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Homotopy Associative Finite H-Spaces and the Mod 3 Reduced Power Operations

Dedicated to Professor Masahiro Sugawara on his 60th birthday

By

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§ 1. Introduction

In this paper, all spaces are assumed to be simply connected.

The actions of the Steenrod operations over the cohomology of an H-space have been studied from various view points. In particular, Thomas [8] determined the action of the squaring operations over the mod 2 cohomology of a finite H-space with primitively generated mod 2 cohomology. Lin [5] also proved the similar result by using a different method. The result of Lin is stated as follows:

(1.1) (Lin [5; Th. 1]) *Let X be an H-space. Suppose that the* mod 2 *cohomology Hopf algebra $H^*(X;\mathbb{Z}/2)$ is finite and primitively generated. Then for any primitive class x of dimension $2^a n-1$ with $n\not\equiv 0$* mod 2 *and $n>2$, we have that*

$$Sq^{2^a}x=0 \text{ and } x=Sq^{2^a}y \text{ for some } y\in H^*(X;\mathbb{Z}/2),$$

where Sq^i is the i-th squaring operation.

On the other hand, we can not get the corresponding result to (1.1) for an odd prime p. In fact, any odd sphere $S^{2n-1}_{(p)}$ localized at an odd prime p is an H-space (cf. [1]). However, if X is a homotopy associative H-space, then under some suitable conditions we can get the similar result to (1.1) about the action of the mod 3 reduced power operation $\mathcal{P}^t$ over $H^*(X;\mathbb{Z}/3)$.

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Let X be a homotopy associative H-space, and let $\iota: \Sigma X \to P_3X$ be the natural inclusion, where P_3X is the projective 3-space of X (Stasheff [7; Def. 13]). Then the classes in the image of $\sigma\iota^*: \tilde{H}^*(P_3X; \boldsymbol{Z}/3) \to \tilde{H}^*(\Sigma X; \boldsymbol{Z}/3) \cong \tilde{H}^{*-1}(X; \boldsymbol{Z}/3)$ are called A_3-primitive, where $\sigma: \tilde{H}^*(\Sigma X; \boldsymbol{Z}/3) \to \tilde{H}^{*-1}(X; \boldsymbol{Z}/3)$ is the suspension isomorphism. If $H^*(X; \boldsymbol{Z}/3)$ is generated by A_3-primitive classes as an algebra, we call $H^*(X; \boldsymbol{Z}/3)$ A_3-primitively generated. Let $P_3H^*(X; \boldsymbol{Z}/3)$ be the module of all A_3-primitive classes in $H^*(X; \boldsymbol{Z}/3)$, i.e., $P_3H^*(X; \boldsymbol{Z}/3) = \operatorname{Im} \sigma\iota^*$. Then the main theorem of this paper is stated as follows:

Theorem 1.2. *Let X be a homotopy associative H-space. Suppose that the* mod 3 *cohomology algebra $H^*(X; \boldsymbol{Z}/3)$ is finite and A_3-primitively generated. Then for any positive integer n with $n \not\equiv 0$* mod 3 *and $n > 3$, if*

(1.3) $$P_3H^{4\cdot 3^a t-1}(X; \boldsymbol{Z}/3) = 0 \text{ for } t \geq n-1,$$

then we have that

$$P_3H^{2\cdot 3^a n-1}(X; \boldsymbol{Z}/3) = \mathscr{P}^{3^a} P_3H^{2\cdot 3^a(n-2)-1}(X; \boldsymbol{Z}/3).$$

The assumption (1.3) in the above theorem cannot be dropped. In fact, we give an example in §7 to show that (1.3) is required.

Theorem 1.2 is deduced from a purely algebraic result.

Let B^* be an augmented graded algebra over $\boldsymbol{Z}/3$. Then we denote the augmentation ideal of B^* by $\tilde{B}^*$, and we define D^iB^* for $i \geq 1$ inductively by

$$D^1B^* = \tilde{B}^* \text{ and } D^iB^* = \tilde{B}^* \cdot D^{i-1}B^* \quad (i \geq 2).$$

Let $\mathscr{A}$ be the mod 3 Steenrod algebra and let $\mathscr{P}^n$ be the n-th mod 3 reduced power operation. Then B^* is called an unstable left $\mathscr{A}$-algebra if B^* is an augmented graded algebra over $\boldsymbol{Z}/3$ with left action of $\mathscr{A}$, such that the Cartan formula and the following unstable conditions hold:

$$\mathscr{P}^n x = 0 \text{ if } \dim x < 2n, \text{ and } \mathscr{P}^n x = x^3 \text{ if } \dim x = 2n.$$

Then we have the following

Theorem 1.4. *Let*

$$H^* = A^*/D^4A^*, \quad A^* = \boldsymbol{Z}/3[x_1, \cdots, x_k], \ \dim x_i\text{: } \textit{even}.$$

Suppose that H^* *is an unstable left* $\mathscr{A}$*-algebra. Then for any positive integer* n *with* $n\not\equiv 0$ mod 3 *and* $n>3$, *if*

$$QH^{4\cdot 3^a t}=0 \text{ for } t\geq n-1, \tag{1.5}$$

then we have that

$$QH^{2\cdot 3^a n}=\mathscr{P}^{3^a}QH^{2\cdot 3^a(n-2)},$$

where $QH^*=H^*/DH^*(DH^*=D^2H^*)$ *is the indecomposable module of* H^*.

