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Diagonal Short Time Asymptotics of Heat Kernels for Certain Degenerate Second Order Differential Operators of Hörmander Type

By

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Introduction

Let $V_0, V_1, \cdots, V_n \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$. Let $\hat{V}_i \in \mathfrak{X}(\mathbb{R}^d)$ be defined by $\hat{V}_i = \sum_{j=1}^d V_i^j \frac{\partial}{\partial x^j}$, $i \in \{0, 1, \cdots, n\}$, and let L denote a second order differential operator written in Hörmander's form

$$L = \frac{1}{2} \sum_{i=1}^n \hat{V}_i^2 + \hat{V}_0 .$$

Assume that at every $x \in \mathbb{R}^d$, $\hat{V}_0, \hat{V}_1, \cdots, \hat{V}_n$ satisfy the Hörmander condition: For some $\nu \geq 1$

$$(1) \quad \begin{cases} \text{linear span } \{[\hat{V}_{i_a}, [\hat{V}_{i_{a-1}}, [\cdots, [\hat{V}_{i_2}, \hat{V}_{i_1}]\cdots](x);\ 1 \leq a \leq \nu, \\ \qquad i_1 \in \{1, \cdots, n\},\ i_2, \cdots, i_a \in \{0, 1, \cdots, n\}\} = T_x(\mathbb{R}^d) . \end{cases}$$

Then it is well-known that the heat equation $\frac{\partial}{\partial t} = L$ has the smooth heat kernel (=fundamental solution) $p(t, x, y)$. We are concerned with the diagonal short time asymptotics of it. In the case when $V_0 \equiv 0$, under the assumption (1), they were obtained by Léandre [11] and Ben Arous [1]:

$$p(t, x, x) \sim t^{-N(x)/2} \sum_{a=0}^\infty b_a t^a \qquad \text{as} \quad t \downarrow 0 .$$

Here $N(x)$ is a positive integer defined in terms of $[\hat{V}_{i_a}, [\hat{V}_{i_{a-1}}, [\cdots, [\hat{V}_{i_2}, \hat{V}_{i_1}]\cdots](x)$, $i_1, i_2, \cdots, i_a \in \{1, \cdots, n\}$, $1 \leq a \leq \nu$ (more precisely, it is defined by (5.14)).

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Previous to them, Bismut [2] discussed the not only diagonal but off diagonal short time asymptotics of $p(t, x, y)$ under the suitable conditions as an application of Wiener functional analysis; the splitting of the Wiener space and the use of an implicit function theorem. This approach by Bismut has been refined and expanded by Kusuoka [8] who introduced the notion of *generalized Malliavin calculus*. On the other hand, S. Watanabe [17], to solve this problem, introduced the notion of *asymptotic expansions of generalized Wiener functionals*. It should be noted that Léandre [11] further discussed the off diagonal short time asymptotics.

In this paper, following the way of S. Watanabe [17], we shall discuss the diagonal short time asymptotics in the general case (i.e. $V_0 \not\equiv 0$). This outline is as follows: Let (W_0^n, P) be the n-dimensional Wiener space. For the operator L, we consider the following stochastic differential equation (SDE) of Stratonovich type on $\mathbb{R}^d$:

$$\begin{cases} dX_t = \varepsilon \sum_{i=1}^{n} V_i(X_t) \circ dw_t^i + \varepsilon^2 V_0(X_t) dt \\ X_0 = x \in \mathbb{R}^d, \end{cases}$$

where $\varepsilon > 0$ and $w = (w_t^i) \in W_0^n$. Then the unique solution $X^\varepsilon(t, x)$ of this SDE is *smooth* in the Malliavin sense, and further, by virtue of the assumption (1), $X^\varepsilon(1, x)$ is *non-degenerate* in the Malliavin sense (cf. [4], [9], [16], [17]). Hence, for the Dirac delta-function δ_x $(\in \mathcal{S}'(\mathbb{R}^d))$, $\delta_x(X^\varepsilon(1, x))$ is defined as a *generalized Wiener functional* and the probabilistic expression of $p(\varepsilon^2, x, x)$ is given:

$$(2) \qquad p(\varepsilon^2, x, x) = E[\delta_x(X^\varepsilon(1, x))]$$

(cf. [4], [16], [17]). First, for the integrand $\delta_x(X^\varepsilon(1, x))$ in (2), we shall show the following asymptotic expansion (cf. [17]):

$$(3)_a \qquad \delta_x(X^\varepsilon(1, x)) \sim \varepsilon^{-N(x)} \sum_{a=0}^{\infty} \varepsilon^a \Theta_a \qquad \text{as} \quad \varepsilon \downarrow 0$$

provided that $\hat{V}_0(x)$ belongs to the linear subspace of $T_x(\mathbb{R}^d)$ spanned by $\hat{V}_i(x)$, $[\hat{V}_j, \hat{V}_k](x)$, $i, j, k \in \{1, \cdots, n\}$ (cf. Theorem (5.31));

$$(3)_b \qquad \delta_x(X^\varepsilon(1, x)) = O(\varepsilon^m) \qquad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for any} \quad m \geq 1$$

provided that $\hat{V}_0(x)$ does not belong to that linear subspace (cf. Theorem (5.34)). Second, from (2), $(3)_a$, $(3)_b$ and some observations, we shall show the short time asymptotic of $p(t, x, x)$ (cf. Theorem (6.8)):

$$(4)\quad \begin{cases} p(t, x, x) \sim t^{-N(x)/2} \sum_{a=0}^{\infty} t^a E[\Theta_{2a}] & \text{as } t \downarrow 0, \text{ in the case } (3)_a, \\ \qquad = O(t^m) & \text{as } t \downarrow 0 \text{ for any } m \geq 1, \text{ in the case } (3)_b. \end{cases}$$

Our argument seems to be simpler than Léandre's and Ben Arous', though it is based on the same idea as them.

As to another study of $p(t, x, y)$, there is the global estimate of it. This problem closely related to the above problem is investigated in many papers [3], [5], [10], [12], [13] etc. Among these, particularly, Kusuoka-Stroock [10] has obtained nice results by using the Malliavin calculus.

The organization of this paper is as follows: In § 1, § 2 and § 3, we shall give some preliminaries for § 4, § 5 and § 6. In particular, "a key" proposition in this paper will be presented in (3.9). In § 4, with the aid of this proposition, we shall prove Proposition (4.4) which gives another look at Taylor's expansion of $X^\varepsilon(1, x)$ with respect to ε. In § 5, by this proposition and by adopting Léandre's idea, the above $(3)_a$ and $(3)_b$ will be proved. In § 6, the above (4) will be proved.

Warning. Throughout this paper, we freely use the notion, notations and the way of representations in [4], [16] and [17]. For details, refer to these papers.

§ 1. Algebraic Preliminaries

Throughout this paper, let $n \geq 1$ be fixed. In this and the next section, we follow Yamato [18]. Set

$$\begin{aligned} &\mathbb{E} := \{0, 1, \cdots, n\}, \\ &\mathbb{E}_a := \{(i_1, \cdots, i_a);\ i_1, \cdots, i_a \in \mathbb{E}\} \qquad a \geq 1\,, \\ &\mathbb{E}(a) := \bigcup_{b=1}^{a} \mathbb{E}_b \qquad 1 \leq a \leq \infty\,. \end{aligned}$$

For $I = (i_1, \cdots, i_a) \in \mathbb{E}(\infty)$, we introduce the following notations:

$$\begin{aligned} &|I| = \text{the length of } I := a,\ \alpha(I) := \sharp\{b \in \{1, \cdots, a\}\,;\ i_b = 0\}\,, \\ &\|I\| := |I| + \alpha(I)\,. \end{aligned}$$

Set

$$\begin{aligned} &\hat{\mathbb{E}}(a) := \mathbb{E}(a) \setminus \{(0)\}, \quad \mathbb{E}((a)) := \{I \in \mathbb{E}(\infty);\ \|I\| \leq a\} \subset \mathbb{E}(a)\,, \\ &\hat{\mathbb{E}}((a)) := \mathbb{E}((a)) \setminus \{(0)\}\,. \end{aligned}$$

Let

$$\begin{aligned}\mathbb{R}(\mathbb{E}) &:= \text{the linear space with basis } \mathbb{E}\,,\\ \mathbb{T}(\mathbb{E}) &:= \text{the tensor algebra generated by } \mathbb{R}(\mathbb{E})\\ &= \mathbb{R}\oplus\mathbb{R}(\mathbb{E})\oplus(\mathbb{R}(\mathbb{E})\otimes\mathbb{R}(\mathbb{E}))\oplus\cdots\,,\\ \mathbb{L}(\mathbb{E}) &:= \text{the Lie subalgebra of } \mathbb{T}(\mathbb{E}) \text{ generated by } \mathbb{E}\,.\end{aligned}$$

Here the bracket product in $\mathbb{T}(\mathbb{E})$ is defined by $[a, b]=a\otimes b-b\otimes a$, $a, b\in \mathbb{T}(\mathbb{E})$. We define $[i_1, \cdots, i_a]\in\mathbb{L}(\mathbb{E})$ for $(i_1, \cdots, i_a)\in\mathbb{E}(\infty)$ by

$$[i_1]:= i_1\,, \quad [i_1, \cdots, i_a]:= [[i_1, \cdots, i_{a-1}], i_a] \qquad a\geq 2$$

inductively. Each $[i_1, \cdots, i_a]$ is expressed as

$$(1.1)\qquad [i_1, \cdots, i_a] = \sum_{(j_1, \cdots, j_b)\in\mathbb{E}(\infty)} c_{i_1,\cdots,i_a}^{j_1,\cdots,j_b}\, j_1\otimes\cdots\otimes j_b$$

and coefficients $c_{i_1,\cdots,i_a}^{j_1,\cdots,j_b}$ are uniquely determined by (1.1). Note that

$$(1.2)\qquad \begin{cases}\text{(i)} & c_i^j = \delta_i^j & \text{for} \quad i, j\in\mathbb{E}\,,\\ \text{(ii)} & c_I^J = 0 & \text{if} \quad |I|\neq|J|\,,\\ \text{(iii)} & c_I^J = 0 & \text{if} \quad \|I\|\neq\|J\|\,.\end{cases}$$

Set $r_a:=\operatorname{rank}[(c_I^J)_{I\in\mathbb{E}_a, J\in\mathbb{E}_a}]$ $a\geq 1$. Then $r_a\geq 1$ for any $a\geq 1$, and it is easy to see that

$$(1.3)\qquad \begin{cases}\textit{For each } a\geq 1, \textit{ there exist } \mathbb{F}_a\subset\mathbb{E}_a \textit{ and } \mathbb{G}_a\subset\mathbb{E}_a \textit{ with}\\ \sharp\mathbb{F}_a=\sharp\mathbb{G}_a=r_a \textit{ such that } (c_I^J)_{I\in G_a, J\in F_a} \textit{ is invertible}\,.\end{cases}$$

Clearly $\mathbb{F}_1=\mathbb{G}_1=\mathbb{E}$ by (1.2) (i). For each $1\leq a\leq\infty$, set

$$\begin{aligned}&\mathbb{G}(a):= \bigcup_{b=1}^{a}\mathbb{G}_b\,, \quad \mathbb{F}(a):= \bigcup_{b=1}^{a}\mathbb{F}_b\,,\\ &\hat{\mathbb{G}}(a):= \mathbb{G}(a)\setminus\{(0)\}\,, \quad \hat{\mathbb{F}}(a):= \mathbb{F}(a)\setminus\{(0)\}\,,\\ &\mathbb{G}((a)):= \mathbb{G}(a)\cap\mathbb{E}((a))\,, \quad \mathbb{F}((a)):= \mathbb{F}(a)\cap\mathbb{E}((a))\,,\\ &\hat{\mathbb{G}}((a)):= \mathbb{G}((a))\setminus\{(0)\}\,, \quad \hat{\mathbb{F}}((a)):= \mathbb{F}((a))\setminus\{(0)\}\,.\end{aligned}$$

From (1.1), (1.2) and (1.3), we have the following, the proof of which is an elementary exercise of the linear algebra:

Proposition 1.4. *For each $a\geq 1$, the following holds:*
(i) *For a pair $(\mathbb{I}, \mathbb{J})=(\mathbb{E}(a), \mathbb{G}(a))$, $(\hat{\mathbb{E}}(a), \hat{\mathbb{G}}(a))$, $(\mathbb{E}((a)), \mathbb{G}((a)))$ and $(\hat{\mathbb{E}}((a)), \hat{\mathbb{G}}((a)))$, respectively, $\{[J];\, J\in\mathbb{J}\}$ form a basis of the linear subspace (of $\mathbb{L}(\mathbb{E})$) spanned by $\{[I];\, I\in\mathbb{I}\}$.*
(ii) *Further, the linear subspace spanned by $\{[J];\, J\in\hat{\mathbb{G}}(a)\}$ coincides with one spanned by $\{[(i, I)];\, i\in\{1, \cdots, n\}, I\in\{\phi\}\cup\mathbb{E}(a-1)\}$. Here $(i, I)\in\mathbb{E}(\infty)$ is*

defined by

$$(i, I) = \begin{cases} (i) & \text{if } I = \phi \\ (i, i_1, \cdots, i_b) & \text{if } I = (i_1, \cdots, i_b) . \end{cases}$$

Let $\mathfrak{X}(\mathbb{R}^r)$ be the totality of C^∞-vector fields on $\mathbb{R}^r$ with the bracket product $[X, Y] = XY - YX$, $X, Y \in \mathfrak{X}(\mathbb{R}^r)$. Let $X_i \in \mathfrak{X}(\mathbb{R}^r)$, $i \in \mathbb{E}$ be given. For $I \in \mathbb{E}(\infty)$, define $X_{[I]} \in \mathfrak{X}(\mathbb{R}^r)$ as follows:

$$X_{[i_1]} := X_{i_1} ,$$
$$X_{[i_1, \cdots, i_a]} := [X_{[i_1, \cdots, i_{a-1}]}, X_{i_a}] \qquad a \geq 2 .$$

Also, we define a differential operator X_I of order $|I|$:

$$X_I := X_{i_1} \cdots X_{i_a} \qquad \text{if} \quad I = (i_1, \cdots, i_a) .$$

Then, as a corollary to (1.1) and (1.4), we have the following:

Corollary 1.5. (i) *For each* $I \in \mathbb{E}(\infty)$,

$$X_{[I]} = \sum_{J \in \mathbb{E}(\infty)} c_I^J X_J .$$

(ii) *For each* $a \geq 1$,

$$\ell.s.\{X_{[I]};\ I \in \mathbb{E}(a)\} = \ell.s.\{X_{[I]};\ I \in \mathbb{G}(a)\} ,$$
$$\ell.s.\{X_{[I]};\ I \in \hat{\mathbb{E}}(a)\} = \ell.s.\{X_{[I]};\ I \in \hat{\mathbb{G}}(a)\}$$
$$= \ell.s.\{X_{[(i,I)]};\ i \in \{1, \cdots, n\},\ I \in \{\phi\} \cup \mathbb{E}(a-1)\} ,$$
$$\ell.s.\{X_{[I]};\ I \in \mathbb{E}((a))\} = \ell.s.\{X_{[I]};\ I \in \mathbb{G}((a))\} ,$$
$$\ell.s.\{X_{[I]};\ I \in \hat{\mathbb{E}}((a))\} = \ell.s.\{X_{[I]};\ I \in \hat{\mathbb{G}}((a))\} .$$

Here "$\ell.s.$" is an abbreviation for "linear span".

