\beta > 0$, we have the assertions by direct calculations. In fact we have

$$\langle A\hat{x}, \hat{x}\rangle = 1 + \langle(A-E)\hat{x}, \hat{x}\rangle,$$

where $E = (\delta_{ij})$ is the $n \times n$ identity matrix,

$$\nabla(\langle A\hat{x}, \hat{x}\rangle) = \langle(\nabla A)\hat{x}, \hat{x}\rangle + 2r^{-1}\{(A-E)\hat{x} - \hat{x}\langle(A-E)\hat{x}, \hat{x}\rangle\},$$

$$\partial_i\partial_j(\langle A\hat{x}, \hat{x}\rangle) = \sum_{k,l=1}^{n}\{(\partial_i\partial_j a_{kl}(x))\hat{x}_k\hat{x}_l + (\partial_i a_{kl}(x))\partial_j(\hat{x}_k\hat{x}_l)$$
$$+ (\partial_j a_{kl}(x))\partial_i(\hat{x}_k\hat{x}_l) + a_{kl}(x)\partial_i\partial_j(\hat{x}_k\hat{x}_l)\},$$

$$\partial_i\hat{x}_k = O(r^{-1}) \text{ and } \partial_i\partial_j\hat{x}_k = O(r^{-2}) \text{ as } |x| \to \infty,$$

$$\operatorname{div}(A\hat{x}) = (n-1)r^{-1} + \operatorname{div}\{(A-E)\hat{x}\},$$

$$\partial_i\{\operatorname{div}(A\hat{x})\} = \sum_{k,l=1}^{n}[\{\partial_i\partial_k a_{kl}(x)\}\hat{x}_l + \{\partial_k a_{kl}(x)\}\partial_i\hat{x}_l$$
$$+ \{\partial_i a_{kl}(x)\}\partial_k\hat{x}_l + a_{kl}(x)\partial_i\partial_k\hat{x}_l]. \qquad \square$$

Lemma 2.2. *For any* $x \in \Omega$ *and any* $\xi \in \mathbf{C}^n$ *we have*

$$\langle A(x)\xi, \bar{\xi}\rangle - \langle A(x)\hat{x}, \hat{x}\rangle^{-1}|\langle A(x)\xi, \hat{x}\rangle|^2$$
$$= \langle A(x)\{\xi - \hat{x}\langle A(x)\hat{x}, \hat{x}\rangle^{-1}\langle A(x)\xi, \hat{x}\rangle\},$$
$$\{\bar{\xi} - \hat{x}\langle A(x)\hat{x}, \hat{x}\rangle^{-1}\langle A(x)\xi, \hat{x}\rangle\}\rangle \geq 0.$$

Proof. By direct calculation we have the assertion. □

Lemma 2. 3. *There exist some constants $R_1 > R_0$ and $C_4 > 0$ such that for any real-valued function $\phi(r) \in C_0^1(R_1, \infty)$ we have*

$$\int_\Omega \phi(r)^2 \{ |Du|^2 + (q_1)_- |u|^2 \} dx \le C_4 \int_\Omega \{ r^{\max\{\delta_1, \delta_2\}} \phi(r)^2 + \phi'(r)^2 \} |u|^2 dx.$$

Proof. By (A3), (A4) and (G1), there exist some $R_1 > R_0$ and $\varepsilon > 0$ such that for any $r \ge R_1$ and any $\xi \in C^n$ we have

$$(1-\varepsilon) |\xi|^2 \le \langle A(x)\xi, \bar{\xi} \rangle \le (1+\varepsilon) |\xi|^2,$$
$$(1-\varepsilon)^2 - (1+\varepsilon) a > 0.$$

Since $u(x)$ satisfies (*), we have by integration by parts for $\phi(r) \in C_0^1(R_1, \infty)$

$$0 = \mathrm{Re} \int_\Omega [-\langle D, ADu \rangle + (q_1 + q_2) u] \phi(r)^2 \overline{u(x)} dx$$

$$= \int_\Omega [\phi^2 \langle ADu, \overline{Du} \rangle + 2\phi\phi' \mathrm{Re}[\langle ADu, \hat{x} \rangle \bar{u}]$$
$$+ \phi^2 (q_1 + \mathrm{Re}[q_2]) |u|^2] dx$$

$$\ge \int_\Omega [\phi^2 \langle ADu, \overline{Du} \rangle - \varepsilon\phi^2 \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADu, \hat{x} \rangle|^2$$
$$- \varepsilon^{-1} \phi'^2 \langle A\hat{x}, \hat{x} \rangle |u|^2 - \phi^2 \{ (q_1)_- + (\mathrm{Re}[q_2])_- \} |u|^2] dx$$

$$\ge (1-\varepsilon)^2 \int_\Omega \phi^2 |Du|^2 dx - \int_\Omega [\varepsilon^{-1} (1+\varepsilon) \phi'^2 |u|^2 + \phi^2 \{ (q_1)_- + C_3 r^{\delta_2} \} |u|^2] dx,$$

where we use Lemma 2.2. For any $\eta > 0$ let $u_\eta(x) = \{ |u(x)|^2 + \eta^2 \}^{1/2}$. Since $|u| \le |u_\eta|$ and $u_\eta \nabla u_\eta = \mathrm{Re}[\bar{u} \nabla u] = \mathrm{Re}[\bar{u} Du]$, we have

$$|\nabla u_\eta| \le |Du|,$$
$$\phi(r) u_\eta(x) \in H^1(\Omega),$$
$$\mathrm{supp}[\phi(r) u_\eta(x)] \text{ is a compact set in } \Omega.$$

By (G1) and limiting procedure we have for any $w \in H^1(\Omega)$ satisfying that $\mathrm{supp}[w]$ is a compact set in Ω

$$\int_\Omega (q_1)_- |w|^2 dx \le a \int_\Omega |\nabla w|^2 dx + C_2 \int_\Omega r^{\delta_1} |w|^2 dx.$$

Then we have

$$\int_\Omega (q_1)_- \phi^2 |u_\eta|^2 dx \le a \int_\Omega |\nabla(\phi u_\eta)|^2 dx + C_2 \int_\Omega r^{\delta_1} |\phi u_\eta|^2 dx$$

$$\le (1+\varepsilon) a \int_\Omega \phi^2 |Du|^2 dx + \int_\Omega \{ \varepsilon^{-1} (1+\varepsilon) a \phi'^2$$
$$+ C_2 r^{\delta_1} \phi^2 \} |u_\eta|^2 dx.$$

Let $\eta \downarrow 0$. Since $q_1 \in L^1_{loc}(\Omega)$ by (C2), we have by Lebesgue's dominated convergence theorem

$$\int_\Omega (q_1)_- \phi^2 |u|^2 dx \leq (1+\varepsilon)a \int_\Omega \phi^2 |Du|^2 dx$$
$$+ \int_\Omega \{\varepsilon^{-1}(1+\varepsilon)a\phi'^2 + C_2 r^{\delta_1}\phi^2\} |u|^2 dx.$$

Then we have

$$\int_\Omega \phi^2 |Du|^2 dx \leq C_5 \int_\Omega \{r^{\max\{\delta_1, \delta_2\}}\phi^2 + \phi'^2\} |u|^2 dx,$$

where $C_5 = \{(1-\varepsilon)^2 - (1+\varepsilon)a\}^{-1} \max\{C_2+C_3, \varepsilon^{-1}(1+\varepsilon)(1+a)\}$. Therefore we have

$$\int_\Omega \phi^2 \{|Du|^2 + (q_1)_- |u|^2\} dx \leq C_4 \int_\Omega \{r^{\max\{\delta_1, \delta_2\}}\phi^2 + \phi'^2\} |u|^2 dx,$$

where $C_4 = C_5 + (1+\varepsilon)aC_5 + \max\{C_2, \varepsilon^{-1}(1+\varepsilon)a\} > 0$. $\qquad\square$

Lemma 2.4. *There exists some constant $C_6 > 0$ such that for any real-valued function $\zeta(r) \in C^1[R_1, \infty)$ satisfying*

$$\int_{|x|>R_1} r^{\max\{\delta_1, \delta_2\}}\zeta(r)^2 |u(x)|^2 dx < \infty,$$

$$\int_{|x|>R_1} \zeta'(r)^2 |u(x)|^2 dx < \infty,$$

$$\liminf_{t\to\infty} \int_{t<|x|<t+1} \zeta(r)^2 |u(x)|^2 dx = 0,$$

and for any $R \geq R_1$ (where R_1 is the one given in Lemma 2.3), we have

$$\int_{|x|>R+1} \zeta(r)^2 \{|Du|^2 + (q_1)_- |u|^2\} dx \leq C_6 \int_{|x|>R} (r^{\max\{\delta_1, \delta_2\}}\zeta^2 + \zeta'^2) |u|^2 dx$$
$$+ C_6 \int_{R<|x|<R+1} \zeta^2 |u|^2 dx.$$

Proof. For any $R \geq R_1$ and $t > R+2$ we choose $\xi_{Rt}(r) \in C_0^\infty(R, t)$ satisfying $\xi_{Rt}(r) = 1$ for $R+1 \leq r \leq t-1$ and $0 \leq \xi_{Rt}(r) \leq 1$ for $R \leq r \leq t$. At the same time we can assume that there exists some constant $C_7 > 0$ such that for any $R \geq R_1$, $t > R+2$ and $r \geq R_1$ we have $|\xi'_{Rt}(r)| \leq C_7$. Then by Lemma 2.3 we have for any $R \geq R_1$ and $t > R+2$, noting $\xi'_{Rt}(r) = 0$ for $R+1 < r < t-1$,

$$\int_{R+1<|x|<t-1} \zeta(r)^2 \{|Du|^2 + (q_1)_- |u|^2\} dx$$

$$\leq \int_{\Omega} (\xi_{Rt}\zeta)^2 \{|Du|^2 + (q_1)_- |u|^2\} dx$$

$$\leq C_4 \int_{\Omega} \{r^{\max\{\delta_1, \delta_2\}} (\xi_{Rt}\zeta)^2 + (\xi'_{Rt}\zeta + \xi_{Rt}\zeta')^2\} |u|^2 dx$$

$$\leq C_4 \int_{R<|x|<t} (r^{\max\{\delta_1, \delta_2\}}\zeta^2 + 2\zeta'^2) |u|^2 dx$$

$$+ 2C_4 C_7^2 \{\int_{R<|x|<R+1} \zeta^2 |u|^2 dx + \int_{t-1<|x|<t} \zeta^2 |u|^2 dx\}.$$

Letting $t \to \infty$ along the suitable subsequence and letting $C_6 = \max \{2C_4, 2C_4 C_7^2\} > 0$, we have the assertion.   □

Definition 2.5. For real-valued functions $\rho(r) \in C^2(R_0, \infty)$, $f(x) \in C^1(\Omega)$ and $g(x) \in C^1(\Omega)$, let

$$v(x) = e^{\rho(r)} u(x),$$
$$k_1(x) = -\{\rho'(r)\}^2 \langle A(x)\hat{x}, \hat{x}\rangle,$$
$$k_2(x) = \rho''(r)\langle A(x)\hat{x}, \hat{x}\rangle + \rho'(r) \operatorname{div} \{A(x)\hat{x}\},$$
$$F(t; \rho, f, g) = \int_{|x|=t} [f(x)\langle A(x)\hat{x}, \hat{x}\rangle\{2\langle A(x)\hat{x}, \hat{x}\rangle^{-1} |\langle ADv, \hat{x}\rangle|^2$$
$$- \langle ADv, \overline{Dv}\rangle - (q_1+k_1) |v|^2\} + g(x) \operatorname{Re}[\langle ADv, \hat{x}\rangle\bar{v}]]dS.$$

The meaning of $F(t; \rho, f, g)$ can be cleared by Lemma 2.8.

