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A Non Spatial Continuous Semigroup of $*$-Endomorphisms of $\mathfrak{B}(\mathfrak{H})$*

By

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Abstract

In this paper a continuous one parameter semigroup α_t of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$ is constructed having the property that there does not exist a strongly continuous one parameter semigroup of intertwining isometries (i. e. there is no strongly continuous semigroup of isometries $U(t)\in\mathfrak{B}(\mathfrak{H})$ so that $U(t)A=\alpha_t(A)U(t)$ for all $A\in\mathfrak{B}(\mathfrak{H})$).

§ I. Introduction

In this paper we construct a continuous one parameter semigroup α_t of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$ having the property that there does not exist a strongly continuous one parameter semigroup of intertwining isometries (i. e. there is no strongly continuous semigroup of isometries $U(t)\in\mathfrak{B}(\mathfrak{H})$ to that $U(t)A=\alpha_t(A)U(t)$ for all $A\in\mathfrak{B}(\mathfrak{H})$). In a previous paper [3 Powers] it was shown how to associate an index with continuous semigroups of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$ having an intertwining semigroup of isometries. This previous paper raised the question of whether such an intertwining semigroup of isometries always existed. The present paper shows that they need not exist.

We will call a continuous one parameter semigroup of $*$-endomorphisms of a von Neumann algebra M an E_0-semigroup of M. The precise definition of an E_0-semigroup is given as follows.

Definition 1.1. We say $\{\alpha_t; t\geq 0\}$ is an E_0-semigroup of a von Neumann algebra M if the following conditions are satisfied.

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i) α_t is a $*$-endomorphism of M for each $t \geq 0$.
ii) α_0 is the identity endomorphism and $\alpha_t \circ \alpha_s = \alpha_{t+s}$ for all $t, s \geq 0$.
iii) For each $f \in M_*$ (the predual of M) and $A \in M$ the function $f(\alpha_t(A))$ is a continuous function of t.

In § II we review some results concerning generalized free state of the CAR algebra and in § III we prove a theorem concerning operators which almost commute with projections onto subspaces of functions with support in $[\lambda, \infty)$. These results are needed in § IV where we prove the main result, Theorem 4. 1.

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§ II. Quasi-Free States of the CAR Algebra

In this section we collect some results concerning generalized free state of the CAR algebra. We refer to [5 Powers, Stormer] for more details. The CAR algebra over $\mathfrak{K}$ denoted $\mathfrak{A}(\mathfrak{K})$ is a C^*-algebra generated by elements $a(f)$ defined for $f \in \mathfrak{K}$ and satisfying the CAR relations

$$a(\alpha f + g) = \alpha a(f) + a(g)$$
$$a(f)a(g) + a(g)a(f) = 0$$
$$a(f)^* a(g) + a(g) a(f)^* = (f, g) I$$

for all $f, g \in \mathfrak{K}$ and complex numbers α.

The gauge invariant generalized free states of $\mathfrak{A}(\mathfrak{K})$ are states of $\mathfrak{A}(\mathfrak{K})$ whose n-point functions satisfy the relations

$$\omega(a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)) = \delta_{nm} \det\{(f_i, W g_j)\}.$$

If ω is any state of $\mathfrak{A}(\mathfrak{K})$ then the two-point function of ω determines an operator $\mathfrak{W}$ on $\mathfrak{K}$ by the relation $\omega(a(f)^* a(g)) = (f, \mathfrak{W} g)$ where $\mathfrak{W}$ satisfies the relation $0 \leq \mathfrak{W} \leq I$. The gauge invariant generalized free states are determined by their two point function.

In the following we denote the trace of an operator A by $\mathrm{tr}(A)$. An operator is of trace class if $\mathrm{tr}(|A|) < \infty$ where $|A| = (A^*A)^{1/2}$ and A is of Hilbert Schmidt class if $\mathrm{tr}(|A|^2) = \mathrm{tr}(A^*A) < \infty$.

Theorem 2. 1. *Suppose ω_A and ω_B are generalized free states of $\mathfrak{A}(\mathfrak{K})$*

and B is a projection (i.e. $B=B^*B$). Then ω_A is a factor state (i.e., ω_A induces a factor representation of $\mathfrak{A}(\mathfrak{K})$) and ω_A induces a type I *factor representation of $\mathfrak{A}(\mathfrak{K})$ if and only if* $\operatorname{tr}(A(I-A))<\infty$. *The states ω_A and ω_B are quasi-equivalent (i.e. these states induce quasi-equivalent representations of $\mathfrak{A}(\mathfrak{K})$) if and only if*

$$(*)\qquad \operatorname{tr}(B(I-A)B+(I-B)A(I-B))<\infty.$$

Furthermore, if ω is any state of $\mathfrak{A}(\mathfrak{K})$ (not necessarily a generalized free state) and ω has a two point function A (i.e., $\omega(a(f)^*a(g))=(f, Ag)$ for $f, g\in\mathfrak{K}$) and A satisfies inequality $(*)$ then ω is a factor state which is quasi-equivalent to ω_B.

Proof. If follows from [5 Powers, Stormer] that ω_A is a factor state and ω_A is of type I if and only if $A=C+E$ where E is a projection and C is a trace class operator. One checks that A can be written in this form if and only if $\operatorname{tr}(A(I-A))<\infty$. It is also shown in [5] that the states ω_A and ω_B are quasi-equivalent if and only if $A^{1/2}-B^{1/2}$ and $(I-A)^{1/2}-(I-B)^{1/2}$ are of Hilbert Schmidt class. We will show that in the case where B is a projection these two differences are of Hilbert Schmidt class if and only inequality $(*)$ is satisfied. To see this let $X=A^{1/2}-B^{1/2}$ and $Y=(I-A)^{1/2}-(I-B)^{1/2}$. Then X and Y are of Hilbert Schmidt class if and only if $\operatorname{tr}(X^2+Y^2)<\infty$. Now we have

$$X^2+Y^2=2I-A^{1/2}B-BA^{1/2}-(I-A)^{1/2}(I-B)-(I-B)(I-A)^{1/2}.$$

Since the trace of a positive operator can be computed using any orthonormal basis we can choose an orthonormal basis of vectors $\{f_i; i=1, 2, \cdots\}$ so that $Bf_i=f_i$ or $Bf_i=0$ for each $i=1, 2, \cdots$. Computing the trace of X^2+Y^2 with this basis we find

$$\operatorname{tr}(X^2+Y^2)=2\operatorname{tr}(B(I-A^{1/2})B)+2\operatorname{tr}((I-B)(I-(I-A)^{1/2}(I-B)).$$

Since for $x\in[0, 1]$ we have $1-x\leq 2-2x^{1/2}\leq 2-2x$, we have $I-A\leq 2(I-A^{1/2})\leq 2(I-A)$. And replacing A by $I-A$ in this inequality we find $A\leq 2(I-(I-A)^{1/2})\leq 2A$. Hence, we have

$$\frac{1}{2}\operatorname{tr}(X^2+Y^2)\leq\operatorname{tr}(B(I-A)B+(I-B)A(I-B))\leq\operatorname{tr}(X^2+Y^2).$$

Hence, ω_A and ω_B are quasi-equivalent if and only if inequlity $(*)$ is satisfied.