H^* in the above theorem is called a truncated polynomial algebra over $\{x_1,\cdots,x_k\}$ of height 4, and it is denoted by $T^4[x_1,\cdots,x_k]$.

Theorem 1.2 is proved from Theorem 1.4 by using the following

Proposition 1.6. *Let* X *be the one in Theorem* 1.2. *Then for the natural inclusion* $\iota:\Sigma X\subset P_3X$ *to the projective* 3*-space* P_3X *of* X, *there are even dimensional classes* $\{y_i\,|\,1\leq i\leq k\}$ *and an ideal* S *in* $H^*(P_3X;\mathbb{Z}/3)$ *so that*

$$\mathscr{A}(S)\subset S,\ \iota^*(S)=0,\ H^*(P_3X;\mathbb{Z}/3)/S\cong T^4[y_1,\cdots,y_k],\ \text{and}$$
$$H^*(X;\mathbb{Z}/3)\cong\bigwedge(x_1,\cdots,x_k)\ \text{with}\ x_i=\sigma\iota^*y_i.$$

Thus, in particular, $H^*=T^4[y_1,\cdots,y_k]$ *is an unstable left* $\mathscr{A}$*-algebra, and* $\sigma\iota^*:QH^{2n}\to P_3H^{2n-1}(X;\mathbb{Z}/3)$ *is an isomorphism for any* $n\geq 1$.

We apply Theorem 1.4 for $H^*=T^4[y_1,\cdots,y_k]$ in Proposition 1.6. Then, by Proposition 1.6, (1.3) implies (1.5), and hence Theorem 1.2 is proved from Theorem 1.4.

In the rest of this paper, §§2–6 are devoted to prove Theorem 1.4 and Proposition 1.6, and we give some examples in §7 to show that the condition (1.3) in Theorem 1.2 is required. To prove Theorem 1.4, we give four propositions. Two of them are proved in §3. The others are proved in §5 by using particular generators for H^* which are given in §4. Finally, in §6, we prove Proposition 1.6.

§2. Reduction of Theorem 1.4

In the rest of this paper we assume that H^* is the augmented graded unstable left $\mathscr{A}$-algebra in Theorem 1.4, where $\mathscr{A}$ is the mod 3 Steenrod algebra. Hereafter we use the following notation for any

$a\geq 0$:

(2. 1) $$d(a)=\max\{0,\ t\,|\,QH^{4\cdot 3^a t}\neq 0,\ t\in\mathbf{Z}\}.$$

Then we have the following

Lemma 2. 2. *$d(a)\geq 3d(a+1)$ for all $a\geq 0$.*

Proof. This is clear if $d(a+1)=0$. While if $d(a+1)>0$, then $QH^{4\cdot 3^a\cdot 3d(a+1)}=QH^{4\cdot 3^{a+1}d(a+1)}\neq 0$, and so $d(a)\geq 3d(a+1)$ by (2. 1).

q. e. d.

Now consider the following statements $S_i(a, m)$ $(1\leq i\leq 4)$ for $a\geq 0$ and $m\geq 0$:

$S_1(a, m)$: *For any positive integer n with $n\not\equiv 0$ mod 3 and $n>3$, if $n\geq m$, then*

$$QH^{2\cdot 3^a n}=\mathcal{P}^{3^a}QH^{2\cdot 3^a(n-2)}.$$

$S_2(a, m)$: *For any positive integer n with $n\not\equiv 0$ mod 3 and $n\neq 2$, if $n\geq m$, then*

$$\mathcal{P}^{3^a}QH^{2\cdot 3^a n}\subset\mathcal{P}^{2\cdot 3^a}QH^{2\cdot 3^a(n-2)}.$$

$S_3(a, m)$: *For any positive integer n with $n\equiv 1$ mod 3, if $n\geq m$, then*

$$\mathcal{P}^{3^a}QH^{2\cdot 3^a n}=0.$$

$S_4(a, m)$: *For any positive integer n with $n\equiv 1$ mod 3 and $n>9$, if $n\geq m$, then*

$$\mathcal{P}^{3^a}H^{2\cdot 3^a n}\subset\sum_{i\leq a}\mathcal{P}^{3^i}DH^{2\cdot 3^i(3^{a-i}(n+2)-2)}.$$

Then we have the following

Lemma 2. 3. *If $m\leq 4$, $S_i(a, m)$ is equivalent to $S_i(a, 4)$ for $1\leq i\leq 3$. If $m\leq 10$, $S_4(a, m)$ is equivalent to $S_4(a, 10)$.*

Proof. It is clear that $S_1(a, m)$ is equivalent to $S_1(a, 4)$ for any $m\leq 4$, and $S_4(a, m)$ is equivalent to $S_4(a, 10)$ for any $m\leq 10$. Furthermore, by the unstable condition, we have that $\mathcal{P}^{3^a}QH^{2\cdot 3^a}=0$. Thus

$S_i(a, m)$ with $m \leq 4$ is also equivalent to $S_i(a, 4)$ for $i=2, 3$.

q. e. d.