Lemma 2.6. *We have for* $t > R_0$

$$F(t; \rho, f, g) = e^{2\rho(t)} \int_{|x|=t} [f(x) \{2 |\langle ADu, \hat{x}\rangle|^2 - \langle A\hat{x}, \hat{x}\rangle\langle ADu, \overline{Du}\rangle\}$$
$$+ \{2\rho' f\langle A\hat{x}, \hat{x}\rangle + g\} \operatorname{Re}[\langle ADu, \hat{x}\rangle\bar{u}]$$
$$+ \{2f\rho'^2\langle A\hat{x}, \hat{x}\rangle^2 + (g\rho' - fq_1)\langle A\hat{x}, \hat{x}\rangle\} |u|^2]dS.$$

Proof. Noting Definition 2.5 we have the assertion by direct calculation.   □

To prove Lemma 2.8 we use the following.

Lemma 2.7. *Let* w, $\partial_i w \in L^1_{\mathrm{loc}}(\Omega)$, *where* $\partial_i w$ *is the derivative in the sense of distribution. Then for any* $t > R_0$, $\int_{|x|=t} \hat{x}_i w dS$ *exists (by choosing suitable representative for* w, *if necessary), and we have for any* $t > s > R_0$

$$\int_{s<|x|<t} \partial_i w \, dx = \int_{|x|=t} \hat{x}_i w \, dS - \int_{|x|=s} \hat{x}_i w \, dS.$$

Proof. See Eastham and Kalf [3, p. 239].   □

Lemma 2.8. *For any* $t>s>R_0$ *we have*

$$F(t;\rho,f,g) - F(s;\rho,f,g)$$
$$= \int_{s<|x|<t} [2\mathrm{Re}[\langle ADv, \nabla f \rangle \langle A\overline{Dv}, \hat{x} \rangle] + \{g - \langle A\nabla f, \hat{x} \rangle$$
$$-f\mathrm{div}(A\hat{x})\} \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2 + \{2r^{-1}f \langle A\hat{x}, \hat{x} \rangle$$
$$+g - \langle A\nabla f, \hat{x} \rangle - f\mathrm{div}(A\hat{x})\} \{\langle ADv, \overline{Dv} \rangle$$
$$-\langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2\} + 2r^{-1}f\{|ADv|^2 - \langle A\hat{x}, \hat{x} \rangle \langle ADv, \overline{Dv} \rangle\}$$
$$+2f\mathrm{Re}[\langle (\langle ADv, \nabla \rangle A) \overline{Dv}, \hat{x} \rangle] - f\,\mathrm{Re}[\langle (\langle \hat{x}, A\nabla \rangle A) Dv, \overline{Dv} \rangle]$$
$$-2f\mathrm{Re}[\sqrt{-1} \langle ABA\hat{x}, (Dv - \hat{x} \langle A\hat{x}, \hat{x} \rangle^{-1} \langle ADv, \hat{x} \rangle) \rangle \bar{v}]$$
$$+2f\mathrm{Re}[\langle ADv, \hat{x} \rangle \overline{q_2 v}] + \mathrm{Re}[\langle ADv, \nabla g \rangle \bar{v}] + \{(g - \langle A\nabla f, \hat{x} \rangle$$
$$-f\mathrm{div}(A\hat{x}))q_1 - f\langle A\nabla q_1, \hat{x} \rangle + g\mathrm{Re}[q_2]\} |v|^2] dx$$
$$+\int_{s<|x|<t} [4\rho'f|\langle ADv, \hat{x} \rangle|^2 + 2(fk_2 + g\rho')\mathrm{Re}[\langle ADv, \hat{x} \rangle \bar{v}]$$
$$+ \{(g - \langle A\nabla f, \hat{x} \rangle - f\mathrm{div}(A\hat{x}))k_1 - f\langle A\nabla k_1, \hat{x} \rangle + gk_2\} |v|^2] dx.$$

Proof. $v(x) \in H^2_{\mathrm{loc}}(E_{R_0})$ (, where $E_{R_0} = \{x \mid |x| > R_0\}$,) given in Definition 2.5 satisfies

$$-\langle D, ADv \rangle + 2\rho'(r) \langle ADv, \hat{x} \rangle$$
$$+ \{(q_1(x) + k_1(x)) + (q_2(x) + k_2(x))\} v(x) = 0$$

for $r>R_0$. Noting (A1), (A2), (B1), (B2), (C1), (C2), (C3), (D1) and (D2) we apply Lemma 2.7 to

$$0 = \mathrm{Re}\int_{s<|x|<t} [-\langle D, ADv \rangle + 2\rho' \langle ADv, \hat{x} \rangle + \{(q_1 + k_1) + (q_2 + k_2)\} v]$$
$$\times [2f(x)\langle A\overline{Dv}, \hat{x} \rangle + g(x)\bar{v}] dx.$$

At the same time we note

$$D_i D_l - D_l D_i = \sqrt{-1} \{\partial_i b_l(x) - \partial_l b_i(x)\},$$
$$2\mathrm{Re}[\langle D_i Dw, A\overline{Dw} \rangle] = \partial_l(\langle ADw, \overline{Dw} \rangle) - \langle (\partial_l A) Dw, \overline{Dw} \rangle,$$
$$\langle ABA\hat{x}, \hat{x} \rangle = 0.$$

Then we have the followings :

$$-2\mathrm{Re}\int_{s<|x|<t}\langle D, ADv\rangle f\langle A\overline{Dv}, \hat{x}\rangle dx$$

$$= -2\left(\int_{|x|=t}-\int_{|x|=s}\right)f|\langle ADv, \hat{x}\rangle|^2 dS$$

$$+\int_{s<|x|<t}[2\mathrm{Re}[\langle ADv, \nabla f\rangle\langle A\overline{Dv}, \hat{x}\rangle]+2r^{-1}f(|ADv|^2$$

$$-|\langle ADv, \hat{x}\rangle|^2)+2f\mathrm{Re}[\langle(\langle ADv, \nabla\rangle A)\overline{Dv}, \hat{x}\rangle]$$

$$+2f\{\sum_{i,k,l=1}^{n}(ADv)_{i}\hat{x}_{k}a_{kl}\overline{D_{i}D_{l}v}\}]dx,$$

$$2\mathrm{Re}\int_{s<|x|<t}f\{\sum_{i,k,l=1}^{n}(ADv)_{i}\hat{x}_{k}a_{kl}\overline{D_{i}D_{l}v}\}dx$$

$$=\int_{s<|x|<t}[f\langle\hat{x}, A\nabla\rangle(\langle ADv, \overline{Dv}\rangle)-f\langle(\langle\hat{x}, A\nabla\rangle A)Dv, \overline{Dv}\rangle$$

$$-2f\mathrm{Re}[\sqrt{-1}\langle ABA\hat{x}, (Dv-\hat{x}\langle A\hat{x}, \hat{x}\rangle^{-1}\langle ADv, \hat{x}\rangle)\rangle\bar{v}]]dx,$$

$$\int_{s<|x|<t}f\langle\hat{x}, A\nabla\rangle(\langle ADv, \overline{Dv}\rangle)dx$$

$$=\left(\int_{|x|=t}-\int_{|x|=s}\right)f\langle A\hat{x}, \hat{x}\rangle\langle ADv, \overline{Dv}\rangle dS$$

$$-\int_{s<|x|<t}\{\langle A\nabla f, \hat{x}\rangle+f\mathrm{div}\,(A\hat{x})\}\langle ADv, \overline{Dv}\rangle dx,$$

$$2\mathrm{Re}\int_{s<|x|<t}(q_1+k_1)vf\langle A\overline{Dv}, \hat{x}\rangle dx$$

$$=\left(\int_{|x|=t}-\int_{|x|=s}\right)f\langle A\hat{x}, \hat{x}\rangle(q_1+k_1)|v|^2 dS$$

$$-\int_{s<|x|<t}[\{\langle A\nabla f, \hat{x}\rangle+f\mathrm{div}\,(A\hat{x})\}(q_1+k_1)$$

$$+f\langle A\nabla(q_1+k_1), \hat{x}\rangle]|v|^2 dx, -\mathrm{Re}\int_{s<|x|<t}\langle D, ADv\rangle g\bar{v}\,dx$$

$$=-\left(\int_{|x|=t}-\int_{|x|=s}\right)g\,\mathrm{Re}[\langle ADv, \hat{x}\rangle\bar{v}]dS$$

$$+\int_{s<|x|<t}[\mathrm{Re}[\langle ADv, \nabla g\rangle\bar{v}]+g\langle ADv, \overline{Dv}\rangle]dx.$$

So we have the assertion.   □

Definition 2.9. Let

$$G(x;v) = (2\alpha-\gamma(x))\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2+(2-\gamma(x))\{\langle ADv, \overline{Dv}\rangle$$

$$-\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2\}+2r\mathrm{Re}[\langle ADv, \nabla(\langle A\hat{x}, \hat{x}\rangle^{-1})\rangle\langle A\overline{Dv}, \hat{x}\rangle]$$

$$+2\{\langle A\hat{x}, \hat{x}\rangle^{-1}|ADv|^2-\langle ADv, \overline{Dv}\rangle\}$$

$$+2r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{Re}[\langle(\langle ADv, \nabla\rangle A)\overline{Dv}, \hat{x}\rangle]$$

$$-r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{Re}[\langle(\langle\hat{x}, A\nabla\rangle A)Dv, \overline{Dv}\rangle]$$

$$-2\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{Re}[\sqrt{-1}\,\langle ABAx, (Dv-\hat{x}\langle A\hat{x}, \hat{x}\rangle^{-1}\langle ADv, \hat{x}\rangle)\rangle\bar{v}]$$

$$+2r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{Re}[\langle ADv, \hat{x}\rangle\overline{q_2 v}]+(\alpha-1)g_0(x)r^{-1}\mathrm{Re}[\langle ADv, \hat{x}\rangle\bar{v}]$$

$$+\mathrm{Re}[\langle ADv, \nabla g_0\rangle\bar{v}]+\{g_0(x)\mathrm{Re}[q_2]-r\langle A\hat{x}, \hat{x}\rangle^{-1}\langle A\nabla q_1, \hat{x}\rangle$$

$$-\gamma(x)q_1\}\,|v|^2,$$

where

$$g_0(x)=\alpha-\gamma(x)+r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})+r\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle.$$

The meanings of $G(x;v)$ and $g_0(x)$ given in Definition 2.9 can be cleared by the following. The choice of $f(x)$ and $g(x)$ given in the following are the same given by Mochizuki [6].

