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Uniqueness of Products in Higher Algebraic K-Theory

Dedicated to Professor Hirosi Toda on his 60th birthday

By

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Introduction

Let $\mathbb{E}$ be a higher algebraic K-theory defined on rings, that is, a functor which assigns to each ring R a spectrum $\mathbb{E}R$ of algebraic K-theory of R. Fiedorowicz uniqueness theorem [2] says that if $\mathbb{E}$ has an external tensor product, then there is a natural map of spectra

$$f: \mathbb{E}R \to \mathbb{G}\mathbb{W}R$$

which induces an equivalence between (-1)-connected covers of $\mathbb{E}R$ and the Gersten-Wagoner spectrum $\mathbb{G}\mathbb{W}R$ ([3] and [13]). May [6] has given a similar uniquenes theorem for higher algebraic K-theories (or, infinite loop space machines) defined on permutative (i. e., symmetric strict monoidal) categories: given an infinite loop space machine $\mathbb{E}$ defined on permutative categories, there exists a natural equivalence of spectra between $\mathbb{E}U$ and the spectrum $\mathbb{S}B\bar{U}$ constructed by Segal [9].

In the present article we study the multiplicativity of such natural transformations between higher algebraic K-theories defined on permutative categories, or exact categories, or rings. Here the term 'multiplicativity' is used in the following sense. Let $\mathbb{E}$ and $\mathbb{E}'$ be functors $\mathscr{C} \to \mathscr{S}$ from permutative categories (or exact categories, or rings) to CW-spectra, and suppose that $\mathbb{E}$ (resp. $\mathbb{E}'$) functorially associates to each pairing $U \times V \to W$ in $\mathscr{C}$ a pairing $\mathbb{E}U \wedge \mathbb{E}V \to \mathbb{E}W$ (resp. $\mathbb{E}'U \wedge \mathbb{E}'V \to \mathbb{E}'W$) of CW-spectra. Then a natural transformation $f: \mathbb{E} \to \mathbb{E}'$ is called multiplicative if the following square commutes

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in the homotopy category $H\mathscr{S}$;

$$\begin{array}{ccc} EU\wedge EV & \longrightarrow & EW \\ {\scriptstyle f\wedge f}\downarrow & & \downarrow{\scriptstyle f} \\ E'U\wedge E'V & \longrightarrow & E'W. \end{array}$$

Notice that most of the constructions of products in higher algebraic K-theory, except for May's [7], provide only weak pairings, i.e., pairings in the sense of G. W. Whitehead. This notion of a weak pairing is inadequate for sophisticated spectrum level analysis. Hence we want to find a condition, as generous as possible, which ensures that a given machine functorially associates 'true' pairings. Thus we introduce a notion of a pairing of S_*-spectra which generalizes May's notion of a pairing of $\mathscr{I}_*$-prespectra [7].

We now state the results of the paper.

A CW-spectrum $E=\{E_n|n\geqq 0\}$ is called an S_*-spectrum if each E_n has an action by the symmetric group S_n (E_n is an S_n-CW complex) which is compatible with the structure maps and restricts to a homotopically trivial A_n-action. (See Section 1.) There is a relevant notion of a pairing of S_*-spectra and we can show that pairings $(E, F)\to G$ of S_*-spectra functorially determine pairings $E\wedge F\to G$ in the stable category.

We use the term *higher algebraic* K*-theory defined on permutative categories* to denote a functor E which assigns to every permutative category U a connective CW-spectrum $EU=\{E_nU|n\geqq 0\}$ together with a natural map $\lambda: BU\to E_0U$ such that the composite $BU\to\Omega^\infty E_\infty U=\bigcup_n \Omega^n E_n U$ is a group completion.

Definition. A higher algebraic K-theory E defined on permutative categories is called *multiplicative* if (i) EU has a natural structure of an S_*-spectrum, and (ii) given a pairing $f: U\times V\to W$ of permutative categories, there exists a natural pairing $Ef=\{E_{m,n}f\}:(EU, EV)\to EW$ of S_*-spectra such that the following square commutes;

$$\begin{array}{ccc} BU\wedge BV & \xrightarrow{Bf} & BW \\ {\scriptstyle \lambda\wedge\lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ E_0U\wedge E_0V & \xrightarrow{E_{0,0}f} & E_0W. \end{array}$$

Thus a multiplicative higher algebraic K-theory E functorially associates a true pairing $\mathbb{E}f: \mathbb{E}U \wedge \mathbb{E}V \to \mathbb{E}W$ of CW-spectra.

It will be shown that both May machine $\mathbb{M}$ [7] and Shimada-Shimakawa machine $\mathbb{C}$ [10] are multiplicative higher algebraic K-theories defined on permutative categories. (But Segal's machine [9] is not.)

Now our first theorem is

Theorem A. *Let $\mathbb{E}$ be a higher algebraic K-theory defined on permutative categories. Then there is a natural equivalence $\gamma: \mathbb{E}U \to \mathbb{C}U$ which is multiplicative when $\mathbb{E}$ is a multiplicative higher algebraic K-theory.*

Note. Because the passage from symmetric monoidal to permutative categories preserves pairings (cf. [7, §2]), every multiplicative higher algebraic K-theory defined on permutative categories (e. g. $\mathbb{M}$) can be canonically regarded as a multiplicative higher algebraic K-theory defined on symmetric monoidal categories. (We omit the obvious definition of the latter notion.) Theorem A holds true for any $\mathbb{E}$ defined on symmetric monoidal categories.

Next let $\mathbb{K}$ denote the Waldhausen machine [14] which assigns to each exact category U a CW-spectrum $\mathbb{K}U = \{BQ^n U^{[n]} \mid n \geqq 0\}$ (cf. [11]). Then $\mathbb{K}$ associates to any biexact functor $f: U \times V \to W$ a pairing $\mathbb{K}f: (\mathbb{K}U, \mathbb{K}V) \to \mathbb{K}W$ of S_*-spectra. (This is essentially the result of [11].) Let us denote by $\mathrm{Is}U$ the subcategory of all isomorphisms in a category U, and consider both $\mathrm{Is}U$ and QU as symmetric monoidal categories.

Theorem B. *There is a multiplicative natural transformation $\kappa: \mathbb{C}\mathrm{Is}U \to \mathbb{K}U$ defined as the composite of a natural equivalence $\eta: \Omega\mathbb{C}QU \cong \mathbb{K}U$ with a natural map $\nu: \mathbb{C}\mathrm{Is}U \to \Omega\mathbb{C}QU$ which deloops the familiar map $B\mathrm{Is}U \to \Omega BQU$.*

Note that by the "$+ = Q$" theorem [4], κ becomes an equivalence if every short exact sequence in U splits.

Finally we consider higher algebraic K-theories defined on rings. We do not know whether Loday's pairing $(\mathbb{G}\mathbb{W}R, \mathbb{G}\mathbb{W}R') \to \mathbb{G}\mathbb{W}(R \otimes R')$ induces a 'true' pairing $\mathbb{G}\mathbb{W}R \wedge \mathbb{G}\mathbb{W}R' \to \mathbb{G}\mathbb{W}(R \otimes R')$ or not. However we have

Theorem C. *There exists a functor A from rings to S_*-spectra which satisfies the followings:*

(1) *There is a natural pairing $\mu:(AR, AR')\to A(R\otimes R')$ of S_*-spectra.*

(2) *For each $n\geqq 1$, there is a natural group completion $f_n: BIsP(S^nR)\to A_nR(\simeq K_0S^nR\times BGLS^nR^+=GW_nR)$ such that*

$$\begin{array}{ccc} BIsP(S^mR)\wedge BIsP(S^nR') & \longrightarrow & BIsP(S^{m+n}(R\otimes R')) \\ \downarrow{\scriptstyle f_m\wedge f_n} & & \downarrow{\scriptstyle f_{m+n}} \\ A_mR\wedge A_nR' & \xrightarrow{\mu_{m,n}} & A_{m+n}(R\otimes R') \end{array}$$

commutes. (Here $P(R)$ denotes the category of finitely generated projective modules over R.)

(3) *The structure map $A_nR\wedge S^1\to A_{n+1}R$ is given by the composite*

$$A_nR\wedge S^1\xrightarrow{1\wedge\iota}A_nR\wedge A_1Z\xrightarrow{\mu_{n,1}}A_{n+1}(R\otimes Z)=A_{n+1}R$$

where $\iota: S^1\to A_1Z$ represents the standard generator of $K_1SZ=Z$ (cf. [5, Chapitre II]).