Now Theorem 1. 4 is a consequence of the following propositions:

Proposition 2. 4. *Let $a \geq 0$. Suppose that if $a \geq 1$ then there are m_i and n_i for any $0 \leq i \leq a-1$ so that $S_3(i, m_i)$ and $S_4(i, n_i)$ are true. Then $S_1(a, m)$ is true for any m with $m \geq \max\{[d(0)/3^a]+2, (m_i+2)/3^{a-i}, (n_i+2)/2 \cdot 3^{a-i}\}$.*

Proposition 2. 5. *Let $a \geq 0$. Suppose that $S_1(a, t)$ is true for some t. Furthermore suppose that if $a \geq 1$ then there are m_i and n_i for any $0 \leq i \leq a-1$ so that $S_3(i, m_i)$ and $S_4(i, n_i)$ are true. Then $S_2(a, m)$ is true for any m with $m \geq \max\{(t+2)/3, (m_i+2)/3^{a-i}, (n_i+2)/2 \cdot 3^{a-i}\}$.*

Proposition 2. 6. Let $a \geq 0$. *Suppose that $S_2(a, t)$ is true for some t. Furthermore suppose that if $a \geq 1$ then there are m_i for any $0 \leq i \leq a-1$ so that $S_3(i, m_i)$ are true. Then $S_3(a, m)$ is true for any m with*

$$m \geq \max\{t+2, (\mathrm{m}_i+2)/3^{a-i}-2\}.$$

Proposition 2. 7. *Let $a \geq 0$. Suppose that $S_1(a, t)$ and $S_2(a, s)$ are true. Furthermore suppose that if $a \geq 1$ then there are m_i for any $0 \leq i \leq a-1$ so that $S_4(i, m_i)$ are true. Then $S_4(a, m)$ is true for any m with*

$$m \geq \max\{t, s+2, (m_i+2)/3^{a-i}-2\}.$$

Then Theorem 1. 4 is proved as follows.

Proof of Theorem 1. 4 *from Propositions* 2. 4–7. First we prove that

(2. 8) $S_1(a, [d(0)/3^a]+2)$ is true for all $a \geq 0$.

Next we prove that

(2. 9) $d(a) \geq [d(0)/3^a]$ for any $a \geq 0$.

Then, by Lemma 2. 2, we have $d(\mathrm{a})=[d(0)/3^a]$, and Theorem 1. 4 follows since it is equivalent to $S_1(a, d(a)+2)$ for all $a \geq 0$.

To prove (2. 8) we put for $a \geq 0$ that

$$m(a)=[d(0)/3^a]+2, \text{ and}$$

$$n(a)=[(d(0)+3^{a+2})/2\cdot 3^a].$$

Then, by simple calculations, we have the following inequalities:

$$\begin{aligned}
&n(a)\geq(m(a)+2)/3 \quad \text{for} \quad a\geq 0,\\
&m(a)\geq\max\{(m(i)+2)/2\cdot 3^{a-i},\ (m(i)+2)/3^{a-i}-2\} \quad \text{for} \quad a>i\geq 0,\\
&n(a)\geq\max\{(n(i)+4)/3^{a-i},\ (m(i)+2)/2\cdot 3^{a-i}\} \quad \text{for} \quad a>i\geq 0,\\
&m(a)\geq(n(i)+4)/3^{a-i} \quad \text{for} \quad a>i\geq 0 \quad \text{if} \quad m(a)\geq 5,\\
&4\geq(n(i)+4)/3^{a-i} \quad \text{for} \quad a>i\geq 0 \quad \text{if} \quad m(a)\leq 4,\\
&m(a)\geq n(a)+2 \quad \text{for} \quad a\geq 0 \quad \text{if} \quad m(a)\geq 11, \text{ and}\\
&10\geq n(a)+2 \quad \text{for} \quad a\geq 0 \quad \text{if} \quad m(a)\leq 10.
\end{aligned}$$

Thus Propositions 2.4–7 give the following implications:

$$\begin{aligned}
&S_1(0,m(0))\Longrightarrow\cdots\Longrightarrow S_4(a-1,m(a-1))\Longrightarrow S_1(a,m(a))\\
&\qquad\Longrightarrow S_2(a,n(a))\Longrightarrow S_3(a,n(a)+2)\Longrightarrow S_4(a,m(a))\Longrightarrow\cdots.
\end{aligned}$$

Here we notice that if $m(a)\geq 5$ (or ≥ 11) then we get $S_1(a,m(a))$ by Proposition 2.4 (or $S_4(a,m(a))$ by Proposition 2.7, resp). While if $m(a)\leq 4$ (or ≤ 10) then we get $S_1(a,4)$ by Proposition 2.4 (or $S_4(a,10)$ by Proposition 2.7, resp.), and hence we get $S_1(a,m(a))$ (or $S_4(a,m(a))$, resp.) by Lemma 2.3.

Now (2.9) for $a=0$ is clear. So we assume (2.9) for all $a\leq b-1$ ($b\geq 1$), and we prove it for $a=b$.

If $d(b-1)\leq 2$, then $d(b)\geq 0=[d(b-1)/3]\geq[d(0)/3^b]$ by (2.9) for $a=b-1$.

Next, if $d(b-1)>2$ and $d(b-1)\equiv 0$ mod 3, then by definition we have $0\neq QH^{4\cdot 3^{b-1}d(b-1)}=QH^{4\cdot 3^b(d(b-1)/3)}$. Thus we have $d(b)\geq d(b-1)/3\geq[d(0)/3^b]$ by (2.1).