§ 2. Regarding $\mathbb{R}^{r(\nu)}$ as a Lie Group

Throughout this section, we take an arbitrary $\nu \geq 1$ and fix it. Set

$$q(\nu) := \sharp \mathbb{E}(\nu) , \quad r(\nu) := \sharp \mathbb{G}(\nu) = \sharp \mathbb{F}(\nu) .$$

We identify the linear spaces (over $\mathbb{R}$)

$$\{(y^I)_{I \in \mathbb{E}(\nu)};\ y^I \in \mathbb{R}^1,\ I \in \mathbb{E}(\nu)\} \quad \text{and} \quad \{(u^I)_{I \in \mathbb{G}(\nu)};\ u^I \in \mathbb{R}^1,\ I \in \mathbb{G}(\nu)\}$$

with $\mathbb{R}^{q(\nu)}$ and $\mathbb{R}^{r(\nu)}$, respectively. The coordinate systems on $\mathbb{R}^{q(\nu)}$ and $\mathbb{R}^{r(\nu)}$ are also denoted by y^I, $I \in \mathbb{E}(\nu)$ and u^I, $I \in \mathbb{G}(\nu)$, respectively. Define $Q_i = Q_i^{(\nu)} \in \mathfrak{X}(\mathbb{R}^{q(\nu)})$, $i \in \mathbb{E}$ by

$$Q_i^{(\nu)} := \frac{\partial}{\partial y^i} + \sum_{\substack{a+1\le\nu \\ j_1,\cdots,j_a\in E}} y^{j_1,\cdots,j_a} \frac{\partial}{\partial y^{j_1,\cdots,j_a,i}} .$$

For $I\in E(\infty)$, $Q_{[I]}\in \mathfrak{X}(R^{q(\nu)})$ is defined in the manner introduced in § 1. Then, owing to Y. Yamato [18], we can state the following:

Proposition 2.1. (i) *For* $(i_1, \cdots, i_a)\in E(\nu)$,

$$Q_{[i_1,\cdots,i_a]} = \sum_{j_1,\cdots,j_a\in E} c_{i_1,\cdots,i_a}^{j_1,\cdots,j_a}\left(\frac{\partial}{\partial y^{j_1,\cdots,j_a}} + \sum_{\substack{b+a\le\nu \\ k_1,\cdots,k_b\in E}} y^{k_1,\cdots,k_b} \frac{\partial}{\partial y^{k_1,\cdots,k_b,j_1,\cdots,j_a}}\right).$$

(ii) *For* $(i_1, \cdots, i_a)\in E(\infty)\setminus E(\nu)$, $Q_{[i_1,\cdots,i_a]}=0$.

Let $\mathfrak{g}=\mathfrak{g}_\nu$ be the Lie subalgebra of $\mathfrak{X}(R^{q(\nu)})$ generated by $Q_i^{(\nu)}$, $i\in E$. Then, from the above proposition, $\mathfrak{g}$ is nilpotent of step ν and $\mathfrak{g}=\mathcal{L}.\mathcal{A}.\ \{Q_{[I]};\ I\in E(\nu)\}$. Further, by (1.4)

(2.2) $Q_{[I]},\ I\in G(\nu)$ *form a basis in* $\mathfrak{g}$.

As one more corollary to (2.1), we have the following: Let η denote the coordinate system on $R^{q(\nu)}$, i.e., $\eta^J((y^I)_{I\in E(\nu)}):=y^J$, $J\in E(\nu)$. Then

Corollary 2.3. *For* $a\ge 1$ *and* $J\in E(\nu)$,

$$Q_{[I_a]}\cdots Q_{[I_1]}\eta^J = \begin{cases} 0 & \text{if } |J|<|I_1|+\cdots+|I_a|, \\ c_{I_a}^{J_a}\cdots c_{I_1}^{J_1} & \text{if } |J|=|I_1|+\cdots+|I_a|, \text{ where } J \text{ is expressed as} \\ & J=(J_a,\cdots,J_1) \text{ with } |J_a|=|I_a|,\cdots,|J_1|=|I_1|, \\ c_{I_a}^{J_a}\cdots c_{I_1}^{J_1}\eta^{K_a} & \text{if } |J|>|I_1|+\cdots+|I_a|, \text{ where } J \text{ is expressed as} \\ & J=(K_a, J_a,\cdots,J_1) \text{ with } |J_a|=|I_a|,\cdots,|J_1|=|I_1|. \end{cases}$$

We denote by $\mathrm{Exp}(tQ)$ the integral curve of a complete vector field Q $(\in\mathfrak{X}(R^{q(\nu)}))$. That is, for each $y=(y^I)_{I\in E(\nu)}\in R^{q(\nu)}$, $\mathrm{Exp}(tQ)(y)$ is the unique solution of

$$\begin{cases} y_t=(y_t^J)_{J\in E(\nu)} \\ \dfrac{d}{dt}y_t^J = Q^J(y_t) \qquad J\in E(\nu) \\ y_0=y \end{cases}$$

where Q^J, $J\in E(\nu)$ stand for the components of Q. The following is a consequence of (2.2) and (2.3):

Corollary 2.4. *If $Q\in\mathfrak{g}$, then, for each $J\in\mathbb{E}(\nu)$ and $y=(y^I)_{I\in E(\nu)}\in\mathbb{R}^{q(\nu)}$,*

$$\mathrm{Exp}(tQ)(y)^J=\sum_{a=0}^{\nu}\frac{t^a}{a!}(Q^a\eta^J)(y).$$

In particular,

$$\mathrm{Exp}(Q)(y)^J=\sum_{a=0}^{\nu}\frac{1}{a!}(Q^a\eta^J)(y), \qquad J\in\mathbb{E}(\nu).$$

We define $\Phi=\Phi_\nu\in C^\infty(\mathbb{R}^{r(\nu)}\times\mathbb{R}^{q(\nu)}, \mathbb{R}^{q(\nu)})$ and $\varphi=\varphi_\nu\in C^\infty(\mathbb{R}^{r(\nu)}, \mathbb{R}^{q(\nu)})$ as follows:

$$\Phi_\nu(u, y):=\mathrm{Exp}\Big(\sum_{I\in G(\nu)}u^I Q_{[I]}\Big)(y),$$
$$\varphi_\nu(u):=\Phi_\nu(u, 0)=\mathrm{Exp}\Big(\sum_{I\in G(\nu)}u^I Q_{[I]}\Big)(0).$$

Then $\Phi(u, \circ)$ is a diffeomorphism on $\mathbb{R}^{q(\nu)}$ for each $u\in\mathbb{R}^{r(\nu)}$, and particularly, $\Phi(0, \circ)$ is the identity mapping. By the Campbell-Hausdorff formula, for $u, v\in\mathbb{R}^{r(\nu)}$ we define a product $u\times v\in\mathbb{R}^{r(\nu)}$ so that $\Phi(u\times v, \circ)=\Phi(u, \Phi(v, \circ))$ holds. With this multiplication, $\mathbb{R}^{r(\nu)}$ can be regarded as a Lie group with 0 as its identity. Let $\mathfrak{h}=\mathfrak{h}_\nu$ denote the right invariant Lie algebra of $\mathbb{R}^{r(\nu)}$ and let $R_i=R_i^{(\nu)}\in\mathfrak{h}_\nu$ be such that $R_i(0)=\left(\frac{\partial}{\partial u^i}\right)_0$, $i\in\mathbb{E}$. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$ under the correspondence: $Q_i\leftrightarrow R_i$. Furthermore, if $R\in\mathfrak{h}$ is an element corresponding to $Q\in\mathfrak{g}$, then it holds that

$$R(f\circ\varphi)=(Qf)\circ\varphi \qquad f\in C^\infty(\mathbb{R}^{q(\nu)}). \tag{2.5}$$

Note that for each $J\in\mathbb{G}(\nu)$, $R_{[J]}\in\mathfrak{h}$ is expressed as

$$R_{[J]}=\sum_{I\in G(\nu)}\frac{\partial}{\partial v^J}(v\times u)^I\Big|_{v=0}\frac{\partial}{\partial u^I}. \tag{2.6}$$

Also, the following holds: For $\lambda\neq 0$, we define an isomorphism $T_{(\nu)}^{(\lambda)}\colon \mathbb{R}^{r(\nu)}\to\mathbb{R}^{r(\nu)}$ by

$$T_{(\nu)}^{((\lambda))}((u)^I{}_{I\in G(\nu)}):=(\lambda^{\|I\|}u^I)_{I\in G(\nu)}.$$

Then, for any $J\in\mathbb{G}(\nu)$

$$(T_{(\nu)}^{((\lambda))})_*R_{[J]}=\lambda^{\|J\|}R_{[J]}. \tag{2.7}$$

Because $Q_{[J]}$ has the same property: For each $J\in\mathbb{E}(\nu)$

$$(S_{(\nu)}^{((\lambda))})_*Q_{[J]}=\lambda^{\|J\|}Q_{[J]} \tag{2.8}$$

where $S^{((\lambda))}_{(\nu)} \in \mathrm{Hom}(\mathbb{R}^{q(\nu)}, \mathbb{R}^{q(\nu)})$ is defined similarly to $T^{((\lambda))}_{(\nu)}$.

To conclude this section, we make some remarks: (i) As a corollary to (2.4), we have that for each $K \in \mathbb{E}(\nu)$

$$(2.9) \qquad \varphi_\nu(u)^K = \sum_{a=1}^{\nu} \frac{1}{a!} \sum_{J_1, \cdots, J_a \in \mathbb{G}(\nu)} u^{J_1} \cdots u^{J_a} (Q^{(\nu)}_{[J_1]} \cdots Q^{(\nu)}_{[J_a]} \eta^K)(0) .$$

(ii) From (2.8) and (2.9), the following is derived by the same way as in Lemma (A.5) of Kusuoka-Stroock [10]:

$$(2.10) \qquad \varphi_\nu \text{ is one-to-one} .$$

(iii) For $\nu' \geq \nu \geq 1$, define $\Pi^{\nu'}_{\nu} \in \mathrm{Hom}(\mathbb{R}^{q(\nu')}, \mathbb{R}^{q(\nu)})$ and $P^{\nu'}_{\nu} \in \mathrm{Hom}(\mathbb{R}^{r(\nu')}, \mathbb{R}^{r(\nu)})$ as follows:

$$\Pi^{\nu'}_{\nu}((y^I)_{I \in \mathbb{E}(\nu')}) := (y^I)_{I \in \mathbb{E}(\nu)} , \quad P^{\nu'}_{\nu}((u^I)_{I \in \mathbb{G}(\nu')}) := (u^I)_{I \in \mathbb{G}(\nu)} .$$

Then it holds that

$$(2.11) \qquad \Pi^{\nu'}_{\nu} \circ \varphi_{\nu'} = \varphi_\nu \circ P^{\nu'}_{\nu} .$$

(iv) Let $R^{(\nu)I}_{[J]}$, $I \in \mathbb{G}(\nu)$ denote the components of $R^{(\nu)}_{[J]} \in \mathfrak{h}_\nu$. Then

$$(2.12) \qquad \inf \{ \sum_{i=1}^{n} \sum_{K \in \{\phi\} \cup \mathbb{E}(\nu-1)} (\sum_{I \in \hat{\mathbb{G}}(\nu)} R^{(\nu)I}_{[(i,K)]}(0) l^I)^2 ; \sum_{I \in \hat{\mathbb{G}}(\nu)} (l^I)^2 = 1 \} > 0 .$$

Here we shall show (2.12) only: Suppose that for any $i \in \{1, \cdots, n\}$ and $K \in \{\phi\} \cup \mathbb{E}(\nu-1)$

$$\sum_{I \in \hat{\mathbb{G}}(\nu)} R^{(\nu)I}_{[(i,K)]}(0) \, l^I = 0 .$$

By (1.5) (ii), this implies that for any $J \in \hat{\mathbb{G}}(\nu)$

$$\sum_{I \in \hat{\mathbb{G}}(\nu)} R^{(\nu)I}_{[J]}(0) \, l^I = 0 .$$

Since, by (2.6), $R^{(\nu)I}_{[J]}(0) = \frac{\partial}{\partial v^J}(v \times 0)^I |_{v=0} = \delta^I_J$, $I, J \in \mathbb{G}(\nu)$, this implies that $l^J = 0$ for $J \in \hat{\mathbb{G}}(\nu)$. Thus (2.12) follows immediately.

§ 3. The Continuous Process $U^{(\nu)}_t$ on $\mathbb{R}^{r(\nu)}$

Let $(W = W^n_0, P)$ be the n-dimensional Wiener space. Then a generic element $w = (w^i_t)$ of W is clearly a realization of an n-dimensional Brownian motion starting at $0 \in \mathbb{R}^n$ under the measure P. Define the *multiple Wiener integrals* w^I_t, $I \in \mathbb{E}(\infty)$ by

$$w_t^{(i_1)} := w_t^{i_1}, \quad w_t^{(i_1,\cdots,i_a)} := \int_0^t w_s^{(i_1,\cdots,i_{a-1})} \circ dw_s^{i_a} \qquad a \geq 2 .$$

Here and hereafter, we set $w_t^0 := t$ for convenience. Then, for each $\nu \geq 1$, the following holds:

Proposition 3.1. (Y. Yamato [18]). *The continuous process* $[(w_t^I)_{I \in E(\nu)};\ t \geq 0]$ *on* $\mathbb{R}^{q(\nu)}$ *is the unique solution of*

$$\begin{cases} dY_t = \sum_{i \in \mathbb{E}} Q_i^{(\nu)}(Y_t) \circ dw_t^i \\ Y_0 = 0 \in \mathbb{R}^{q(\nu)} . \end{cases}$$

The proof of (3.1) is obvious from the definition of $Q_i^{(\nu)}$, $i \in \mathbb{E}$.