(4) *There is a multiplicative natural transformation $\alpha: CIsP(R)\to AR$ such that the induced map $\Omega^\infty C_\infty IsP(R)\to\Omega^\infty A_\infty R$ is an equivalence.*

Note that the condition (3) is similar to the description of the structure map of GWR given by Loday [5]. From (2) we see that $\mu_{m,n}$ is weakly homotopic to Loday's map $GW_mR\wedge GW_nR'\to GW_{m,n}(R\otimes R')$. As a consequence we have

Corollary. (*Cf. Weibel* [15].) *The product structures in higher algebraic K-theory of rings constructed by Waldhausen* [14], *May* [7], *Shimada-Shimakawa* [10], *and Loday* [5] (*modified as in Theorem C*) *all agree with each other.*

The proofs of the above theorems are given in the final section. In Section 1 we introduce a notion of a pairing of S_*-spectra and prove that pairings of S_*-spectra functorially determine pairings in the stable category. Section 2 illustrates how the machines of Waldhausen, Shimada-Shimakawa, and May associate pairings of

S_*-spectra, and Section 3 provides a key tool on which our proofs of the theorems are based, that is to say, a multiplicative version of the 'up and across theorem' [2, 8].

§ 1. Pairings of S_*-Spectra

Throughout the paper we regard S^1 as the one-point compactification of R ($\{\infty\}$ is the base-point), and denote by S^n the smash product of n copies of S^1. Each S^n is an S_n-CW complex equipped with the standard S_n-action; $\sigma(s_1, \ldots, s_n) = (s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(n)})$ for every permutation $\sigma \in S_n$.

Let $E = \{E_n | n \geqq 0\}$ be a CW-spectrum. We say that E is an S_*-*spectrum* if each E_n is a based S_n-CW complex, and if the following two conditions hold: (i) the diagram

$$\begin{array}{ccc} E_n \wedge S^k & \longrightarrow & E_{n+k} \\ \Big\downarrow{\scriptstyle \sigma\wedge\tau} & & \Big\downarrow{\scriptstyle \sigma+\tau} \\ E_n \wedge S^k & \longrightarrow & E_{n+k} \end{array}$$

commutes for all $\sigma \in S_n$ and $\tau \in S_k$; and (ii) for every even permutation $\sigma \in A_n$ the map $\sigma: E_n \to E_n$ is homotopic to the identity. Given S_*-spectra E and F, a function $f: E \to F$ is called a *function of S_*-spectra* if each $f_n: E_n \to F_n$ is S_n-equivariant. A *map of S_*-spectra* is a map $f: E \to F$ which is represented by a function $f': E' \to F$ of S_*-spectra for some cofinal subspectrum E' such that each E'_n is invariant under the S_n-action on E_n.

If we consider $S_* = \amalg_n S_n$ as the skeletal category of finite sets and their isomorphims, then each S_*-spectrum E can be regarded as a functor from S_* to CW-complexes. Moreover the structure maps $E_m \wedge S^n \to E_{m+n}$ constitute a natural transformation $E \wedge S^0 \stackrel{\cdot}{\longrightarrow} E \circ \oplus$ where $S^0: \mathfrak{n} \mapsto S^n$ denotes the sphere-valued functor. It follows that an $\mathscr{I}_*$-prespectrum in the sense of May [7] restricts, via the canonical embedding $S_* \to \mathscr{I}_*$, to an S_*-spectrum in our sense. (Strictly speaking, $\mathscr{I}_*$-prespectra are not supposed to have a structure of a CW-spectrum. But this is not serious because the passage from $\mathscr{I}_*$-prespectra to the stable category is equivalent to the process of replacing spectra by CW-approximations.)

Note that in the definition of an S_*-spectrum the condition (ii) follows from (i) if the S_n-action on E_n extends to an $O(n)$-action (e. g., E is an $\mathscr{I}_*$-prespectrum), or if E is an almost Ω-spectrum, that is, the maps $E_n \to \Omega E_{n+1}$ are homotopy equivalences for $n \geqq 1$ (cf. [11, Lemma 4. 1]).

Now let E, F and G be S_*-spectra.

Definition 1. 1. A pairing of S_*-spectra $\mu: (E, F) \to G$ is a family of maps

$$\mu_{m,n}: E_m \wedge F_n \longrightarrow G_{m+n}; \qquad m, n \geqq 0$$

such that the following diagram commutes;

$$\begin{array}{ccc} E_m \wedge F_n \wedge S^k \wedge S^l \xrightarrow{\mu \wedge 1} G_{m+n} \wedge S^{k+l} & \longrightarrow & G_{m+n+k+l} \\ \downarrow{\scriptstyle 1\wedge T \wedge 1} & & \downarrow{\scriptstyle 1+\tau+1} \\ E_m \wedge S^k \wedge F_n \wedge S^l \longrightarrow E_{m+k} \wedge F_{n+l} & \xrightarrow{\mu} & G_{m+k+n+l} \end{array}$$

where $1+\tau+1$ denotes the permutation

$$\begin{pmatrix} m+1 & \dots & m+n & m+n+1 & \dots & m+n+k \\ m+k+1 & \dots & m+k+n & m+1 & \dots & m+k \end{pmatrix}.$$

Example 1. 2. Let E be an S_*-spectrum. Then the canonical pairing $\varepsilon: (E, S^0) \to E$ consisting of the maps $E_m \wedge S^n \to E_{m+n}$ is a pairing of S_*-spectra.

We now describe the passage from pairings of S_*-spectra to pairings in the stable category $H\mathscr{S}$.

Given an S_*-spectrum G we construct a sort of double telescope WG as follows. For every $n \geqq 0$, WG_n is defined as an identification space of the union

$$\begin{aligned} &\vee_{i+j\leqq n}([i]\times[j])_+ \wedge G_{i+j} \wedge S^{n-i-j} \\ &\vee \vee_{i+j\leqq n-1}([i]\times[j, j+1])_+ \wedge G_{i+j} \wedge S^{n-i-j} \\ &\vee \vee_{i+j\leqq n-1}([i, \ i+1]\times[j])_+ \wedge G_{i+j} \wedge S^{n-i-j} \\ &\vee \vee_{i+j\leqq n-2} G_{i+j} \wedge M(\tau) \wedge S^{n-i-j-2} \end{aligned}$$

where $M(\tau)$ denotes the Thom space of a certain $SO(2)$-bundle τ over the 2-cell $[i, \ i+1]\times[j, \ j+1]$ and we identify

$$((i,j), g, s) \in ([i] \times [j,j+1])_+ \wedge G_{i+j} \wedge S^{n-i-j} \quad \text{or}$$
$$\in ([i, i+1] \times [j])_+ \wedge G_{i+j} \wedge S^{n-i-j}$$

with

$$((i,j), g, s) \in ([i] \times [j])_+ \wedge G_{i+j} \wedge S^{n-i-j};$$
$$((i,j+1), g, s, t) \in ([i] \times [j,j+1])_+ \wedge G_{i+j} \wedge S^1 \wedge S^{n-i-j-1}$$

with

$$((i,j+1), [g, s], t) \in ([i] \times [j+1])_+ \wedge G_{i+j+1} \wedge S^{n-i-j-1};$$

and

$$((i+1,j), g, s, t) \in ([i, +1] \times [j])_+ \wedge G_{i+j} \wedge S^1 \wedge S^{n-i-j-1}$$

with

$$((i+1,j), \sigma_{i,j}[g, (-1)^j s], t) \in ([i+1] \times [j])_+ \wedge G_{i+1+j} \wedge S^{n-i-1-j}$$

where

$$\sigma_{i,j} = \begin{pmatrix} i+1 & i+2 \cdots i+j+1 \\ i+2 & i+3 \cdots \ i+1 \end{pmatrix} \in S_{i+j+1}.$$

(The identification of $G_{i+j} \wedge M(\tau) \wedge S^{n-i-j-2}$ with the part already constructed is quite similar to that described in [1, p. 175].) The structure maps are obvious. (Compare with the definition of the smash product of spectra [1, §4].)

If $\mu: (E, F) \to G$ is a pairing of S_*-spectra, then we have

$$\mu_{i+1,j}([e, (-1)^j s], f) = \sigma_{i,j}[\mu_{i,j}(e,f), (-1)^j s]$$

for all $(e, f, s) \in E_i \wedge F_j \wedge S^1$. Hence there is a well-defined map

$$\bar{\mu}: E \wedge F \to WG.$$

Lemma 1.3. *Pairings $\mu: (E, F) \to G$ of S_*-spectra functorially determine pairings*

$$\tilde{\mu}: E \wedge F \to G$$

in the stable category $H\mathscr{S}$.