Finally suppose that $d(b-1)>2$ and $d(b-1)\not\equiv 0$ mod 3. Since $2d(b-1)\not\equiv 0$ mod 3, $2d(b-1)>3$, and $2d(b-1)\geq[d(0)/3^{b-1}]+2$ by (2.9) for $a=b-1$, $S_1(b-1, [d(0)/3^{b-1}]+2)$ implies that $0\neq QH^{4\cdot 3^{b-1}d(b-1)}=\mathscr{P}^{3^{b-1}}QH^{4\cdot 3^{b-1}(d(b-1)-1)}$. In particular, $QH^{4\cdot 3^{b-1}(d(b-1)-1)}\neq 0$. Thus, if $d(b-1)\equiv 1$ mod 3, then by (2.1), we have $d(b)\geq(d(b-1)-1)/3=[d(b-1)/3]\geq[d(0)/3^b]$. While if $d(b-1)\equiv 2$ mod 3, then $2(d(b-1)-1)\not\equiv 0$ mod 3, $2(d(b-1)-1)>3$, and $2(d(b-1)-1)\geq[d(0)/3^{b-1}]+2$ by (2.9) for $a=b-1$. (We note that $d(b-1)\equiv 2$ mod 3 implies that $d(b-1)\geq 5$.) Thus, also $S_1(b-1, [d(0)/3^{b-1}]+2)$ implies that $0\neq QH^{4\cdot 3^{b-1}(d(b-1)-1)}\subset\mathscr{P}^{3^{b-1}}QH^{4\cdot 3^{b-1}(d(b-1)-2)}$. So $d(b)\geq(d(b-1)-2)/3=[d(b-1)/3]\geq[d(0)/3^b]$.

Thus we have (2.9) for all $a\geq 0$, and Theorem 1.4 is proved.
q. e. d.

§ 3. Proofs of Propositions 2.6 and 2.7

In this section we prove Propositions 2.6 and 2.7.

Proof of Proposition 2.6. Let $n\equiv 1 \bmod 3$ with $n\geq m$. Then, $S_2(a, t)$ implies that

$$\mathscr{P}^{3^a}QH^{2\cdot 3^a n}\subset \mathscr{P}^{2\cdot 3^a}QH^{2\cdot 3^a(n-2)}.$$

Here, for $n=1$ and 4, the unstable condition implies that $\mathscr{P}^{3^a}QH^{2\cdot 3^a}=0$ and $\mathscr{P}^{3^a}QH^{8\cdot 3^a}\subset \mathscr{P}^{2\cdot 3^a}QH^{4\cdot 3^a}=0$. So we assume that $n\geq 7$. Thus also $S_2(a, t)$ implies that

$$\mathscr{P}^{3^a}QH^{2\cdot 3^a(n-2)}\subset \mathscr{P}^{2\cdot 3^a}QH^{2\cdot 3^a(n-4)}.$$

Now we use the following relations given from the Adem relation:

$$\mathscr{P}^{3^k s}\mathscr{P}^{3^k t}=(-1)^s\binom{2t-1}{s}\mathscr{P}^{3^k(s+t)}+\textstyle\sum_{i\leq k-1}\mathscr{P}^{3^i}\alpha_i\ (\alpha_i\in\mathscr{A}),$$

(3.1) for $s=1, 2$, and

$$\mathscr{P}^m=\textstyle\sum_{i\leq k}\mathscr{P}^{3^i}\beta_i\ (\beta_i\in\mathscr{A})\ \text{if}\ m\not\equiv 0 \bmod 3^{k+1}.$$

Then we have that

$$\begin{aligned}\mathscr{P}^{3^a}QH^{2\cdot 3^a n}&\subset \mathscr{P}^{2\cdot 3^a}QH^{2\cdot 3^a(n-2)}\\ &\subset (\mathscr{P}^{3^a})^2QH^{2\cdot 3^a(n-2)}+\textstyle\sum_{i\leq a-1}\mathscr{P}^{3^i}QH^{2\cdot 3^i n_i}\\ &\subset \mathscr{P}^{3^a}\mathscr{P}^{2\cdot 3^a}QH^{2\cdot 3^a(n-4)}+\textstyle\sum_{i\leq a-1}\mathscr{P}^{3^i}QH^{2\cdot 3^i n_i}\\ &\subset \textstyle\sum_{i\leq a-1}\mathscr{P}^{3^i}QH^{2\cdot 3^i n_i},\end{aligned}$$

where $n_i=3^{a-i}(n+2)-2$. Then since $S_3(i, m_i)$ implies $\mathscr{P}^{3^i}QH^{2\cdot 3^i n_i}=0$ for $i\leq a-1$, we have $\mathscr{P}^{3^a}QH^{2\cdot 3^a n}=0$, and $S_3(a, m)$ is true. q. e. d.

Proof of Proposition 2.7. We use the similar method as above.

Let $n\equiv 1 \bmod 3$ with $n\geq m$ and $n>9$. Then $S_1(a, t)$ implies that

$$QH^{2\cdot 3^a n}=\mathscr{P}^{3^a}QH^{2\cdot 3^a(n-2)}.$$

Next we use $S_2(a, s)$ to get that $QH^{2\cdot 3^a n}=\mathscr{P}^{3^a}QH^{2\cdot 3^a(n-2)}\subset \mathscr{P}^{2\cdot 3^a}QH^{2\cdot 3^a(n-4)}$, and hence

$$H^{2\cdot 3^a n}\subset \mathcal{P}^{2\cdot 3^a}H^{2\cdot 3^a(n-4)}+DH^{2\cdot 3^a n}.$$