Let $\nu \geq 1$ be fixed arbitrarily. We consider the following stochastic differential equation (SDE) on $\mathbb{R}^{r(\nu)}$:

$$\begin{cases} dU_t = \sum_{i \in \mathbb{E}} R_i^{(\nu)}(U_t) \circ dw_t^i \\ U_0 = 0 \in \mathbb{R}^{r(\nu)} . \end{cases} \tag{3.2}$$

We denote by $U_t^{(\nu)}$ the unique solution of this SDE. Then $U_t^{(\nu)} \in \mathbb{D}^\infty(\mathbb{R}^{r(\nu)})$ for each $t \geq 0$. Let Y_t and Z_t be the unique solutions of the following SDE's on $\mathbb{R}^{r(\nu)} \otimes \mathbb{R}^{r(\nu)}$, respectively:

$$\begin{cases} dY_t = \sum_{i \in \mathbb{E}} \partial R_i^{(\nu)}(U_t^{(\nu)}) Y_t \circ dw_t^i \\ Y_0 = (\delta_J^I)_{I,J \in G(\nu)} , \end{cases} \tag{3.3}$$

$$\begin{cases} dZ_t = -\sum_{i \in \mathbb{E}} Z_t \partial R_i^{(\nu)}(U_t^{(\nu)}) \circ dw_t^i \\ Z_0 = (\delta_J^I)_{I,J \in G(\nu)} . \end{cases} \tag{3.4}$$

Then $Z_t Y_t = (\delta_J^I)_{I,J \in G(\nu)}$. Further the following is well-known (cf. [4], [9], [16]): Let $\sigma_{(\nu)} = (\sigma_\nu^{IJ})_{I,J \in G(\nu)}$ be the Malliavin covariance of $U_1^{(\nu)}$:

$$\sigma_\nu^{IJ} := \langle DU_1^{(\nu)I}, DU_1^{(\nu)J} \rangle \qquad I, J \in \mathbb{G}(\nu) .$$

Set $\tau_{(\nu)} = (\tau_\nu^{IJ})_{I,J \in G(\nu)}$:

$$\tau_\nu^{IJ} := \sum_{i=1}^n \int_0^1 (Z_s R_i^{(\nu)}(U_s^{(\nu)}))^I (Z_s R_i^{(\nu)}(U_s^{(\nu)}))^J ds \qquad I, J \in \mathbb{G}(\nu) .$$

Then $\sigma_{(\nu)} = Y_1 \tau_{(\nu)} (Y_1)^*$. Note that for any $u, v \in \mathbb{R}^{r(\nu)}$

$$(u \times v)^i = u^i + v^i \qquad i \in \mathbb{E} . \tag{3.5}$$

This is easily verified by viewing the Campbell-Hausdorff series:

$$\sum_{I\in G(\nu)}(u\times v)^I Q^{(\nu)}_{[I]} = \sum_{I\in G(\nu)}(v^I+u^I)Q^{(\nu)}_{[I]} + \frac{1}{2}\Big[\sum_{I\in G(\nu)} v^I Q^{(\nu)}_{[I]}, \sum_{I\in G(\nu)} u^I Q^{(\nu)}_{[I]}\Big]+\cdots .$$

Combining (3.5) with (2.6), we see that $R^{(\nu)j}_i(\circ)=\delta^j_i$, $i,j\in E$. Hence, in view of (3.2) and (3.3)

$$U^{(\nu)j}_t = w^j_t\,,\quad Y^{jJ}_t = \delta^j_J \qquad j\in E,\ J\in G(\nu)\,. \tag{3.6}$$

Since $Z_t Y_t = Y_t Z_t = (\delta^I_J)_{I,J\in G(\nu)}$, we further see that $Z^{jJ}_t=\delta^j_J$, $j\in E$, $J\in G(\nu)$. Thus, if we set

$$\hat{Y}_t := (Y^{IJ}_t)_{I,J\in\hat{G}(\nu)}\,,\quad \hat{Z}_t := (Z^{IJ}_t)_{I,J\in\hat{G}(\nu)}\,,$$
$$\hat{\sigma}_{(\nu)} := (\sigma^{IJ}_\nu)_{I,J\in\hat{G}(\nu)}\,,\quad \hat{\tau}_{(\nu)} := (\tau^{IJ}_\nu)_{I,J\in\hat{G}(\nu)}\,,$$

then we have

$$\begin{cases} Y_t = \begin{bmatrix} 1 & 0 \\ * & \hat{Y}_t \end{bmatrix}, \quad Z_t = \begin{bmatrix} 1 & 0 \\ ** & \hat{Z}_t \end{bmatrix}, \\ \sigma_{(\nu)} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\sigma}_{(\nu)} \end{bmatrix}, \quad \tau_{(\nu)} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\tau}_{(\nu)} \end{bmatrix}, \\ \hat{Z}_t\hat{Y}_t = (\delta^I_J)_{I,J\in\hat{G}(\nu)}\,,\quad \hat{\sigma}_{(\nu)} = \hat{Y}_1\hat{\tau}_{(\nu)}(\hat{Y}_1)^*\,. \end{cases} \tag{3.7}$$

Now, owing to Kusuoka-Stroock [9], we can state the following: Set

$$\hat{\pi}_{(\nu)} := \inf\{\langle\hat{\sigma}_{(\nu)}\hat{l}, \hat{l}\rangle;\ \hat{l}=(l^I)_{I\in\hat{G}(\nu)} \text{ such as } \sum_{I\in\hat{G}(\nu)}(l^I)^2=1\}.$$

Then

Proposition 3.8. *It holds that* $\hat{\pi}_{(\nu)}>0$ *a.s.* (P) *and*

$$(\hat{\pi}_{(\nu)})^{-1}\in L_{\infty-} := \bigcap_{p>1} L_p\,.$$

Proof. First of all, set

$$\hat{\rho}_{(\nu)} := \inf\{\langle\hat{\tau}_{(\nu)}\hat{l}, \hat{l}\rangle;\ \hat{l}=(l^I)_{I\in\hat{G}(\nu)} \text{ such as } \sum_{I\in\hat{G}(\nu)}(l^I)^2=1\}.$$

It is sufficient to show (3.8) for $\hat{\rho}_{(\nu)}$. For, we observe by (3.7) that for $\hat{l}=(l^I)_{I\in G(\nu)}$ such as $\sum_{I\in\hat{G}(\nu)}(l^I)^2=1$

$$\langle\hat{\sigma}_{(\nu)}\hat{l}, \hat{l}\rangle = \langle\hat{Y}_1\hat{\tau}_{(\nu)}(\hat{Y}_1)^*\hat{l}, \hat{l}\rangle = \langle\hat{\tau}_{(\nu)}(\hat{Y}_1)^*\hat{l}, (\hat{Y}_1)^*\hat{l}\rangle$$
$$= |(\hat{Y}_1)^*\hat{l}|^2\Big\langle\hat{\tau}_{(\nu)}\frac{(\hat{Y}_1)^*\hat{l}}{|(\hat{Y}_1)^*\hat{l}|}, \frac{(\hat{Y}_1)^*\hat{l}}{|(\hat{Y}_1)^*\hat{l}|}\Big\rangle$$

$$\geq |(\hat{Y}_1)^* \hat{l}|^2 \hat{\rho}_{(\nu)} \geq ||(\hat{Z}_1)^*||^{-2} \hat{\rho}_{(\nu)};$$

Noticing that $||(\hat{Z}_1)^*|| \in L_{\infty-}$ by (3.4), we see (3.8) for $\hat{\pi}_{(\nu)}$ at once.

Now we shall prove (3.8) for $\hat{\rho}_{(\nu)}$. In view of (3.2) and (3.4), we observe by Itô's formula that

$$\begin{aligned} &Z_t R_i^{(\nu)}(U_t^{(\nu)}) \\ &= R_i^{(\nu)}(0) + \sum_{i_1 \in E} [R_{i_1}^{(\nu)}, R_i^{(\nu)}](0) w_t^{i_1} + \sum_{i_1, i_2 \in E} [R_{i_1}^{(\nu)}, [R_{i_2}^{(\nu)}, R_i^{(\nu)}]](0) w_t^{i_1, i_2} \\ &+ \cdots + \sum_{i_1, \cdots, i_{\nu-1} \in E} [R_{i_1}^{(\nu)}, [R_{i_2}^{(\nu)}, \cdots, [R_{i_{\nu-1}}^{(\nu)}, R_i^{(\nu)}] \cdots](0) w_t^{i_1, \cdots, i_{\nu-1}}, \end{aligned}$$

where we have used the fact: $[R_{j_1}^{(\nu)}, [R_{j_2}^{(\nu)}, \cdots, [R_{j_{a-1}}^{(\nu)}, R_{j_a}^{(\nu)}] \cdots] = 0$ for any $a \geq \nu + 1$. Since, in general, it holds that

$$[R_{i_1}^{(\nu)}, [R_{i_2}^{(\nu)}, \cdots, [R_{i_a}^{(\nu)}, R_i^{(\nu)}] \cdots] = (-1)^a R_{[i, i_a, \cdots, i_1]}^{(\nu)},$$

the above is equal to

$$\sum_{I \in \{\phi\} \cup E(\nu-1)} (-1)^{|I|} R_{[(i, \check{I})]}^{(\nu)}(0) w_t^I .$$

Here, for convenience sake, we set that $|\phi| := 0$, $w_s^\phi := 1$ and

$$\check{I} := \begin{cases} \phi & \text{if } I = \phi \\ (i_a, \cdots, i_1) & \text{if } I = (i_1, \cdots, i_a). \end{cases}$$

Hence, recalling (3.7) and the definition of $\tau_{(\nu)}$, we see that for $\hat{l} = (l^I)_{I \in \hat{G}(\nu)}$

$$\langle \hat{\tau}_{(\nu)} \hat{l}, \hat{l} \rangle = \sum_{i=1}^n \int_0^1 \Big(\sum_{K \in \{\phi\} \cup E(\nu-1)} \sum_{I \in \hat{G}(\nu)} (-1)^{|K|} R_{[(i, \check{K})]}^{(\nu)I}(0) l^I w_s^K \Big)^2 ds .$$

Thus, setting

$$\mathcal{V}_\nu(\hat{l}) := \sum_{i=1}^n \sum_{K \in \{\phi\} \cup E(\nu-1)} \Big(\sum_{I \in \hat{G}(\nu)} R_{[i, K]}^{(\nu)I}(0) l^I \Big)^2, \qquad \hat{l} = (l^I)_{I \in \hat{G}(\nu)}$$

we have

$$\begin{aligned} \hat{\rho}_{(\nu)} \geq & \frac{1}{n} \inf \{ \mathcal{V}_\nu(\hat{l}); \sum_{I \in \hat{G}(\nu)} (l^I)^2 = 1 \} \\ & \times \inf \left\{ \int_0^1 \Big(\sum_{K \in \{\phi\} \cup E(\nu-1)} b_K w_s^K \Big)^2 ds; \sum_{K \in \{\phi\} \cup E(\nu-1)} (b_K)^2 = 1 \right\}. \end{aligned}$$

Consequently, from Theorem (A.6) in Kusuoka-Stroock [9] and (2.12), (3.8) for $\hat{\rho}_{(\nu)}$ follows. //

The following proposition is a key in this paper, though its proof is easy.

Proposition 3.9. *For each $I\in E(\nu)$,*

$$w_t^I=\sum_{a=1}^{\nu}\frac{1}{a!}\sum_{J_1,\cdots,J_a\in G(\nu)}U_t^{(\nu)J_1}\cdots U_t^{(\nu)J_a}(Q_{[J_1]}^{(\nu)}\cdots Q_{[J_a]}^{(\nu)}\eta^I)(0)\,.$$

Proof. Let $f\in C^\infty(\mathbf{R}^{q(\nu)})$. By (3.2) and (2.5), we observe

$$df(\varphi_\nu(U_t^{(\nu)}))=\sum_{i\in E}R_i^{(\nu)}(f\circ\varphi_\nu)(U_t^{(\nu)})\circ dw_t^i=\sum_{i\in E}(Q_i^{(\nu)}f)(\varphi_\nu(U_t^{(\nu)}))\circ dw_t^i\,.$$

By (3.1), this implies

$$(w_t^I)_{I\in E(\nu)}=\varphi_\nu(U_t^{(\nu)})\,, \tag{3.10}$$

from which and (2.9), (3.9) follows at once. //

Let $\nu'\geq\nu\geq 1$. Recalling (2.11), we can state a relation between $U_t^{(\nu')}$ and $U_t^{(\nu)}$.

Proposition 3.11. $P_\nu^{\nu'}U_t^{(\nu')}=U_t^{(\nu)}\quad t\geq 0$.

Proof. By (2.11), we observe

$$\Pi_\nu^{\nu'}(\varphi_{\nu'}(U_t^{(\nu')}))=\varphi_\nu(P_\nu^{\nu'}U_t^{(\nu')})\,.$$

By (3.10), this implies that $\varphi_\nu(U_t^{(\nu)})=\varphi_\nu(P_\nu^{\nu'}U_t^{(\nu')})$. Hence, (3.11) follows from (2.10). //

By virtue of (3.11), we can define a continuous process $[U_t^{(\infty)};\ t\geq 0]$ on $\mathbf{R}^\infty\simeq\{(u^I)_{I\in G(\infty)};\ u^I\in\mathbf{R}^1,\ I\in G(\infty)\}$ so that $P_\nu^\infty U_t^{(\infty)}=U_t^{(\nu)}$ for any $\nu\geq 1$. Let U_t^I, $I\in G(\infty)$ be the components of $U_t^{(\infty)}$. For $\lambda\neq 0$, an isomorphism $T^{((\lambda))}\colon\mathbf{R}^\infty\to\mathbf{R}^\infty$ is defined by

$$T^{((\lambda))}((u^I)_{I\in G(\infty)}):=(\lambda^{\|I\|}u^I)_{I\in G(\infty)}\,.$$

Then

Proposition 3.12. (i) *$[T^{((\lambda))}U_t^{(\infty)};\ t\geq 0]$ is equivalent in law to $[U_{\lambda^2 t}^{(\infty)};\ t\geq 0]$.*
(ii) *For any $I\in G(\infty)$, $U_t^I(-w)=(-1)^{\|I\|}U_t^I(w)$.*

Proof. Let $\nu\geq 1$ be fixed arbitrarily. Let $V_t^{((\lambda))}$ denote the unique solution of

$$\begin{cases} dV_t=\lambda\sum_{i=1}^{n}R_i^{(\nu)}(V_t)\circ dw_t^i+\lambda^2R_0^{(\nu)}(V_t)dt \\ V_0=0\in\mathbf{R}^{r(\nu)}\,. \end{cases}$$

Then, from (2.7) and (3.2), it is easy to see that

$$V_t^{((\lambda))}=T_{(\nu)}^{((\lambda))}U_t^{(\nu)}\qquad t\geq 0\,. \tag{3.13}$$

On the other hand, from the scaling property of $(w_t^i)_{i\in\{1,\cdots,n\}}$, it is clear that $[V^{((\lambda))}_{t/\lambda^2};\ t\geq 0]$ is equivalent in law to $[U_t^{(\nu)};\ t\geq 0]$. Hence, combining this and (3.13), we have the assertion (i). For (ii), if we take $\lambda=-1$, then $V_t^{((-1))}(w)=U_t^{(\nu)}(-w)$, and thus, this, together with (3.13), implies the assertion (ii). //

We end this section with the following remark: Set

$$m_{\nu,\kappa} := \max_{|u|\leq\kappa}\left|\frac{1}{2}\sum_{i=1}^{n}\partial R_i^{(\nu)}(u)\circ R_i^{(\nu)}(u)+R_0^{(\nu)}(u)\right|+\max_{|u|\leq\kappa}\sum_{i=1}^{n}|R_i^{(\nu)}(u)|^2.$$

Note that $m_{\nu,\kappa}>0$, since $R_i^{(\nu)I}(0)=\delta_i^I$, $i\in\mathbb{E}$, $I\in\mathbb{G}(\nu)$. Then, by the standard procedure due to Stroock-Varadahn [15], we can obtain that for any $0<t\leq \frac{1}{2}\frac{\kappa}{r(\nu)m_{\nu,\kappa}}$

$$P(\max_{0\leq s\leq t}|U_s^{(\nu)}|\geq\kappa)\leq 2r(\nu)\exp\left(-\frac{1}{8}\frac{\kappa^2}{r(\nu)^2m_{\nu,\kappa}}\frac{1}{t}\right).$$

Thus, by putting this and (3.12) (i) together, the following estimate holds: For $t>0$ and $0<\varepsilon\leq\left(\frac{1}{2}\frac{\kappa}{r(\nu)m_{\nu,\kappa}t}\right)^{1/2}$,

$$P(\max_{0\leq s\leq t}|T^{((\varepsilon))}_{(\nu)}U_s^{(\nu)}|\geq\kappa)\leq 2r(\nu)\exp\left(-\frac{1}{8}\frac{\kappa^2}{r(\nu)^2m_{\nu,\kappa}}\frac{1}{t}\frac{1}{\varepsilon^2}\right). \tag{3.14}$$