Proof. There is a sequence of natural maps of spectra

$$E \wedge F \xrightarrow{\bar{\mu}} WG \xleftarrow{\bar{\varepsilon}} G \wedge S^0 \xleftarrow{\simeq} TG \xrightarrow{\simeq} G$$

where TG denotes the telescope of G. To define $\tilde{\mu}$ we have only to prove that $\bar{\varepsilon}: G \wedge S^0 \to WG$ is a homotopy equivalence.

Take a partition $A=B\cup C$ of an ordered set $A\cong N$, and define a spectrum $W_{BC}G$ as follows: for every $a\in A$ we put

$$W_{BC}G_{\alpha(a)}=G_{\beta(a)+\gamma(a)}\,,$$

where $\alpha(a)=\#\{x\in A\,|\,x<a\}$ etc., and identify $(g,s)\in W_{BC}G_{\alpha(a)}\wedge S^1$ with

$$\begin{array}{ll} [g,s]\in G_{\beta(a)+(\gamma(a)+1)} & \text{if } a\in C \\ \sigma_{\beta(a),\gamma(a)}[g,(-1)^{\gamma(a)}s]\in G_{(\beta(a)+1)+\gamma(a)} & \text{if } a\in B\,. \end{array}$$

(Compare with the definition of naive smash product [1].) Now suppose that both B and C are infinite and that $\gamma(a)$ is even (and hence $\sigma_{\beta(a),\gamma(a)}\simeq\mathrm{id}$) whenever $a\in B$. Then we have a commutative diagram

$$\begin{array}{ccc} G\wedge S^0 & \xrightarrow{\bar{\varepsilon}} & WG \\ \simeq\uparrow & & \uparrow\simeq \\ T(G\wedge_{BC}S^0) & \longrightarrow & TW_{BC}G \\ \simeq\downarrow & & \downarrow\simeq \\ G\wedge_{BC}S^0 & \longrightarrow & W_{BC}G \end{array}$$

in which every vertical map is a homotopy equivalence. Since $W_{BC}G_n\wedge S^1\to W_{BC}G_{n+1}$ are homotopic to the original structure maps $G_n\wedge S^1\to G_{n+1}$, the bottom map becomes a weak homotopy equivalence. Thus we see that $\bar{\varepsilon}$ is a homotopy equivalence.

Notation. In what follows we use the same letter μ to denote the pairing $E\wedge F\to G$ induced from a pairing $\mu\colon(E,F)\to G$ of S_*-spectra.

§2. Multiplicative Higher Algebraic K-Theories

2.1. Waldhausen machine. For every exact category U we have a CW-spectrum $\boldsymbol{K}U=\{\boldsymbol{K}_nU\,|\,n\geqq 0\}$ where

$$\boldsymbol{K}_nU=BQ^nU^{[n]}=BQ_1\cdots Q_nU^{[n]}$$

denotes the classifying space of the n-fold category obtained by applying Q on every component of the n-fold exact category $U^{[n]}$ of commutative n-cubes in U (cf. [11]). Note that $\boldsymbol{K}_0U$ is the set oU of all objects of U. (This differs from the definition given in [11] in which $\boldsymbol{K}_0U$ is defined to be ΩBQU.)

We have shown in [11] that the evident S_n-action on $U^{[n]}$ induces an S_n-action on $BQ^nU^{[n]}$ with respect to which $\mathbb{K}U$ becomes an S_*-spectrum. Moreover any biexact functor $f: U\times V\to W$ induces a natural map

$$\mathbb{K}_{m,n}f: BQ^mU^{[m]}\wedge BQ^nV^{[n]}\to BQ^{m+n}W^{[m+n]}$$

for each pair of integers m and n. The diagram (4.2) in [11] shows that $\mathbb{K}_{m,n}f$ define a pairing $\mathbb{K}f: (\mathbb{K}U, \mathbb{K}V)\to \mathbb{K}W$ of S_*-spectra. Thus f functorially associates a pairing

$$\mathbb{K}f: \mathbb{K}U\wedge \mathbb{K}V\to \mathbb{K}W$$

in the stable category.

2.2. Shimada-Shimakawa machine. In [10] we have associated to any symmetric monoidal category $U=\langle U, \oplus\rangle$ a spectrum $\mathbb{C}U=\{B\mathscr{B}^nU\,|\,n\geqq 0\}$ where $\mathscr{B}$ is a functor which assigns to each symmetric monoidal (topological) category C a symmetric monoidal category $\mathscr{B}C$ together with a natural map $BC\wedge S^1\to B\mathscr{B}C$. By extending the arguement of [10, Lemma 2.6] we see that $\mathbb{C}_nU=B\mathscr{B}^nU$ is identical with the geometric realization of a Γ^n-space $B\hat{B}^nU$ defined as follows.

For each $(\mathfrak{r}_1, \dots, \mathfrak{r}_n)\in\Gamma^n$, denote by $\hat{B}^nU(\mathfrak{r}_1, \dots, \mathfrak{r}_n)$ the symmetric monoidal category with objects

$$\langle a; \alpha^1, \dots, \alpha^n\rangle$$

where a is a function which assigns to each n-tuple $(T_1, \dots, T_n)$ of subsets $T_i\subset\mathfrak{r}_i$ an object $a(T_1, \dots, T_n)$ of U, α^i is a family of isomorphisms

$$a(T_1, \dots, T_i\amalg T_i', \dots, T_n)\xrightarrow{\cong} a(T_1, \dots, T_i, \dots, T_n)\oplus a(T_1, \dots, T_i', \dots, T_n)$$

satisfying the conditions similar to those of [10, Definition 2.1 (i)], and for any T of the form $(T_1, \dots, T_i^0\amalg T_i^1, \dots, T_j^0\amalg T_j^1, \dots, T_n)$ the following diagram commutes;

$$\begin{array}{ccc} a(T) & \xrightarrow{(\alpha^j\oplus\alpha^j)\alpha^i} & a(T_{00})\oplus a(T_{01})\oplus a(T_{10})\oplus a(T_{11}) \\ \| & & \downarrow\cong \\ a(T) & \xrightarrow{(\alpha^i\oplus\alpha^i)\alpha^j} & a(T_{00})\oplus a(T_{10})\oplus a(T_{01})\oplus a(T_{11}) \end{array}$$

in which $T_{\varepsilon\delta}=(T_1, \dots, T_i^\varepsilon, \dots, T_j^\delta, \dots, T_n)$ $(0\leqq\varepsilon, \delta\leqq 1)$. Given objects $\langle a; \alpha^1, \dots, \alpha^n\rangle$ and $\langle b; \beta^1, \dots, \beta^n\rangle$ a morphism $f: \langle a; \alpha^1, \dots, \alpha^n\rangle\to\langle b; \beta^1, \dots, \beta^n\rangle$ is a family of morphisms $f(T_1, \dots, T_n): a(T_1, \dots, T_n)\to b(T_1,$

$\dots, T_n)$ compatible with all α^i and β^i in the sense of [10, Definition 2.1]. (Note that $\hat{B}U=\hat{B}^1U$ coincides with $\bar{U}$ of May [6].)

There is a canonical isomorphism

$$\hat{B}^{n+1}U(\mathbf{r}_1, \dots, \mathbf{r}_{n+1}) \cong \hat{B}(\hat{B}^nU(\mathbf{r}_1, \dots, \mathbf{r}_n))(\mathbf{r}_{n+1})$$

natural in both U and $(\mathbf{r}_1, \dots, \mathbf{r}_{n+1})$. Hence, as in [10, §2], we can inductively prove that $C_nU=B\mathscr{B}^nU$ is isomorphic to the geometric realization of the Γ^n-space $B\hat{B}^nU$. Moreover the structure map $C_nU\wedge S^1\to C_{n+1}U$ is described as the inclusion

$$|B\hat{B}^nU|\wedge S^1\cong|B\hat{B}^{n+1}U(\cdots,1)|\wedge S^1\to|B\hat{B}^{n+1}U|.$$