Thus, we have that

$$\begin{aligned}\mathcal{P}^{3^a}H^{2\cdot 3^a n}&\subset \mathcal{P}^{3^a}\mathcal{P}^{2\cdot 3^a}H^{2\cdot 3^a(n-4)}+\mathcal{P}^{3^a}DH^{2\cdot 3^a n}\\ &\subset \textstyle\sum_{i\le a-1}\mathcal{P}^{3^i}H^{2\cdot 3^i n_i}+\mathcal{P}^{3^a}DH^{2\cdot 3^a n}\end{aligned}$$

by (3. 1), where $n_i=3^{a-i}(n+2)-2$. Then since $S_4(i, m_i)$ implies that $\mathcal{P}^{3^i}H^{2\cdot 3^i n_i}\subset\sum_{j\le i}\mathcal{P}^{3^j}DH^*$ for $i\le a-1$, we have $\mathcal{P}^{3^a}H^{2\cdot 3^a n}\subset\sum_{i\le a}\mathcal{P}^{3^i}DH^*$, and $S_4(a, m)$ is true. q. e. d.

§ 4. Particular Generators for H^*

In this section we give particular generators for H^*. First we prove the following

Lemma 4. 1. *Let M^* be a finite dimensional graded vector space over $\mathbf{Z}/p$, where p is a prime. Let f: $M^*\to M^*$ be an endomorphism of degree d with $f^q=0$ for some $q>0$, where $f^q=f\cdots f$ (q fold) and $f^0=\mathrm{id}$. Then there is a homogeneous basis $\mathcal{Y}=\{y(t, u; i)\mid t\ge 0,\ u\ge 0,\ t+u\le q-1,\ 1\le i\le r(t+u)\}$ for M^* so that*

$$f(y(t,u;i))=y(t+1,u-1;i)\ \textit{for}\ u\ge 1,\ \textit{and}\ f(y(t,0;i))=0,$$

where r is a certain integer valued function of non-negative integers.

Proof. Put $\bar{M}^*=M^*/\mathrm{Ker}\, f$. Then f induces an endomorphism $\bar{f}$: $\bar{M}^*\to\bar{M}^*$ of degree d with $\bar{f}^{q-1}=0$. So we can prove the lemma by induction on q. The details are left to the reader. q. e. d.

By using the above lemma, we give particular generators for H^*.

Lemma 4. 2. *Let H^* be the algebra in Theorem 1. 4. Then for any $r\ge 0$, there is a system of algebra generators $\mathcal{X}=\{x_1, \cdots, x_k\}$ for H^* (i. e., $H^*=T^4[x_1,\cdots, x_k]$), such that the following conditions hold:*

$$\textit{If}\ \mathcal{P}^{3^r}x_i\notin DH^*,\ \textit{then}\ \mathcal{P}^{3^r}x_i=x_j\ \textit{for some}\ x_j\in\mathcal{X}.$$

$$\mathcal{P}^{3^r}x_i=\mathcal{P}^{3^r}x_j\in\mathcal{X}\ \textit{implies}\ x_i=x_j\in\mathcal{X}.$$

Proof. Put $M^*=QH^*$ and $f=\mathcal{P}^{3^r}$. Then M^* together with f

satisfies the conditions in Lemma 4. 1 since QH^* is finite. Let $\mathscr{Y}$ be the basis for M^* given in Lemma 4. 1. We choose any representatives $x(0,t;i)\in H^*$ for $y(0,t;i)\in M$. Put $x(t,u;i)=(\mathscr{P}^{3^r})^t x(0,t+u;i)$ for $t\geq 1$. Then $x(t,u;i)$ is a representative for $y(t,u;i)$, and $\mathscr{X}=\{x(t,u;i)\}$ is a system of algebra generators for H^*. Clearly, $\mathscr{X}$ satisfies the desired properties. q. e. d.

§5. Proof of Propositions 2. 4 and 2. 5

In this section we prove Propositions 2. 4 and 2. 5. To prove them we fix a system of algebra generators $\mathscr{X}$ for H^* given in Lemma 4. 2 for $r=a$, and express all element in H^* as a polynomial of $\mathscr{X}$. Then, for any $u\in H^*$ and for any monomial v of $\mathscr{X}$, v is said to be contained in u provided that the coefficient of v in u is not zero. In this case we denote that $v\in u$.

First we prove Proposition 2. 4.

Proof of Proposition 2. 4. First we note that if n is great enough then $QH^{2\cdot 3^a n}=0$. Thus $S_1(a,m)$ is true for great m. So we prove that $S_1(a,m)$ is true for $m\geq\max\{[d(0)/3^a]+2, (m_i+2)/3^{a-i}, (n_i+2)/2\circ 3^{a-i}\}$ under the inductive assumption that $S_1(a,m+1)$ is true. Furthermore if $m+1\leq 4$ then $S_1(a,m)$ is equivalent to $S_1(a,m+1)$ by Lemma 2. 3. So we assume that $m>3$.

Let n be an integer with $n\not\equiv 0$ mod 3 and $n\geq m$, and let $x\in\mathscr{X}$ be a generator with dim $x=2\circ 3^a n$. ($n\geq m$ implies $n>3$.) Then by the unstable condition and (3. 1), we have that $x^3=\mathscr{P}^{3^a n}x=\sum_{i\leq a}\mathscr{P}^{3^i}\alpha_i x$ for some $\alpha_i\in\mathscr{A}$, and so

(5. 1) $\quad x^3\in\mathscr{P}^{3^i}y_i$ for some $y_i\in H^{2\cdot 3^i(3^{a-i+1}n-2)}$ with $i\leq a$.