Lemma 2.10. *we have*

(1) *for* $t>s>R_0$

$$F(t;\rho, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0(x)r^{\alpha-1})-F(s;\rho, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0(x)r^{\alpha-1})$$

$$=\int_{s<|x|<t}r^{\alpha-1}[G(x;v)+4\rho'r\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2$$

$$+2\{r\rho''+r\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})+g_0\rho'\}\mathrm{Re}[\langle ADv, \hat{x}\rangle\bar{v}]$$

$$+\langle A\hat{x}, \hat{x}\rangle\{\gamma(x)\rho'^2+2r\rho'\rho''-r\rho'^2\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle$$

$$+g_0\rho''+g_0\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})\}\,|v|^2]dx,$$

(2) $g_0(x)=n+\alpha-1-\gamma(x)+o(1)$ *as* $|x|\to\infty$,

(3) $\nabla g_0(x)=o(r^{\beta-1})$ *as* $|x|\to\infty$.

Proof. Noting Definitions 2.5, 2.9 and Lemma 2.8, we have (1) by direct calculation. (2) and (3) are also obtained by $\beta>0$, (A5), (F5), Lemma 2.1 and Definition 2.9.   □

Lemma 2.11. *There exist some constants* $C_8>0$ *and* $R_2>R_0$ *such that for any* $r>R_2$ *we have*

$$G(x;v)\geq C_8(|\langle ADv, \hat{x}\rangle|^2+r^{2\beta-2}|v|^2)$$

and for any $t>s>R_2$ *we have*

$$F(t;0, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0(x)r^{\alpha-1})-F(s;0, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0(x)r^{\alpha-1})$$

$$\geq C_8\int_{s<|x|<t}r^{\alpha-1}[|\langle ADv, \hat{x}\rangle|^2+r^{2\beta-2}|v|^2]dx.$$

Proof. Noting Lemma 2.10, we have only to show the former

part of the assertion. In the sequel $\varepsilon_i(r)$ $(i=1, 2, \ldots)$ means a positive function for $r > R_0$ which tends to 0 as $r \to \infty$. Noting (A1)$\sim$(A5), Lemmas 2.1, 2.2, and 2.10 and inequality

$$|\langle A(x)\xi, \eta\rangle| \leq \langle A(x)\xi, \bar{\xi}\rangle^{1/2}\langle A(x)\eta, \bar{\eta}\rangle^{1/2} \quad \text{for any } x \in \Omega, \xi, \eta \in \mathbf{C}^n,$$

we have the followings:

$$2r\text{Re}[\langle ADv, \nabla(\langle A\hat{x}, \hat{x}\rangle^{-1})\rangle\langle A\overline{Dv}, \hat{x}\rangle] \geq -\varepsilon_1(r)\langle ADv, \overline{Dv}\rangle,$$

$$2\{\langle A\hat{x}, \hat{x}\rangle^{-1}|ADv|^2 - \langle ADv, \overline{Dv}\rangle\} \geq -\varepsilon_2(r)\langle ADv, \overline{Dv}\rangle,$$

$$2r\langle A\hat{x}, \hat{x}\rangle^{-1}\text{Re}[\langle(\langle ADv, \nabla\rangle A)\overline{Dv}, \hat{x}\rangle] \geq -\varepsilon_3(r)\langle ADv, \overline{Dv}\rangle,$$

$$-r\langle A\hat{x}, \hat{x}\rangle^{-1}\text{Re}[\langle(\langle \hat{x}, A\nabla\rangle A)Dv, \overline{Dv}\rangle] \geq -\varepsilon_4(r)\langle ADv, \overline{Dv}\rangle,$$

$$-2\langle A\hat{x}, \hat{x}\rangle^{-1}\text{Re}[\sqrt{-1}\langle ABAx, (Dv - \hat{x}\langle A\hat{x}, \hat{x}\rangle^{-1}\langle ADv, \hat{x}\rangle)\rangle\bar{v}]$$

$$\geq -2|v|\langle A\hat{x}, \hat{x}\rangle^{-1}\langle ABAx, BAx\rangle^{1/2}\langle A(Dv - \hat{x}\langle A\hat{x}, \hat{x}\rangle^{-1}\langle ADv, \hat{x}\rangle),$$
$$(\overline{Dv} - \hat{x}\langle A\hat{x}, \hat{x}\rangle^{-1}\langle A\overline{Dv}, \hat{x}\rangle)\rangle^{1/2}$$

$$\geq -\frac{2-\gamma(x)}{1+\eta}\{\langle ADv, \overline{Dv}\rangle - \langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2\}$$

$$-\frac{1+\eta}{2-\gamma(x)}\langle A\hat{x}, \hat{x}\rangle^{-2}\langle ABAx, BAx\rangle|v|^2,$$

$$2r\langle A\hat{x}, \hat{x}\rangle^{-1}\text{Re}[\langle ADv, \hat{x}\rangle\overline{q_2\bar{v}}] \geq -\frac{2\alpha-\gamma(x)}{1+(\eta/2)}\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2$$

$$-\frac{1+(\eta/2)}{2\alpha-\gamma(x)}\langle A\hat{x}, \hat{x}\rangle^{-1}|rq_2|^2|v|^2,$$

$$(\alpha-1)g_0 r^{-1}\text{Re}[\langle ADv, \hat{x}\rangle\bar{v}] \geq -r^{-\beta}\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2$$
$$-4^{-1}(\alpha-1)^2|g_0|^2r^{\beta-2}\langle A\hat{x}, \hat{x}\rangle|v|^2,$$

$$\text{Re}[\langle ADv, \nabla g_0\rangle\bar{v}] \geq -\varepsilon_5(r)\{\langle ADv, \overline{Dv}\rangle + r^{2\beta-2}|v|^2\},$$

$$g_0\text{Re}[q_2] \geq -\frac{\eta}{2(2\alpha-\gamma(x))}\langle A\hat{x}, \hat{x}\rangle^{-1}|rq_2|^2 - \frac{2\alpha-\gamma(x)}{2\eta}\langle A\hat{x}, \hat{x}\rangle|g_0|^2r^{-2}.$$

By (F), (F1), (F2), (F3), and Lemmas 2.1 and 2.10, there exist some constants $C_8 > 0$ and $R_2 > R_0$ such that for any $r > R_2$ we have

$$2\alpha - \gamma(x) - \frac{2\alpha-\gamma(x)}{1+(\eta/2)} - r^{-\beta} - \sum_{i=1}^{5}\varepsilon_i(r) \geq C_1 C_8,$$

$$2 - \gamma(x) - \frac{2-\gamma(x)}{1+\eta} - \sum_{i=1}^{5}\varepsilon_i(r) \geq C_1 C_8,$$

$$-\left[r\langle A\hat{x}, \hat{x}\rangle^{-1}\langle A\nabla q_1, \hat{x}\rangle + \gamma(x)q_1\right.$$

$$+ (1+\eta) \left\{ \frac{1}{2\alpha - \gamma(x)} \langle A\hat{x}, \hat{x} \rangle^{-1} |rq_2|^2 \right.$$

$$\left. + \frac{1}{2-\gamma(x)} \langle A\hat{x}, \hat{x} \rangle^{-2} \langle ABAx, BAx \rangle \right\} \right]$$

$$- 4^{-1}(\alpha-1)^2 |g_0|^2 \langle A\hat{x}, \hat{x} \rangle r^{\beta-2} - \varepsilon_5(r) r^{2\beta-2}$$

$$- \frac{2\alpha - \gamma(x)}{2\eta} \langle A\hat{x}, \hat{x} \rangle |g_0|^2 r^{-2} \geq C_8 r^{2\beta-2},$$

where C_1 is the one given in (A4). So, by Definition 2.9 and $\langle ADv, \overline{Dv} \rangle = \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2 + \{\langle ADv, \overline{Dv} \rangle - \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2\}$, we have the assertion. □

Remark 2.12. Under the assumptions in Theorem 1.1 we have also $G(x;v) \geq C_8(|Dv|^2 + r^{2\beta-2}|v|^2)$. But considering the proof of Theorem 1.4, we formalize the statement given in Lemma 2.11.

Lemma 2.13. *For any constant* $l > 2^{-1}(2 - \lim\inf_{r\to\infty} \gamma(x)) \geq \delta/2 > 0$, *there exists some constant* $R_3 \geq R_2($, *where* R_2 *is the one given in Lemma 2.11,) such that for any* $t > s > R_3$ *and any* $m \geq 1$ *we have*

$$F(t;mr^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0(x) r^{\alpha-1}) \geq F(s;mr^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0(x) r^{\alpha-1}).$$

Proof. Let

$$\rho(r) = mr^l, \quad f(x) = \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, \quad g(x) = g_0(x) r^{\alpha-1}.$$

Then we have

$$4\rho' r \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2 = 4mlr^l \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2,$$

$$2(r\rho'' + r\rho' \langle A\hat{x}, \hat{x} \rangle^{-1} \text{div}(A\hat{x}) + g_0\rho') \text{Re}[\langle ADv, \hat{x} \rangle \bar{v}]$$

$$= 2mlr^{l-1}\{l-1 + r\langle A\hat{x}, \hat{x} \rangle^{-1} \text{div}(A\hat{x}) + g_0\} \text{Re}[\langle ADv, \hat{x} \rangle \bar{v}]$$

$$\geq -mlr^l \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2$$

$$- mlr^{l-2} \langle A\hat{x}, \hat{x} \rangle \{l-1 + r\langle A\hat{x}, \hat{x} \rangle^{-1} \text{div}(A\hat{x}) + g_0\}^2 |v|^2,$$

$$\gamma\rho'^2 + 2r\rho'\rho'' - r\rho'^2 \langle A\nabla(\langle A\hat{x}, \hat{x} \rangle^{-1}), \hat{x} \rangle + g_0\rho'' + g_0\rho' \langle A\hat{x}, \hat{x} \rangle^{-1} \text{div}(A\hat{x})$$

$$= m^2 l^2 r^{2l-2} \{2l + \gamma(x) - 2 - r\langle A\nabla(\langle A\hat{x}, \hat{x} \rangle^{-1}), \hat{x} \rangle\}$$

$$+ g_0 mlr^{l-2} \{l-1 + r\langle A\hat{x}, \hat{x} \rangle^{-1} \text{div}(A\hat{x})\}.$$

By Lemmas 2.1 and 2.10, there exists some constant $R_3 \geq R_2$ such that for any $r > R_3$ and any $m \geq 1$ we have

$$mlr^l \{2l + \gamma(x) - 2 - r\langle A\nabla(\langle A\hat{x}, \hat{x} \rangle^{-1}), \hat{x} \rangle\}$$

$$+g_0\{l-1+r\langle A\hat{x},\,\hat{x}\rangle^{-1}\mathrm{div}\,(A\hat{x})\}$$
$$-\langle A\hat{x},\,\hat{x}\rangle\{l-1+r\langle A\hat{x},\,\hat{x}\rangle^{-1}\mathrm{div}\,(A\hat{x})+g_0\}^2\geq0.$$

So, by Lemmas 2.10 and 2.11, we have the assertion. $\square$

§3. Proofs of Theorem 1.1 and Corollary 1.2

In this section we assume Assumptions (A)∼(G). And let $u(x)$ satisfy (*). Keep in our mind that we intend to prove Theorem 1.1 by contradiction.