We now define an S_n-action on C_nU. Given a Γ^n-category E and $\sigma\in S_n$, denote by E^σ the Γ^n-category such that $E^\sigma(\mathbf{r}_1, \dots, \mathbf{r}_n)=E(\mathbf{r}_{\sigma^{-1}(1)}, \dots, \mathbf{r}_{\sigma^{-1}(n)})$. Clearly we have $|BE^\sigma|=|BE|$ (cf. [11, 1.14]). Returning to our case, for every $\sigma\in S_n$ there is a natural isomorphism $\Theta(\sigma):\hat{B}^nU\to(\hat{B}^nU)^\sigma$ which assigns to each $\boldsymbol{a}=\langle a;\alpha^1, \dots, \alpha^n\rangle\in\hat{B}^nU(\mathbf{r}_1, \dots, \mathbf{r}_n)$,

$$\Theta(\sigma)\boldsymbol{a}=\langle\tilde{a}:\tilde{\alpha}^1, \dots, \tilde{\alpha}^n\rangle\in\hat{B}^nU(\mathbf{r}_{\sigma^{-1}(1)}, \dots, \mathbf{r}_{\sigma^{-1}(n)})$$

where $\tilde{a}(T_{\sigma^{-1}(1)}, \dots, T_{\sigma^{-1}(n)})=a(T_1, \dots, T_n)$ and $\tilde{\alpha}^{\sigma^{-1}(i)}=\alpha^i$, $1\leqq i\leqq n$. It is easy to see that the induced maps

$$\begin{array}{ccc} |B\hat{B}^nU| & \xrightarrow{|B\Theta(\sigma)|} & |B(\hat{B}^nU)^\sigma|=|B\hat{B}^nU| \\ \Downarrow & & \Downarrow \\ [\boldsymbol{a}, s_1, \dots, s_n] & \longmapsto & [\Theta(\sigma)\boldsymbol{a}, s_{\sigma^{-1}(1)}, \dots, s_{\sigma^{-1}(n)}] \end{array}$$

define an S_n-action on C_nU with respect to which CU becomes an S_*-spectrum.

Let $f:U\times V\to W$ be a pairing of symmetric monoidal categories. Then we define a map

$$f_{m,n}:\hat{B}^mU(\mathbf{r}_1, \dots, \mathbf{r}_m)\times\hat{B}^nV(\mathbf{r}_{m+1}, \dots, \mathbf{r}_{m+n})\to\hat{B}^{m+n}W(\mathbf{r}_1, \dots, \mathbf{r}_{m+n})$$

of Γ^{m+n}-categories by

$$\begin{aligned} &f_{m,n}(\langle a;\alpha^1, \dots, \alpha^m\rangle, \langle b;\beta^1, \dots, \beta^n\rangle)= \\ &\qquad\langle f(a\times b);\delta(\alpha^1\times 1), \dots, \delta(\alpha^m\times 1), \delta'(1\times\beta^1), \dots, \delta'(1\times\beta^n)\rangle \end{aligned}$$

where $\delta(\alpha^i\times 1)$ denotes the family of composite isomorphisms

$$\begin{aligned} &f(a(T_1, \dots, T_i\amalg T_i', \dots, T_m), b(T_{m+1}, \dots, T_{m+n})) \\ &\quad\cong f(a(T_1, \dots, T_i, \dots, T_m)\oplus a(T_1, \dots, T_i', \dots, T_m), b(T_{m+1}, \dots, T_{m+n})) \\ &\quad\cong f(a(T_1, \dots, T_i, \dots, T_m), b(T_{m+1}, \dots, T_{m+n})) \\ &\qquad\qquad\oplus f(a(T_1, \dots, T_i', \dots, T_m), b(T_{m+1}, \dots, T_{m+n})) \end{aligned}$$

and $\delta'(1\times\beta^i)$ the family of isomorphisms

$$f(a(T_1,\dots,T_m),b(T_{m+1},\dots,T_{m+i}\amalg T'_{m+i},\dots,T_{m+n}))$$
$$\cong f(a(T_1,\dots,T_m),\ b(T_{m+1},\dots,T_{m+i},\dots,T_{m+n}))\oplus$$
$$f(a(T_1,\dots,T_m),\ b(T_{m+1},\dots,T'_{m+i},\dots,T_{m+n}))$$

similarly defined.

One easily checks that the induced maps

$$\boldsymbol{C}_{m,n}f:\boldsymbol{C}_mU\wedge\boldsymbol{C}_nV\to\boldsymbol{C}_{m+n}W$$

satisfy the condition of a pairing of S_*-spectra, and that $\boldsymbol{C}_{0,0}f$ coincides with Bf: $BU\wedge BV\to BW$. Thus we see that the functor $U\mapsto\boldsymbol{C}U$ is a multiplicative higher algebraic K-theory defined on symmetric monoidal (or permutative) categories.

2.3. May machine. May [7] has defined a functor from permutative categories to $\mathscr{I}_*$-prespectra by the composite

$$U\mapsto\tilde{U}\mapsto B\tilde{U}\mapsto TB\tilde{U}$$

where $\tilde{U}$ denotes the functor $\mathscr{F}\to\mathbf{Cat}$ obtained by applying the Street's first construction [12] on a lax functor $\mathbf{n}\mapsto U^n$, and T assigns to every $\mathscr{F}$-space X an $\mathscr{I}_*$-prespectrum $TX:V\mapsto B(\Sigma^V,\hat{C}_V,X)$. Let us denote by $\boldsymbol{M}U$ the associated S_*-spectrum $\{\boldsymbol{M}_nU=TB\tilde{U}(\boldsymbol{R}^n)\mid n\geqq0\}$.

The canonical inclusion $U\to\tilde{U}(\mathbf{1})$ induces a natural map $\lambda:BU\to B\tilde{U}(\mathbf{1})\to TB\tilde{U}(\{0\})=\boldsymbol{M}_0U$ such that the composite $BU\to\Omega^\infty\boldsymbol{M}_\infty U$ is a group completion.

By Theorems 2.1 and 6.2 of [7] we see that pairings $f:(U,V)\to W$ of permutative categories functorially determine pairings $\boldsymbol{M}f:(\boldsymbol{M}U,\boldsymbol{M}V)\to\boldsymbol{M}W$ of S_*-spectra, and it is easy to see that the following square commutes;

$$\begin{array}{ccc} BU\wedge BV & \longrightarrow & BW \\ {\scriptstyle\lambda\wedge\lambda}\downarrow & & \downarrow{\scriptstyle\lambda} \\ \boldsymbol{M}_0U\wedge\boldsymbol{M}_0V & \longrightarrow & \boldsymbol{M}_0W. \end{array}$$

Therefore the functor $U\mapsto\boldsymbol{M}U$ also becomes a multiplicative higher algebraic K-theory defined on permutative categories.

Remark. Unfortunately, Segal machine applied to the Γ-space $B\hat{B}U=B\bar{U}$ does not provide an S_*-spectrum. In fact, Segal-Woolfson approach to constructing pairings of spectra is much more complicated than that described here. (Compare [16] and [9].)

§ 3. S_*-Bispectra and Their Pairings

A *CW-bispectrum* is a family $X=\{X_{n,q}\mid n\geqq 0, q\geqq 0\}$ of CW-complexes $X_{n,q}$ equipped with cellular embeddings

$$X_{n,q}\wedge S^1\to X_{n+1,q}, \quad (x, s)\mapsto [x, s]$$
$$X_{n,q}\wedge S^1\to X_{n,q+1}, \quad (x, s)\mapsto [x, s]'$$

such that

$$[[x, s], t]'=[[x, t]', s]$$

holds for all $(x, s, t)\in X_{n,q}\wedge S^1\wedge S^1$. Every CW-bispectrum X determines (and is uniquely determined by) CW-spectra

$$X_{n*}=\{X_{n,q}\mid q\geqq 0\}, \quad X_{*q}=\{X_{n,q}\mid n\geqq 0\}$$

and functions of spectra

$$X_{n*}\wedge S^1\longrightarrow X_{n+1*}, \quad X_{*q}\wedge S^1\longrightarrow X_{*q+1}.$$

For any bispectrum X we define two diagonal spectra DX and $D'X$ as follows (cf. [8]):

$$D_nX=\Omega^n X_{n,n}=D'_nX;$$

and the structure maps $\delta: D_nX\wedge S^1\to D_{n+1}X$ and $\delta': D'_nX\wedge S^1\to D'_{n+1}X$ are given by

$$\delta(x, s)(t_1, \dots, t_n, t_{n+1})=[[x(t_1, \dots, t_n), t_{n+1}], s]'$$
$$\delta'(x, s)(t_1, \dots, t_n, t_{n+1})=[[x(t_1, \dots, t_n), t_{n+1}]', s]$$

for all $(x, s)\in\Omega^n X_{n,n}\wedge S^1$ and $(t_1, \dots, t_n, t_{n+1})\in S^{n+1}$. Then there are maps of spectra (natural in $H\mathscr{S}$) $e: X_{0*}\to DX$ given by

$$e_qx(t_1, \dots, t_q)=[x, t_1, \dots, t_q]$$

for every $x\in X_{0,q}$; $e': X_{*0}\to D'X$ given by

$$e'_ny(t_1, \dots, t_n)=[y, t_1, \dots, t_n]'$$

for every $y\in X_{n,0}$; and $c=d'\rho^{-1}: DX\xrightarrow{\simeq}D'X$ where ρ^{-1} is the homotopy inverse of the canonical map $\rho: TDX\to DX$ and $d': TDX\to D'X$ is induced from the preternatural weak map $d: DX\to D'X$;

$$d_nx(t_1, \dots, t_n)=x(-t_1, \dots, -t_n)$$

for every $x \in D_n X$ and $(t_1, \ldots, t_n) \in S^n$. (Compare [8, Appendix A].) Explicitly we fix a homotopy $k_t = (f_t, g_t): S^2 \to S^2$ such that $k_0(u, v) = (u, v)$, $k_1(u, v) = (v, -u)$ and use the following homotopies $h_n: d_{n+1}\delta_n \simeq \delta'_n(d_n \wedge 1)$ to define d';

$$(h_n)_t(x, s)(t_1, \ldots, t_n, t_{n+1}) = d_{n+1}\delta_n(x, f_t(s, t_{n+1}))(t_1, \ldots, t_n, g_t(s, t_{n+1}))$$