Next suppose ω is an arbitrary state of $\mathfrak{A}(\mathfrak{K})$ (not necessarily a generalized free state) and $\omega(a(f)^*a(g))=(f, Ag)$ for all $f, g\in\mathfrak{K}$. Suppose B is a projection and $\mathrm{tr}(B(I-A)B+(I-B)A(I-B))<\infty$. One may see that ω is a factor state $\mathfrak{A}(\mathfrak{K})$ which is quasi-equivalent to ω_B as follows. Let $b(f)=a((I-B)f)+a(BSf)^*$ for $f\in\mathfrak{K}$ where S is a conjugation which commutes with B (i.e. S is a conjugate linear isometry of $\mathfrak{K}$ onto $\mathfrak{K}$ so that $SSf=f$ for all $f\in\mathfrak{K}$ and $SBS=B$). One easily checks that the $b(f)$ satisfy the CAR relations given at the beginning of this section and the $b(f)$ generate $\mathfrak{A}$. Now let $\{f_i; i=1,2,\cdots\}$ be an orthonormal basis for $\mathfrak{K}$ chosen so that $Bf_i=f_i$ or $Bf_i=0$ for all $i=1,2,\cdots$. Then one finds

$$\sum_{i=1}^{\infty}\omega(b(f_i)^*b(f_i))=\mathrm{tr}(B(I-A)B+(I-B)A(I-B)).$$

Then as shown in [2 Gårding, Wightman] for pure states and [1 Dell'Antonio, Doplicher] for arbitrary states of $\mathfrak{A}(\mathfrak{K})$ if

$$\sum_{i=1}^{\infty}\omega(b(f_i)^*b(f_i))<\infty$$

then ω is quasi-equivalent to the Fock state ρ_0 defined by the property that $\rho_0(b(f)^*b(f))=0$ for all $f\in\mathfrak{K}$ (see also [4 Powers]). One checks that if $\rho_0(b(f)^*b(f))=0$ for all $f\in\mathfrak{K}$ then $\rho_0=\omega_B$. Hence if ω is a state with two-point function $\omega(a(f)^*a(g))=(f, Ag)$ and $\mathrm{tr}(B(I-A)B+(I-B)A(I-B))<\infty$ then the state ω is quasi-equivalent to ω_B. □

Theorem 2.2. *Suppose ω_1 and ω_2 are factor states of $\mathfrak{A}(\mathfrak{K})$ with two point functions A and B (so $\omega_1(a(f)^*a(g))=(f, Ag)$ and $\omega_2(a(f)^*a(g))=(f, Bg)$ for all $f, g\in\mathfrak{K}$). Then $A-B$ is a compact operator.*

Proof. See ([4], Theorem 2-1).

If $\mathfrak{M}$ is a linear subspace of $\mathfrak{K}$ we denote by $\mathfrak{A}(\mathfrak{M})$ the C^*-subalgebra of $\mathfrak{A}(\mathfrak{K})$ generated by the $a(f)$ with $f\in\mathfrak{M}$.

Theorem 2.3. *Suppose ω_P is a generalized free state of $\mathfrak{A}(\mathfrak{K})$ with two-point function a projection P and $(\pi, \mathfrak{H}, \Omega_0)$ is a cyclic $*$-representation of $\mathfrak{A}(\mathfrak{K})$ induced by ω_P on a Hilbert space $\mathfrak{H}$ with cyclic vector Ω_0. Suppose $\mathfrak{M}$ is a closed subspace of $\mathfrak{K}$ and E is the orthogonal projection* *of $\mathfrak{H}$ onto $\mathfrak{M}$. Suppose* $\operatorname{tr}(EPE(I-P)E)<\infty$. *Then* $\pi(\mathfrak{A}(\mathfrak{M}))''$ *is a type I factor and there is a unitary operator* $S\in\pi(\mathfrak{A}(\mathfrak{M}))''$ *so that* $S^2=I$ *and* $S\pi(a(f))S=\pi(a((I-2E)f))$ *for all* $f\in\mathfrak{M}$. *Furthermore, the commutant* $\pi(\mathfrak{A}(\mathfrak{M}))'$ *is generated by the elements* $Sa(f)$ *with* $f\in\mathfrak{M}^{\perp}$ *(with* $\mathfrak{M}^{\perp}$ *the orthogonal complement of* $\mathfrak{M}$*).*

Proof. Suppose the hypothesis and notation of the theorem are satisfied. Let θ_t be the one parameter group of $*$-automorphisms of $\mathfrak{A}(\mathfrak{K})$ defined by the relation $\theta_t(a(f))=a((I-E)f+e^{it}Ef)$ for all $f\in\mathfrak{K}$. Let $A_0=EPE$. Since $\operatorname{tr}(A_0(I-A_0))<\infty$ there is an orthonormal basis $\{f_i; i=1,2,\cdots\}$ for $\mathfrak{M}$ so that $A_0f_i=\lambda_i f_i$ and $\sum_{i=1}^{\infty}\lambda_i-\lambda_i^2<\infty$. Let $N_n=\sum_{i=1}^{n}a(f_i)^*a(f_i)-\lambda_i I$. Clearly we have $N_n\in\pi_0(\mathfrak{A}(\mathfrak{M}))''$. Let $V_n(t)=\pi(\exp(itN_n))$. First we show $V_n(t)\Omega_0$ converges strongly as $n\to\infty$. We have for $n>m$

$$\begin{aligned}\|(V_n(t)-V_m(t))\Omega_0\|^2&=2-2\operatorname{Re}(V_m(t)\Omega_0,\ V_n(t)\Omega_0)\\&=2-2\operatorname{Re}(\omega_0(\exp(it(N_n-N_m)))\\&\le t^2\omega_0((N_n-N_m)^2)=t^2\sum_{i=m+1}^{n}\lambda_i-\lambda_i^2.\end{aligned}$$

Since $\sum_{i=1}^{\infty}\lambda_i-\lambda_i^2<\infty$ we have from the above inequality that $V_n(t)\Omega_0$ is a Cauchy sequence in norm. Hence $V_n(t)\Omega_0\to\Omega_t$ in norm as $n\to\infty$. One may calculate that

$$V_n(t)\pi(a(f))V_n(t)^{-1}=\pi(a((I-E_n)f+e^{it}E_nf))$$

for all $f\in\mathfrak{K}$ where E_n is the projection onto the space spanned by $\{f_1,\cdots,f_n\}$. Since $E_n\to E$ as $n\to\infty$ it follows then that $V_n(t)\pi(p)\Omega_0\to\pi(\theta_t(p))\Omega_t$ as $n\to\infty$ where p is a polynomial in the $a(f)$ and $a(g)^*$. Hence, it follows that $V_n(t)$ converges strongly to a strongly continuous one parameter unitary group $V(t)$ as $n\to\infty$ and $V(t)\pi(A)V(t)^{-1}=\pi(\theta_t(A))$ for all $A\in\mathfrak{A}(\mathfrak{K})$.