Lemma 3.1. *If*

$$\liminf_{R\to\infty} R^\alpha\int_{|x|=R}[\,|\langle ADu,\,\hat{x}\rangle|^2+\{r^{-2}+(q_1)_-\}\,|u|^2]dx=0,$$

then we have for any $m>0$

$$\int_{|x|>R_0}r^m[\,|Du|^2+\{1+(q_1)_-\}\,|u|^2]dx<\infty,$$

$$\liminf_{R\to\infty} R^m\int_{|x|=R}[\,|Du|^2+\{1+(q_1)_-\}\,|u|^2]dS=0.$$

Proof. We have only to prove the former part of the assertion. At first we show

$$(\dagger):\int_{|x|>R_0}[r^{m\beta+\alpha-1}\,|\langle ADu,\,\hat{x}\rangle|^2+r^{(m+2)\beta+\alpha-3}\,|u|^2]dx<\infty$$

for $m=0,1,2,\ldots$, by mathematical induction. By Definition 2.5 and Lemmas 2.1 and 2.10, there exists some constant $C_9>0$ such that for any $t\geq R_0$ we have

$$F(t;0,\langle A\hat{x},\,\hat{x}\rangle^{-1}r^\alpha,\,g_0(x)r^{\alpha-1})$$
$$=t^\alpha\int_{|x|=t}[2\langle A\hat{x},\,\hat{x}\rangle^{-1}\,|\langle ADu,\,\hat{x}\rangle|^2-\langle ADu,\,\overline{Du}\rangle$$
$$+g_0r^{-1}\mathrm{Re}[\langle ADu,\,\hat{x}\rangle\bar{u}]-q_1|u|^2]dS$$
$$\leq C_9t^\alpha\int_{|x|=t}[\,|\langle ADu,\,\hat{x}\rangle|^2+\{r^{-2}+(q_1)_-\}\,|u|^2]dS.$$

Then by assumption we have

$$\liminf_{t\to\infty} F(t;0,\langle A\hat{x},\,\hat{x}\rangle^{-1}r^\alpha,\,g_0(x)r^{\alpha-1})\leq0.$$

Therefore, letting $t \to \infty$ along the suitable subsequence in Lemma 2.11, we have for any $s \geq R_2$

$$C_8 \int_{|x|>s} [r^{\alpha-1}|\langle ADu, \hat{x}\rangle|^2 + r^{\alpha+2\beta-3}|u|^2]dx \leq -F(s; 0, \langle A\hat{x}, \hat{x}\rangle^{-1}r^{\alpha}, g_0 r^{\alpha-1}),$$

which shows that $(\dagger)$ holds for $m=0$. Now we assume that $(\dagger)$ holds for m. Then by above inequality, Definition 2.5 and Lemma 2.8 we have for any $t \geq R_2$

$$C_8 \int_{R_2}^{t} s^{(m+1)\beta-1}ds \int_{|x|>s} [r^{\alpha-1}|\langle ADu, \hat{x}\rangle|^2 + r^{\alpha+2\beta-3}|u|^2]dx$$

$$\leq -\int_{R_2}^{t} s^{(m+1)\beta-1}F(s; 0, \langle A\hat{x}, \hat{x}\rangle^{-1}r^{\alpha}, g_0 r^{\alpha-1})ds$$

$$= \int_{R_2<|x|<t} r^{(m+1)\beta+\alpha-1}[\langle ADu, \overline{Du}\rangle + q_1|u|^2 - g_0 r^{-1}\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}]$$

$$-2\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADu, \hat{x}\rangle|^2]dx$$

$$= F(t; 0, 0, r^{(m+1)\beta+\alpha-1}) - F(R_2; 0; 0, r^{(m+1)\beta+\alpha-1})$$

$$-\int_{R_2<|x|<t} r^{(m+1)\beta+\alpha-1}[2\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADu, \hat{x}\rangle|^2$$

$$+ \{g_0 + (m+1)\beta + \alpha - 1\}r^{-1}\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] + \mathrm{Re}[q_2]|u|^2]dx.$$

Noting $\beta>0$, (G2), Lemmas 2.1 and 2.10, and $(\dagger)$ for m, there exists some constant $C_{10}>0$ such that for any $t>R_2$ we have

$$-\int_{R_2<|x|<t} r^{(m+1)\beta+\alpha-1}[2\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADu, \hat{x}\rangle|^2$$

$$+ \{g_0 + (m+1)\beta + \alpha - 1\}r^{-1}\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] + \mathrm{Re}[q_2]|u|^2]dx$$

$$\leq \int_{|x|>|R_2} r^{(m+1)\beta+\alpha-1}[8^{-1}\langle A\hat{x}, \hat{x}\rangle\{g_0 + (m+1)\beta + \alpha - 1\}^2 r^{-2} + C_3 r^{\beta-2}]|u|^2 dx$$

$$\leq C_{10}\int_{|x|>R_2} r^{(m+2)\beta+\alpha-3}|u|^2 dx < \infty.$$

By Definition 2.5. we have

$$F(t; 0, 0, r^{(m+1)\beta+\alpha-1})$$

$$\leq 2^{-1}\int_{|x|=t} [r^{m\beta+\alpha}|\langle ADu, \hat{x}\rangle|^2 + r^{(m+2)\beta+\alpha-2}|u|^2]ds.$$

Then by $(\dagger)$ for m, we have

$$\liminf_{t\to\infty} F(t; 0, 0, r^{(m+1)\beta+\alpha-1}) \leq 0.$$

Therefore

$$+\infty > \int_{R_2}^{\infty} s^{(m+1)\beta-1} ds \int_{|x|>s} [r^{\alpha-1} |\langle ADu, \hat{x}\rangle|^2 + r^{\alpha+2\beta-3} |u|^2] dx$$

$$= \frac{1}{(m+1)\beta} \int_{|x|>R_2} (r^{(m+1)\beta} - R_2^{(m+1)\beta}) [r^{\alpha-1} |\langle ADu, \hat{x}\rangle|^2 + r^{\alpha+2\beta-3} |u|^2] dx,$$

which shows that (†) holds for $m+1$. Letting in Lemma 2.4 $\zeta(r)$ $= r^{m/2}$, and noting (†), we have the former half of the assertion.   □

Lemma 3.2. *Let*

$$\lim_{R\to\infty} \inf R^\alpha \int_{|x|=R} [|\langle ADu, \hat{x}\rangle|^2 + \{r^{-2} + (q_1)_-\} |u|^2] dx = 0.$$

Then for any $0 < l < \beta$ and any $m > 0$ there exists some constant $R_4 \geq R_2$ (, where R_2 is the one given in Lemma 2.11) such that for any $t > s > R_4$ we have

$$e^{mt^l} \int_{|x|=t} \langle A\hat{x}, \hat{x}\rangle |u|^2 dS \leq e^{ms^l} \int_{|x|=s} \langle A\hat{x}, \hat{x}\rangle |u|^2 dS.$$

Proof. For $0 < l < \beta$ and $m > 0$ let $a(r) = 2^{-1}(mlr^l + n)$. By Lemma 2.1 there exists some constant $R_5 \geq R_0$ such that for any $r \geq R_5$ we have $n \geq r\langle A\hat{x}, \hat{x}\rangle^{-1} \mathrm{div}(A\hat{x})$. Then for any $t > s > R_5$ we have by integration by parts

$$\left(\int_{|x|=t} - \int_{|x|=s}\right) e^{mr^l} \langle A\hat{x}, \hat{x}\rangle |u|^2 dS$$

$$= \int_{s<|x|<t} e^{mr^l} [2\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] + \{mlr^{l-1}\langle A\hat{x}, \hat{x}\rangle + \mathrm{div}(A\hat{x})\} |u|^2] dx$$

$$\leq 2\int_s^t e^{m\tau^l} d\tau \int_{|x|=\tau} [\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] + r^{-1} a(r)\langle A\hat{x}, \hat{x}\rangle |u|^2] dS.$$

So we have only to show that there exists some $R_4 \geq R_5$ such that for any $\tau > R_4$ we have

$$\int_{|x|=\tau} [\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] + r^{-1} a(r)\langle A\hat{x}, \hat{x}\rangle |u|^2] dS \leq 0.$$

For $\rho(r) = a(\tau) \log r$ and $g(r) = r^{-2a(\tau)}$ we have $e^{2\rho(r)} g(r) = 1$, $e^{2\rho(r)} g(r) \rho'(r) = r^{-1} a(\tau)$, $g'(r) + 2g(r)\rho'(r) = 0$, $k_1(x) = -a(\tau)^2 r^{-2}\langle A\hat{x}, \hat{x}\rangle$ and $k_2(x) = -a(\tau) r^{-2}\langle A\hat{x}, \hat{x}\rangle + a(\tau) r^{-1} \mathrm{div}(A\hat{x})$ by Definition 2.5. Therefore by Lemmas 2.6 and 2.8 we have for $t_1 > \tau > R_5$

$$\left(\int_{|x|=t_1} - \int_{|x|=\tau}\right) [\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] + r^{-1} a(\tau)\langle A\hat{x}, \hat{x}\rangle |u|^2] dS$$

$$= F(t_1; a(\tau) \log r, 0, r^{-2a(\tau)}) - F(\tau; a(\tau) \log r, 0, r^{-2a(\tau)})$$

$$= \int_{\tau < |x| < t_1} r^{-2a(\tau)} [\langle ADv, \overline{Dv} \rangle + \{q_1 + \mathrm{Re}[q_2]\} |v|^2$$
$$+ a(\tau) r^{-2} \{r \ \mathrm{div}(A\hat{x}) - \langle A\hat{x}, \hat{x} \rangle - a(\tau) \langle A\hat{x}, \hat{x} \rangle \} |v|^2] dx$$

$$= - \int_\tau^{t_1} s_1^{-(a+2a(\tau))} \{F(s_1; a(\tau) \log r, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1})$$

$$+ 2 C_3 s_1^{\alpha+\beta-2} \int_{|x|=s_1} \langle A\hat{x}, \hat{x} \rangle |v|^2 dS \} ds_1$$

$$+ \int_{\tau < |x| < t_1} r^{-2a(\tau)} [2 \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2 + g_0 r^{-1} \mathrm{Re}[\langle ADv, \hat{x} \rangle \bar{v}]$$
$$+ \{\mathrm{Re}[q_2] + a(\tau) r^{-2} (r \ \mathrm{div}(A\hat{x}) - \langle A\hat{x}, \hat{x} \rangle)$$
$$+ 2 C_3 r^{\beta-2} \langle A\hat{x}, \hat{x} \rangle \} |v|^2] dx,$$