Definition 3.1. A CW-bispectrum X is called an S_*-bispectrum if every X_{m*} and X_{*p} have a structure of an S_*-spectrum, and if $X_{m*} \wedge S^1 \to X_{m+1*}$ and $X_{*p} \wedge S^1 \to X_{*p+1}$ are functions of S_*-spectra. (Thus each $X_{n,q}$ has an $S_n \times S_q$-action such that

$$(\sigma+\sigma', \tau+\tau')[[x, s], t]' = [[(\sigma, \tau)x, \sigma's], \tau't]'$$

for every $\sigma \in S_n$, $\tau \in S_q$, $\sigma' \in S_k$, $\tau' \in S_r$ and $(x, s, t) \in X_{n,q} \wedge S^k \wedge S^r$.) Given S_*-bispectra X, Y and Z, a pairing $\mu: (X, Y) \to Z$ of S_*-bispectra is a family of maps

$$X_{m,q} \wedge Y_{n,q} \to Z_{m+n, p+q}$$

which restricts to pairings of S_*-spectra

$$(X_{m*}, Y_{n*}) \to Z_{m+n*} \text{ and } (X_{*p}, Y_{*q}) \to Z_{*p+q}.$$

For example, the CW-bispetrum $\Delta S = \{S^n \wedge S^q \mid n \geqq 0, q \geqq 0\}$ equipped with the structure maps

$$[(x,y), s] = ((x, s), y), \quad [(x,y), s]' = (x, (y, s)) \quad \text{for } ((x,y), s) \in \Delta S_{n,q} \wedge S^1$$

canonically has a structure of an S_*-bispectrum, and for every S_*-bispectrum X we have a natural pairing $\tilde{\varepsilon}: (X, \Delta S) \to X$ of S_*-bispectra;

$$\begin{array}{ccc} X_{m,p} \wedge (S^n \wedge S^q) & \longrightarrow & X_{m+n, p+q}. \\ \cup\!\!\!\cup & & \cup\!\!\!\cup \\ (x, (s, t)) & \longmapsto & [[x, s], t]' \end{array}$$

It is easy to see that if X is an S_*-bispectrum then both DX and $D'X$ are S_*-spectra with the S_n-action

$$(\sigma, S^n \xrightarrow{x} X_{n,n}) \longmapsto (S^n \xrightarrow{\sigma^{-1}} S^n \xrightarrow{x} X_{n,n} \xrightarrow{(\sigma,\sigma)} X_{n,n})$$

on each $D_n X = \Omega^n X_{n,n} = D'_n X$, and also that $e: X_{0*} \to DX$ and $e': X_{*0} \to D'X$ are functions of S_*-spectra. (However $c: DX \to D'X$ is not a function of S_*-spectra.) Now let $\mu: (X, Y) \to Z$ be a pairing of S_*-bispectra. Then the diagram

$$\begin{array}{ccc} \Omega^m X_{m,m}\wedge\Omega^n Y_{n,n}\wedge S^k\wedge S^l & \xrightarrow{\Omega^{m+n}\mu\wedge 1} & \Omega^{m+n}Z_{m+n,m+n}\wedge S^{k+l} \\ \downarrow{\scriptstyle 1\wedge T\wedge 1} & & \downarrow{\scriptstyle \delta} \\ \Omega^m X_{m,m}\wedge S^k\wedge\Omega^n Y_{n,n}\wedge S^l & & \Omega^{m+n+k+l}Z_{m+n+k+l,m+n+k+l} \\ \downarrow{\scriptstyle \delta\wedge\delta} & & \downarrow{\scriptstyle 1+\tau+1} \\ \Omega^{m+k}X_{m+k,m+k}\wedge\Omega^{n+l}Y_{n+l,n+l} & \xrightarrow{\Omega^{m+k+n+l}\mu} & \Omega^{m+k+n+l}Z_{m+k+n+l,m+k+n+l} \end{array}$$

commutes. Hence the maps $\Omega^{m+n}\mu:\Omega^m X_{m,m}\wedge\Omega^n Y_{n,n}\to\Omega^{m+n}Z_{m+n,m+n}$ define a pairing $D\mu:(DX, DY)\to DZ$ of S_*-spectra, and similarly $D'\mu$: $(D'X, D'Y)\to D'Z$. The following proposition is a multiplicative version of the up and across theorem (cf. [2] and [8]).

Proposition 3.2. *Let $\mu:(X, Y)\to Z$ be a pairing of S_*-bispectra. Then the following diagram commutes in the stable category;*

$$(*)\quad \begin{array}{ccccccc} X_{0*}\wedge Y_{0*} & \xrightarrow{e\wedge e} & DX\wedge DY & \xrightarrow{c\wedge c} & D'X\wedge D'Y & \xleftarrow{e'\wedge e'} & X_{*0}\wedge Y_{*0}. \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle D\mu} & & \downarrow{\scriptstyle D'\mu} & & \downarrow{\scriptstyle \mu} \\ Z_{0*} & \xrightarrow{e} & DZ & \xrightarrow{c} & D'Z & \xleftarrow{e'} & Z_{*0}. \end{array}$$

Proof. For every m and n, we have a commutative diagram

$$\begin{array}{ccccc} X_{0,m}\wedge Y_{0,n} & \xrightarrow{\mu} & Z_{0,m+n} & \xleftarrow{\varepsilon} & Z_{0,m}\wedge S^n \\ \downarrow{\scriptstyle e_m\wedge e_n} & & \downarrow{\scriptstyle e_{m+n}} & & \downarrow{\scriptstyle e_m\wedge 1} \\ D_m X\wedge D_n Y & \xrightarrow{D\mu} & \mathrm{D}_{m+n}Z & \xleftarrow{\varepsilon} & \mathrm{D}_m Z\wedge S^n. \end{array}$$

Noting that e is a map of S_*-spectra (and hence induces We: $WZ_{0*}\to WDZ$), we can easily show that the left-hand square in $(*)$ commutes.

Quite similarly we can prove the commutativity of the right-hand square in $(*)$.

Finally, to prove the commutativity of the middle square, let us take a partition of $N=B\cup C$, and denote $n'=\beta(n)$, $n''=\gamma(n)$ for every $n\in N$. Then there is a canonical inclusion $T(DX\wedge_{BC}DY)\to TDX\wedge_{BC} TDY$ which sends $[k]_+\wedge(D_{k'}X\wedge D_{k''}Y)\wedge S^{n-k}\subset T_n(DX\wedge_{BC}DY)$ to $([k']_+\wedge D_{k'}X\wedge S^{n'-k'})\wedge([k'']_+\wedge D_{k''}Y\wedge S^{n''-k''})\subset T_{n'}DX\wedge T_{n''}DY$, and we have a commutative diagram

$$\begin{array}{ccccc}
DX\wedge_{BC}DY & \xrightarrow{\rho\wedge\rho} & TDX\wedge_{BC}TDY & \xrightarrow{d'\wedge d'} & D'X\wedge_{BC}D'Y \\
 & \nwarrow & \cup & \nearrow & \\
\Big\downarrow{\scriptstyle D\mu} & & T(DX\wedge_{BC}DY) & & \Big\downarrow{\scriptstyle D'\mu} \\
 & & \downarrow & & \\
W_{BC}DZ & \xleftarrow{\rho} & TW_{BC}DZ & \xrightarrow{W_{BC}d'} & W_{BC}D'Z
\end{array}$$

in which $W_{BC}d'$ is defined by using the natural homotopies

$$d_{n+1}(W_{BC}\delta)_n\simeq(W_{BC}\delta')_n(d_n\wedge 1)\,;$$

$$\tilde{h}_n=\begin{cases} h_n & \text{if } n\in C \\ \sigma_{n',n''}h_n(1\wedge(-1)^{n''}) & \text{if } n\in B\,. \end{cases}$$