First suppose that

$$x^3\in\mathscr{P}^{3^a}y_a \text{ for some } y_a\in H^{2\cdot 3^a(3n-2)}.$$

Then $S_1(a,m+1)$ implies that $y_a=\mathscr{P}^{3^a}y'_a+d'_a$ for some $y'_a\in H^{2\cdot 3^a(3n-4)}$ and $d'_a\in DH^*$. Also $S_1(a,m+1)$ implies that $y'_a=\mathscr{P}^{3^a}y''_a+d''_a$ for some $y''_a\in H^{2\cdot 3^{a+1}(n-2)}$ and $d''_a\in DH^*$. Thus, by using (3. 1), we have that

$$x^3\in(\mathscr{P}^{3^a})^3y''_a+\mathscr{P}^{3^a}d_a=\sum_{i\leq a-1}\mathscr{P}^{3^i}\beta_i y''_a+\mathscr{P}^{3^a}d_a\,(\beta_i\in\mathscr{A}, d_a=d'_a+\mathscr{P}^{3^a}d''_a\in DH^*).$$

Thus for any case in (5. 1), we have that

$$x^3 \in \mathscr{P}^{3^i} y_i + \mathscr{P}^{3^a} d_a$$

for some $y_i \in H^{2\cdot 3^i(3^{a-i+1}n-2)}$ with $i \leq a-1$ and for some $d_a \in DH^*$.

Next we use $S_4(i, n_i)$ to get $\mathscr{P}^{3^i} y_i \in \sum_{j\leq i} \mathscr{P}^{3^j} DH^*$. Thus we have $x^3 \in \mathscr{P}^{3^i} d_i$ for some $d_i \in DH^*$ with $i \leq a$. This means that

(5. 2) $\quad x^2 \in \mathscr{P}^{3^i} w_i$ for some $w_i \in H^{2\cdot 3^i(2\cdot 3^{a-i}n-2)}$ $(i \leq a)$, or

(5. 3) $\quad x \in \mathscr{P}^{3^i} z_i$ for some $z_i \in H^{2\cdot 3^i(3^{a-i}n-2)}$ $(i \leq a)$.

First consider the case of (5. 2). If $i \leq a-1$ then $\mathscr{P}^{3^i} w_i \in \sum_{j\leq i} \mathscr{P}^{3^j} DH^*$ by $S_4(i, n_i)$. While if $i=a$ then $w_a \in H^{4\cdot 3^a(n-1)} = DH^{4\cdot 3^a(n-1)}$ by (2. 1) since $3^a(n-1) \geq 3^a(m-1) \geq 3^a([d(0)/3^a]+1) \geq d(0)+1$. Thus (5. 2) implies $x^2 \in \sum_{i\leq a} \mathscr{P}^{3^i} DH^*$, and so (5. 3) holds in any case.

Now $S_3(i, m_i)$ implies that $\mathscr{P}^{3^i} z_i \in DH^*$ for $i \leq a-1$. So we have $x \in \mathscr{P}^{3^a} z_a$ for some $z_a \in H^{2\cdot 3^a}$, and hence $x = \mathscr{P}^{3^a} x'$ for some $x' \in \mathscr{X}$ by the definition of $\mathscr{X}$. This proves that $S_1(a, m)$ is true since $\mathscr{X}$ gives a basis for QH^*. q. e. d.

To prove Proposition 2. 5, we prepare the following

Lemma 5. 4. *Let a and n be non-negative integers with* $n \not\equiv 0 \bmod 3$. *Let $x \in \mathscr{X}$ be a generator with* $\dim x = 2\cdot 3^a n$ *and $\mathscr{P}^{3^a} x = y \in \mathscr{X}$. Then under the assumption of Proposition* 2. 5, *if $n \geq m$ then x^2 is not contained in $\mathscr{P}^{3^a} u$ for any $u \in H^{4\cdot 3^a(n-1)}$.*

Proof. Suppose contrarily that

$$x^2 \in \mathscr{P}^{3^a} u \text{ for some } u \in H^{4\cdot 3^a(n-1)}.$$

Then, by using (3. 1), we have that

$$\begin{aligned} \mathscr{P}^{2\cdot 3^a} x^2 &= y^2 + 2\sum_{i<3^a} (\mathscr{P}^i x)(\mathscr{P}^{2\cdot 3^a - i} x) \\ &\in \mathscr{P}^{2\cdot 3^a} \mathscr{P}^{3^a} u = \sum_{i\leq a-1} \mathscr{P}^{3^i} \alpha_i u \quad (\alpha_i \in \mathscr{A}). \end{aligned}$$

Now $y^2 \notin 2\sum_{i<3^a} (\mathscr{P}^i x)(\mathscr{P}^{2\cdot 3^a - i} x)$ by dimensional reason. So we have that

$$y^2 \in \mathscr{P}^{3^i} u_i \text{ for some } u_i \in H^{2\cdot 3^i(3^{a-i}(2n+4)-2)} \text{ with } i \leq a-1.$$

Then $S_4(i, n_i)$ implies that $\mathscr{P}^{3^i} u_i \in \sum_{j \leq i} \mathscr{P}^{3^j} DH^*$, and so we have that

$$y \in \mathscr{P}^{3^i} w_i \text{ for some } w_i \in H^{2\cdot 3^i(3^{a-i}(n+2)-2)} \text{ with } i \leq a-1.$$

Then $S_3(i, m_i)$ implies that $\mathscr{P}^{3^i} w_i \in DH^*$, and this is a contradiction. Thus x^2 is not contained in $\mathscr{P}^{3^a} u$ for any $u \in H^{4\cdot 3^a(n-1)}$. q. e. d.