where C_3 is the one given in (G2). By (G2), Lemmas 2.1 and 2.10, and $0 < l < \beta$, there exist some constants $R_6 \geq R_5$ and $C_{11} > 0$ such that for any $r > \tau \geq R_6$ we have

$$2 \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2 + g_0 r^{-1} \mathrm{Re}[\langle ADv, \hat{x} \rangle \bar{v}]$$
$$+ \{\mathrm{Re}[q_2] + a(\tau) r^{-2} (r \ \mathrm{div}(A\hat{x}) - \langle A\hat{x}, \hat{x} \rangle) + 2 C_3 r^{\beta-2} \langle A\hat{x}, \hat{x} \rangle \} |v|^2$$

$$\geq \{-8^{-1} g_0^2 r^{-2} \langle A\hat{x}, \hat{x} \rangle + 2 C_3 \langle A\hat{x}, \hat{x} \rangle r^{\beta-2} - (\mathrm{Re}[q_2])_-$$
$$+ a(\tau) r^{-2} (r \ \mathrm{div}(A\hat{x}) - \langle A\hat{x}, \hat{x} \rangle) \} |v|^2$$

$$\geq (2^{-1} C_3 r^{\beta-2} - C_{11} r^{l-2}) |v|^2 \geq 0.$$

Noting for $t_2 > s_1 > R_6$

$$(\int_{|x|=t_2} - \int_{|x|=s_1}) r^{\alpha+\beta-2} \langle A\hat{x}, \hat{x} \rangle |v|^2 dS$$

$$= \int_{s_1 < |x| < t_2} r^{\alpha+\beta-2} [2 \mathrm{Re}[\langle ADv, \hat{x} \rangle \bar{v}] + \{\mathrm{div}(A\hat{x})$$
$$+ (\alpha+\beta-2) \langle A\hat{x}, \hat{x} \rangle r^{-1}\} |v|^2] dx,$$

we have by Lemma 2.10 for $t_1, t_2 > s_1 > \tau > R_6$

$$F(t_2; a(\tau) \log r, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1}) - F(s_1; a(\tau) \log r, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1})$$

$$+ 2 C_3 (\int_{|x|=t_2} - \int_{|x|=s_1}) r^{\alpha+\beta-2} \langle A\hat{x}, \hat{x} \rangle |v|^2 dS$$

$$= \int_{s_1 < |x| < t_2} r^{\alpha-1} [G(x; v) + 4 a(\tau) \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2$$
$$+ 2 a(\tau) r^{-1} \{g_0 + r \langle A\hat{x}, \hat{x} \rangle^{-1} \mathrm{div}(A\hat{x}) - 1\} \mathrm{Re}[\langle ADv, \hat{x} \rangle \bar{v}]$$
$$+ a(\tau) \langle A\hat{x}, \hat{x} \rangle r^{-2} \{(\gamma(x) - 2 - r \langle A\nabla(\langle A\hat{x}, \hat{x} \rangle^{-1}), \hat{x} \rangle) a(\tau)$$
$$- g_0 + r g_0 \langle A\hat{x}, \hat{x} \rangle^{-1} \mathrm{div}(A\hat{x})\} |v|^2 + 4 C_3 r^{\beta-1} \mathrm{Re}[\langle ADv, \hat{x} \rangle \bar{v}]$$
$$+ 2 C_3 \langle A\hat{x}, \hat{x} \rangle \{r \langle A\hat{x}, \hat{x} \rangle^{-1} \mathrm{div}(A\hat{x}) + \alpha+\beta-2\} r^{\beta-2} |v|^2] dx.$$

Noting $0<l<\beta$, Lemmas 2.1, 2.10 and 2.11, there exist some constants $R_7\geq\max\{R_2, R_6\}$ and $C_{12}>0$ such that for any $r>\tau>R_7$ we have

$$G(x;v)\geq C_8 r^{2\beta-2}|v|^2,$$

$$2a(\tau)r^{-1}\{g_0+r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})-1\}\mathrm{Re}[\langle ADv, \hat{x}\rangle\bar{v}]$$
$$\geq -a(\tau)\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2-C_{12}r^{l-2}|v|^2,$$

$$a(\tau)\langle A\hat{x}, \hat{x}\rangle r^{-2}\{(\gamma-2-r\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle)a(\tau)$$
$$-g_0+rg_0\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})\}|v|^2$$
$$\geq -C_{12}r^{2l-2}|v|^2,$$

$$2C_3\langle A\hat{x}, \hat{x}\rangle\{r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})+\alpha+\beta-2\}r^{\beta-2}|v|^2\geq -C_{12}r^{\beta-2}|v|^2,$$

$$4C_3 r^{\beta-1}\mathrm{Re}[\langle ADv, \hat{x}\rangle\bar{v}]\geq -a(\tau)\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2-C_{12}r^{2\beta-2}\tau^{-l}|v|^2.$$

There exists some constant $R_4>R_7\geq R_2$ such that for any $r>\tau>R_4$ we have

$$C_8 r^{2\beta-2}-C_{12}r^{2\beta-2}\tau^{-l}-C_{12}(r^{l-2}+r^{2l-2}+r^{\beta-2})>0.$$

Therefore for any $t_2>s_1>\tau>R_4$ we have

$$F(t_2;a(\tau)\log\ r, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0 r^{\alpha-1})+2C_3\int_{|x|=t_2}r^{\alpha+\beta-2}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS$$

$$\geq F(s_1;a(\tau)\log\ r, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0 r^{\alpha-1})+2C_3\int_{|x|=s_1}r^{\alpha+\beta-2}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS.$$

By Lemmas 2.1, 2.6 and 2.10, for $\tau\geq R_4$ there exist some constant $R_8\geq R_4$ and $C_{13}>0$ such that for any $t_2\geq R_8$ we have

$$F(t_2;a(\tau)\log\ r, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0 r^{\alpha-1})+2C_3\int_{|x|=t_2}r^{\alpha+\beta-2}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS$$

$$\leq C_{13}t_2^{\alpha+2a(\tau)}\int_{|x|=t_2}[|\langle ADu, \hat{x}\rangle|^2+\{r^{\beta-2}+(q_1)_-\}|u|^2]dS.$$

Therefore by Lemma 3.1 we have

$$\liminf_{t_2\to\infty}[F(t_2;a(\tau)\log\ r, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0 r^{\alpha-1})$$
$$+2C_3\int_{|x|=t_2}r^{\alpha+\beta-2}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS]\leq 0.$$

So for any $s_1\geq\tau\geq R_4$ we have

$$F(s_1;a(\tau)\log\ r, \langle A\hat{x}, \hat{x}\rangle^{-1}r^\alpha, g_0 r^{\alpha-1})+2C_3\int_{|x|=s_1}r^{\alpha+\beta-2}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS\leq 0,$$

and then for any $t_1 \geq \tau \geq R_4$ we have

$$\int_{|x|=t_1} [\text{Re}[\langle ADu, \hat{x} \rangle \bar{u}] + a(\tau) r^{-1} \langle A\hat{x}, \hat{x} \rangle |u|^2] dS$$

$$\geq \int_{|x|=\tau} [\text{Re}[\langle ADu, \hat{x} \rangle \bar{u}] + a(\tau) r^{-1} \langle A\hat{x}, \hat{x} \rangle |u|^2] dS.$$

Letting $t_1 \to \infty$ along the suitable subsequence, we have, by Lemma 3.1., for any $\tau \geq R_4$

$$\int_{|x|=\tau} [\text{Re}[\langle ADu, \hat{x} \rangle \bar{u}] + a(\tau) r^{-1} \langle A\hat{x}, \hat{x} \rangle |u|^2] dS \leq 0. \qquad \square$$

Lemma 3.3. *If*

$$\liminf_{R \to \infty} R^\alpha \int_{|x|=R} [|\langle ADu, \hat{x} \rangle|^2 + \{r^{-2} + (q_1)_-\} |u|^2] dS = 0,$$

then for any $0 < l < \beta$ *and any* $m > 0$ *we have*

$$\int_{|x| > R_0} e^{mr^l} [|Du|^2 + \{1 + (q_1)_-\} |u|^2] dx < \infty.$$

Proof. We apply Lemma 3.2 replacing m wfth $m+2$. Then for any $r \geq R_4$ we have

$$\int_{|x|=r} |u|^2 dS \leq C_{14} e^{-(m+2)r^l},$$

where $C_{14} = e^{(m+2)R_4^l} \int_{|x|=R_4} |u|^2 dS \geq 0$. Then we have

$$\int_{|x| \geq R_0} e^{(m+1)r^l} |u|^2 dx \leq e^{(m+1)R_4^l} \int_{R_0 < |x| < R_4} |u|^2 dx + C_{14} \int_{R_4}^\infty e^{-r^l} dr < \infty.$$

Letting $\zeta(r) = e^{(m/2)r^l}$ in Lemma 2.4, we have the assertion. $\qquad \square$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. At first we shall prove the former half of the statement. If we assume that the statement is false, we have

$$\liminf_{R \to \infty} R^\alpha \int_{|x|=R} [|\langle ADu, \hat{x} \rangle|^2 + \{r^{-2} + (q_1)_-\} |u|^2] dS = 0.$$

By (F) and (F2) we can choose a constant l satisfying $0 < 2^{-1}\delta \leq 2^{-1}(2 - \liminf_{r \to \infty} \gamma(x)) < l < \beta$. By Lemma 2.12 we have for any $t > s \geq$

R_3 and any $m \geq 1$

$$F(t; mr^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1}) \geq F(s; mr^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1}).$$

By Lemmas 2.1, 2.6 and 2.10, for $m \geq 1$ there exists some constant $R_9 \geq R_3$ such that for any $t \geq R_9$ we have

$$F(t; mr^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1})$$

$$\leq e^{(2m+1)t^l} \int_{|x|=t} [|\langle ADu, \hat{x} \rangle|^2 + \{1 + (q_1)_-\} |u|^2] dS.$$

So by Lemma 3.3 we have

$$\liminf_{t \to \infty} F(t; mr^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1}) \leq 0,$$

and then we have for any $s \geq R_3$ and any $m \geq 1$

$$F(s; mr^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1}) \leq 0.$$

Since supp$[u]$ is not a compact set in $\bar{\Omega}$ by (*), there exists some constant $R_{10} \geq R_3$ such that $\int_{|x|=R_{10}} \langle A\hat{x}, \hat{x} \rangle |u|^2 dS > 0$. By Lemma 2.6, $e^{-2mR_{10}^l} F(R_{10}; mr^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1})$ is a quadratic in m, whose coefficient of m^2 is $2l^2 R_{10}^{2l+\alpha-2} \int_{|x|=R_{10}} \langle A\hat{x}, \hat{x} \rangle |u|^2 dS > 0$. Therefore there exists some constant $m_0 \geq 1$ such that

$$F(R_{10}; m_0 r^l, \langle A\hat{x}, \hat{x} \rangle^{-1} r^\alpha, g_0 r^{\alpha-1}) > 0,$$

which is the contradiction.