From this and the similar diagram with μ replaced by $\tilde{\varepsilon}:(Z,\Delta S)\to Z$, we see that

$$\begin{array}{ccccc}
DX\wedge DY & \xrightarrow{D\mu} & WDZ & \xrightarrow{D\tilde{\varepsilon}^{-1}} & DZ\wedge D\Delta S \\
\Big\downarrow{\scriptstyle c\wedge c} & & & & \Big\downarrow{\scriptstyle c\wedge c} \\
D'X\wedge D'Y & \xrightarrow{D'\mu} & WD'Z & \xrightarrow{D'\tilde{\varepsilon}^{-1}} & D'Z\wedge D'\Delta S
\end{array}$$

commutes in the stable category. Because the composite $TDZ\to DZ\wedge D\Delta S\to WDZ$ (resp. $TD'Z\to D'Z\wedge D'\Delta S\to WD'Z$) coincides with $TDZ\to DZ\wedge S^0\to WDZ$ (resp. $TD'Z\to D'Z\wedge S^0\to WD'Z$), we conclude that the middle square in (*) commutes.

§ 4. Proofs of the Theorems

4.1. Proof of Theorem A. Let U be a permutative category. Then we have a bispectrum $XU=\{X_{n,q}U\,|\,n\geqq 0,\ q\geqq 0\}$ defined as follows:

$X_{n,q}U$ is the geometric realization of the Γ^q-space

$$E_n\hat{B}^qU:(\mathfrak{r}_1,\ldots,\mathfrak{r}_q)\longmapsto E_n\hat{B}^qU(\mathfrak{r}_1,\ldots,\mathfrak{r}_q)\,;$$

the structure maps $X_{n,q}U\wedge S^1\to X_{n+1,q}U$ and $X_{n,q}U\wedge S^1\to X_{n,q+1}U$ are given by the evident maps

$$|E_n\hat{B}^qU|\wedge S^1=|E_n\hat{B}^qU\wedge S^1|\to|E_{n+1}\hat{B}^qU|,$$
$$|E_n\hat{B}^qU|\wedge S^1=|E_n\hat{B}^{q+1}U(\ldots,1)|\wedge S^1\to|E_n\hat{B}^{q+1}U|$$

respectively.

By the definition we have $X_{*0}U=EU$ and there is a natural map $CU\to X_{0*}U$ given by

$$C_nU=|B\hat{B}^nU|\xrightarrow{|\lambda|}|E_0\hat{B}^nU|=X_{0,n}U.$$

Because $|E_n\hat{B}^qU|$ is connected when $n\geqq 1$, the canonical map $X_{n,q}U\to\Omega X_{n,q+1}U$ is a homotopy equivalence for all $n\geqq 1$. Hence $e':EU\to D'XU$ becomes an equivalence. Moreover, because $B(\ )\to\Omega^\infty E_\infty(\ )$ and $B(\ )\to\Omega^\infty C_\infty(\ )$ are group completions, the composite $k:CU\to X_{0*}U\xrightarrow{e}DXU$ also becomes an equivalence. Thus we have an equivalence

$$\gamma=k^{-1}c^{-1}e':EU\to CU$$

natural in U.

Now suppose that E is a multiplicative higher algebraic K-theory. Then XU becomes an S_*-bispectrum with an $S_n\times S_q$-action on $X_{n,q}U$ induced from the S_n-action on E_n and the S_q-action on $\hat{B}^q$.

Let $f:U\times V\to W$ be a pairing of permutative categories. Then we have a natural pairing $Xf:(XU,XV)\to XW$ of S_*-bispectra consisting of the maps $X_{m,p}U\wedge X_{n,q}V\to X_{m+n,p+q}W$ induced by the map of Γ^{p+q}-spaces

$$\begin{aligned}E_{m,n}f_{p,q}:E_m\hat{B}^pU(\mathfrak{r}_1,\dots,\mathfrak{r}_p)\wedge E_n\hat{B}^qV(\mathfrak{r}_{p+1},\dots,\mathfrak{r}_{p+q})\\ \longrightarrow E_{m+n}\hat{B}^{p+q}W(\mathfrak{r}_1,\dots,\mathfrak{r}_p,\mathfrak{r}_{p+1},\dots,\mathfrak{r}_{p+q}).\end{aligned}$$

By Proposition 3.2 and by the multiplicativity of $\lambda:B(\ )\to E_0(\ )$, we see that the following diagram commutes in $H\mathcal{S}$;

$$\begin{array}{ccc} EU\wedge EV & \xrightarrow{Ef} & EW \\ \downarrow & & \downarrow \\ X_{0*}U\wedge X_0V & \longrightarrow & X_{0*}W \\ \uparrow & & \uparrow \\ CU\wedge CV & \xrightarrow{Cf} & CW. \end{array}$$

Q.E.D.

4.2. Proof of Theorem B. Let U be an exact category. We first define an equivalence $\eta:\Omega CQU\to KU$. By [10, Lemma 4.2] every $\hat{B}^qU(\mathfrak{r}_1,\dots,\mathfrak{r}_q)$ has a natural structure of an exact category. Hence we can define an S_*-bispectrum XU by

$$X_{n,q}U=|K_n\hat{B}^qU|=|(\mathfrak{r}_1,\dots,\mathfrak{r}_q)\longmapsto BQ^n(\hat{B}^qU(\mathfrak{r}_1,\dots,\mathfrak{r}_q))^{[n]}|.$$

Clearly $X_{*0}U=KU$, and by [10, Corollary 4.5] we have

$$\begin{aligned}X_{1*}U&=\{|BQ\hat{B}^nU|\,|n\geqq 0\}\\&=\{|B\hat{B}^nQU|\,|n\geqq 0\}=CQU.\end{aligned}$$

Now let us endow $\mathbb{X}'U=\{\Omega\mathbb{X}_{1+n,q}U\,|\,n\geqq 0,\ q\geqq 0\}$ with the structure of an S_*-bispectrum such that the maps

$$\mathbb{X}_{n,q}U\longrightarrow\Omega\mathbb{X}_{1+n,q}U,\quad x\mapsto(s\mapsto\sigma_{0,n}[x,s])$$

give rise to a function $\phi:\mathbb{X}U\to\mathbb{X}'U$ of S_*-bispectra. Then $\mathbb{X}'_{0*}U=\Omega\mathbb{X}_{1*}U=\Omega\mathbb{C}QU$ and it is easy to see that the maps $e:\mathbb{X}'_{0*}U\to D\mathbb{X}'U$, $e':\mathbb{X}'_{*0}U\to D'\mathbb{X}'U$ and $\phi:\mathbb{X}_{*0}U\to\mathbb{X}'_{*0}U$ are homotopy equivalences. Thus we have a natural equivalence $\eta:\Omega\mathbb{C}QU\to\mathbb{K}U$ defined as the composite

$$\mathbb{X}'_{0*}U\xrightarrow{e}D\mathbb{X}'U\xrightarrow{c}D'\mathbb{X}'U\xrightarrow{e'^{-1}}\mathbb{X}'_{*0}U\xrightarrow{\phi^{-1}}\mathbb{X}_{*0}U.$$