Note that $V(2\pi)$ commutes with $\pi(A)$ for $A\in\mathfrak{A}(\mathfrak{K})$ and since $\pi(\mathfrak{A}(\mathfrak{K}))''=\mathfrak{B}(\mathfrak{H})$ since ω_P is a pure state we have $V(2\pi)=\lambda I$. It will be convenient to have $V(2\pi)=I$. This can be arranged by redefining $V'(t)=e^{ist}V(t)$ with $e^{-2\pi is}=\lambda$. From now on we will assume that the group $V(t)$ has been redefined so that $V(2\pi)=I$. Then we define $S=V(\pi)=V(\pi)^*$. From the construction of S we have $S\in\pi(\mathfrak{A}(\mathfrak{M}))''$, $S^2=I$ and $S\pi(a(f))S=\pi(a((I-2E)f))$ for all $f\in\mathfrak{K}$.

Next we show $\pi(\mathfrak{A}(\mathfrak{M}))''$ is a factor of type I. Let N be the von Neumann algebra generated by the elements $S\pi(a(f))$ with $f\in\mathfrak{M}^{\perp}$. Since N is generated by elements which commutes with the $a(f)$ with $f\in\mathfrak{M}$ we have that $N\subset\pi(\mathfrak{A}(\mathfrak{M}))'$. Let R be the von Neumann algebra generated by $\pi(\mathfrak{A}(\mathfrak{M}))$ and N. Since $S\in\pi(\mathfrak{A}(\mathfrak{M}))''$ it follow that $\pi(a(f))=SS\pi(a(f))\in R$ for $f\in\mathfrak{M}^{\perp}$ and $\pi(a(f))\in M\subset R$ for $f\in\mathfrak{M}$. Hence, $\pi(a(f))\in R$ for all for all $f\in\mathfrak{K}$. Since the $a(f)$ generate $\mathfrak{A}(\mathfrak{K})$ we have $R=\pi(\mathfrak{A}(\mathfrak{K}))''=\mathfrak{B}(\mathfrak{H})$. Suppose $C\in\pi(\mathfrak{A}(\mathfrak{M}))''\cap\pi(\mathfrak{A}(\mathfrak{M}))'$. Since $C\in\pi(\mathfrak{A}(\mathfrak{M}))''$ we have $C\in N'$. Hence, C commutes with both $\pi(\mathfrak{A}(\mathfrak{M}))''$ and N we have $C\in R'$. Hence, $C=\lambda I$. Hence, $\pi(\mathfrak{A}(\mathfrak{M}))''$ is a factor.

Consider the state $(\Omega_0 B\Omega_0)$ for $B\in\pi(\mathfrak{A}(\mathfrak{M}))''$. This state is the weakly continuous extension of the state $(\Omega_0, \pi(B)\Omega_0)=\omega_P(B)$ for $B\in\mathfrak{A}(\mathfrak{M})$. Let ω_0 be the restriction of ω_P to $\mathfrak{A}(\mathfrak{M})$. The state ω_0 is a generalized free state of $\mathfrak{A}(\mathfrak{M})$ whose n-point functions are given by

$$\omega_0(a(f_n)^*\cdots\cdot a(f_1)^*a(g_1)\cdots\cdot a(g_m))=\det(f_i, A_0 g_j)$$

where $A_0=EPE$ is the restriction of P to $\mathfrak{M}$. As we have seen this state is of type I if and only if $A_0-A_0^2$ is of trace class. Since $\operatorname{tr}(A_0-A_0^2)=\operatorname{tr}(EPE(I-P))<\infty$ we have ω_0 is a type I state. Hence, $\pi(\mathfrak{A}(\mathfrak{M}))''$ is a factor of type I and $\pi(\mathfrak{A}(\mathfrak{M}))'$ is generated by the elements $S\pi(a(f))$ with $f\in M^{\perp}$. $\square$

§ III. Almost Multiplication Operators

Let $\mathfrak{K}=L^2(-\infty, \infty)\oplus L^2(-\infty, \infty)$ be the Hilbert space of square integrable two component functions on the real line. Let P_λ be the orthogonal projection of $\mathfrak{K}$ onto the subspace $\mathfrak{M}_\lambda$ of $\mathfrak{K}$ of functions with support in $[\lambda, \infty)$ (i, e., $(P_\lambda f)=f(x)$ for $x\geq\lambda$ and $(P_\lambda f)(x)=0$ for $x<\lambda$). For $a<b$ let $P_{[a,b]}=P_a-P_b$ and let $\mathfrak{M}[a, b]$ be the range of $P_{[a,b]}$. The main result of this section is the following theorem.

Theorem 3.1. *Suppose $a<b$ and A is a positive compact operator on $\mathfrak{K}$ with the property that $(I-P_\lambda)AP_\lambda$ is of rank not more than one for all $\lambda\in[a, b]$. Then there are numbers c and d so that $a\leq c<d\leq b$ so that $\operatorname{tr}(P_{[c,d]}AP_{[c,d]})<\infty$.*

Before we prove this it is useful to prove the following.

Lemma 3.2. *Suppose A is a compact hermitian operator on the acting on $\mathfrak{R}$ and $(I-P_\lambda)AP_\lambda=0$ for $\lambda\in[a,b]$. Then $P_{[a,b]}AP_{[a,b]}=0$.*

Proof. Suppose A satisfies the hypothesis of the lemma. Since A is hermitian we have A commutes with P_λ for $\lambda\in[a,b]$. It follows that $B=P_{[a,b]}AP_{[a,b]}$ commutes with the operation of multiplication by functions of x. Suppose $f\in\mathfrak{M}[a,b]$ and θ_n is the operator of multiplication by e^{inx}. Then $Bf=\theta_n^*B\theta_nf$ for all $n=1,2,\cdots$ and as $n\to\infty$, θ_nf tends weakly to zero. Since B is compact we have $B\theta_nf$ tends to zero in norm as $n\to\infty$. Hence, $Bf=0$ and $B=0$. □