Now we prove Proposition 2. 5.

Proof of Proposition 2. 5. By the same reason as in the proof of Proposition 2. 4, we have only to prove that $S_2(a, m)$ is true for $m \geq \max\{(t+2)/3,\ (m_i+2)/3^{a-i},\ (n_i+2)/2\cdot 3^{a-i},\ 4\}$ under the inductive assumption that $S_2(a, m+1)$ is true.

Let n be an integer with $n \not\equiv 0 \bmod 3$ and $n \geq m$, and let $x \in \mathscr{X}$ be a generator with $\dim x = 2\cdot 3^a n$. If $\mathscr{P}^{3^a} x \in DH^*$, then $\mathscr{P}^{3^a} x = 0 \in \operatorname{Im} \mathscr{P}^{2\cdot 3^a}$ in QH^*. So we assume that

$$\mathscr{P}^{3^a} x = y \in \mathscr{X}.$$

Now we use the same method as in the proof of Proposition 2. 4 to get (5. 2) or (5. 3). Here we notice that we use $S_1(a, t)$ to show that $y_a = \mathscr{P}^{3^a} y'_a + d'_a$ and $S_2(a, m+1)$ to show that $\mathscr{P}^{3^a} y'_a = \mathscr{P}^{2\cdot 3^a} y''_a + d''_a$ ($d''_a \in DH^*$) instead of $S_1(a, m+1)$. Then we use $S_4(i, n_i)$ and Lemma 5. 4 to get (5. 3). Thus by using the same method as in the proof of Proposition 2. 4, we have $x = \mathscr{P}^{3^a} x'$ for some $x' \in \mathscr{X}$, and hence $\mathscr{P}^{3^a} x = (\mathscr{P}^{3^a})^2 x' = \mathscr{P}^{2\cdot 3^a}(2x') + \sum_{i \leq a-1} \mathscr{P}^{3^i} \gamma_i x'$ ($\gamma \in \mathscr{A}$) by (3. 1). Finally we use $S_3(i, m_i)$ to get that $\mathscr{P}^{3^i} \gamma_i x' \in DH^*$ ($i \leq a-1$) and hence $\mathscr{P}^{3^a} QH^{2\cdot 3^a n} \subset \mathscr{P}^{2\cdot 3^a} QH^{2\cdot 3^a(n-2)}$. This proves that $S_2(a, m)$ is true since $\mathscr{X}$ gives a basis for QH^*. q. e. d.

§ 6. Proof of Proposition 1. 6

Let X be the homotopy associative H-space in Theorem 1. 2, and let $P_3H^*(X; \mathbb{Z}/3)$ be the module of all A_3-primitive classes in $H^*(X; \mathbb{Z}/3)$. Then we have the natural inclusion $\varepsilon : P_3H^*(X; \mathbb{Z}/3) \subset PH^*(X; \mathbb{Z}/3)$, where $PH^*(X; \mathbb{Z}/3)$ is the module of all primitive classes in $H^*(X;\mathbf{Z}/3)$. Let $\rho: PH^*(X;\mathbf{Z}/3)\to QH^*(X;\mathbf{Z}/3)$ be the natural projection. Since $H^*(X;\mathbf{Z}/3)$ is A_3-primitively generated, $\rho\varepsilon$ is an epimorphism, and so is ρ. Thus the natural projection $\rho_*: PH_*(X;\mathbf{Z}/3)\to QH_*(X;\mathbf{Z}/3)$, which is the dual of ρ, is a monomorphism, and so the Pontrjagin product on $H_*(X;\mathbf{Z}/3)$ is commutative by [6; 4. 20]. This implies by Zabrodsky [9; 3. 2 (d)] that $H^*(X;\mathbf{Z}/3)$ is a free algebra. But since $H^*(X;\mathbf{Z}/3)$ is finite, it is an exterior algebra generated by finitely many odd dimensional classes, and hence ρ is an isomorphism by [6; 4. 21]. Thus ε is also an isomorphism, and we have the following

Lemma 6. 1. $H^*(X;\mathbf{Z}/3)=\Lambda(x_1,\cdots,x_k)$,

where $x_i(1\leq i\leq k)$ are A_3-primitive odd dimensional classes.

Now we can prove Proposition 1. 6 by the same method as Iwase [4].

Proof of Proposition 1. 6. Let P_tX be the projective t-space for $X(t=2,3)$, and let $\iota_2:\Sigma X\subset P_2X$ and $\iota_3: P_2X\subset P_3X$ be the natural inclusions. Then the composition $\iota_3\iota_2$ is equal to the inclusion $\iota:\Sigma X\subset P_3X$ in §1. Now we have the following diagram (see [4]):

$$\begin{array}{ccccc}
\tilde{H}^*(\Sigma X;\mathbf{Z}/3) & \xleftarrow{\iota_2^*} & \tilde{H}^*(P_2X;\mathbf{Z}/3) & \xleftarrow{\iota_3^*} & \tilde{H}^*(P_3X;\mathbf{Z}/3)\\
\alpha_1\nearrow \quad \beta_1\searrow & & \alpha_2\nearrow \quad \beta_2\searrow & & \alpha_3\nearrow\\
\tilde{H}^*(X;\mathbf{Z}/3) \quad \tilde{H}^*(X;\mathbf{Z}/3)\otimes\tilde{H}^*(X;\mathbf{Z}/3) & & & & \tilde{H}^*(X;\mathbf{Z}/3)\otimes\tilde{H}^*(X;\mathbf{Z}/3)\otimes\tilde{H}^*(X;\mathbf{Z}/3),
\end{array}$$