§ 4. The Smooth (in the Malliavin Sense) Wiener Functional $X^\varepsilon(1, x)$

Let $V_i\in C_b^\infty(\mathbb{R}^d,\mathbb{R}^d)$, $i\in\mathbb{E}$. Define $\hat{V}_i\in\mathfrak{X}(\mathbb{R}^d)$, $i\in\mathbb{E}$ and a second order differential operator L on $\mathbb{R}^d$ as follows:

$$\hat{V}_i := \sum_{j=1}^{d}V_i^j\frac{\partial}{\partial x^j}\qquad i\in\mathbb{E},$$

$$L := \frac{1}{2}\sum_{i=1}^{n}\hat{V}_i^2+\hat{V}_0.$$

Let $(W=W_0^n, P)$ be, as before, the n-dimensional Wiener space. For $\varepsilon>0$, we consider the following SDE on $\mathbb{R}^d$:

$$\begin{cases} dX_t = \varepsilon\sum_{i=1}^{n}V_i(X_t)\circ dw_t^i+\varepsilon^2V_0(X_t)dt \\ X_0 = x\in\mathbb{R}^d. \end{cases} \tag{4.1}$$

We denote by $X^\varepsilon(t, x)$ the unique solution of this SDE. Then, for each $t\geq 0$ and $x\in\mathbb{R}^d$, $X^\varepsilon(t, x)\in\mathbb{D}^\infty(\mathbb{R}^d)$. Further, the following is well-known as Taylor's expansion of $X^\varepsilon(t, x)$ with respect to ε (cf. [9], [17]): Let ζ be the coordinate system on $\mathbf{R}^d$, i.e., $\zeta^i(x)=x^i$ for $i\in\{1, \cdots, d\}$. Then, for each $a\geq 1$

$$(4.2)\quad \begin{cases} X^\varepsilon(t, x) = x+\sum_{I\in\mathbf{E}(a)} (\hat{V}_I\zeta)(x)\varepsilon^{\|I\|}w_t^I \\ \quad+\sum_{(i_1,\cdots,i_{a+1})\in\mathbf{E}_{a+1}} \varepsilon^{\|(i_1,\cdots,i_{a+1})\|}\int_0^t \circ dw_{t_{a+1}}^{i_{a+1}}\int_0^{t_{a+1}}\circ dw_{t_a}^{i_a}\cdots \\ \qquad\cdots\int_0^{t_2}(\hat{V}_{(i_1,\cdots,i_{a+1})}\zeta)(X^\varepsilon(t_1, x))\circ dw_{t_1}^{i_1}. \end{cases}$$

By using (3.9), we shall rewrite (4.2). For this, we introduce $F_\nu^\varepsilon(t, x)$, $R_\nu^\varepsilon(t, x)\in\mathbf{D}^\infty(\mathbf{R}^d)$, $\nu\geq 1$:

$$(4.3)\quad \begin{cases} F_\nu^\varepsilon(t, x) := \sum_{a=1}^{\nu}\frac{1}{a!}\sum_{I_1,\cdots,I_a\in\mathbf{G}((\nu))}(\varepsilon^{\|I_1\|}U_t^{I_1})\cdots(\varepsilon^{\|I_a\|}U_t^{I_a})(\hat{V}_{[I_1]}\cdots\hat{V}_{[I_a]}\zeta)(x), \\ R_\nu^\varepsilon(t, x) := \sum_{a=1}^{\nu}\frac{1}{a!}\sum_{(I_1,\cdots,I_a)\in\prod_{b=1}^{a}\mathbf{G}(\nu)\setminus\prod_{b=1}^{a}\mathbf{G}((\nu))}\varepsilon^{\|I_1\|+\cdots+\|I_a\|} \\ \qquad\times U_t^{I_1}\cdots U_t^{I_a}(\hat{V}_{[I_a]}\cdots\hat{V}_{[I_1]}\zeta)(x) \\ \quad-\sum_{a=1}^{\nu}\frac{1}{a!}\sum_{\substack{I_1,\cdots,I_a\in\mathbf{G}(\nu)\\ |I_1|+\cdots+|I_a|\geq\nu+1}}\varepsilon^{\|I_1\|+\cdots+\|I_a\|}U_t^{I_1}\cdots U_t^{I_a}(\hat{V}_{[I_a]}\cdots\hat{V}_{[I_1]}\zeta)(x) \\ \quad+\sum_{(i_1,\cdots,i_{\nu+1})\in\mathbf{E}_{\nu+1}}\varepsilon^{\|(i_1,\cdots,i_{\nu+1})\|}\int_0^t\circ dw_{t_{\nu+1}}^{i_{\nu+1}}\int_0^{t_{\nu+1}}\circ dw_{t_\nu}^{i_\nu}\cdots \\ \qquad\cdots\int_0^{t_2}(\hat{V}_{(i_1,\cdots,i_{\nu+1})}\zeta)(X^\varepsilon(t_1, x))\circ dw_{t_1}^{i_1}. \end{cases}$$

Here recall that U_t^I, $I\in\mathbf{G}(\infty)$ are the components of $U_t^{(\infty)}$ defined in § 3. Then

Proposition 4.4. *It holds that*

$$X^\varepsilon(t, x) = x+F_\nu^\varepsilon(t, x)+R_\nu^\varepsilon(t, x).$$

Proof. Let $\nu\geq 1$ be fixed arbitrarily. By virtue of (3.9), we observe that

$$\begin{aligned} &\sum_{J\in\mathbf{E}(\nu)}(\hat{V}_I\zeta)(x)\varepsilon^{\|I\|}w_t^I \\ &= \sum_{I\in\mathbf{E}(\nu)}(\hat{V}_J\zeta)(x)\varepsilon^{\|J\|}\sum_{a=1}^{\nu}\frac{1}{a!}\sum_{\substack{I_1,\cdots,I_a\in\mathbf{G}(\nu)\\ |I_1|+\cdots+|I_a|\leq\nu}}U_t^{I_1}\cdots U_t^{I_a}(Q_{[I_a]}^{(\nu)}\cdots Q_{[I_1]}^{(\nu)}\eta^J)(0) \\ &= \sum_{a=1}^{\nu}\frac{1}{a!}\sum_{\substack{I_1,\cdots,I_a\in\mathbf{G}(\nu)\\ |I_1|+\cdots+|I_a|\leq\nu}}U_t^{I_1}\cdots U_t^{I_a}\sum_{J\in\mathbf{E}(\nu)}\varepsilon^{\|J\|}(\hat{V}_J\zeta)(x)(Q_{[I_a]}^{(\nu)}\cdots Q_{[I_1]}^{(\nu)}\eta^J)(0). \end{aligned}$$

Since, by (2.3)

$$(Q^{(\nu)}_{[I_a]}\cdots Q^{(\nu)}_{[I_1]}\eta^J)(0)$$
$$=\begin{cases}0 & \text{if } |J|\neq|I_1|+\cdots+|I_a| \\ c^{J_a}_{I_a}\cdots c^{J_1}_{I_1} & \text{if } |J|=|I_1|+\cdots+|I_a|, \text{ where } J \text{ is expressed as} \\ & J=(J_a,\cdots,J_1) \text{ with } |J_a|=|I_a|,\cdots,|J_1|=|I_1|,\end{cases}$$

the above is equal to

$$\sum_{a=1}^{\nu}\frac{1}{a!}\sum_{\substack{I_1,\cdots,I_a\in\mathbb{G}(\nu)\\|I_1|+\cdots+|I_a|\leq\nu}}U^{I_1}_t\cdots U^{I_a}_t$$
$$\times\sum_{\substack{J_1,\cdots,J_a\in\mathbb{E}(\nu)\\|I_1|=|J_1|,\cdots,|I_a|=|J_a|}}\varepsilon^{\|(J_a,\cdots,J_1)\|}(\hat{V}_{(J_a,\cdots,J_1)}\zeta)(x)c^{J_1}_{I_1}\cdots c^{J_a}_{I_a}$$
$$=\sum_{a=1}^{\nu}\frac{1}{a!}\sum_{\substack{I_1,\cdots,I_a\in\mathbb{G}(\nu)\\|I_1|+\cdots+|I_a|\leq\nu}}\varepsilon^{\|I_1\|+\cdots+\|I_a\|}U^{I_1}_t\cdots U^{I_a}_t$$
$$\times\sum_{J_1,\cdots,J_a\in\mathbb{E}(\nu)}((c^{J_a}_{I_a}\hat{V}_{J_a})\cdots(c^{J_1}_{I_1}\hat{V}_{J_1})\zeta)(x)$$
$$=\sum_{a=1}^{\nu}\frac{1}{a!}\sum_{\substack{I_1,\cdots,I_a\in\mathbb{G}(\nu)\\|I_1|+\cdots+|I_a|\leq\nu}}\varepsilon^{\|I_1\|+\cdots+\|I_a\|}U^{I_1}_t\cdots U^{I_a}_t(\hat{V}_{[I_a]}\cdots\hat{V}_{[I_1]}\zeta)(x)\,.$$

Here the last equality has come from (1.5) (i). Thus, putting (4.2), (4.3) and the above together, we obtain (4.4) immediately. //

For $I\in\mathbb{E}(\infty)$, let $V_{[I]}=(V^i_{[I]})_{i\in\{1,\cdots,d\}}\in C^\infty_b(\mathbb{R}^d,\mathbb{R}^d)$ denote the components of $\hat{V}_{[I]}\in\mathfrak{X}(\mathbb{R}^d)$: $\hat{V}_{[I]}=\sum_{j=1}^{d}V^j_{[I]}\frac{\partial}{\partial x^j}$. Define $\mathbb{V}_{I_1,\cdots,I_a}=(\mathbb{V}^{ij}_{I_1,\cdots,I_a})_{i,j\in\{1,\cdots,d\}}\in C^\infty_b(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^d)$ for $I_1,\cdots,I_a\in\mathbb{E}(\infty)$ and $a\geq1$ by

$$\mathbb{V}^{ij}_{I_1,\cdots,I_a}:=\partial_j(\hat{V}_{[I_1]}\cdots\hat{V}_{[I_a]}\zeta^i)\qquad i,j\in\{1,\cdots,d\}\,.$$

For convenience, we set $\mathbb{V}_\phi:=I_{\mathbb{R}^d}$ if $I=\phi$. Then we easily see that for $i\in\{1,\cdots,d\}$ and $I_1,\cdots,I_a\in\mathbb{E}(\infty)$

$$\hat{V}_{[I_1]}\cdots\hat{V}_{[I_a]}\zeta^i=\sum_{j=1}^{d}V^j_{[I_1]}\mathbb{V}^{ij}_{I_2,\cdots,I_a}\,.\tag{4.5}$$

For each $\nu\geq1$, we define $\alpha_\nu=(\alpha^{iI}_\nu)_{i\in\{1,\cdots,d\},I\in\mathbb{G}((\nu))}\in C^\infty_b(\mathbb{R}^d,\mathbb{R}^d\otimes\mathbb{R}^{r((\nu))})$ and $M_\nu=(M^{ij}_\nu)_{i,j\in\{1,\cdots,d\}}\in C^\infty(\mathbb{R}^d\times\mathbb{R}^{r((\nu))},\mathbb{R}^d\otimes\mathbb{R}^d)$ as follows:

$$\alpha^{iI}_\nu(x):=V^i_{[I]}(x)\,,$$
$$M^{ij}_\nu(x,(u^I)_{I\in\mathbb{G}((\nu))}):=\delta^i_j+\sum_{a=1}^{\nu-1}\frac{1}{(a+1)!}\sum_{I_1,\cdots,I_a\in\mathbb{G}((\nu))}\mathbb{V}^{ij}_{I_1,\cdots,I_a}(x)u^{I_1}\cdots u^{I_a}\,.$$

Here, as before, we identify

$$\{(u^I)_{I\in G((\nu))};\ u^I\in \mathbb{R}^1,\ I\in G((\nu))\}$$

with $\mathbb{R}^{r((\nu))}=\mathbb{R}^{\sharp G((\nu))}$. Set $F_\nu(x, u):=M_\nu(x, u)\alpha_\nu(x)u\in C^\infty(\mathbb{R}^d\times\mathbb{R}^{r((\nu))}, \mathbb{R}^d)$. Then, from (4.3) and (4.5), we have

Proposition 4.6. *It holds that*

$$F_\nu^\varepsilon(t, x) = F_\nu(x, T_{((\nu))}^{((\varepsilon))}U_t^{((\nu))})\,.$$

Here $U_t^{((\nu))}:=(U_t^I)_{I\in G((\nu))}$ *and* $T_{((\nu))}^{((\varepsilon))}\in \mathrm{Hom}(\mathbb{R}^{r((\nu))}, \mathbb{R}^{r((\nu))})$ *is defined by* $T_{((\nu))}^{((\varepsilon))}((u^I)_{I\in G((\nu))}):=(\varepsilon^{||I||}u^I)_{I\in G((\nu))}$.

Now we consider the following condition: For some integer $\nu\geq 1$,

$$\mathcal{L.A.}\{\hat{V}_{[I]}(x);\ I\in\hat{E}((\nu))\} = \mathcal{L.A.}\{\hat{V}_{[I]}(x);\ I\in\hat{G}((\nu))\} = T_x(\mathbb{R}^d)\,. \tag{4.7}$$

Note that the first equality in (4.7) always holds from (1.5) (ii). We denote by $\nu_0=\nu_0(x)$ the smallest ν satisfying (4.7). The following proposition is due to Kusuoka-Stroock ([9]):

Proposition 4.8. *If the condition* (4.7) *is satisfied at* $x\in\mathbb{R}^d$, *then* $X^\varepsilon(1, x)\in \mathbb{D}^\infty(\mathbb{R}^d)$ *is non-degenerate in the Malliavin sense. More precisely, there exist a positive integer* k *depending only on* $\nu_0=\nu_0(x)$ *and, for each* $p\geq 1$, *a positive constant* $c=c(p, x)$ *such that*

$$||(\det \sigma_{X^\varepsilon(1,x)})^{-1}||_p\leq c\varepsilon^{-2k} \qquad \textit{for all} \quad \varepsilon>0\,.$$

Here $\sigma_{X^\varepsilon(1,x)}$ *stands for the Malliavin covariance of* $X^\varepsilon(1, x)$.

Thus, if the condition (4.7) is satisfied at $x\in\mathbb{R}^d$, then for any $T\in\mathcal{S}'(\mathbb{R}^d)$, $T(X^\varepsilon(1, x))\in\mathbb{D}^{-\infty}$ is defined for every $\varepsilon>0$ (cf. [4], [16], [17]). In particular, $\delta_y(X^\varepsilon(1, x))$ is defined for every $y\in\mathbb{R}^d$ and the generalized expectation $E[\delta_y(X^\varepsilon(1, x))]$ coincides with $p(\varepsilon^2, x, y)$, where $p(t, x, y)$ is the fundamental solution of the heat equation $\dfrac{\partial}{\partial t}=L$.