Proof of Theorem 3.1. Suppose A satisfies the hypothesis of the lemma. Suppose $(I-P_\lambda)AP_\lambda=0$ for all $\lambda\in[a,b]$. Then by the previous lemma we have $P_{[a,b]}AP_{[a,b]}=0$ and, therefore, the pair $(c,d)=(a,b)$ satisfies the conclusion of the theorem. Suppose then there is a $\lambda\in(a,b)$ so that $(I-P_\lambda)AP_\lambda$ is a rank one operator. Let $A_0=P_{[a,b]}AP_{[a,b]}$. There are functions h_0, $k_0\in\mathfrak{M}[a,b]$ so that $(I-P_\lambda)A_0P_\lambda f=(k_0,f)h_0$. Let $d\in(\lambda,b)$ so that $P_dk_0\neq0$ and $c\in(a,\lambda)$ so that $(I-P_c)h_0\neq0$. Note the following. Suppose $x,y\in[c,d]$ and $x<y$. There are functions $h_x,h_y,k_x,k_y\in\mathfrak{M}[a,b]$ so that $(I-P_x)A_0P_xf=(k_x,f)h_x$ and $(I-P_y)A_0P_yf=(k_y,f)h_y$ for all $f\in\mathfrak{R}$ and the functions must satisfy the relations

$$(*)\qquad k_y=\alpha P_yk_x \quad\text{and}\quad h_x=\bar{\alpha}(I-P_x)h_y$$

where α is a complex number. The truth of the above statement follows immediately from the fact that the operators $(I-P_x)A_0P_x$ and $(I-P_y)A_0P_y$ are of rank one and these operators are equal when sandwiched between $(I-P_x)$ on the left and P_y on the right.

Applying this statement to the numbers c and d we obtain functions h_c,h_d,k_c,k_d satisfying $(*)$. The functions h_d and k_d are unique up to the transformation $h_d'=\lambda h_d$ and $k_d'=\bar{\lambda}^{-1}k_d$. We can then choose the functions h_d and k_d so that the α of $(*)$ is one. We assume the h_d and k_d have been so chosen. Then we have $k_d=P_dk_c$ and $h_c=(I-P_c)h_d$.

Now suppose $s\in[c,d]$. Then we have $(I-P_s)A_0P_sf=(k_s,f)h_s$ for all $f\in\mathfrak{M}[a,b]$. We have h_s and k_s are related to h_c and k_c by $(*)$ and with an appropriate choice of h_s and k_s we can arrange it so that the α of (*) is one. Then with this choice of the h_s and k_s we have

$$\begin{array}{ll} k_d = P_d k_c & h_c = (I-P_c)\,h_d \\ k_s = P_s k_c & h_c = (I-P_c)\,h_s \\ k_d = \alpha P_d k_s & h_s = \bar{\alpha}(I-P_s)\,h_d\,. \end{array}$$

We will show that the α in the above equation is one. Since $(I-P_c)A_0P_c$ and $(I-P_s)A_0P_s$ are equal when sandwiched between $(I-P_c)$ on the left and P_d on the right we have for $f\in\mathfrak{K}$

$$(P_dk_s, f)(I-P_c)h_s = (P_dk_c, f)h_c\,.$$

Since $(I-P_c)h_s = h_c$ it follows that $P_dk_s = P_dk_c$. But $P_dk_c = k_d$ so $k_d = P_dk_s$. Hence, the above $\alpha = 1$ and $h_s = (I-P_s)h_d$. Hence, we have for $s\in[c, d]$

$$(I-P_s)A_0P_sf = (P_sk_c, f)(I-P_s)h_d$$

for all $f\in\mathfrak{K}$. Now let B be an operator $\mathfrak{K}$ with kernel $K_{ij}(x,y)$ given by

$$K_{ij}(x,y) = h_{di}(x)\overline{k_{cj}(y)} \qquad \text{for } x\leq y$$

and

$$K_{ij}(x,y) = k_{ci}(x)\overline{h_{dj}(y)} \qquad \text{for } x>y$$

where

$$(Bf)_i(x) = \sum_{j=1}^{2}\int_a^b K_{ij}(x,y)f_j(y)\,dy\,.$$

Clearly, B is a compact hermitian operator (in fact, B is a Hilbert Schmidt class operator) and from the construction of B we have $(I-P_s)(A_0-B)P_s = 0$ for all $s\in[c, d]$. Since A_0-B is a compact hermitian operator it follows from Lemma 3.2 that $P_{[c,d]}A_0P_{[c,d]} = P_{[c,d]}BP_{[c,d]}$. Let $Q = P_{[c,d]}BP_{[c,d]}$.

We show Q is of trace class. Since $A_0\geq 0$ we have $Q\geq 0$ and Q is given by a kernel $K_{ij}(x,y)$. Suppose C is a positive compact operator so that $Cf = \sum_{i=1}^{\infty}\lambda_i(h_i, f)h_i$ where $\lambda_i>0$ and the $\{h_i\}$ are an orthonormal set of vectors. Then

$$(Cf)_i(x) = \sum_{j=1}^{2}\int_a^b J_{ij}(x, y)f_j(y)\,dy$$

with

$$J_{ij}(x, y) = \sum_{k=1}^{\infty}\lambda_k h_{ki}(x)\overline{h_{kj}(y)}.$$

We see that the trace of C is given by

$$\operatorname{tr}(C)=\sum_{k=1}^{\infty}\lambda_k=\sum_{i=1}^{2}\int_a^b J_{ii}(x, x)\,dx$$

where the integral diverges if the trace of C is not finite. This formula for the trace of C must be used with some care since the kernel $J_{ij}(x, y)$ is only defined up to sets of measure zero and the set of (x, y) with $x=y$ is a set of measure zero.

To calculate the trace of a positive operator C with kernel $J_{ij}(x, y)$ on $\mathfrak{M}[a, b]$ one may proceed as follows. Consider the functions $e_{ki}(x)$ for $i=1, 2$ and $k=1, \cdots, n$ given by $(e_{ki})_j(x)=(n/(b-a))^{1/2}$ if $i=j$ and $(k-1)(b-a)/n\leq x-a<k(b-a)/n$ and $(e_{ki})_j(x)=0$ otherwise. Note the e_{ki} are an orthonormal set of step functions. Then we have

$$\sum_{i=1}^{2}\sum_{k=1}^{n}(e_{ki}, Ce_{ki})=\sum_{i=1}^{2}\int_a^b J_{ii}(x, y)\,\theta_n(x, y)\,dx\,dy$$

where $\theta_n(x,y)$ is a positive function with vanishes when $|x-y|>(b-a)/n$. One can show that the above expression converges to the trace of C as $n\to\infty$ where the expression diverges if the trace of C is not finite. This follows from the facts that the trace of a positive operator can be computed using any orthonormal basis and as $n\to\infty$ any function in $\mathfrak{M}[a, b]$ can be approximated in norm by linear combinations of the e_{ki}. Calculating the trace of Q by such a procedure we find

$$\begin{aligned}(e_{ki}, Qe_{ki})=&\frac{n}{b-a}\iint_{x,y\in I_k x\leq y}h_{di}(x)\overline{k_{ci}(y)}\,dx\,dy\\&+\frac{n}{b-a}\iint_{x,y\in I_k x>y}k_{ci}(x)\overline{h_{di}(y)}\,dx\,dy\,.\end{aligned}$$