where α_i and β_i are $\mathbf{Z}/3$-module homomorphisms of degree i and $-i$, respectively, with $\beta_1\alpha_1(u)=-\tilde{\mu}^*(u)=-\mu^*(u)+1\otimes u+u\otimes 1$ and $\beta_2\alpha_2(u\otimes v)=-\tilde{\mu}^*(u)\otimes v+u\otimes\tilde{\mu}^*(v)$. We notice that α_1^{-1} is the suspension isomorphism σ. Since $x_i\in H^*(X;\mathbf{Z}/3)$ are A_3-primitive, we have $y_i\in H^*(P_3X;\mathbf{Z}/3)$ with $\dim y_i=\dim x_i+1$ so that

$$\sigma\iota^*y_i=\alpha_1^{-1}\iota_2^*\iota_3^*y_i=x_i.$$

Put $S=\alpha_3(DH^*(X;\mathbf{Z}/3)\otimes\tilde{H}^*(X;\mathbf{Z}/3)\otimes\tilde{H}^*(X;\mathbf{Z}/3)+\tilde{H}^*(X;\mathbf{Z}/3)\otimes DH^*(X;\mathbf{Z}/3)\otimes\tilde{H}^*(X;\mathbf{Z}/3)+\tilde{H}^*(X;\mathbf{Z}/3)\otimes\tilde{H}^*(X;\mathbf{Z}/3)\otimes DH^*(X;$

$\mathbb{Z}/3)$). Then, clearly, $\mathscr{A}(S)\subset S$ and $\iota^*(S)=0$. Furthermore, by the same reason as [4], we have $H^*(P_3X;\mathbb{Z}/3)/S\cong T^4[y_1,\cdots y_k]$. q. e. d.

§7. Example

In this section, we give an example to show that (1.3) in Theorem 1.2 is required.

Consider the spinor group $Spin(2k)$. Since $Spin(2k)$ is a Lie group, it is a homotopy associative H-space. Furthermore, $H^*(Spin(2k);\mathbb{Z}/3)\cong H^*(SO(2k);\mathbb{Z}/3)$ as $\mathscr{A}$-algebras. Thus, by [2; Prop. 10.2] and [3; Cor. 14.3], we have that

(7.1) $$H^*(Spin(2k);\mathbb{Z}/3)=\Lambda(x_3,x_7,\cdots,x_{4k-5},e),\text{ and}\\ e\notin\tilde{\mathscr{A}}(H^*(Spin(2k);\mathbb{Z}/3)),$$

for some universal transgressive elements x_i and e with $\dim x_i=i$ and $\dim e=2k-1$, where $\tilde{\mathscr{A}}$ is the augmentation ideal of $\mathscr{A}$. Since universal transgressive elements are A_3-primitive by definition, $H^*(Spin(2k);\mathbb{Z}/3)$ satisfies the conditions in Theorem 1.2. We consider the case of $k=3^a n$ with $n\not\equiv 0 \bmod 3$ and $n>3$. Then $4\cdot 3^a(n-1)-1\leq 4k-5$ and $4\cdot 3^a n-1>4k-5$. Thus, by Milnor-Moore [6; Prop. 4.21], we have that

$$P_3H^{4\cdot 3^a t-1}(Spin(2k);\mathbb{Z}/3)=PH^{4\cdot 3^a t-1}(Spin(2k);\mathbb{Z}/3)=0\quad\text{if and only if } t\geq n.$$

On the other hand, $e\notin\operatorname{Im}\mathscr{P}^{3^a}$ by (7.1). These show that (1.3) is required.

We can also show that (1.5) in Theorem 1.4 is required by considering $H^*=A^*/D^4A^*$, where $A^*=H^*(BSpin(2k);\mathbb{Z}/3)$.

References

[1] J. F. Adams, The sphere, considered as an H-space mod p, *Quart. J. Math. Oxford* (2) **12** (1961), 52-60.

[2] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, *Ann. of Math.* **57** (1953), 115-207.

[3] A. Borel et J.-P. Serre, Groupes de Lie et puissances réduites de Steenrod, *Amer. J. Math.* **75** (1953), 409-448.

[4] N. Iwase, On the K-ring structure of X-projective n-space, *Memoirs of the Faculty of Science, Kyushu Univ. Ser. A* **38** (1984), 285-297.

[5] J. P. Lin, Steenrod squares in the mod 2 cohomology of a finite H-space, *Comment.*

Math. Helv. **55** (1980), 398–412.
[6] J. W. Milnor and J. C. Moore, On the structure of Hopf algebras, *Ann. of Math.* **81** (1965), 211-264.
[7] J. D. Stasheff, Homotopy associativity of H-spaces. I, *Trans. Amer. Math. Soc.* **108** (1963), 275-292.
[8] E. Thomas, Steenrod squares and H-spaces: II, *Ann. of Math.* **81** (1965), 473–495.
[9] A. Zabrodsky, Implications in the cohomology of H-spaces, *Illinois J. Math.* **14** (1970), 363–375.