§ 5. The Asymptotic Expansion of $\delta_x(X^\varepsilon(1, x))$ as $\varepsilon\downarrow 0$

We shall continue working in the preceding section. Throughout this section, we fix $x_0\in\mathbb{R}^d$ and set for simplicity:

$$X^\varepsilon(t) := X^\varepsilon(t, x_0)\,,\quad F_\nu^\varepsilon(t) := F_\nu^\varepsilon(t, x_0)\,,\quad R_\nu^\varepsilon(t):= R_\nu^\varepsilon(t, x_0)\,,$$
$$\alpha_\nu := \alpha_\nu(x_0)\in\mathbb{R}^d\otimes\mathbb{R}^{r((\nu))}\,,\quad M_\nu(\cdot) := M_\nu(x_0, \cdot)\in C^\infty(\mathbb{R}^{r((\nu))}, \mathbb{R}^d\otimes\mathbb{R}^d)\,,$$
$$F_\nu(\cdot) := F_\nu(x_0, \cdot) = M_\nu(\cdot)\alpha_\nu\cdot\in C^\infty(\mathbb{R}^{r((\nu))}, \mathbb{R}^d).$$

Suppose that *for some* $\nu\geq 1$, *the condition* (4.7) *is satisfied at* $x_0\in\mathbb{R}^d$. In this section, we study the asymptotic expansion of $\delta_{x_0}(X^\varepsilon(1))$ in $\widetilde{\mathbb{D}^{-\infty}}$ as $\varepsilon\downarrow 0$. $\nu_0=\nu_0(x_0)$ is the smallest integer ν satisfying (4.7), i.e., it is a natural number such that

$$\mathcal{L.A.}\{\hat{V}_{[I]}(x_0);\, I\in\hat{G}((\nu_0-1))\}\subsetneq\mathcal{L.A.}\{\hat{V}_{[I]}(x_0);\, I\in\hat{G}((\nu_0))\} = T_{x_0}(\mathbb{R}^d)\,.$$

From this, we can find an $\mathbb{H}\subset\hat{G}((\nu_0))$ with $\sharp\mathbb{H}=d$ such that for each $a=1,\cdots,\nu_0$

$$\mathcal{L.A.}\{\hat{V}_{[I]}(x_0);\, I\in\hat{G}((a))\} = \mathcal{L.A.}\{\hat{V}_{[I]}(x_0);\, I\in\hat{G}((a))\cap\mathbb{H}\}\,. \tag{5.1}$$

We fix such an $\mathbb{H}$ to proceed with our discussion. Set

$$\beta := (V^i_{[I]}(x_0))_{i\in\{1,\cdots,d\},I\in\mathbb{H}}\in\mathbb{R}^d\otimes\mathbb{R}^d\,. \tag{5.2}$$

Clearly β is invertible. Define $\gamma=(\gamma^{IJ})_{I\in\mathbb{H},J\in G((\infty))}\in\mathbb{R}^d\otimes\mathbb{R}^\infty$ as follows:

$$\gamma := \beta^{-1}(V^i_{[I]}(x_0))_{i\in\{1,\cdots,d\},I\in G((\infty))}\,.$$

Then, from the choice of $\mathbb{H}$, γ has the following properties:

$$\begin{cases}\gamma^{IJ}=\delta^I_J & \text{if}\quad J\in\mathbb{H}\,,\\ \gamma^{IJ}=0 & \text{if}\quad \|I\|>\|J\| \text{ and } J\in\hat{G}((\nu_0))\,.\end{cases} \tag{5.3}$$

In the following, unless otherwise stated, we assume that $\nu\geq\nu_0$. Set

$$\gamma_{((\nu))}:=(\gamma^{IJ})_{I\in\mathbb{H},J\in G((\nu))}\,,\quad \hat{\gamma}_{((\nu))}:=(\gamma^{IJ})_{I\in\mathbb{H},J\in\hat{G}((\nu))}\,.$$

Then $\hat{\gamma}_{((\nu))}(\hat{\gamma}_{((\nu))})^*>0$ by (5.3). Recalling the definition of $M_\nu(\circ)$ and $F_\nu(\circ)$ given in § 4, we easily see that

$$M_\nu(0)=I_{\mathbb{R}^d}\,,\quad (\partial_I F^i_\nu(0))_{i\in\{1,\cdots,d\},I\in\hat{G}((\nu_0))}=\beta\hat{\gamma}_{((\nu_0))}\,.$$

Thus, we can choose a small $\kappa_\nu>0$ such that for any $u\in\mathbb{R}^{r((\nu))}$ such as $|u|\leq\kappa_\nu$

$$\begin{cases}(\partial_I F^i_\nu(u))_{i\in\{1,\cdots,d\},I\in\hat{G}((\nu_0))}((\partial_I F^i_\nu(u)_{i\in\{1,\cdots,d\},I\in\hat{G}((\nu_0))})^*\\ \geq\dfrac{1}{2}\beta\hat{\gamma}_{((\nu_0))}(\beta\hat{\gamma}_{((\nu_0))})^*>0\,,\end{cases} \tag{5.4}$$

$$\det M_\nu(u)\geq\frac{1}{2}\,. \tag{5.5}$$

Firstly, from (5.4) we shall present the following lemma: For this, let $\hat{U}^{((\nu))}_t:=(U^I_t)_{I\in\hat{G}((\nu))}\in\mathbb{D}^\infty(\mathbb{R}^{r((\nu))-1})$ and set

$$\hat{\pi}_{((\nu))}:=\inf\{\langle\hat{\sigma}_{((\nu))}\hat{l},\hat{l}\rangle;\,\hat{l}=(l^I)_{I\in\hat{G}((\nu))}\text{ such as }\sum_{I\in\hat{G}((\nu))}(l^I)^2=1\}\,.$$

Here $\hat{\sigma}_{((\nu))}$ is the Malliavin covariance of $\hat{U}_1^{((\nu))}$. Since $\hat{\sigma}_{((\nu))}=(\sigma_\nu^{IJ})_{I,J\in\hat{G}((\nu))}$, we see from (3.8) that $\hat{\pi}_{((\nu))}>0$ a.s. (P) and

$$(\hat{\pi}_{((\nu))})^{-1}\in L_{\infty-}\,. \tag{5.6}$$

Also, from (3.6) we see

$$U_t^{((\nu))}=\begin{bmatrix} t \\ \hat{U}_t^{((\nu))}\end{bmatrix}. \tag{5.7}$$

Now, for simplicity, let us denote by σ_ν^ε the Malliavin covariance of $F_\nu^\varepsilon(1)$. Then

Lemma 5.8. *For* $0<\varepsilon\leq 1$, *it holds that*

$$\sigma_\nu^\varepsilon\geq\frac{1}{2}\varepsilon^{2\nu_0}\hat{\pi}_{((\nu))}\beta\hat{\tau}_{((\nu_0))}(\beta\hat{\tau}_{((\nu_0))})^* \qquad a.s.\ on \quad \{|T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))}|\leq\kappa_\nu\}.$$

Proof. Let $0<\varepsilon\leq 1$. Since, by (4.6) and (5.7)

$$\begin{aligned} D(F_\nu^{\varepsilon,i}(1)) &= \sum_{I\in G((\nu))}\partial_I F_\nu^i(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))})\varepsilon^{\|I\|}DU_1^I \\ &= \sum_{I\in\hat{G}((\nu))}\partial_I F_\nu^i(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))})\varepsilon^{\|I\|}DU_1^I \qquad i\in\{1,\cdots,d\}, \end{aligned}$$

we observe that for $i,j\in\{1,\cdots,d\}$

$$\begin{aligned} (\sigma_\nu^\varepsilon)^{ij} &= \langle D(F_\nu^{\varepsilon,i}(1)),\, D(F_\nu^{\varepsilon,j}(1))\rangle \\ &= \sum_{I,J\in\hat{G}((\nu))}\partial_I F_\nu^i(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))})\partial_J F_\nu^j(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))})\varepsilon^{\|I\|+\|J\|}\sigma_\nu^{IJ}\,. \end{aligned}$$

Hence, for any $l\in\boldsymbol{R}^d$ it holds that

$$\begin{aligned} \langle\sigma_\nu^\varepsilon l,\, l\rangle &= \sum_{i,j=1}^d(\sigma_\nu^\varepsilon)^{ij}l^i l^j \\ &= \sum_{I,J\in\hat{G}((\nu))}\sigma_\nu^{IJ}\Big(\sum_{i=1}^d l^i\varepsilon^{\|I\|}\partial_I F_\nu^i(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))})\Big)\Big(\sum_{j=1}^d l^j\varepsilon^{\|J\|}\partial_J F_\nu^j(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))})\Big) \\ &\geq\hat{\pi}_{((\nu))}\sum_{I\in\hat{G}((\nu))}\varepsilon^{2\|I\|}\Big(\sum_{i=1}^d l^i\partial_I F_\nu^i(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))})\Big)^2 \\ &\geq\hat{\pi}_{((\nu))}\varepsilon^{2\nu_0}\sum_{I\in\hat{G}((\nu_0))}\Big(\sum_{i=1}^d l^i\partial_I F_\nu^i(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))})\Big)^2 \\ &= \varepsilon^{2\nu_0}\hat{\pi}_{((\nu))}|((\partial_I F_\nu^i(T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))}))_{i\in\{1,\cdots,d\},I\in\hat{G}((\nu_0))})^*l|^2\,. \end{aligned}$$

Consequently, combining this with (5.4), we obtain (5.8). //

The following is an immediate consequence of (5.6) and (5.8): For $0<\varepsilon\leq 1$, $\det\sigma_\nu^\varepsilon>0$ a.s. on $\{|T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))}|\leq\kappa_\nu\}$ and

$$(5.9)\qquad \begin{cases} (E[(\det \sigma_\nu^\varepsilon)^{-p};\ |T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))}|\leq \kappa_\nu])^{1/p} \\ \leq 2^d(\det \beta\hat{r}_{((\nu_0))}(\beta\hat{r}_{((\nu_0))})^*)^{-1}\|(\hat{\pi}_{((\nu))})^{-1}\|_{pd}^d\,\varepsilon^{-2d\nu_0} \qquad p\geq 1\,. \end{cases}$$

Secondly, we choose an $h\in C^\infty(\mathbb{R}^1)$ such that $0\leq h\leq 1$, $h(x)=1$ if $|x|\leq\frac{1}{2}$ and $h(x)=0$ if $|x|\geq 1$. Set $h_\nu(x):=h(x/(\kappa_\nu/2)^2)$ and define $\chi_\nu^\varepsilon\in\mathbb{D}^\infty$ by

$$\chi_\nu^\varepsilon := h_\nu(|T_{((\nu))}^{((\varepsilon))}U_1^{((\nu))}|^2)\,.$$

Let δ_0 be the Dirac delta-function at $0\in\mathbb{R}^d$. Then $\delta_0(X^\varepsilon(1)-x_0)$, $\chi_\nu^\varepsilon\cdot\delta_0(X^\varepsilon(1)-x_0)\in\widetilde{\mathbb{D}^{-\infty}}$ (in fact, $\in\bigcap_{p>1}\mathbb{D}_p^{-2([d/2]+1)}$). By (4.8) and (3.14), we can prove the following in the same way as in [17]:

Lemma 5.10. *For any $\nu\geq\nu_0$ and $p>1$, there exist positive constants c_1 and c_2 independent of ε such that*

$$\|\delta_0(X^\varepsilon(1)-x_0)-\chi_\nu^\varepsilon\cdot\delta_0(X^\varepsilon(1)-x_0)\|_{p,-2([d/2]+1)}\leq c_1\exp\left(-\frac{c_2}{\varepsilon^2}\right)\qquad as\quad \varepsilon\downarrow 0.$$

By virtue of (5.9), for any $T\in\mathcal{S}'(\mathbb{R}^d)$, $\chi_\nu^\varepsilon\cdot T(F_\nu^\varepsilon(1))\in\widetilde{\mathbb{D}^{-\infty}}$ is defined similarly to $T(X^\varepsilon(1)-x_0)$. More precisely, the mapping $\phi\in\mathcal{S}(\mathbb{R}^d)\mapsto\chi_\nu^\varepsilon\cdot\phi(F_\nu^\varepsilon(1))\in\mathbb{D}^\infty$ can be extended uniquely to a linear mapping

$$T\in\mathcal{S}'(\mathbb{R}^d)\mapsto\chi_\nu^\varepsilon\cdot T(F_\nu^\varepsilon(1))\in\mathbb{D}^{-\infty}$$

such that its restriction $T\in\mathcal{T}_{-2m}\mapsto\chi_\nu^\varepsilon\cdot T(F_\nu^\varepsilon(1))\in\mathbb{D}_p^{-2m}$ is continuous for every $p\in(1,\infty)$ and $m=0, 1, 2, \cdots$ (cf. [4], [16]). In fact, $\chi_\nu^\varepsilon\cdot T(F_\nu^\varepsilon(1))\in\widetilde{\mathbb{D}^{-\infty}}$ for every $T\in\mathcal{S}'(\mathbb{R}^d)$. In particular, if we take $T=\delta_0$, $\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))\in\bigcap_{p>1}\mathbb{D}_p^{-2([d/2]+1)}$. From (4.3) and (4.4), we note that

$$X^\varepsilon(1)-x_0-F_\nu^\varepsilon(1)=O(\varepsilon^{\nu+1})\qquad\text{in }\mathbb{D}^\infty(\mathbb{R}^d)\quad\text{as }\varepsilon\downarrow 0$$

(cf. [17]). By this, (5.9) and (4.8), we can also prove the following in the same way as in [17]:

Lemma 5.11. *There exists an increasing sequence $\{l_\nu=l_\nu(d,\nu_0)\}_{\nu\geq\nu_0}$ such that*

(i) $\lim_{\nu\uparrow\infty} l_\nu=+\infty$,

(ii) *for any $p>1$,* $\|\chi_\nu^\varepsilon\cdot\delta_0(X^\varepsilon(1)-x_0)-\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))\|_{p,-2([d/2]+2)}=O(\varepsilon^{l_\nu})$ *as* $\varepsilon\downarrow 0$.

Thus, from (5.10) and (5.11), it follows that for any $p>1$ and $\nu\geq\nu_0$

$$(5.12)\qquad \|\delta_0(X^\varepsilon(1)-x_0)-\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))\|_{p,-2([d/2]+2)}=O(\varepsilon^{l_\nu})\qquad\text{as}\quad\varepsilon\downarrow 0\,.$$

Thirdly, we shall present an available expression of $\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))$. As before, it is assumed that $\nu\geq\nu_0$. First of all, we note that for any $G\in\mathbf{D}^\infty$,

$$E[G\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))]=\lim_{\psi\to\delta_0}E[G\chi_\nu^\varepsilon\cdot\psi(F_\nu^\varepsilon(1))]\,. \tag{5.13}$$

For $\varepsilon>0$, the following matrices are defined:

$$\begin{aligned}
A^\varepsilon_{((\nu))}&:=(\varepsilon^{||J||-||I||}\gamma^{IJ})_{I\in\boldsymbol{H},J\in G((\nu))}\,,\\
\hat{A}^\varepsilon_{((\nu))}&:=(\varepsilon^{||J||-||I||}\gamma^{IJ})_{I\in\boldsymbol{H},J\in\hat{G}((\nu))}\,,\\
a^\varepsilon&:=(A^{\varepsilon,I(0)}_{((\nu))})_{I\in\boldsymbol{H}}=(\varepsilon^{2-||I||}\gamma^{I(0)})_{I\in\boldsymbol{H}}\,,\\
\hat{A}_{((\nu))}&:=(\delta^{||I||}_{||J||}\gamma^{IJ})_{I\in\boldsymbol{H},J\in\hat{G}((\nu))}\,,\\
B_{((\nu))}&:=(\delta^I_J)_{I\in G((\nu))\setminus\boldsymbol{H},J\in G((\nu))}\,,\\
\hat{B}_{((\nu))}&:=(\delta^I_J)_{I\in\hat{G}((\nu))\setminus\boldsymbol{H},J\in\hat{G}((\nu))}\,,\\
T^{((\varepsilon))}_{\boldsymbol{H}}&:=(\varepsilon^{||J||}\delta^I_J)_{I\in\boldsymbol{H},J\in\boldsymbol{H}}\,.
\end{aligned}$$

Set

$$\begin{cases}
N=N(x_0):=\sum_{I\in\boldsymbol{H}}||I|| \\
=\sum_{a=1}^{\infty}a(\dim \mathcal{L.S.}\{\hat{V}_{[I]}(x_0);\,I\in\hat{\boldsymbol{E}}((a))\} \\
\qquad -\dim \mathcal{L.S.}\{\hat{V}_{[I]}(x_0);\,I\in\hat{\boldsymbol{E}}((a-1))\})\,.
\end{cases} \tag{5.14}$$

Then, by recalling (5.3), the following is easily verified:

Lemma 5.15. *The following holds*:

(i) $\alpha_\nu=\beta\gamma_{((\nu))}$.