Hence,

$$(e_{ki}, Qe_{ki})\leq\frac{2n}{b-a}\int_{x\in I_k}|h_{di}(x)|\,dx\int_{x\in I_k}|k_{ci}(x)|\,dx,$$

where I_k is the interval of support for e_{ki}. Then we have from the Schwarz's inequality that

$$\sum_{i=1}^{2}\sum_{k=1}^{n}(e_{ki}, Qe_{ki})\leq 2\|h_d\|\,\|k_c\|\,.$$

Taking the limit as $n\to\infty$ we have $\operatorname{tr}(Q)\leq 2\|h_d\|\,\|k_c\|$. Hence,

$$\operatorname{tr}(P_{[c,d]}AP_{[c,d]})=\operatorname{tr}(P_{[c,d]}A_0P_{[c,d]})<\infty. \qquad \square$$

§IV. An E_0-Semigroup with No Intertwining Semigroup of Isometries

In this section we construct an example of an E_0-semigroup of $\mathfrak{B}(\mathfrak{H})$ for which there is no strongly continuous one parameter semigroup of intertwining isometries. The construction uses the CAR algebra $\mathfrak{A}=\mathfrak{A}(\mathfrak{K})$ over a Hilbert space $\mathfrak{K}=L^2(-\infty, \infty)\oplus L^2(-\infty, \infty)$ the space of square integrable two component functions on the real line. If $f\in\mathfrak{K}$ we denote by $\tilde{f}$ the Fourier transform of f given by

$$\tilde{f}(p)=(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ipx}f(x)\,dx.$$

Let E_0 be the projection on $\mathfrak{K}$ given by $(E_0\tilde{f})(p)=e(p)\tilde{f}(p)$ where $e(p)$ is a (2×2) matrix with entries

$$e(p)=\begin{bmatrix} \frac{1}{2} & \frac{1}{2}e^{i\theta(p)} \\ \frac{1}{2}e^{-i\theta(p)} & \frac{1}{2} \end{bmatrix} \quad \text{where } \theta(p)=(1+p^2)^{-(1/5)}.$$

Let σ be the (2×2) matrix

$$\sigma=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then in x-space E_0 acts as follows.

$$(4.1)\qquad (E_0 f)(x)=\sigma f(x)+\int_{-\infty}^{\infty}\Gamma(x-y)f(y)\,dy$$

where $\Gamma(x)$ is a (2×2) matrix with entries

$$(4.2)\qquad \Gamma(x)=\begin{bmatrix} 0 & \gamma(x) \\ \overline{\gamma(-x)} & 0 \end{bmatrix} \text{ and } \gamma(x)=\frac{1}{4\pi}\int_{-\infty}^{\infty}(e^{i\theta(p)}-1)e^{ipx}dp.$$

We now state our main theorem.

Theorem 4.1. *Let ω_0 be the gauge invariant generalized free state of $\mathfrak{A}=\mathfrak{A}(\mathfrak{K})$ with two-point function E_0 defined above (i.e. $\omega_0(a(f)^*a(g))=(f, E_0 g)$ for $f, g\in\mathfrak{K}$). Let $(\pi_0, \mathfrak{H}, \Omega_0)$ be a cyclic $*$-representation of $\mathfrak{A}$ induced by ω_0 on a Hilbert space $\mathfrak{H}$ with cyclic vector Ω_0. Let T_t be the unitary group of translations on $\mathfrak{K}$ so $(T_t f)(x)=f(x-t)$ for $f\in\mathfrak{K}$ and let*

β_t be the group of $*$-automorphisms of $\mathfrak{A}$ defined by the requirement $\beta_t(a(f)) = a(T_t f)$ for all $f \in \mathfrak{K}$. Since ω_0 is a β_t invariant state there is a unitary group $W(t)$ acting on $\mathfrak{H}$ defined by the requirements $W(t)\Omega_0 = \Omega_0$ and $\pi_0(\beta_t(A)) = W(t)\pi_0(A)W(t)^{-1}$ for all $A \in \mathfrak{A}$ and $t \geq 0$. Let $\mathfrak{M}_+$ be the subspace of $\mathfrak{K}$ of all functions f with support in $[0, \infty)$ and let $\mathfrak{B} = \mathfrak{A}(\mathfrak{M}_+)$ be the C^*-subalgebra of $\mathfrak{A}$ generated by the $a(f)$ with $f \in \mathfrak{M}_+$. Let $M = \pi_0(\mathfrak{B})''$ and for $A \in M$ and $t \geq 0$ we define $\alpha_t(A) = W(t)AW(t)^{-1}$. Then M is a type I factor and α_t is an E_0-semigroup of M. Furthermore, there does not exist a strongly continuous one parameter semigroup of isometries $U(t) \in M$ with the property that $U(t)A = \alpha_t(A)U(t)$ for all $A \in M$ and $t \geq 0$.

The proof of this theorem will be based on the following lemmas.

Lemma 4.2. *The function γ defined above has the property that $\gamma(x) = iK_1|x|^{-3/5} - K_2|x|^{-1/5} + h(x)$ where K_1 and K_2 are positive constants and h is a bounded function of x. Furthermore, it is true that*

$$\int_0^\infty x|\gamma(x)|^2 dx < \infty.$$

Proof. We have

$$(*) \qquad \gamma(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{ipx}\left(i|p|^{-2/5} - \frac{1}{2}|p|^{-4/5} + a(p) + b(p) + c(p)\right)dp$$

where

$$a(p) = i\theta(p) - i|p|^{-2/5},\ b(p) = -\frac{1}{2}\theta(p)^2 + \frac{1}{2}|p|^{-4/5},$$
$$c(p) = e^{i\theta(p)} - 1 - i\theta(p) + \frac{1}{2}\theta(p)^2.$$

Routine estimates show that $|a(p)| < p^{-2}$, $|b(p)| < p^{-2}$. Hence, these functions are in L^1 for large $|p|$ and one sees by inspection these functions are in L^1 for small p. Hence, we have $a, b \in L^1(-\infty, \infty)$. Since $|e^{ix} - 1 - ix + \frac{1}{2}x^2| \leq |x|^3/6$ for all real x if follows that

$$|c(p)| \leq \theta(p)^3/6 = (1/6)(1+p^2)^{-3/5}$$

for all p. Hence, we have $c \in L^1(-\infty, \infty)$. Hence $d = a + b + c \in L^1(-\infty, \infty)$. Now we have by a change of variable $y = px$ for $0 < s < 1$ that