Now we shall prove the latter half of Theorem 1.1. Let $\varepsilon > 0$. We take $\xi_R(r) \in C_0^1(R, R+\varepsilon)$ satisfying $\xi_R(r) = 1$ for $R + (\varepsilon/3) \leq r \leq R + (2\varepsilon/3)$ and $0 \leq \xi_R(r) \leq 1$ for $R \leq r \leq R+\varepsilon$. Moreover we can admit that there exists some constant $C_{15} > 0$ such that for any $R \geq R_0$ and any $r \geq R$ we have $|\xi_R'(r)| \leq C_{15}$. Let $\sigma = \max\{0, \delta_1, \delta_2\} \geq 0$. By the conclusion of the former half of Theorem 1.1, there exist some constants $C_{16} > 0$ and $R_{11} \geq R_0$ such that for any $R \geq R_{11}$ we have

$$C_{16} R^{-\alpha-\sigma} \leq \int_{R+(\varepsilon/3)}^{R+(2\varepsilon/3)} r^{-\alpha-\sigma} dr,$$

$$C_{16} R^{-\alpha} \leq \int_{|x|=R} [|\langle ADu, \hat{x} \rangle|^2 + \{r^{-2} + (q_1)_-\} |u|^2] dS.$$

Then by (A4), Lemmas 2.2, 2.3, $-\sigma \leq 0$ and $-\sigma + \max\{\delta_1, \delta_2\} \leq 0$, we have for any $R \geq R_{11}$

$$C_{16}^2 R^{-\alpha-\sigma} \leq \int_{R+(\varepsilon/3)}^{R+(2\varepsilon/3)} C_{16} r^{-\alpha-\sigma} dr,$$

$$\leq \int_{R+(\varepsilon/3)<|x|<R+(2\varepsilon/3)} r^{-\sigma} [\, |\langle ADu, \hat{x}\rangle|^2 + \{r^{-2} + (q_1)_-\} \, |u|^2] dx$$

$$\leq C_1^2 \int_{R<|x|<R+\varepsilon} (\xi_R(r) r^{-\sigma/2})^2 [\, |Du|^2 + (q_1)_- \, |u|^2] dx$$

$$+ \int_{R<|x|<R+\varepsilon} r^{-\sigma-2} |u|^2 dx$$

$$\leq C_1^2 C_4 \int_{R<|x|<R+\varepsilon} \{r^{\max\{\delta_1, \delta_2\}} \xi_R^2 r^{-\sigma} + 2\xi_R'^2 r^{-\sigma}$$

$$+ 2(-\sigma/2)^2 \xi_R^2 r^{-\sigma-2}\} \, |u|^2 dx + \int_{R<|x|<R+\varepsilon} r^{-\sigma-2} |u|^2 dx$$

$$\leq C_{17} \int_{R<|x|<R+\varepsilon} |u|^2 dx,$$

where C_1 is the one given in (A4) and $C_{17} = C_1^2 C_4\left(1 + 2C_{15}^2 + \dfrac{\sigma^2}{2}\right) + 1 > 0.$ $\qquad\square$

Proof of Corollary 1.2. By Theorem 1.1 there exist some constant $C_{18} > 0$ and some integer $N_0 \geq R_0$ such that for any integer $N \geq N_0$ we have

$$\int_{N<|x|<N+1} |u|^2 dx \geq C_{18} N^{-\alpha-\max\{0, \delta_1, \delta_2\}} \geq C_{18} N^{-1}.$$

Then we have

$$\int_{N_0<|x|<N+1} |u|^2 dx = \sum_{n=N_0}^{N} \int_{n<|x|<n+1} |u|^2 dx \geq C_{18} \sum_{n=N_0}^{N} n^{-1},$$

and then $\displaystyle\int_{|x|\geq R_0} |u|^2 dx = \infty.$ $\qquad\square$

§4. Proofs of Theorems 1.3 and 1.4

In this section we assume (B) $\sim$ (G) replacing $a_{ij}(x)$ with δ_{ij} except for (C3). Instead of (C3) we assume (C3)$'$. Let $u(x)$ satisfy (∗∗). In this case we use the following.

Lemma 4.1. *Let* $w, \partial_r w \in L^1_{\mathrm{loc}}(\Omega)$, *where* $\partial_r w$ *is the derivative in the distribution sense. Then for any* $t > R_0$, $\displaystyle\int_{|x|=t} w \, dS$ *exists (by choosing suitable representative for* w, *if necessary), and for any* $t > s > R_0$ *we have*

$$\int_{s<|x|<t} \partial_r w \, dx = (\int_{|x|=t} - \int_{|x|=s}) w \, dS - \int_{s<|x|<t} (n-1) r^{-1} w \, dx.$$

Proof. See Lemma 4.1 of Uchiyama [11].   □

Remark 4.2. If $a_{ij}(x) = \delta_{ij}$ and if we assume (C3)′ instead of (C3), then Lemma 2.8 with $a_{ij}(x) = \delta_{ij}$ also holds. In fact by (C3)′ we have $f(q_1+k_1) |v|^2$, $\partial_r \{f(q_1+k_1) |v|^2\} \in L^1_{\mathrm{loc}}(\Omega)$, and then by Lemma 4.1 we have

$$2\mathrm{Re}\int_{s<|x|<t} (q_1+k_1) v f \langle \overline{Dv}, \hat{x} \rangle dx$$

$$= (\int_{|x|=t} - \int_{|x|=s}) f(q_1+k_1) |v|^2 dS$$

$$- \int_{s<|x|<t} [\{\partial_r f + (n-1) r^{-1} f\} (q_1+k_1) + f \partial_r (q_1+k_1)] |v|^2 dx.$$

Now we can prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Since Lemma 2.8 with $a_{ij}(x) = \delta_{ij}$ is also true by Remark 4.2, we can follow the proofs of Theorem 1.1 and Corollary 1.2.   □

Proof of Theorem 1.4. Since Lemma 2.8 with $a_{ij}(x) = \delta_{ij}$ and $b_i(x) = 0$ is also true by Remark 4.2, we have for $t > s > R_0$ by Definition 2.9 and Lemma 2.10

$$F(t; \rho, r^\alpha, g_0 r^{\alpha-1}) - F(s; \rho, r^\alpha, g_0 r^{\alpha-1})$$

$$= \int_{s<|x|<t} r^{\alpha-1} [G(x;v) + 4\rho' r |\partial_r v|^2 + 2\{r\rho'' + (n-1)\rho' + g_0\rho'\} \mathrm{Re}[\bar{v}\partial_r v]$$

$$+ \{\gamma(x)\rho'^2 + 2r\rho'\rho'' + g_0\rho'' + (n-1) g_0\rho' r^{-1}\} |v|^2] dx,$$

where

$$g_0(x) = \alpha + n - 1 - \gamma(x) = O(1) \quad \text{as } |x| \to \infty,$$

$$G(x;v) = (2\alpha - \gamma(x)) |\partial_r v|^2 + (2 - \gamma(x)) (|\nabla v|^2 - |\partial_r v|^2) + 2r \, \mathrm{Re}[(\partial_r v) \overline{q_2 v}]$$

$$+ (\alpha - 1) g_0 r^{-1} \mathrm{Re}[\bar{v}\partial_r v] + \mathrm{Re}[\langle \nabla v, \nabla g_0 \rangle \bar{v}]$$

$$+ \{g_0 \mathrm{Re}[q_2] - r\partial_r q_1 - \gamma(x) q_1\} |v|^2.$$

By (F2)′ and (F5)′ we have

$$\langle \nabla v, \nabla g_0 \rangle = (\partial_r g_0) \partial_r v + \langle (\nabla - \hat{x}\partial_r) v, (\nabla - \hat{x}\partial_r) g_0 \rangle,$$

$$\partial_r g_0 = o(r^{\beta-1}) \quad \text{as} \quad |x| \to \infty,$$

$$|(\nabla - \hat{x}\partial_r)g_0| = |(\nabla - \hat{x}\partial_r)\gamma(x)| \leq (2-\gamma(x))^{1/2}p(x) \quad \text{for } r > R_0,$$

$$p(x) = o(r^{\beta-1}) \quad \text{as} \quad |x| \to \infty.$$

So we have

$$2r \ \mathrm{Re}\,[(\partial_r v)\overline{q_2 v}] \geq -\frac{2\alpha-\gamma(x)}{1+(\eta/2)}\,|\partial_r v|^2 - \frac{1+(\eta/2)}{2\alpha-\gamma(x)}\,|rq_2|^2|v|^2,$$

$$(\alpha-1)g_0 r^{-1}\mathrm{Re}[\bar{v}\partial_r v] \geq -r^{-\beta}|\partial_r v|^2 - 4^{-1}(\alpha-1)^2|g_0|^2 r^{\beta-2}|v|^2,$$

$$\mathrm{Re}[\langle \nabla v, \nabla g_0\rangle\bar{v}] \geq -\varepsilon_6(r)\{|\partial_r v|^2 + r^{2\beta-2}|v|^2\} - (2-\gamma(x))\{|\nabla v|^2 - |\partial_r v|^2\},$$

$$g_0\mathrm{Re}[q_2] \geq -\frac{\eta}{2(2\alpha-\gamma(x))}\,|rq_2|^2 - \frac{2\alpha-\gamma(x)}{2\eta}\,|g_0|^2 r^{-2}.$$

By (F), (F1) and (F3), there exist some constant $C_{19} > 0$ and $R_{12} > R_0$ such that for any $r > R_{12}$ we have

$$2\alpha-\gamma(x) - \frac{2\alpha-\gamma(x)}{1+(\eta/2)} - r^{-\beta} - \varepsilon_6(r) \geq C_{19},$$

$$-\{r\partial_r q_1 + \gamma(x)q_1 + \frac{1+\eta}{2\alpha-\gamma(x)}\,|rq_2|^2\} - 4^{-1}(\alpha-1)^2|g_0|^2 r^{\beta-2}$$

$$-\varepsilon_6(r)r^{2\beta-2} - \frac{2\alpha-\gamma(x)}{2\eta}\,|g_0|^2 r^{-2}$$

$$\geq C_{19}r^{2\beta-2}.$$

So for any $r \geq R_{12}$ we have

$$G(x;v) \geq C_{19}\{|\partial_r v|^2 + r^{2\beta-2}|v|^2\},$$

which corresponds to Lemma 2.11. Therefore we can follow the proofs of Theorem 1.1 and Corollary 1.2. ☐

§5. Examples

In this section we give several applications of Theorem 1.3. Let $u(x)$ satisfy (**), where (**) is given in Theorem 1.3. In the sequel we treat special $b_i(x)$, $q_1(x)$ and $q_2(x)$ in (**). So it is easy to check that all assumptions (B), (C)′, (D) and (E) are satisfied, where (C)′ is the condition given by a replacement (C3) with (C3)′ in (C). So we are mainly interested in checking (F) and (G). In (F) we choose $\gamma(x)$ as a constant function, and we rewrite its constant as γ again. After checking that (F) and (G) hold, we have by Theorem 1.3

$$(\#) \quad \varliminf_{R\to\infty} R^{\mu} \int_{R<|x|<R+1} |u(x)|^2 dx > 0,$$

where

$$\mu = \alpha + \max\{0, \delta_1, \delta_2\},$$

and α, δ_1, δ_2 are the ones given in (F) and (G). Here we note that the choice of α in (F) has some freedom. The smaller we choose α, the better results for μ we have.