Next consider the natural sequence of spectra

$$\mathbb{C}\mathrm{Is}U\longrightarrow\mathbb{C}LU\longrightarrow\mathbb{C}QU$$

associated with the sequence of symmetric monoidal categories $\mathrm{Is}U\to LU\to QU$ (cf. [14, §9]). As in the proof of [11, Theorem 3. 1], there are adjunctions

$$\hat{B}^nLU(\mathfrak{r}_1,\dots,\mathfrak{r}_n)\underset{h}{\overset{t}{\rightleftarrows}}\hat{B}^nJU(\mathfrak{r}_1,\dots,\mathfrak{r}_n)\rightleftarrows 0$$

natural in both U and $(\mathfrak{r}_1,\dots,\mathfrak{r}_n)$. Therefore we have a null homotopy on every $\mathbb{C}_nLU=|B\hat{B}^nLU|$ which is compatible with the structure maps of $\mathbb{C}LU$, and is natural in U. Since the composite $\mathbb{C}_n\mathrm{Is}U\to\mathbb{C}_nQU$ is the constant map, we have a natural map $\nu:\mathbb{C}\mathrm{Is}U\to\Omega\mathbb{C}QU$, and hence the composite

$$\kappa=\eta\nu:\mathbb{C}\mathrm{Is}U\longrightarrow\mathbb{K}U.$$

We now prove that κ is multiplicative. Let $f:U\times V\to W$ be a biexact functor. Then, as in the proof of Theorem A, there is a natural pairing $\mathbb{X}f:(\mathbb{X}U,\mathbb{X}V)\to\mathbb{X}W$ of S_*-bispectra such that $\mathbb{X}_{m,p}U\wedge\mathbb{X}_{n,q}V\to\mathbb{X}_{m+n,p+q}W$ is induced from the $(m+n)$-fold functors

$$\begin{aligned}Q^m(\hat{B}^pU(\mathfrak{r}_1,\dots,\mathfrak{r}_p))^{[m]}\amalg Q^n(\hat{B}^qV\{\mathfrak{r}_{p+1},\dots,\mathfrak{r}_{p+q}))^{[n]}\\ \longrightarrow Q^{m+n}(\hat{B}^{p+q}W(\mathfrak{r}_1,\dots,\mathfrak{r}_{p+q}))^{[m+n]}\end{aligned}$$

associated with the biexact functor $f_{p,q}:\hat{B}^pU(\mathfrak{r}_1,\dots,\mathfrak{r}_p)\times\hat{B}^qV(\mathfrak{r}_{p+1},\dots,\mathfrak{r}_{p+q})\to\hat{B}^{p+q}W(\mathfrak{r}_1,\dots,\mathfrak{r}_{p+q})$ (cf. [11, §4]). It is easy to see that the composite maps

$$\Omega\mathbb{X}_{1+m,p}U\wedge\Omega\mathbb{X}_{1+n,q}V\xrightarrow{\Omega^2\mathbb{X}f}\Omega^2\mathbb{X}_{1+m+1+n,p+q}W\longrightarrow\Omega^2\mathbb{X}_{2+m+n,p+q}W$$

define a pairing of S_*-bispectra $\mathbb{X}'f:(\mathbb{X}'U,\mathbb{X}'V)\to\mathbb{X}''W$ where

$X''W=\{\Omega^2X_{2+n,q}W\,|\,n\geqq 0,\ q\geqq 0\}$ is equipped with the structure of an S_*-bispectrum evidently defined. Thus we have a commutative diagram

$$\begin{array}{ccc} \Omega CQU\wedge\Omega CQV & \longrightarrow & \Omega^2CQ^2W^{[2]} \\ {\scriptstyle \eta\wedge\eta}\downarrow & & \downarrow{\scriptstyle \bar{\eta}} \\ KU\wedge KV & \xrightarrow{Kf} & KW \end{array}$$

where $\Omega^2CQ^2W^{[2]}$ denotes $X''_{0*}W=\Omega^2X_{2*}W$, and $\bar{\eta}$ the composite

$$X''_{0*}W\xrightarrow{\simeq}X''_{*0}W\xrightarrow{\simeq}X_{*0}W.$$

We now define a natural map $\bar{\nu}:C\mathrm{Is}W\to\Omega^2CQ^2W^{[2]}$ such that the following diagram commutes in the stable category (cf. [14, 9.2]);

$$(4.1)\qquad \begin{array}{ccc} C\mathrm{Is}U\wedge C\mathrm{Is}V & \longrightarrow & C\mathrm{Is}W \\ {\scriptstyle \nu\wedge\nu}\downarrow & & \downarrow{\scriptstyle \bar{\nu}} \\ \Omega CQU\wedge\Omega CQV & \longrightarrow & \Omega^2CQ^2W^{[2]} \end{array}$$

As stated in [11, §3], any biexact functor $g:C\times D\to E$ defines a 2-fold functor $C\Pi D\to E^{[2]}$ which induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{Is}C\Pi\mathrm{Is}D & \longrightarrow & LC\Pi\mathrm{Is}D & \longrightarrow & QC\Pi\mathrm{Is}D \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Is}^2E^{[2]} & \longrightarrow & L\mathrm{Is}E^{[2]} & \longrightarrow & Q\mathrm{Is}E^{[2]} \end{array}$$

natural in g. (We denote $\mathrm{Is}^2=\mathrm{Is}_1\mathrm{Is}_2$, $L\mathrm{Is}=L_1\mathrm{Is}_2$ and $Q\mathrm{Is}=Q_1\mathrm{Is}_2$. Cf. [11].) Applying this construction to biexact functors

$$f_{p,q}:\hat{B}^pU(\mathfrak{r}_1,\dots,\mathfrak{r}_p)\times\hat{B}^qV(\mathfrak{r}_{p+1},\dots,\mathfrak{r}_{p+q})\to\hat{B}^{p+q}W(\mathfrak{r}_1,\dots,\mathfrak{r}_{p+q}),$$

and then realizing the associated Γ^{p+q}-spaces, we have a commutative diagram

$$\begin{array}{ccccc} C_p\mathrm{Is}U\wedge C_q\mathrm{Is}V & \longrightarrow & C_pLU\wedge C_q\mathrm{Is}V & \longrightarrow & C_pQU\wedge C_q\mathrm{Is}V \\ {\scriptstyle \rho_{p,q}}\downarrow & & {\scriptstyle \rho''_{p,q}}\downarrow & & \downarrow{\scriptstyle \rho'_{p,q}} \\ C_{p+q}\mathrm{Is}^2W^{[2]} & \longrightarrow & C_{p+q}L\mathrm{Is}W^{[2]} & \longrightarrow & C_{p+q}Q\mathrm{Is}W^{[2]} \\ \| & & \| & & \| \\ |B\mathrm{Is}^2(\hat{B}^{p+q}W)^{[2]}| & & |BL\mathrm{Is}(\hat{B}^{p+q}W)^{[2]}| & & |BQ\mathrm{Is}(\hat{B}^{p+q}W)^{[2]}| \end{array}$$

for all $p,\ q\geqq 0$. Since $\rho''_{p,q}$ is compatible with the null homotopies, the following diagram commutes;

$$\begin{array}{ccc} C_p\mathrm{Is}U\wedge C_q\mathrm{Is}V & \longrightarrow & \Omega C_pQU\wedge_q\mathrm{Is}V \\ \rho_{p,q}\downarrow & & \downarrow \Omega\rho'_{p,q} \\ C_{p+q}\mathrm{Is}^2W^{[2]} & \longrightarrow & \Omega C_{p+q}Q\mathrm{Is}\,W^{[2]}. \end{array}$$

By the definition, $\{\rho_{p,q}\}$ and $\{\rho'_{p,q}\}$ are pairings of S_*-spectra, and hence we have a diagram

(4.2)
$$\begin{array}{ccc} C\mathrm{Is}U\wedge C\mathrm{Is}V & \longrightarrow & C\mathrm{Is}^2W^{[2]} \\ \nu\wedge 1\downarrow & & \downarrow\nu' \\ \Omega CQU\wedge C\mathrm{Is}V & \longrightarrow & \Omega CQ\mathrm{Is}\,W^{[2]} \end{array}$$

which commutes in the stable category.

Similarly we have a commutative diagram

(4.3)
$$\begin{array}{ccc} CQU\wedge C\mathrm{Is}V & \longrightarrow & CQ\mathrm{Is}\,W^{[2]} \\ 1\wedge\nu\downarrow & & \downarrow\nu'' \\ CQU\wedge\Omega CQV & \longrightarrow & \Omega CQ^2W^{[2]} \end{array}$$

associated with the natural sequence

$$\begin{array}{ccccc} QU\amalg\mathrm{Is}V & \longrightarrow & QU\amalg LV & \longrightarrow & QU\amalg QV \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathrm{Is}\,W^{[2]} & \longrightarrow & QLW^{[2]} & \longrightarrow & Q^2W^{[2]}. \end{array}$$

By [14, 9.2.3] we see that the following diagram commutes up to natural homotopy

(4.4)
$$\begin{array}{ccc} C\mathrm{Is}U\wedge C\mathrm{Is}V & \longrightarrow & C\mathrm{Is}W \\ \| & & \downarrow u \\ C\mathrm{Is}U\wedge C\mathrm{Is}V & \longrightarrow & C\mathrm{Is}^2W^{[2]} \end{array}$$

where $u_n: C_n\mathrm{Is}\,W\xrightarrow{\simeq}C_n\mathrm{Is}^2W^{[2]}$ denotes the canonical inclusion

$$|B\mathrm{Is}\hat{B}^nW| = |B\mathrm{Is}_{1O_2}(\hat{B}^nW)^{[2]}|\rightarrow|B\mathrm{Is}^2(\hat{B}^nW)^{[2]}|.$$