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{ipx}|p|^{-s}dp=\frac{|x|^{s-1}}{4\pi}\int_{-\infty}^{\infty} e^{iy}|y|^{-s}dy=K(s)\,|x|^{s-1},$$

where $K(s)$ is positive. Then we have from equation (*)

$$\gamma(x)=iK(2/5)\,|x|^{-3/5}-\frac{1}{2}K(4/5)\,|x|^{-1/5}+\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{ipx}d(p)\,dp.$$

Since $d\in L^1(-\infty,\infty)$ we have shown that $\gamma(x)=iK_1|x|^{-3/5}-K_2|x|^{-1/5}+h(x)$ where K_1 and K_2 are positive constants and h is a bounded function. Hence, $|x|^{1/2}\gamma(x)$ is in L^2 for small x and since

$$(d^n/dp^n)\,(e^{i\theta(p)}-1)\in L^1(-\infty,\infty)\quad \text{for } n=1,2,\cdots$$

it follows that $|x|^n\gamma(x)\to 0$ as $x\to\infty$ so $|x|^{1/2}\gamma(x)$ is in L^2 for large x. Hence, $x|\gamma(x)|^2\in L^1[0,\infty)$. □

Lemma 4.3. *The von Neumann algebra $M=\pi_0(\mathfrak{B})''$ define in Theorem 4.1 is a type I factor and α_t is an E_0-semigroup of M.*

Proof. It follows from Theorem 2.3 that $M=\pi_0(\mathfrak{B})''$ is a type I factor if $\mathrm{tr}(P_+E_0P_+(I-E_0)P_+)<\infty$ where P_+ is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{M}_+$. Now $P_+E_0P_+-P_+E_0P_+E_0P_+=P_+E_0P_-EP_+=Q^*Q$. To compute the trace of Q^*Q where Q in an operator with a kernel $K_{ij}(x,y)$ one has

$$\mathrm{tr}\,(Q^*Q)=\sum_{ij=1}^{22}\iint_{-\infty}^{\infty}|K_{ij}(x,y)|^2\,dx\,dy.$$

Hence,

$$\mathrm{tr}\,(P_+E_0P_+(I-E_0)P_+)=2\int_{-\infty}^{0}\int_{0}^{\infty}|\gamma(x-y)|^2dx\,dy=2\int_{0}^{\infty}x|\gamma(x)|^2dx<\infty$$

where the last integral converges by Lemma 4.2. Then by Theorem 2.1 $M=\pi_0(\mathfrak{B})''$ is a type I factor.

Since β_t maps $\mathfrak{B}$ into itself for $t\geq 0$ and $\pi_0(\mathfrak{B})$ is strongly dense in M we have α_t maps M into itself and from the form of α_t $(\alpha_t(A)=W(t)AW(t)^{-1})$ it is clear that α_t is an E_0-semigroup of M. □

Lemma 4.4. *Let $\mathfrak{M}[a,b]$ be the subspace of $\mathfrak{K}$ of functions f having support in the interval $[a,b]$ and let $\mathfrak{B}[a,b]=\mathfrak{A}(\mathfrak{M}[a,b])$ be the C^*-subalgebra of $\mathfrak{A}$ generated by the $a(f)$ with $f\in\mathfrak{M}[a,b]$. Then $\pi_0(\mathfrak{B}[a,b])''$ is a type I factor.*

Proof. Assume the hypothesis and notation of the lemma holds. Let P be the orthogonal projection $\mathfrak{K}$ onto $\mathfrak{M}[a,b]$. Then from Theorem 2.3 it follows that $\pi_0(\mathfrak{B}[a,b])''$ is a type I factor if $\mathrm{tr}(PE_0P(I-E_0)P)<\infty$. Now we have $PE_0P(I-E_0)P=PE_0(I-P)E_0P=Q^*Q$. Hence,

$$\mathrm{tr}(PE_0P(I-E_0)P)=2\iint_{\substack{x\in[0,1]\\ y\notin[0,1]}}|\gamma(x-y)|^2dx\ dy<4\int_0^\infty x|\gamma(x)|^2dx<\infty.$$

Hence, $\pi_0(\mathfrak{B}[a,b])''$ is a type I factor.

Lemma 4.5. *Suppose there is a strongly continuous one parameter semigroup of isometries $U(t)\in M$ so that $U(t)A=\alpha_t(A)U(t)$ for all $A\in M$. Suppose e_0 is a minimal projection in M (which exists since M is a type I factor) and F_0 is a unit vector in the range of e_0. Let $F_1=U(1)F_0$ and let $\rho(A)=(F_1,\ \pi_0(A)F_1)$ for $A\in\mathfrak{A}$. Let W be the two-point function of ρ (i.e., $\rho(a(f)^*a(g))=(f,\ Wg)$ for $f,g\in\mathfrak{K}$). Let P_λ be the projection on $\mathfrak{K}$ given by $(P_\lambda f)(x)=f(x)$ for $x\geq\lambda$ and $(P_\lambda f)(x)=0$ for $x<\lambda$. Then $(I-P_\lambda)WP_\lambda$ is an operator of rank at most one for all $0\leq\lambda\leq1$.*

Proof. Suppose the hypothesis and notation of the lemma are satisfied. Since F_0 is in the range of a minimal projection $e_0\in M$ the state $(F_0,\ AF_0)$ is pure on M. Let $\mathfrak{B}_t=\beta_t(\mathfrak{B})$ for $t\geq0$. We note the restriction of ρ to $\mathfrak{B}_t$ is pure for $0\leq t\leq1$. This may be seen as follows. We have $U(t)\pi_0(A)=\alpha_t(\pi_0(A))U(t)=\pi_0(\beta_t(A))U(t)$ for $A\in\mathfrak{B}$. Hence, we have $\pi_0(A)=U(t)^*\pi_0(\beta_t(A))U(t)$. Hence, for $0\leq t\leq1$ and $A\in\mathfrak{B}$ we have

$$\begin{aligned}\rho(\beta_t(A))&=(U(1)F_0,\ \pi_0(\beta_t(A))U(1)F_0)\\&=(U(1-t)F_0,\ U(t)^*\pi_0(\beta_t(A))U(t)U(1-t)F_0)\\&=(U(1-t)F_0,\ \pi_0(A)U(1-t)F_0).\end{aligned}$$

Since $U(1-t)\in M$ we have ρ is a pure state of $\mathfrak{B}_t$ for all $0\leq t\leq1$. We will show that $(I-P_\lambda)WP_\lambda$ is an operator of rank at most one for $0\leq\lambda\leq1$.