Example 5.1. In (**) let

$$q_1(x) = -h(x) + V(x) - \lambda,$$

$$q_2(x) = o(r^{(\theta/2)-1}) \quad \text{as } r\to\infty,$$

$$b_i(x) = 0,$$

where $h(x)$ is a positive continuous homogeneous function of degree $\theta > 0$, $V(x)$ is a real-valued function satisfying

$$(\#\#) \quad V(x) = o(r^\theta) \quad \text{and} \quad \partial_r V(x) = o(r^{\theta-1}) \quad \text{as } r\to\infty,$$

and λ is a real constant.

Note $r\partial_r q_1 + \gamma q_1 \leq -\{\theta + \gamma - o(1)\} r^\theta \min_{|x|=1} h(x)$. So $\beta = 2^{-1}(2+\theta) > 0$ and $\min\{2, 2\alpha\} > \gamma > 2 - 2\beta = -\theta$ satisfy (F). $\delta_1 = \theta > \delta_2 = (\theta/2) - 1 = \beta - 2$ satisfy (G). Then in (#) $\mu > \theta/2$ is arbitrary. Moreover if $0 < \theta < 2$, we have $u(x) \notin L^2(\Omega)$. Compare with Agmon [2, Theorem 5] and Uchiyama [8, Example 5.2].

Example 5.2. In (**) let

$$q_1(x) = -h(x) + V(x),$$

$$q_2(x) = o(r^{(\theta/2)-1}) \quad \text{as } r\to\infty,$$

$$b_i(x) = 0,$$

where $h(x)$ is a positive continuous homogeneous function of degree $-2 < \theta < 0$ and $V(x)$ is a real-valued function satisfying (##).

In this case $\beta = 2^{-1}(2+\theta) > 0$ and $\min\{2, 2\alpha\} > \gamma > 2 - 2\beta = -\theta$ satisfy (F), and $0 > \delta_1 = \theta > \delta_2 = (\theta/2) - 1 = \beta - 2$ satisfy (G). Then in (#) $\mu > -(\theta/2)$ is arbitrary and $u(x) \notin L^2(\Omega)$. Compare with Agmon [1, Theorem 4 without its proof] and Uchiyama [8, Example 5.2].

Example 5.3. In (**) let

$$q_1(x) = h_1(x) - h_2(x) + V(x) - \lambda,$$

$$q_2(x) = o(r^\sigma) \quad \text{as } r \to \infty,$$

$$b_i(x) = 0,$$

where $h_1(x)$ is a real-valued homogeneous function of degree $-\gamma > -2$ satisfying

(###) there exist some constant $0 < a < 1$ and $C > 0$ such that for any $w(x) \in C_0^\infty(\Omega)$ we have

$$\int_\Omega |h_1(x)| \, |w(x)|^2 dx \leq a \int_\Omega |\nabla w(x)|^2 dx + C \int_\Omega r^{\max\{0, -\gamma\}} |w(x)|^2 dx,$$

$h_2(x)$ is a real-valued continuous homogeneous function of degree θ, $V(x)$ is a real-valued function satisfying (##), and λ is a real constant. Here we do not assume that $h_1(x)$ has a constant sign.

If $h_2(x)$ is positive, $\theta > \max\{0, -\gamma\}$ and $\sigma = (\theta/2) - 1$, then in (#) $\mu > (\gamma/2) + \theta$ is arbitrary, since $\beta = 2^{-1}(2 + \theta)$ and $2\alpha > \gamma > 2 - 2\beta = -\theta$ satisfy (F) and $\delta_1 = \theta > \delta_2 = (\theta/2) - 1 = \beta - 2$ satisfy (G). So if $h_2(x)$ is positive, $-2 < \gamma < 2$, $\max\{0, -\gamma\} < \theta < 1 - (\gamma/2)$ and $\sigma = (\theta/2) - 1$, then $u(x) \notin L^2(\Omega)$. On the other hand if $\gamma > 0$, $\lambda > 0$, $\theta < 0$ and $\sigma = -1$, then in (#) $\mu > \gamma/2$ is arbitrary and $u(x) \notin L^2(\Omega)$, since $\beta = 1$, $2\alpha > \gamma > 2 - 2\beta = 0$, $\delta_1 = 0$ and $\delta_2 = -1 = \beta - 2$ satisfy (F) and (G). Compare with Uchiyama [8, Example 5.6 and Remark 5.3].

Remark 5.4. We note that

$$h_1(x) = -\sum_{1 \leq i \leq N} \frac{c_i}{r_i} + \sum_{1 \leq i < j \leq N} \frac{c_{ij}}{r_{ij}}$$

is a homogeneous function of degree -1, does not have a constant sign and satisfies the condition (###) given in Example 5.3, where c_i, c_{ij} are positive constants, $n = 3N$, $x = (x^{(1)}, \ldots, x^{(N)}) \in \mathbb{R}^{3N}$, $x^{(i)} \in \mathbb{R}^3$, $r_i = |x^{(i)}|$ and $r_{ij} = |x^{(i)} - x^{(j)}|$.

Example 5.5. In (**) let $n = 2$ and

$$q_1(x) = cr^\theta + V(x) - \lambda,$$

$$q_2(x) = o(r^\sigma) \quad \text{as } r \to \infty,$$

$$b_1(x) = -\frac{1}{2} b_0 x_2 f(r), \quad b_2(x) = \frac{1}{2} b_0 x_1 f(r),$$

where $V(x)$ is a real-valued function satisfying (##), $f(r) = r^\delta$ or $f(r) = (1+r)^\delta$, $b_0 \neq 0$, c, λ, θ, σ, δ are real constants.

Noting

$$|B(x)x|^2 = \frac{1}{4}b_0^2\{rf'(r) + 2f(r)\}^2 r^2$$

we have the followings:

(i) If $c<0$, $\theta > \max\{0, 2(\delta+1)\}$ and $\sigma = (\theta/2) - 1$, then in (#) $\mu > \theta/2$ is arbitrary, since $\beta = 2^{-1}(2+\theta) > 0$, $2 > \gamma > 2 - 2\beta = -\theta$ and $\alpha > \gamma/2$ satisfy (F), and $\delta_1 = \theta > \delta_2 = \sigma = (\theta/2) - 1 = \beta - 2$ satisfy (G). So if $c<0$, $2 > \theta > \max\{0, 2(\delta+1)\}$ and $\sigma = (\theta/2) - 1$, then $u(x) \notin L^2(\Omega)$.

(ii) If $\theta = 2(\delta+1) > 0$, $\sigma = (\theta/2) - 1$ and $b_0^2 + 4c < 0$, then in (#) $\mu > \frac{1}{4}[2 - \theta - (2+\theta)\{1 + b_0^2(4c)^{-1}\}^{1/2}] + \theta$ is arbitrary. In fact for any γ satisfying $(2 - 2\beta = -\theta <) \frac{1}{2}[2 - \theta - (2+\theta)\{1 + b_0^2(4c)^{-1}\}^{1/2}] < \gamma < \frac{1}{2}[2 - \theta + (2+\theta)\{1 + b_0^2(4c)^{-1}\}^{1/2}]$ (<2), we have $c(\theta+\gamma) + \frac{1}{2-\gamma}\frac{1}{4}b_0^2(\delta+2)^2 < 0$, which shows that $\beta = 2^{-1}(2+\theta)$, γ given above and $2\alpha > \gamma$ satisfy (F). $\delta_1 = \theta > \delta_2 = (\theta/2) - 1 = \beta - 2$ satisfy (G). So if $0 < \theta = 2(\delta+1) < 2[1 + \{1 + b_0^2(4c)^{-1}\}^{1/2}][3 - \{1 + b_0^2(4c)^{-1}\}^{1/2}]^{-1}$ and $\sigma = (\theta/2) - 1$, then $u(x) \notin L^2(\Omega)$.

(iii) If $\theta < 0$, $\sigma = \delta = -1$ and $4\lambda > b_0^2$, then in (#) $\mu > \frac{1}{2}[1 - \{1 - b_0^2(4\lambda)^{-1}\}^{1/2}]$ is arbitrary and $u(x) \notin L^2(\Omega)$. In fact for any γ satisfying $(2 - 2\beta = 0 <) 1 - \{1 - b_0^2(4\lambda)^{-1}\}^{1/2} < \gamma < 1 + \{1 - b_0^2(4\lambda)^{-1}\}^{1/2} (<2)$, we have $-\gamma\lambda + \frac{1}{2-\gamma}\frac{1}{4}b_0^2 < 0$, which appears in (F). (F) is satisfied by $\beta = 1$, γ given above and $2\alpha > \gamma$. (G) is satisfied by $\delta_1 = 0$ and $\delta_2 = -1 = \beta - 2$.

(iv) If $\theta < 0$, $\delta < -1$, $\sigma = -1$ and $\lambda > 0$, then in (#) $\mu > 0$ is arbitrary and $u(x) \notin L^2(\Omega)$. In fact $\beta = 1$, $0 < \gamma < 2$ and $2\alpha > \gamma$ satisfy (F) and $\delta_1 = 0$ and $\delta_2 = -1 = \beta - 2$ satisfy (G). Compare with Uchiyama [10, Example 7].

Example 5.6. In (**) let $n = 3$ and

$$q_1(x) = cr^\theta + V(x) - \lambda,$$

$$q_2(x) = o(r^\sigma) \quad \text{as } r \to \infty,$$

$$b_1(x) = -\frac{1}{2}b_0 x_2 f(x), \quad b_2(x) = \frac{1}{2}b_0 x_1 f(x), \quad b_3(x) = 0,$$

where $f(x) = r^\delta (x_1^2 + x_2^2)^{\delta'/2}$, $b_0 \neq 0$, $\delta' > -1$, δ, c, θ, σ are real constants and $V(x)$ is a real-valued function satisfying (##). The condition $\delta' > -1$ is imposed to satisfy (B2).

Noting

$$|B(x)x|^2 = \frac{1}{4} b_0^2 (x_1^2 + x_2^2)(r\partial_r f + 2f)^2 = \frac{1}{4} b_0^2 (\delta + \delta' + 2)^2 r^{2\delta}(x_1^2 + x_2^2)^{\delta'+1},$$

we have the same results as Example 5.5 (i)$\sim$(iv), where we replace δ with $\delta + \delta'$. Compare with Uchiyama [10, Example 10].