From (4.2), (4.3) and (4.4), we see that (4.1) commutes if we put $\tilde{\nu}=(\Omega\nu'')\nu'u$. Moreover, from the commutative diagram

$$\begin{array}{ccccccccc} C\mathrm{Is}W & \xrightarrow{\nu} & \Omega CQW & = & X'_{0*}W & \xrightarrow{\simeq} & X'_{*0}W & & \\ u\downarrow & & \downarrow u' & & \downarrow\simeq & & \downarrow\simeq & \overset{\psi}{>} & X_{*0}W \\ C\mathrm{Is}^2W^{[2]} & \xrightarrow{\nu'} & \Omega CQ\mathrm{Is}\,W^{[2]} & \xrightarrow{\Omega\nu''} & X''_{0*}W & \xrightarrow{\simeq} & X''_{*0}W & & \end{array}$$

in which u' consists of the canonical inclusions $u'_n\colon |BQo(\hat{B}^nW)^{[2]}|\to |BQ\mathrm{Is}(\hat{B}^nW)^{[2]}|$, we see that the composite $\tilde{\eta}\tilde{\nu}\colon C\mathrm{Is}W\to\Omega^2CQ^2W^{[2]}\to KW$ coincides with $\kappa=\eta\nu$. Therefore the diagram

$$\begin{array}{ccc} C\mathrm{Is}U\wedge C\mathrm{Is}V & \xrightarrow{Cf} & C\mathrm{Is}W \\ \downarrow{\scriptstyle \kappa\wedge\kappa} & & \downarrow{\scriptstyle \kappa} \\ KU\wedge KV & \xrightarrow{Kf} & KW \end{array}$$

commutes in the stable category. Q. E. D.

4.3. Proof of Theorem C. Given a ring R, we define AR as follows. For each $n\geqq 0$,

$$A_nR=\Omega^nC_n\mathrm{Is}P(S^nR)$$

where $S^nR=R\otimes(\overset{n}{\otimes}SZ)$ (cf. [5]); and the structure map $A_nR\wedge S^1\to A_{n+1}R$ is defined as the composite

$$A_nR\wedge S^1\xrightarrow{1\wedge\iota}\Omega^nC_n\mathrm{Is}P(S^nR)\wedge\Omega C_1\mathrm{Is}P(SZ)$$
$$\xrightarrow{\Omega^{n+1}C_{n,1}f}\Omega^{n+1}C_{n+1}\mathrm{Is}P(S^{n+1}R)=A_{n+1}R$$

where f denotes the evident pairing

$$\mathrm{Is}P(S^nR)\times\mathrm{Is}P(SZ)\longrightarrow\mathrm{Is}P(S^nR\otimes SZ)=\mathrm{Is}P(S^{n+1}R),$$

and

$$\iota\colon S^1\longrightarrow B\mathrm{Is}P(SZ)\subset\Omega C_1\mathrm{Is}P(SZ)$$

the cellular inclusion corresponding to the '1-cell'

$$\begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}\in GL_1SZ.$$

Note that ι represents a generator of $K_1SZ=K_0Z=Z$ (cf. [5]).

Using the standard S_n-action on $\overset{n}{\otimes}SZ$, we define an S_n-action on A_nR by

$$(\tau,S^n\xrightarrow{x}C_n\mathrm{Is}P(S^nR))\mapsto(S^n\xrightarrow{\tau^{-1}}S^n\xrightarrow{x}C_n\mathrm{Is}P(R\otimes(\overset{n}{\otimes}SZ))$$
$$\xrightarrow{\tau C_n\mathrm{Is}P(1\otimes\tau)}C_n\mathrm{Is}P(R\otimes(\overset{n}{\otimes}SZ))).$$

Then it is easy to see that AR becomes an S_*-spectrum with respect to this action.

Given rings R and R' we have a natural pairing

$$\mathrm{Is}P(S^m R)\times \mathrm{Is}P(S^n R')\to \mathrm{Is}P(S^m R\otimes S^n R')\cong \mathrm{Is}P(S^{m+n}(R\otimes R'))$$

of symmetric monoidal categories, and this in turn induces a map

$$\begin{aligned}A_m R\wedge A_n R' &= \Omega^m C_m \mathrm{Is}P(S^m R)\wedge \Omega^n C_n \mathrm{Is}P(S^n R')\\ &\longrightarrow \Omega^{m+n} C_{m+n}\mathrm{Is}P(S^{m+n}(R\otimes R'))=A_{m+n}(R\otimes R').\end{aligned}$$

Thus we have a natural pairing

$$\mu\colon (AR, AR')\longrightarrow A(R\otimes R')$$

of S_*-spectra.

If $n\geqq 1$, then the canonical inclusion $f_n\colon B\mathrm{Is}P(S^n R)\to \Omega^n C_n \mathrm{Is}P(S^n R)$ is a group completion, and hence there is a homotopy equivalence

$$A_n R\simeq K_0 S^n R\times BGLS^n R^+ = \mathbb{G}\mathbb{W}_n R.$$

By the definition we see that the square

$$\begin{array}{ccc} B\mathrm{Is}P(S^m R)\wedge B\mathrm{Is}P(S^n R') & \longrightarrow & B\mathrm{Is}P(S^{m+n}(R\otimes R')) \\ \Big\downarrow{\scriptstyle f_m\wedge f_n} & & \Big\downarrow{\scriptstyle f_{m+n}} \\ A_m R\wedge A_n R' & \xrightarrow{\mu_{m,n}} & A_{m+n}(R\otimes R') \end{array}$$

commutes, and that the structure map $A_n R\wedge S^1\to A_{n+1}R$ coincides with the composite $\mu_{n,1}(1\wedge\iota)$. Hence the conditions (2) and (3) hold.

We now define a CW-bispectrum $\mathbb{X}R$ as follows.

$$\mathbb{X}_{n,q}R=\Omega^q C_{n+q}\mathrm{Is}P(S^q R)\,;$$

and the structure maps are given by

$$\begin{aligned}\mathbb{X}_{n,q}R\wedge S^1 &= \Omega^q C_{n+q}\mathrm{Is}P(S^q R)\wedge S^1\\ &\longrightarrow \Omega^q C_{n+q+1}\mathrm{Is}P(S^q R)\\ &\xrightarrow{\Omega^q\sigma_{n,q}} \Omega^q C_{n+1+q}\mathrm{Is}P(S^q R)=\mathbb{X}_{n+1,q}R\end{aligned}$$

and

$$\begin{aligned}\mathbb{X}_{n,q}R\wedge S^1 &\xrightarrow{1\wedge\iota} \Omega^q C_{n+q}\mathrm{Is}P(S^q R)\wedge \Omega C_1 \mathrm{Is}(S\mathbb{Z})\\ &\longrightarrow \Omega^{q+1}C_{n+q+1}\mathrm{Is}P(S^{q+1}R)=\mathbb{X}_{n,q+1}R.\end{aligned}$$

It is easy to see that $\mathbb{X}R$ becomes an S_*-bispectrum if each $\mathbb{X}_{n,q}R$ is endowed with the $S_n\times S_q$-action

$$\begin{aligned}((\sigma,\tau), S^q\xrightarrow{x} C_{n+q}\mathrm{Is}P(S^q R)) &\longmapsto (S^q\xrightarrow{\tau^{-1}} S^q\xrightarrow{x} C_{n+q}\mathrm{Is}P(S^q R)\\ &\xrightarrow{(\sigma+\tau)C_{n+q}\mathrm{Is}P(1\otimes\tau)} C_{n+q}\mathrm{Is}P(S^q R)).\end{aligned}$$

Since $e: X_{0*} = AR \to DXR$ is a homotopy equivalence, we have a natural map

$$\alpha: C\mathrm{Is}P(R) = X_{*0}R \longrightarrow AR$$

(natural in $H\mathscr{S}$).

Finally the maps $X_{m,p}R \wedge X_{n,q}R' \to X_{m+n,p+q}(R \otimes R')$ defined as the composite

$$\begin{aligned} &\Omega^p C_{m+p}\mathrm{Is}P(S^p R) \wedge \Omega^q C_{n+q}\mathrm{Is}P(S^q R') \\ &\quad \longrightarrow \Omega^{p+q} C_{m+p+n+q}\, \mathrm{Is}P(S^{p+q}(R \otimes R')) \\ &\quad \xrightarrow{\Omega^{p+q}(1+\tau+1)} \Omega^{p+q} C_{m+n+p+q}\, \mathrm{Is}P(S^{p+q}(R \otimes R')) \end{aligned}$$

determine a natural pairing $(XR, XR') \to X(R \otimes R')$ of S_*-bispectra. Hence α becomes a multiplicative natural transformation.

Q. E. D.

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