In Lemma 4.2 we saw that $\mathrm{tr}(P_+E_0P_+(I-E_0)P_+)<\infty$ and, hence, by Theorem 2.3 it follows that there is a unitary $S\in M$ so that $S^2=I$ and $S\pi_0(a(f))S=\pi_0(a((I-2P_+)f))$ for all $f\in\mathfrak{K}$ and M' is generated by the elements $S\pi_0(a(f))$ for all $f\in\mathfrak{M}_-$ where $\mathfrak{M}_-$ is the orthogonal complement of $\mathfrak{M}_+$ in $\mathfrak{K}$. Let f be a unit vector with support in $(-\infty, \lambda]$ so $P_\lambda f=0$ and *let* $C=\frac{1}{2}I+\frac{1}{2}\pi(a(f)+a(f)^*)\alpha_\lambda(S)$. One checks that $C^*=C$ and $C^2=C$. Let τ be the functional on $\mathfrak{B}_\lambda$ given by $\tau(A)=(F_1, \pi_0(A)CF_1)$. Note $\pi_0(a(g))$ and $\pi_0(a(g))^*$ commute with C for g in the range of P_λ. Since such elements generate $\pi_0(\mathfrak{B}_\lambda)$ we have that $C\in\pi_0(\mathfrak{B}_\lambda)'$. Hence, for $A\in\mathfrak{B}_\lambda$ we have

$$\begin{aligned}\tau(A^*A) &= (F_1,\ \pi_0(A^*A)CF_1)=(F_1, \pi_0(A^*A)C^2F_1)\\ &= (CF_1, \pi_0(A^*A)CF_1)\geq 0\end{aligned}$$

and

$$\begin{aligned}\rho(A^*A)-\tau(A^*A) &= (F_1, \pi_0(A^*A)\,(I-C)\,F_1)\\ &= ((I-C)\,F_1,\ \pi(A^*A)\,(I-C)\,F_1)\geq 0.\end{aligned}$$

Hence, $0\leq\tau\leq\rho\,|\,\mathfrak{B}_\lambda$. Since ρ is pure on $\mathfrak{B}_\lambda$ we have that $\tau(A)=\alpha\rho(A)$ for $A\in\mathfrak{B}_\lambda$. Setting $A=I$ we can evaluate the constant α. Then we obtain $(F_1, \pi_0(A)CF_1)=(F_1C\pi_0(A)F_1)=(F_1, CF_1)\,(F_1, \pi_0(A)F_1)$ for $A\in\mathfrak{B}_\lambda$. By weak continuity this relation extends to all of $\pi_0(\mathfrak{B}_\lambda)''$. Recalling the definition of C and subtracting of the identity term from both sides of this equation we find

$$\begin{aligned}&(F_1, \pi_0(a(f)+a(f)^*)\alpha_\lambda(S)AF_1)\\ &\quad=(F_1, \pi_0(a(f)+a(f)^*)\alpha_\lambda(S)F_1)\,(F_1, AF_1)\end{aligned}$$

for all $A\in\pi_0(\mathfrak{B}_\lambda)''$. Replacing f by if in this equation and combining the two equations we find

$$(F_1, \pi_0(a(f)^*)\alpha_\lambda(S)AF_1)=(F_1, \pi_0(a(f)^*)\alpha_\lambda(S)F_1)\,(F_1, AF_1)$$

for all $A\in\pi_0(\mathfrak{B}_\lambda)''$. Let $A=\alpha_\lambda(S)\pi_0(a(g))$ with g in the range of P_λ. Then $A\in\pi_0(\mathfrak{B}_\lambda)''$ and we have

$$\begin{aligned}(f, Wg) &= \rho(a(f)^*a(g))=(F_1, \pi_0(a(f)^*a(g)F_1)\\ &= (F_1, \pi_0(a(f)^*)\alpha_\lambda(S)\alpha_\lambda(S)\pi_0(a(g))F_1)\\ &= (F_1, \pi_0(a(f)^*)\alpha_\lambda(S)F_1)\,(F_1, \alpha_\lambda(S)\pi_0(a(g))F_1).\end{aligned}$$

Hence, (f, Wg) is of the form $(f, Wg)=\overline{Q(f)}R(g)$ where Q and R are linear functionals on the range of $I-P_\lambda$ and the range of P_λ, respectively. Hence, $(I-P_\lambda)WP_\lambda$ is an operator of rank at most one for all $\lambda\in[0, 1]$. □

Lemma 4.6. *Suppose the hypothesis and notation of Lemma* 4.5 *is satisfied. Let* B_1 *be the projection on* $\mathfrak{K}$ *given by* $(B_1f)(x)=\sigma f(x)$ *where* σ *is the* (2×2)*-matrix defined at the beginning of this section and let* ω_1

be the generalized free state of $\mathfrak{A}$ *with two-point function* B_1. *Then there are numbers* a *and* b *so that* $0 \leq a < b \leq 1$ *and the restrictions of* ω_0 *and* ω_1 *to* $\mathfrak{B}[a, b]$ *are quasi-equivalent. (Recall that* $\mathfrak{B}[a, b]$ *is the the* C^*-*subalgebra of* $\mathfrak{A}$ *generated by the* $a(f)$ *with* f *having support in* $[a, b]$.)

Proof. Assume the hypothesis and notation of the lemma are satisfied. We will first show that there are numbers a and b so that $0 \leq a < b \leq 1$ and the restrictions of ρ and ω_1 to $\mathfrak{B}[a, b]$ are quasi-equivalent. Let W be the two-point function of ρ. Let the projections P_λ and $P_{[a,b]}$ be defined as in § III. We first show that $Q = P_{[0,1]}(W - B_1)P_{[0,1]}$ is a compact operator. Note $\omega_0(A) = (\Omega_0, \pi_0(A)\Omega_0)$ and $\rho(A) = (F_1, \pi_0(A)F_1)$ for all $A \in \mathfrak{B}[0, 1]$. From Lemma 4.4 we have $\pi_0(\mathfrak{B}[0, 1])''$ is a type I factor. Since $(\Omega_0, A\Omega_0)$ and (F_1, AF_1) are normal state of $\pi_0(\mathfrak{B}[0, 1])''$ and since $\pi_0(\mathfrak{B}[0, 1])''$ is a type I factor it follows restrictions of ω_0 and ρ to $\mathfrak{B}[0, 1]$ are quasi-equivalent type I states. Hence, $P_{[0,1]}(E_0 - W)P_{[0,1]}$ is compact by Theorem 2.2. We show $P_{[0,1]}(E_0 - B_1)P_{[0,1]}$ is compact. For $f \in \mathfrak{K}$ we have

$$(P_{[0,1]}(E_0 - B_1)P_{[0,1]}f)(x) = \int_0^1 K(x)\Gamma(x-y)f(y)\,dy$$

where Γ is defined in equation (4.2) in terms of the function γ. And $K(x) = 1$ for $x \in [0, 1]$ and $K(x) = 0$ for $x \notin [0, 1]$. Let