Example 5.7. Let $N \geq 1$ be integer, c_i, $c_{i,j}$, c, $b_0 \neq 0$, δ, δ', θ, σ and λ be real constants. And in (**) let $n = 3N$ and

$$q_1(x) = -\sum_{1 \leq i \leq N} \frac{c_i}{r_i} + \sum_{1 \leq i < j \leq N} \frac{c_{i,j}}{r_{i,j}} + cr^\theta + V(x) - \lambda,$$

$$q_2(x) = o(r^\sigma) \quad \text{as } r \to \infty,$$

$$b_{3i-2}(x) = -\frac{1}{2} b_0 x_{3i-1} f_i(x), \quad b_{3i-1}(x) = \frac{1}{2} b_0 x_{3i-2} f_i(x), \quad b_{3i}(x) = 0,$$

where $f_i(x) = r^\delta r_i^{\delta'}$ $(\delta' \geq -1)$ or $f_i(x) = r^\delta (x_{3i-2}^2 + x_{3i-1}^2)^{\delta'/2}$ $(\delta' > -1)$ and $V(x)$ is a real-valued function satisfying (##).

Since

$$\max \{s_1^\tau + \cdots + s_N^\tau \mid s_1^2 + \cdots + s_N^2 = 1, \ s_i \geq 0, \ 1 \leq i \leq N\}$$
$$= \max \{N^{1-(\tau/2)}, 1\} \quad \text{for } \tau \geq 0,$$

we have

$$|B(x)x|^2 = \frac{1}{4} b_0^2 \sum_{1 \leq i \leq N} (x_{3i-2}^2 + x_{3i-1}^2)(r\partial_r f_i + 2f_i)^2$$

$$= \frac{1}{4} b_0^2 (\delta + \delta' + 2)^2 \sum_{1 \leq i \leq N} (x_{3i-2}^2 + x_{3i-1}^2) f_i^2,$$

$$\limsup_{r \to \infty} r^{-2(\delta'+1)} \sum_{1 \leq i \leq N} (x_{3i-2}^2 + x_{3i-1}^2) r_i^{2\delta'} = \max \{N^{-\delta'}, 1\},$$

$$\limsup_{r \to \infty} r^{-2(\delta'+1)} \sum_{1 \leq i \leq N} (x_{3i-2}^2 + x_{3i-1}^2)^{\delta'+1} = \max \{N^{-\delta}, 1\}.$$

Noting Remark, 5.4, we must choose $2 - 2\beta < \gamma = 1 < 2$ in (F), and we have the followings:

(i) If $c < 0$, $\theta > \max \{0, 2(\delta + \delta' + 1)\}$ and $\sigma = (\theta/2) - 1$, then in (#) $\mu > \frac{1}{2} + \theta$ is arbitrary. In fact $\beta = 2^{-1}(2 + \theta) > 0$ and $\min\{2, 2\alpha\} > \gamma = 1 > 2 - 2\beta = -\theta$ satisfy (F), and $\delta_1 = \theta > \delta_2 = \sigma = \beta - 2$ satisfy (G). So if

$c<0$, $\dfrac{1}{2}>\theta>\max\{0, 2(\delta+\delta'+1)\}$ and $\sigma=(\theta/2)-1$, then $u(x)\notin L^2(\Omega)$.

(ii) If $\theta=2(\delta+\delta'+1)>0$, $\sigma=(\theta/2)-1$ and $16c(\theta+1)+b_0^2(\theta+2)^2\max\{N^{-\delta'}, 1\}<0$, then in (#) $\mu>\dfrac{1}{2}+\theta$ is arbitrary. In fact the choice of α, β, γ, δ_1, δ_2 given in (i) also satisfies (F) and (G) in this case. So if $\dfrac{1}{2}>\theta=2(\delta+\delta'+1)>0$, $\sigma=(\theta/2)-1$ and $16c(\theta+1)+b_0^2(\theta+2)^2\max\{N^{-\delta'}, 1\}<0$, then $u(x)\notin L^2(\Omega)$.

(iii) If $\theta<0$, $\delta+\delta'=\sigma=-1$ and $4\lambda>b_0^2\max\{N^{-\delta'}, 1\}$, then in (#) $\mu>1/2$ is arbitrary and $u(x)\notin L^2(\Omega)$, since $\beta=1$, $2\alpha>\gamma=1>2-2\beta=0$ satisfy (F) and $\delta_1=0>\delta_2=\sigma=-1=\beta-2$ satisfy (G).

(iv) If $\theta<0$, $\delta+\delta'<-1$, $\sigma=-1$ and $\lambda=0$, then in (#) $\mu>1/2$ is arbitrary and $u(x)\notin L^2(\Omega)$, since $\beta=1$, $2\alpha>\gamma=1>2-2\beta=0$ satisfy (F), and $\delta_1=0$ and $\delta_2=\sigma=-1=\beta-2$ satisfy (G). Compare with Eastham-Kalf [3, Example 5.3.3, p. 134] and Uchiyama [10, Example 11].

Example 5.8. In (**) let

$$q_1(x)+q_2(x)=\frac{b\sin 2ar}{r}+V(x)+o(r^{-1})-\lambda \quad \text{as } r\to\infty,$$

$$b_i(x)=0,$$

where $a>0$, $b\neq0$ and $\lambda>0$ are real constants, and real-valued function $V(x)$ satisfies (\#\#) with $\theta=0$.

We shall show that $u(x)\notin L^2(\Omega)$, if $\lambda>2^{-1}a[\{a^2+4b^2\}^{1/2}-a]$. For simplicity let $b=\pm\tau a$ ($\tau>0$) and $\lambda=\nu a^2$ ($\nu>0$). And let

$$q_1(x)=\pm\tau a(1-s)\frac{\sin 2ar}{r}+V(x)-\nu a^2,$$

$$q_2(x)=\pm\tau as\frac{\sin 2ar}{r}+o(r^{-1}) \quad \text{as } r\to\infty,$$

where $-\infty<s<\infty$ is a parameter. Let $\alpha=\beta=1$, $\delta_1=0$, $\delta_2=-1=\beta-2$ and $2-2\beta=0<\gamma=2-t<2\alpha=2$ ($0<t<2$), which satisfy (G). If (F) holds for some $-\infty<s_0<\infty$ and $0<t_0<2$, then $\mu=1$ satisfy (#) and $u(x)\notin L^2(\Omega)$. In order to show that (F) holds for some $-\infty<s_0<\infty$ and $0<t_0<2$, it is enough to show

$$\inf\{H(s, t) \mid -\infty<s<\infty, 0<t<2\}<0,$$

where

$$H(s, t) = a^{-2} \lim_{r \to \infty} \sup [r \partial_r q_1(x) + (2-t) q_1(x) + t^{-1} |rq_2(x)|^2].$$

Now we calculate $H(s, t)$ as follows.

$$H(s, t) = \lim_{r \to \infty} \sup [\tau^2 s^2 t^{-1} + \nu t - \tau^2 s^2 t^{-1} \{\cos 2ar \mp (1-s) t \tau^{-1} s^{-2}\}^2$$
$$+ t(1-s)^2 s^{-2} - 2\nu]$$
$$= \begin{cases} \tau^2 s^2 t^{-1} + \nu t + t(1-s)^2 s^{-2} - 2\nu, & \text{if } t \leq \tau s^2 |1-s|^{-1}, \\ 2\tau |1-s| + \nu t - 2\nu, & \text{if } t > \tau s^2 |1-s|^{-1}. \end{cases}$$

$H(s, t)$ is continuous in $(s, t) \in \mathbb{R} \times (0, 2)$, and there exists some $(s_1, t_1) \in \mathbb{R} \times [0, 2]$ such that $t_1 = \tau s_1^2 |1-s_1|^{-1}$ and $H(s_1, t_1) = \inf \{H(s, t) \mid t > \tau s^2 |1-s|^{-1}, \ 0 < t < 2, \ s \in \mathbb{R}\}$. Noting $\tau^2 s^2 t^{-1} + t(1-s)^2 s^{-2} \geq 2\tau |1-s|$ for $(s, t) \in \mathbb{R} \times (0, 2)$, we have

$$\inf \{H(s, t) \mid s \in \mathbb{R}, 0 < t < 2\}$$
$$= \inf \{\tau^2 s^2 t^{-1} + \nu t + t(1-s)^2 s^{-2} - 2\nu \mid s \in \mathbb{R}, 0 < t < 2\}.$$

Here we have

$$\tau^2 s^2 t^{-1} + \nu t + t(1-s)^2 s^{-2} - 2\nu$$
$$= t^{-1} s^{-2} \{\nu s^2 + (1-s)^2\} [t - \nu s^2 \{\nu s^2 + (1-s)^2\}^{-1}]^2$$
$$+ t^{-1} s^2 \tau^2 (\nu+1) \{\nu s^2 + (1-s)^2\}^{-1} \{s - (\nu+1)^{-1}\}^2$$
$$+ t^{-1} s^2 \{\nu s^2 + (1-s)^2\}^{-1} \nu (\nu+1)^{-1} \{\tau^2 - \nu(\nu+1)\}.$$

Then $\inf \{H(s, t) \mid s \in \mathbb{R}, 0 < t < 2\} < 0$ is equivalent to $\tau^2 < \nu(\nu+1)$, which leads to $\lambda > 2^{-1} a [\{a^2 + 4b^2\}^{1/2} - a]$.

References

[1] Agmon, S., Lower bounds for solutions of Schrödinger-type equations in unbounded domains, *Proceedings of the International Conference of Functional Analysis and Related Topics*, Tokyo, (1969), 216-224.

[2] ————, Lower bounds for solutions of Schrödinger equations, *J. Analyse Math.* **23** (1970), 1-25.

[3] Eastham, M. S. P. and Kalf, H., *Schrödinger-type operators with continuous spectra*, Research notes in mathematics **65**, Pitman Advanced Publishing Program, 1982.

[4] Ikebe, T. and Uchiyama, J., On the asymptotic behavior of eigenfunctions of second-order elliptic operators, *J. Math. Kyoto Univ.* **11** (1971), 425-448.

[5] Kato, T., Growth properties of solutions of the reduced wave equation with a variable coefficient, *Comm. Pure Appl. Math.* **12** (1959), 403-425.

[6] Mochizuki, K., Growth properties of solutions of second-order elliptic differential equations, *J. Math. Kyoto Univ.* **16** (1976), 351-373.

[7] Uchiyama, J., Lower bounds of growth order of solutions of Schrödinger equations with homogeneous potentials, *Publ. RIMS*, Kyoto Univ. **10** (1975), 425-444.

[8] ————, Decay order of eigenfunctions of second-order elliptic operators in an

unbounded domain, and its applications, *Publ. RIMS*, Kyoto Univ. **22** (1986), 1079–1104.

[9]   Uchiyama, J., Decay order of eigenfunctions of second-order elliptic operators in an unbounded domain, and its applications, *Proceedings of Conference on Spectral and Scattering theory for differential operators*, edited by S. Ito et al. (1986), 206–213.

[10]   ————, Examples of exponential decay of eigenfunctions of magnetic Schrödinger operators, to appear on *Proceedings of UAB / UA International Conference on Differential equations and mathematical Physics*, edited by I. Knowles and Y. Saito.

[11]   ————, Lower bounds of decay order of eigenfunctions of second-order elliptic operators, *Publ. RIMS*, Kyoto Univ. **21** (1985), 1281–1297.

[12]   Yamada, O., A note on the non-existence of eigenvalues of Schrödinger equations, *Memoirs of the Research Institute of Science and Engineering*, Ritsumeikan Univ., **32** (1977), 1–6.