(ii) $A^\varepsilon_{((\nu))}=[a^\varepsilon\quad\hat{A}^\varepsilon_{((\nu))}]$, $B_{((\nu))}=\begin{bmatrix}1 & 0\\ 0 & \hat{B}_{((\nu))}\end{bmatrix}$ *and* $\gamma_{((\nu))}T^{((\varepsilon))}_{((\nu))}=T^{((\varepsilon))}_{\boldsymbol{H}}A^\varepsilon_{((\nu))}$.

(iii) $\lim_{\varepsilon\downarrow 0}\hat{A}^\varepsilon_{((\nu))}=\hat{A}_{((\nu))}$.

(iv) $\hat{A}_{((\nu))}(\hat{A}_{((\nu))})^*>0$ *and* $\begin{bmatrix}\hat{A}_{((\nu))}\\ \hat{B}_{((\nu))}\end{bmatrix}$ *is invertible.*

(v) $T^{((\varepsilon))}_{\boldsymbol{H}}$ *is invertible and* $\det T^{((\varepsilon))}_{\boldsymbol{H}}=\varepsilon^N$.

Thus, from the above (iii) and (iv), there exists an $\varepsilon_0=\varepsilon_0(\nu)>0$ such that for every $0<\varepsilon\leq\varepsilon_0$

$$\begin{cases}
\hat{A}^\varepsilon_{((\nu))}(\hat{A}^\varepsilon_{((\nu))})^*\geq\dfrac{1}{2}\hat{A}_{((\nu))}(\hat{A}_{((\nu))})^*>0\,,\\
\begin{bmatrix}\hat{A}^\varepsilon_{((\nu))}\\ \hat{B}_{((\nu))}\end{bmatrix}\begin{bmatrix}\hat{A}^\varepsilon_{((\nu))}\\ \hat{B}_{((\nu))}\end{bmatrix}^*\geq\dfrac{1}{2}\begin{bmatrix}\hat{A}_{((\nu))}\\ \hat{B}_{((\nu))}\end{bmatrix}\begin{bmatrix}\hat{A}_{((\nu))}\\ \hat{B}_{((\nu))}\end{bmatrix}^*>0\,.
\end{cases} \tag{5.16}$$

And, from (ii), it is easy to see that

$$(5.17)\quad \begin{cases} \begin{bmatrix} A^{\varepsilon}_{((\nu))} \\ B_{((\nu))} \end{bmatrix} U_1^{((\nu))} = \begin{bmatrix} A^{\varepsilon}_{((\nu))} U_1^{((\nu))} \\ 1 \\ \hat{B}_{((\nu))} \hat{U}_1^{((\nu))} \end{bmatrix}, \quad A^{\varepsilon}_{((\nu))} U_1^{((\nu))} = a^{\varepsilon} + \hat{A}^{\varepsilon}_{((\nu))} \hat{U}_1^{((\nu))}, \\ \begin{bmatrix} A^{\varepsilon}_{((\nu))} U_1^{((\nu))} \\ \hat{B}_{((\nu))} \hat{U}_1^{((\nu))} \end{bmatrix} = \begin{bmatrix} a^{\varepsilon} \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{A}^{\varepsilon}_{((\nu))} \\ \hat{B}_{((\nu))} \end{bmatrix} \hat{U}_1^{((\nu))}. \end{cases}$$

Consequently, putting (5.6), (5.16) and (5.17) together, we have that

$$(5.18)\quad A^{\varepsilon}_{((\nu))} U_1^{((\nu))} \textit{ and } \begin{bmatrix} A^{\varepsilon}_{((\nu))} U_1^{((\nu))} \\ \hat{B}_{((\nu))} \hat{U}_1^{((\nu))} \end{bmatrix} \textit{ are uniformly non-degenerate}$$

(cf. [17]). Also, from (4.6), (5.15) (i), (ii), (5.16) and (5.17), it is easily seen that

$$(5.19)\quad F^{\varepsilon}_{\nu}(1) = M_{\nu}(T^{((\varepsilon))}_{((\nu))}(C^{\varepsilon}_{((\nu))})^{-1} \begin{bmatrix} A^{\varepsilon}_{((\nu))} U_1^{((\nu))} \\ 1 \\ \hat{B}_{((\nu))} \hat{U}_1^{((\nu))} \end{bmatrix}) \beta T^{((\varepsilon))}_{H} A^{\varepsilon}_{((\nu))} U_1^{((\nu))} \qquad 0<\varepsilon\leq\varepsilon_0$$

where $C^{\varepsilon}_{((\nu))} := \begin{bmatrix} A^{\varepsilon}_{((\nu))} \\ B_{((\nu))} \end{bmatrix}$ is invertible for $0<\varepsilon\leq\varepsilon_0$ by (5.16).

Now, as to $\chi^{\varepsilon}_{\nu}\cdot\delta_0(F^{\varepsilon}_{\nu}(1))$, we present the following: Set $f_{\nu}\in C_0^{\infty}(\mathbb{R}^{r((\nu))})$ as follows:

$$f_{\nu}(u) := \frac{h_{\nu}(|u|^2)}{\det M_{\nu}(u)} \qquad u\in\mathbb{R}^{r((\nu))},$$

which is well-defined from (5.5) and the definition of h_{ν}. Then we have

Lemma 5.20. *For each* $0<\varepsilon\leq\varepsilon_0$

$$\chi^{\varepsilon}_{\nu}\cdot\delta_0(F^{\varepsilon}_{\nu}(1)) = \varepsilon^{-N}|\det\beta|^{-1} f_{\nu}(T^{((\varepsilon))}_{((\nu))} U_1^{((\nu))})\cdot\delta_0(A^{\varepsilon}_{((\nu))} U_1^{((\nu))}),$$

where δ_0 *is the Dirac delta-function at* $0\in\mathbb{R}^d=\mathbb{R}^{\sharp H}$.

Proof. Let $0<\varepsilon\leq\varepsilon_0$. For simplicity, set

$$\begin{aligned} T &:= T^{((\varepsilon))}_{((\nu))}, \quad T_H := T^{((\varepsilon))}_{H}, \quad C := C^{\varepsilon}_{((\nu))}, \\ V &:= A^{\varepsilon}_{((\nu))} U_1^{((\nu))} \in \{(v^I)_{I\in H}\} \simeq \mathbb{R}^d, \\ W &:= \hat{B}_{((\nu))} \hat{U}_1^{((\nu))} \in \{(w^I)_{I\in\hat{G}((\nu))\setminus H}\} \simeq \mathbb{R}^{r((\nu))-1-d}. \end{aligned}$$

Choose a $\psi\in C_0^{\infty}(\mathbb{R}^d)$ so that $\psi\geq 0$ and $\int_{\mathbb{R}^d}\psi(v)dv=1$. For $\lambda>0$, set $\psi_{\lambda}(v) := 1/\lambda^d\,\psi(v/\lambda)$. Let $G\in\mathbb{D}^{\infty}$ be fixed arbitrarily. By (5.13),

$$E[G\chi^{\varepsilon}_{\nu}\cdot\delta_0(F^{\varepsilon}_{\nu}(1))] = \lim_{\lambda\downarrow 0} E[G\chi^{\varepsilon}_{\nu}\cdot\psi_{\lambda}(F^{\varepsilon}_{\nu}(1))].$$

On the other hand, by (5.19) we observe that

$$E[G\chi_\nu^\varepsilon\cdot\psi_\lambda(F_\nu^\varepsilon(1))]$$

$$= E[Gh_\nu(|TC^{-1}\begin{bmatrix}V\\1\\W\end{bmatrix}|^2)\psi_\lambda(M_\nu(TC^{-1}\begin{bmatrix}V\\1\\W\end{bmatrix})\beta T_H V)]$$

$$= \int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^{r(\langle\nu\rangle)-1-d}} h_\nu(|TC^{-1}\begin{bmatrix}v\\1\\w\end{bmatrix}|^2)\psi_\lambda(M_\nu(TC^{-1}\begin{bmatrix}v\\1\\w\end{bmatrix})\beta T_H v)$$

$$\times\langle G,\ \delta_{\left[\begin{smallmatrix}v\\w\end{smallmatrix}\right]}\begin{pmatrix}V\\W\end{pmatrix}\rangle dv\,dw$$

$$= \varepsilon^{-N}\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^{r(\langle\nu\rangle)-1-d}} h_\nu(|TC^{-1}\begin{bmatrix}(T_H)^{-1}v\\1\\w\end{bmatrix}|^2)$$

$$\times\psi_\lambda(M_\nu(TC^{-1}\begin{bmatrix}(T_H)^{-1}v\\1\\w\end{bmatrix})\beta v)\langle G,\ \delta_{\left[\begin{smallmatrix}(T_H)^{-1}v\\w\end{smallmatrix}\right]}\begin{pmatrix}V\\W\end{pmatrix}\rangle dv\,dw$$

(by a change of variables: $v\mapsto(T_H)^{-1}v$ and (5.15) (v))

$$= \varepsilon^{-N}\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^{r(\langle\nu\rangle)-1-d}} h_\nu(|TC^{-1}\begin{bmatrix}(T_H)^{-1}v\\1\\w\end{bmatrix}|^2)$$

$$\times\frac{1}{\lambda^d}\psi(M_\nu(TC^{-1}\begin{bmatrix}(T_H)^{-1}v\\1\\w\end{bmatrix})\beta\frac{v}{\lambda})\langle G,\ \delta_{\left[\begin{smallmatrix}(T_H)^{-1}v\\w\end{smallmatrix}\right]}\begin{pmatrix}V\\W\end{pmatrix}\rangle dv\,dw$$

(by the definition of ψ_λ)

$$= \varepsilon^{-N}\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^{r(\langle\nu\rangle)-1-d}} h_\nu(|TC^{-1}\begin{bmatrix}(T_H)^{-1}\lambda v\\1\\w\end{bmatrix}|^2)$$

$$\times\psi(M_\nu(TC^{-1}\begin{bmatrix}(T_H)^{-1}\lambda v\\1\\w\end{bmatrix})\beta v)\langle G,\ \delta_{\left[\begin{smallmatrix}(T_H)^{-1}\lambda v\\w\end{smallmatrix}\right]}\begin{pmatrix}V\\W\end{pmatrix}\rangle dv\,dw$$

(by a change of variables: $v\mapsto\lambda v$).

Hence, letting $\lambda\downarrow 0$, we see

$$E[G\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))]$$

$$= \varepsilon^{-N}\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^{r(\langle\nu\rangle)-1-d}} h_\nu(|TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix}|^2)$$

$$\times\psi(M_\nu(TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix})\beta v)\langle G,\ \delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\begin{pmatrix}V\\W\end{pmatrix}\rangle dv\,dw$$

$$= \varepsilon^{-N}\int_{\boldsymbol{R}^{r(\langle\nu\rangle)-1-d}} h_\nu(|TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix}|^2)\langle G,\ \delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\begin{pmatrix}V\\W\end{pmatrix}\rangle dw$$

$$\times \int_{R^d} \psi(M_\nu(TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix})\beta v)\,dv$$

$$= \varepsilon^{-N} \int_{R^{r((\nu))-1-d}} h_\nu(|TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix}|^2)\langle G, \delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\binom{V}{W}\rangle$$

$$\times |\det \beta|^{-1}(\det M_\nu(TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix}))^{-1} dw$$

(by a change of variables: $v \mapsto (M_\nu(TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix})\beta)^{-1}v$ and by the fact: $\int_{R^d}\psi(v)dv=1$)

$$= \varepsilon^{-N}|\det\beta|^{-1}\int_{R^{r((\nu))-1-d}} f_\nu(TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix})\langle G, \delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\binom{V}{W}\rangle dw$$

(by the definition of f_ν)

$$= \varepsilon^{-N}|\det\beta|^{-1}\langle G, f_\nu(TU_1^{((\nu))})\cdot\int_{R^{r((\nu))-1-d}}\delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\binom{V}{W}dw\rangle .$$

Here the last equality has come from the fact:

$$f_\nu(TC^{-1}\begin{bmatrix}0\\1\\w\end{bmatrix})\langle G, \delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\binom{V}{W}\rangle = \langle G, f_\nu(TC^{-1}\begin{bmatrix}V\\1\\W\end{bmatrix})\delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\binom{V}{W}\rangle$$

$$= \langle G, f_\nu(TU_1^{((\nu))}\,\delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\binom{V}{W}\rangle .$$

Thus, since $G \in D^\infty$ is arbitrary, we obtain

$$\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1)) = \varepsilon^{-N}|\det\beta|^{-1} f_\nu(TU_1^{((\nu))})\cdot\int_{R^{r((\nu))-1-d}}\delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\binom{V}{W}dw ,$$

from which and the fact:

$$\int_{R^{r((\nu))-1-d}}\delta_{\left[\begin{smallmatrix}0\\w\end{smallmatrix}\right]}\binom{V}{W}dw = \delta_0(V),$$

(5.20) follows at once. //

Noting that $\hat{V}_0(x_0)=\sum_{I\in H}\gamma^{I(0)}\hat{V}_{[I]}(x_0)$, we define

$$\mu_0 := \max\{||I||;\ \gamma^{I(0)}\neq 0\} . \tag{5.21}$$

Here $\max\{\phi\}:=0$ for convenience. From its definition, the following is clear:

(5.22) $\begin{cases} \text{(i)} & \gamma^{I(0)}=0 \text{ for } I\in\boldsymbol{H} \text{ such as } \|I\|>\mu_0 . \\ \text{(ii)} & \text{If } \mu_0\geq 1, \text{ then } \gamma^{I(0)}\neq 0 \text{ for some } I\in\boldsymbol{H} \text{ with } \|I\|=\mu_0 . \end{cases}$

In the following, individually, we think of the case (A) and (B):

$$\begin{cases} \text{(A)} & \text{The case where } \mu_0\leq 2 . \\ \text{(B)} & \text{The case where } \mu_0\geq 3 . \end{cases}$$

First, we consider the case (A). From (5.20) we shall present the asymptotic expansion of $\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))$ in $\widetilde{\boldsymbol{D}^{-\infty}}$ as $\varepsilon\downarrow 0$ (cf. [17]): Note that by (5.22) (i), $\lim_{\varepsilon\downarrow 0} a^\varepsilon=a:=(\delta_2^{\|I\|}\gamma^{I(0)})_{I\in\boldsymbol{H}}$. We define $v_a=(v_a^I)_{I\in\boldsymbol{H}}$, $v_a^{((\nu))}=(v_a^{((\nu))I})_{I\in\boldsymbol{H}}\in\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ and $\Xi_a\in\widetilde{\boldsymbol{D}^{-\infty}}$, $a\geq 0$ as follows:

(5.23) $$\begin{cases} v_a^I := \sum_{J\in G((\infty))\,;\, \|J_\|\|=\|I\|+a} \gamma^{IJ} U_1^J & I\in\boldsymbol{H}, \\ v_a^{((\nu))I} := \sum_{J\in G((\nu))\,;\, \|J\|=\|I\|+a} \gamma^{IJ} U_1^J & I\in\boldsymbol{H}, \\ \Xi_0 := \delta_0(v_0) , & \\ \Xi_a := \sum_{l=1}^{a} \frac{1}{l!} \sum_{I_1,\cdots,I_l\in\boldsymbol{H}} \sum_{\substack{a_1\cdots a_l\geq 1 \\ a_1+\cdots+a_l=a}} v_{a_1}^{I_1}\cdots v_{a_l}^{I_l}\cdot(\partial_{I_1}\cdots\partial_{I_l}\delta_0)(v_0) & a\geq 1 . \end{cases}$$

Putting $v_a^{((\nu))}$ in place of v_a, we similarly define $\Xi_a^{((\nu))}\in\widetilde{\boldsymbol{D}^{-\infty}}$, $a\geq 0$. Since

$$v_0 = v_0^{((\nu))} = a+\hat{A}_{((\nu_0))}\hat{U}_1^{((\nu_0))} ,$$

$v_0=v_0^{((\nu))}\in\boldsymbol{D}^\infty(\boldsymbol{R}^d)$ is non-degenerate by (5.6) and (5.15) (iv), and so, Ξ_a, $\Xi_a^{((\nu))}$, $a\geq 0$ are well-defined. Note that $v_a^{((\nu))}=v_a$ for $0\leq a\leq\nu-\nu_0$ and hence

(5.24) $$\Xi_a^{((\nu))} = \Xi_a \qquad \textit{for any} \quad 0\leq a\leq\nu-\nu_0 .$$

From (5.3), (5.22) (i) and the definition of $A_{((\nu))}^\varepsilon$, it is easy to see that

$$A_{((\nu))}^\varepsilon U_1^{((\nu))} = \sum_{a=0}^{\nu-1} \varepsilon^a v_a^{((\nu))} .$$

Thus, by applying the general theory due to S. Watanabe ([17]), from this and (5.18), it follows that

(5.25) $$\delta_0(A_{((\nu))}^\varepsilon U_1^{((\nu))})\sim\Xi_0^{((\nu))}+\varepsilon\Xi_1^{((\nu))}+\varepsilon^2\Xi_2^{((\nu))}+\cdots \qquad \text{in } \widetilde{\boldsymbol{D}^{-\infty}} \text{ as } \varepsilon\downarrow 0 .$$

Next, as to the asymptotic expansion of $f_\nu(T_{((\nu))}^{((\varepsilon))} U_1^{((\nu))})$, we easily see that it is given by

(5.26) $$f_\nu(T_{((\nu))}^{((\varepsilon))} U_1^{((\nu))})\sim\xi_0^{((\nu))}+\varepsilon\xi_1^{((\nu))}+\varepsilon^2\xi_2^{((\nu))}+\cdots \qquad \text{in } \boldsymbol{D}^\infty \text{ as } \varepsilon\downarrow 0 ,$$

where $\xi_a^{((\nu))}\in\boldsymbol{D}^\infty$, $a\geq 0$ are defined as follows:

$$\xi_0^{((\nu))} := 1 ,$$

$$\xi_a^{((\nu))} := \sum_{l=1}^{a} \frac{1}{l!} \sum_{\substack{I_1,\cdots,I_l \in G((\nu)) \\ \|I_1\|+\cdots+\|I_l\|=a}} (\partial_{I_1} \cdots \partial_{I_l} f_\nu)(0)\, U_1^{I_1} \cdots U_1^{I_l}$$

$$= \sum_{l=1}^{a} \frac{1}{l!} \sum_{\substack{I_1,\cdots,I_l \in G((\nu)) \\ \|I_1\|+\cdots+\|I_l\|=a}} \left(\partial_{I_1} \cdots \partial_{I_l} \frac{1}{\det M_\nu}\right)(0)\, U_1^{I_1} \cdots U_1^{I_l} \qquad a \geq 1 .$$

Therefore, putting (5.20), (5.25) and (5.26) together, we have *the asymptotic expansion of* $\chi_\nu^\varepsilon \circ \delta_0(F_\nu^\varepsilon(1))$:

$$\text{(5.27)} \qquad \chi_\nu^\varepsilon \circ \delta_0(F_\nu^\varepsilon(1)) \sim \varepsilon^{-N} |\det \beta|^{-1} (\Xi_0^{((\nu))} + \sum_{a=1}^{\infty} \varepsilon^a \sum_{\substack{b,c \geq 0 \\ b+c=a}} \xi_b^{((\nu))} \circ \Xi_c^{((\nu))})$$

$$\textit{in } \widetilde{\mathbb{D}^{-\infty}} \textit{ as } \varepsilon \downarrow 0 .$$

Now, as to M_ν, we make a few remarks: Here, for a moment let $\nu \geq 1$. Define $e_{J_1,\cdots,J_d}(I)$ for $I \in \mathbb{E}(\infty)$, $J_1, \cdots, J_d \in \{\phi\} \cup \mathbb{E}(\infty)$ by

$$e_{J_1,\cdots,J_d}(I) := \begin{cases} 1 & \text{if } J_i = I \text{ and } J_1, \cdots, J_{i-1}, J_{i+1}, \cdots, J_d = \phi \\ & \text{for some } i \in \{1, \cdots, d\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that for any $I_1, \cdots, I_a \in \mathbb{E}(\infty)$ and $a \geq 1$

$$\text{(5.28)} \qquad \begin{cases} \partial_{I_1} \cdots \partial_{I_a} \det M_\nu \\ = \sum_{J_1^{(1)},\cdots,J_d^{(1)},\cdots,J_1^{(a)},\cdots,J_d^{(a)} \in \{\phi\} \cup \mathbb{E}(\infty)} e_{J_1^{(1)},\cdots,J_d^{(1)}}(I_1) \cdots e_{J_1^{(a)},\cdots,J_d^{(a)}}(I_a) \\ \times \det [\partial_{J_1^{(1)}} \cdots \partial_{J_1^{(a)}} M_{\nu,1}, \cdots, \partial_{J_d^{(1)}} \cdots \partial_{J_d^{(a)}} M_{\nu,d}] . \end{cases}$$

Here $M_{\nu,j}$ denotes the j-th column of M_ν and we set $\partial_\phi :=$ the identity operator if $J = \phi$. On the other hand, from the definition of M_ν, we also easily see that for $0 \leq a \leq \nu - 1$, $J_1, \cdots, J_a \in \{\phi\} \cup \mathbb{E}((\nu))$ and $\nu' \geq \nu$

$$(\partial_{J_1} \cdots \partial_{J_a} M_\nu)(0) = (\partial_{J_1} \cdots \partial_{J_a} M_{\nu'})(0) .$$

Hence, from this and (5.28) it follows that for $0 \leq a \leq \nu - 1$, $I_1, \cdots, I_a \in \mathbb{E}((\nu))$ and $\nu' \geq \nu$,

$$(\partial_{I_1} \cdots \partial_{I_a} \det M_\nu)(0) = (\partial_{I_1} \cdots \partial_{I_a} \det M_{\nu'})(0) .$$

In view of the explicit expression of $\partial_{I_1} \cdots \partial_{I_a} \dfrac{1}{\det M_\nu}$ in terms of $\partial_{J_1} \cdots \partial_{J_b} \times \det M_\nu$, $1 \leq b \leq a$, $J_1, \cdots, J_b \in \{I_1, \cdots, I_a\}$, this implies that for $0 \leq a \leq \nu - 1$, $I_1, \cdots, I_a \in \mathbb{E}((\nu))$ and $\nu' \geq \nu$

$$(5.29)\qquad \left(\partial_{I_1}\cdots\partial_{I_a}\frac{1}{\det M_\nu}\right)(0)=\left(\partial_{I_1}\cdots\partial_{I_a}\frac{1}{\det M_{\nu'}}\right)(0).$$

We continue the discussion in the case (A). We shall present the asymptotic expansion of $\delta_{x_0}(X^\varepsilon(1))=\delta_0(X^\varepsilon(1)-x_0)$ in $\widetilde{D^{-\infty}}$ as $\varepsilon\downarrow 0$. Let $\nu\geq\nu_0$ again. Set $f_{I_1,\cdots,I_a}\in\boldsymbol{R}^1$, $I_1,\cdots,I_a\in\boldsymbol{E}(\infty)$, $a\geq 1$ by

$$f_{I_1,\cdots,I_a}:=\left(\partial_{I_1}\cdots\partial_{I_a}\frac{1}{\det M_{\|I_1\|+\cdots+\|I_a\|+1}}\right)(0),$$

and define $\xi_a\in\boldsymbol{D}^\infty$, $a\geq 0$ by

$$(5.30)\qquad \begin{cases}\xi_0:=1,\\ \xi_a:=\sum_{l=1}^{a}\frac{1}{l!}\sum_{\substack{I_1,\cdots,I_l\in\boldsymbol{G}((\infty))\\ \|I_1\|+\cdots+\|I_l\|=a}} f_{I_1,\cdots,I_l}U_1^{I_1}\cdots U_1^{I_l}\qquad a\geq 1.\end{cases}$$

Then, from (5.29) and the definition of $\xi_a^{((\nu))}$, we see that

$$\xi_a^{((\nu))}=\xi_a\qquad \textit{for any}\quad 0\leq a\leq\nu-\nu_0.$$

Hence, by virtue of this and (5.24), (5.27) implies that

$$\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))=\varepsilon^{-N}|\det\beta|^{-1}(\Xi_0+\sum_{a=1}^{\nu-\nu_0}\varepsilon^a\sum_{\substack{b,c\geq 0\\ b+c=a}}\xi_b\cdot\Xi_c)+O(\varepsilon^{\nu-\nu_0+1-N})\quad \textit{in }\widetilde{D^{-\infty}}\textit{ as }\varepsilon\downarrow 0,$$

that is, there exists an $s=s(\nu)>0$ such that for any $p>1$

$$\|\chi_\nu^\varepsilon\cdot\delta_0(F_\nu^\varepsilon(1))-\varepsilon^{-N}|\det\beta|^{-1}(\Xi_0+\sum_{a=1}^{\nu-\nu_0}\varepsilon^a\sum_{\substack{b,c\geq 0\\ b+c=a}}\xi_b\cdot\Xi_c)\|_{p,-s}$$
$$=O(\varepsilon^{\nu-\nu_0+1-N})\qquad as\quad \varepsilon\downarrow 0.$$

Thus, combining this and (5.12), we have that for any $p>1$

$$\|\delta_{x_0}(X^\varepsilon(1))-\varepsilon^{-N}|\det\beta|^{-1}(\Xi_0+\sum_{a=1}^{\nu-\nu_0}\varepsilon^a\sum_{\substack{b,c\geq 0\\ b+c=a}}\xi_b\cdot\Xi_c)\|_{p,-(s\vee 2([d/2]+2))}$$
$$=O(\varepsilon^{(l_\nu+N)\wedge(\nu-\nu_0+1)-N})\qquad as\quad \varepsilon\downarrow 0.$$

Consequently, noting that by (5.11) (i), $(l_\nu+N)\wedge(\nu-\nu_0+1)$ tends to infinity as $\nu\uparrow\infty$, we obtain the following theorem:

Theorem 5.31. *Let $V_0, V_1,\cdots,V_n\in C_b^\infty(\boldsymbol{R}^d,\boldsymbol{R}^d)$ and $x_0\in\boldsymbol{R}^d$. Let $X^\varepsilon(t,x_0)$ denote the unique solution of* (4.1) (*for these V_i, $i=0,1,\cdots,n$*) *starting at x_0. Suppose that the condition* (4.7) *is satisfied at x_0 and that μ_0 defined by* (5.21) *is* *less than or equal to* 2. *Then* $\delta_{x_0}(X^{\varepsilon}(1, x_0))$ *has the following asymptotic expansion in* $\widetilde{\boldsymbol{D}^{-\infty}}$ *as* $\varepsilon \downarrow 0$:

$$\delta_{x_0}(X^{\varepsilon}(1, x_0)) \sim \varepsilon^{-N} |\det \beta|^{-1} \sum_{a=0}^{\infty} \varepsilon^{a} \Psi_a$$

and $\Psi_a \in \widetilde{\boldsymbol{D}^{-\infty}}$, $a \geq 0$ *are given by*

$$\Psi_0 = \Xi_0, \quad \Psi_a = \sum_{\substack{b,c\geq 0 \\ b+c=a}} \xi_b \circ \Xi_c \qquad a \geq 1 .$$

Here N, β, ξ_a *and* Ξ_a *are defined by* (5.14), (5.2), (5.30) *and* (5.23), *respectively.*

Second, we consider the case (B). In this case, we observe by (5.22) that

$$\begin{aligned} b^{\varepsilon} := \varepsilon^{\mu_0-2} a^{\varepsilon} &= \varepsilon^{\mu_0-2} \sum_{a=1}^{\mu_0} \varepsilon^{2-a} (\delta_a^{||I||} \gamma^{I(0)})_{I\in H} \\ &= \sum_{a<\mu_0} \varepsilon^{\mu_0-a} (\delta_a^{||I||} \gamma^{I(0)})_{I\in H} + (\delta_{\mu_0}^{||I||} \gamma^{I(0)})_{I\in H} \\ &\to (\delta_{\mu_0}^{||I||} \gamma^{I(0)})_{I\in H} = :b \neq 0 \qquad \text{as } \varepsilon \downarrow 0 . \end{aligned}$$

So we can take an $\varepsilon_1 > 0$ such that

$$(5.32) \qquad |b^{\varepsilon}| > \frac{1}{2} |b| > 0 \qquad \text{for any} \quad 0 < \varepsilon \leq \varepsilon_1 .$$

By (5.20) and (5.17),

$$(5.33) \qquad \chi_{\nu}^{\varepsilon} \cdot \delta_0(F_{\nu}^{\varepsilon}(1)) = \varepsilon^{-N} |\det \beta|^{-1} f_{\nu}(T_{((\nu))}^{((\varepsilon))} U_1^{((\nu))}) \delta_{-\varepsilon^{-(\mu_0-2)} b^{\varepsilon}} (\hat{A}_{((\nu))}^{\varepsilon} \hat{U}_1^{((\nu))}) \qquad 0 < \varepsilon \leq \varepsilon_0 .$$

Recall that $\hat{A}_{((\nu))}^{\varepsilon} \hat{U}_1^{((\nu))}$ is uniformly non-degenerate. From this fact, the following is easily seen: For any $p > 1$ and $m \geq 1$,

$$\sup_{v \in \boldsymbol{R}^d} \sup_{0<\varepsilon\leq\varepsilon_0} (1+|v|^2)^m ||\delta_v(\hat{A}_{((\nu))}^{\varepsilon} \hat{U}_1^{((\nu))})||_{p,-2([d/2]+1)} < +\infty .$$