$$\gamma_n(x) = \frac{1}{4\pi}\int_{-n}^{+n} (e^{i\theta(p)} - 1)e^{ipx}dp$$

and let Γ_n be defined in terms of γ_n as Γ was defined in terms of γ. Let J_n be the operator on $\mathfrak{K}$ given by

$$(J_n f)(x) = \int_0^1 K(x)\Gamma_n(x-y)f(y)\,dy.$$

A straight forward computation shows that J_n is a Hilbert Schmidt class operator (in fact, $\operatorname{tr}(J_n^* J_n) < 4n$) and since $\theta(p) \to 0$ as $|p| \to \infty$ we have $\|J_n - P_{[0,1]}(E_0 - B_1)P_{[0,1]}\| \to 0$ as $n \to \infty$. Since the norm limit of compact operators is compact $P_{[0,1]}(E_0 - B_1)P_{[0,1]}$ is compact. Since $P_{[0,1]}(E_0 - W)P_{[0,1]}$ and $P_{[0,1]}(E_0 - B_1)P_{[0,1]}$ are compact we have $Q = P_{[0,1]}(W - B_1)P_{[0,1]}$ is compact.

Let $Q_1 = (I - B_1)Q(I - B_1)$ and $Q_2 = -B_1 QB_1$. Since $(I - P_\lambda)B_1 P_\lambda = 0$ and since from Lemma 4.5 $(I - P_\lambda)WP_\lambda$ is of rank not greater than one for $\lambda \in [0, 1]$, we find $(I - P_\lambda)Q_i P_\lambda$ is of rank not greater than one for $i=1, 2$ and $\lambda \in [0, 1]$. Since Q_1 is positive and compact we have from Theorem 3.1 there exist numbers c and d so that $0 \leq c < d \leq 1$ so that $\mathrm{tr}(P_{[c,d]} Q_1 P_{[c,d]}) < \infty$. Since Q_2 is positive and compact and $(I-P_\lambda) Q_2 P_\lambda$ is of rank not greater than one for $\lambda \in [c, d]$ we have from Theorem 3.1 that there are numbers a and b so that $c \leq a < b \leq d$ and $\mathrm{tr}(P_{[a,b]} Q_2 P_{[a,b]}) < \infty$. Since $[a, b] \subset [c, d]$ we have $\mathrm{tr}(P_{[a,b]} Q_i P_{[a,b]}) < \infty$ for $i=1, 2$. Now ω_1 restricted to $\mathfrak{B}[a, b]$ is a generalized free state of $\mathfrak{B}[a, b]$ with two-point function (the restriction of B_1 to $\mathfrak{M}[a, b]$) a projection. Hence, it follows from Theorem 2.1 that the restrictions ω_1 and ρ to $\mathfrak{B}[a, b]$ are quasi-equivalent if

$$\mathrm{tr}(P_{[a,b]}(B_1(I-W)B_1 + (I-B_1)W(I-B_1))P_{[a,b]}) < \infty.$$

But the above expression is equal to $\mathrm{tr}(P_{[a,b]}(Q_2+Q_1)P_{[a,b]})$ which is less than infinity. Hence, the restrictions of ω_1 and ρ to $\mathfrak{B}[a, b]$ are quasi-equivalent.

Since $\pi_0(\mathfrak{B}[a, b])''$ is a type I factor by Lemma 4.4 and since ω_0 and ρ arise from normal states of $\pi_0(\mathfrak{B}[a, b])''$ (recall $\rho(A) = (F_1, \pi_0(A)F_1)$ and $\omega_0(A) = (\Omega_0, \pi_0(A)\Omega_0)$) it follows that the restrictions of ρ and ω_0 to $\mathfrak{B}[a, b]$ are quasi-equivalent. Hence, the restrictions of ω_1 and ω_0 to $\mathfrak{B}[a, b]$ are quasi-equivalent.

Lemma 4.7. *If $a<b$ then the restrictions of ω_0 and ω_1 to $\mathfrak{B}[a, b]$ are not quasi-equivalent.*

Proof. Suppose $a<b$. Let E_0^0 and B_1^0 be the restriction of E_0 and B_1 to $\mathfrak{M}[a, b]$ (i.e., $E_0^0 f = P_{[a,b]} E_0 f$ and $B_1^0 f = P_{[a,b]} B_1 f = B_1 f$ for $f \in \mathfrak{M}[a, b]$). Since the restrictions of ω_0 and w_1 to $\mathfrak{B}[a, b]$ are gauge invariant generalized free states of $\mathfrak{B}[a, b]$ and B_1^0 is a projection it follows from Theorem 2.1 that the restrictions of these states to $\mathfrak{B}[a, b]$ are quasi-equivalent if and only if $\mathrm{tr}(D) < \infty$ where $D = B_1^0(I-E_0^0)B_1^0 + (I-B_1^0)E_0^0(I-B_1^0)$ where the trace is taken to be the trace on $\mathfrak{M}[a, b]$. Calculating D we find

$$(Df)(x) = \int_a^b K(x-y) f(y)\, dy$$

where

$$K(x) = -\frac{1}{2}\gamma(x) - \frac{1}{2}\overline{\gamma(-x)} = -\mathrm{Re}(\gamma(x)).$$

From the discussion at the end of Theorem 3.1 we recall that the trace of a positive operator D with kernel $K_{ij}(x, y)$ is given by

$$\operatorname{tr}(D)=\int_a^b K_{11}(x, x)+K_{22}(x, x)\, dx$$

where the integral is suitably interpreted as a limit of other integrals. In our case $K_{11}(x, y)=K_{22}(x, y)=-\operatorname{Re}(\gamma(x-y))$ and from Lemma 4. 2 we have

$$-\operatorname{Re}(\gamma(x))=K_2|x|^{-1/5}+\text{a bounded function of } x.$$

Then one can show that $\sum_{i=1}^{2}\sum_{k=1}^{n}(e_{ki}, De_{ki})$ diverges like $n^{1/5}$ as $n\to\infty$ (where the e_{ki} are the functions constructed in §3). Hence, $\operatorname{tr}(D)=\infty$.□

Proof of Theorem 4. 1. Assume the hypothesis and notation of Theorem 4. 1. From Lemma 4. 3 we have that $M=\pi_0(\mathfrak{B})''$ is a type I factor and α_t is an E_0-semigroup of M. Assume there is a strongly continuous semigroup of isometries $U(t)\in M$ so that $U(t)A=\alpha_t(A)U(t)$ for all $A\in M$. Then the hypothesis of Lemma 4. 5 is satisfied as is the hypothesis of Lemma 4. 6 and, therefore, there are numbers a and b so that $a<b$ and the restrictions of ω_0 and ω_1 to $\mathfrak{B}[a, b]$ are quasi-equivalent. But by Lemma 4. 7 this is not possible. Hence, it follows that there is no intertwining semigroup of isometries $U(t)\in M$.□

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