Hence, combining this with (5.32) and (5.33), we see that for any $p > 1$, $m \geq 1$ and $0 < \varepsilon \leq \varepsilon_0 \wedge \varepsilon_1$

$$\begin{aligned} &||\chi_{\nu}^{\varepsilon} \cdot \delta_0(F_{\nu}^{\varepsilon}(1))||_{p,-2([d/2]+1)} \\ &\leq \varepsilon^{-N} |\det \beta|^{-1} C_{p,d} ||f_{\nu}(T_{((\nu))}^{((\varepsilon))} U_1^{((\nu))})||_{2p,2([d/2]+1)} \\ &\qquad \times ||\delta_{-\varepsilon^{-(\mu_0-2)} b^{\varepsilon}} (\hat{A}_{((\nu))}^{\varepsilon} \hat{U}_1^{((\nu))})||_{2p,-2([d/2]+1)} \\ &\leq \varepsilon^{2m(\mu_0-2)-N} |\det \beta|^{-1} \left(\frac{2}{|b|}\right)^{2m} C_{p,d} \sup_{0<\varepsilon\leq\varepsilon_0} ||f_{\nu}(T_{((\nu))}^{((\varepsilon))} U_1^{((\nu))})||_{2p,2([d/2]+1)} \\ &\qquad \times \sup_{v\in\boldsymbol{R}^d} \sup_{0<\varepsilon\leq\varepsilon_0} (1+|v|^2)^m ||\delta_v(\hat{A}_{((\nu))}^{\varepsilon} \hat{U}_1^{((\nu))})||_{2p,-2([d/2]+1)} . \end{aligned}$$

This asserts that $||\chi_{\nu}^{\varepsilon} \cdot \delta_0(F_{\nu}^{\varepsilon}(1))||_{p,-2([d/2]+1)} = O(\varepsilon^m)$ as $\varepsilon \downarrow 0$ for any $p > 1$ and

$m \geq 1$.Thus, putting this and (5.12) together, we obtain the following theorem:

Theorem 5.34. *Let $V_0, V_1, \cdots, V_n \in C_b^\infty(\mathbf{R}^d, \mathbf{R}^d)$ and $x_0 \in \mathbf{R}^d$. Let $X^\varepsilon(t, x_0)$ denote the unique solution of* (4.1) *starting at x_0. Suppose that the condition* (4.7) *is satisfied at x_0 and that μ_0 is more than* 2. *Then it holds that for any $p>1$ and $m \geq 1$*

$$\|\delta_{x_0}(X^\varepsilon(1, x_0))\|_{p,-2([d/2]+1)} = O(\varepsilon^m) \qquad as\ \varepsilon \downarrow 0\,.$$

§ 6. Diagonal Short Time Asymptotics

Suppose that $V_0, V_1, \cdots, V_n \in C_b^\infty(\mathbf{R}^d, \mathbf{R}^d)$ satisfy the condition (4.7) at $x_0 \in \mathbf{R}^d$. Let $X^\varepsilon(t, x_0)$ be the unique solution of (4.1) for V_i, $i=0, 1, \cdots, n$ starting at $x_0 \in \mathbf{R}^d$. By (4.8),

$$p(\varepsilon^2, x_0, x_0) = E[\delta_{x_0}(X^\varepsilon(1, x_0))\,, \tag{6.1}$$

where $p(t, x, y)$ is the fundamental solution of $\dfrac{\partial}{\partial t} = \dfrac{1}{2}\sum_{i=1}^n \hat{V}_i^2 + \hat{V}_0$. In this section, we study the short time asymptotic of $p(t, x_0, x_0)$. Let ν_0 be the smallest integer ν satisfying (4.7) at x_0 and let μ_0 be a nonnegative integer defined by (5.21).

First, in the case (B), i.e., the case where $\mu_0 \geq 3$, we easily see from (6.1) and Theorem (5.34) that

$$p(t, x_0, x_0) = O(t^m) \qquad \text{as } t \downarrow 0 \quad \text{for any } m \geq 1\,.$$

Next, as to the case (A), i.e., the case where $\mu_0 \leq 2$. By (6.1) and Theorem (5.31), we have obviously

$$p(\varepsilon^2, x_0, x_0) \sim \varepsilon^{-N} |\det \beta|^{-1} \sum_{a=0}^\infty \varepsilon^a E[\Psi_a] \qquad \text{as } \varepsilon \downarrow 0\,. \tag{6.2}$$

Recall (3.12) (ii). From this fact, we easily see that

$$\begin{cases} \text{(i)} & \xi_a(-w) = (-1)^a \xi_a(w) \qquad a \geq 0\,, \\ \text{(ii)} & v_a(-w) = (-1)^a T_H^{((-1))} v_a(w) \qquad a \geq 0\,, \end{cases} \tag{6.3}$$

where $T_H^{((\lambda))} \in \mathrm{Hom}(\mathbf{R}^d, \mathbf{R}^d)$ is defined by $T_H^{((\lambda))}((v^I)_{I \in H}) := (\lambda^{\|I\|} v^I)_{I \in H}$. Here, note that in general, it holds that for $l \geq 0$, $I_1, \cdots, I_l \in H$ and $v \in \mathbf{R}^d$

$$(\partial_{I_1} \cdots \partial_{I_l} \delta_v)(T_H^{((\lambda))} v_0) = |\lambda|^{-N} \lambda^{-(\|I_1\| + \cdots + \|I_l\|)} (\partial_{I_1} \cdots \partial_{I_l} \delta_{T_H^{((\lambda^{-1}))} v})(v_0) \qquad \lambda \neq 0\,.$$

Hence, combining this with (6.3) (ii), we also see that

$$\Xi_a(-w) = (-1)^a \Xi_a(w) \qquad a \geq 0 .$$

Thus, from this and (6.3) (i), it follows that $\Psi_a(-w)=(-1)^a\Psi_a(w)$ for $a\geq 0$, and as its consequence, we have that

$$E[\Psi_a] = 0 \qquad \text{if } a \text{ is odd}, \tag{6.4}$$

because the mapping $w \mapsto -w$ preserves the measure P. Therefore, by (6.2) and (6.4), we obtain that

$$p(t, x_0, x_0) \sim t^{-N/2} \sum_{b=0}^{\infty} |\det \beta|^{-1} E[\Psi_{2b}] t^b \qquad \text{as } t \downarrow 0 .$$

It remains to study the positivity of the first constant c_0 appearing in this asymptotic expansion, that is, the positivity of $E[\delta_0(v_0)]$. But, under the case where $\mu_0 \leq 2$ only, we are not able to show that. For this, we consider the following strong condition (A)′ or (A)″:

(A)′ $\begin{cases} \text{On some neighborhood } W \text{ of } x_0, \hat{V}_0 \text{ is represented as } \hat{V}_0 = \sum_{i=1}^{n} g^i \hat{V}_i \\ \text{where } g^i \in C^\infty(W), \quad 1 \leq i \leq n . \end{cases}$

(A)″ $\nu_0 \leq 3$ and $\mu_0 \leq 2$.

In what follows, under the condition (A)′ or (A)″, we shall show the positivity of $E[\delta_0(v_0)]$: First of all, recall $v_0 \in \mathbb{D}^\infty(\mathbb{R}^{\sharp H})$ given in (5.23):

$$\begin{cases} v_0 = a + \hat{A}_{((\nu_0))} \hat{U}_1^{((\nu_0))} = [a, \hat{A}_{((\nu_0))}] \begin{bmatrix} 1 \\ \hat{U}_1^{((\nu_0))} \end{bmatrix}, \\ [a, \hat{A}_{((\nu_0))}] = [(\delta_2^{\|I\|} r^{I(0)})_{I\in H}, (\delta_{\|J\|}^{\|I\|} r^{IJ})_{I\in H, J\in \hat{G}((\nu_0))}] . \end{cases} \tag{6.5}$$

First we consider the case (A)′. Set $\mathbb{E}^0(\nu) := \{I \in \mathbb{E}(\nu); \alpha(I)=0\}$ and $\mathbb{G}^0(\nu) := \mathbb{E}^0(\nu) \cap \mathbb{G}(\nu)$. In this case, it is easy to see that for each $I \in \mathbb{E}(\infty)$, there exist $g^{IJ} \in C^\infty(W)$, $J \in \mathbb{E}^0(|I|)$ such that

$$\hat{V}_{[I]} = \sum_{J \in \mathbb{E}^0(|I|)} g^{IJ} \hat{V}_{[J]} \qquad \text{on } W . \tag{6.6}$$

And, we can see from (6.6) that for each $\nu \geq 1$

$$\text{l.s.}\{\hat{V}_{[I]}(x_0); I \in \hat{\mathbb{G}}((\nu))\} = \text{l.s.}\{\hat{V}_{[I]}(x_0); I \in \mathbb{G}^0(\nu)\} .$$

Since, in particular, it follows from this that

$$T_{x_0}(\mathbb{R}^d) = \text{l.s.}\{\hat{V}_{[I]}(x_0); I \in \mathbb{G}^0(\nu_0)\} ,$$

an $\mathbb{H}$ appearing in (5.1) can be chosen in $\mathbb{G}^0(\nu_0)$: $\mathbb{H} \subset \mathbb{G}^0(\nu_0)$. From (6.6) and this choice of $\mathbb{H}$, (5.3) is rewritten as follows:

$$(6.7)\qquad \begin{cases} \text{(i)} & \gamma^{IJ} = \delta^I_{F} \quad \text{if } J \in \boldsymbol{H}, \\ \text{(ii)} & \gamma^{IJ} = 0 \quad \text{if } |I| \geq |J|+1 . \end{cases}$$

Hence, combining (6.7) (ii) with (6.5), we have that

$$v_0 = (\delta^{|I|}_{|J|}\,\gamma^{IJ})_{I\in \boldsymbol{H}, J\in G^0(\nu_0)}(U^J_1)_{J\in G^0(\nu_0)} \, .$$

Note that rank $[(\delta^{|I|}_{|J|}\,\gamma^{IJ})_{I\in \boldsymbol{H}, J\in G^0(\nu_0)}] = \sharp\boldsymbol{H} = d$ by (6.7) (i). Thus, from this fact and Proposition (A.13), the positivity of $E[\delta_0(v_0)]$ follows immediately.

The positivity of $E[\delta_0(v_0)]$ in the case (A)″ can be shown in the same way as above by noting that rank $\hat{A}_{((\nu_0))} = \sharp\boldsymbol{H} = d$.

Consequently, summarizing all the above, we have the following theorem:

Theorem 6.8. *The short time asymptotic of* $p(t, x, y)$ *at the diagonal* (x_0, x_0) *is as follows:*

(i) *If* $\mu_0 \leq 2$, *then*

$$p(t, x_0, x_0) \sim t^{-N/2} \sum_{b=0}^{\infty} c_b t^b \qquad \text{as } t \downarrow 0$$

with $c_b = |\det \beta|^{-1} E[\Psi_{2b}]$, $b \geq 0$. *Here* N *and* β *are defined by* (5.14) *and* (5.2), *respectively, and* Ψ_a *is given in Theorem* (5.31). *Further, if either the condition* (A)′ *or* (A)″ *holds, then* c_0 *is positive.*

(ii) *If* $\mu_0 \geq 3$, *then* $p(t, x_0, x_0) = O(t^m)$ *as* $t \downarrow 0$ *for any* $m \geq 1$.

The condition (A)′ on $\hat{V}_0$ is just one in Kusuoka-Stroock [10], and it clearly contains the case where $\hat{V}_0 \equiv 0$, which was treated by Léandre [11] and Ben Arous [1]. Also, in the elliptic case, i.e., the case where $\hat{V}_1(x_0), \cdots, \hat{V}_n(x_0)$ span the tangent space at x_0, this condition is automatically satisfied for any $\hat{V}_0$. The following is a simple example satisfying the condition (A)″ but not the condition (A)′:

Example. (Kolmogorov [6]). Let $d=2$, $n=1$ and $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $V_0 = \begin{bmatrix} 0 \\ x^1 \end{bmatrix}$. Then it is easily checked that for every $x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \in \boldsymbol{R}^2$, (i) the condition (A)′ is not satisfied, (ii) $N(x) = 4$, $\nu_0(x) = 3$ and

$$\mu_0(x) = \begin{cases} 0 & \text{if } x^1 = 0 \\ 3 & \text{if } x^1 \neq 0 . \end{cases}$$

Hence, by applying Theorem (6.8), we can obtain the diagonal short time asymptotic of the heat kernel $p(t, x, y)$ of $\dfrac{\partial}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2}{\partial (x^1)^2} + x^1 \dfrac{\partial}{\partial x^2}$. But, in this case,

$p(t, x, x)$ is concretely evaluated:

$$p(t, x, x) = \frac{\sqrt{3}}{\pi}\frac{1}{t^2}\exp\left(-\frac{6(x^1)^2}{t}\right) \qquad t>0,\ x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}.$$

So, when $x^1 \neq 0$, Theorem (6.8) (ii) is, for example, restated as follows:

$$\lim_{t \downarrow 0} -t \log p(t, x, x) = 6(x^1)^2 > 0 .$$

This example suggests to us that Theorem (6.8) (ii) has room for improvement.

Appendix

First of all, following Kusuoka-Stroock [9], we introduce the following notations for $I=(i_1, \cdots, i_a) \in \boldsymbol{E}(\infty)$:

$$I_* := i_a\,, \quad I' := \begin{cases} \phi & \text{if } a=1 \\ (i_1, \cdots, i_{a-1}) & \text{if } a \geq 2 . \end{cases}$$

Let us fix $\nu \geq 1$ and let $\boldsymbol{I} \subset \boldsymbol{E}(\nu)$ be a non-empty set satisfying the following:

$$\begin{cases} \text{If } (i_1, \cdots, i_a) \in \boldsymbol{I}, \text{ then } (i_1, \cdots, i_b), (i_{b+1}, \cdots, i_a) \in \boldsymbol{I} \text{ for any} \\ 1 \leq b \leq a-1, \text{ and } (i_{\sigma(1)}, \cdots, i_{\sigma(a)}) \in \boldsymbol{I} \text{ for any permutation } \sigma \in \mathfrak{S}_a . \end{cases}$$

Putting $\boldsymbol{I}$ in place of $\boldsymbol{E}(\nu)$, we trace our discussions in § 1, § 2 and § 3: Set

$$\boldsymbol{G}^I(\nu) := \boldsymbol{I} \cap \boldsymbol{G}(\nu)\,, \quad \hat{\boldsymbol{G}}^I(\nu) := \boldsymbol{G}^I(\nu) \setminus \{(0)\} .$$

Note that

(A.1) $$\boldsymbol{G}^I(\nu) = \hat{\boldsymbol{G}}^I(\nu) \qquad \text{if } (0) \notin \boldsymbol{I} .$$

As before, we identify

$$\{(y^I)_{I \in \boldsymbol{I}};\ y^I \in \boldsymbol{R}^1,\ I \in \boldsymbol{I}\} \simeq \boldsymbol{R}^{\sharp \boldsymbol{I}},$$

and the coordinate system on $\boldsymbol{R}^{\sharp \boldsymbol{I}}$ is also denoted by y^I, $I \in \boldsymbol{I}$. We understand $\boldsymbol{R}^{\sharp \boldsymbol{G}^I(\nu)}$ and $\boldsymbol{R}^{\sharp \hat{\boldsymbol{G}}^I(\nu)}$ similarly to $\boldsymbol{R}^{\sharp \boldsymbol{I}}$. Define $Q_i^{\boldsymbol{I}} \in \mathfrak{X}(\boldsymbol{R}^{\sharp \boldsymbol{I}})$, $i \in \boldsymbol{E}$ by

$$Q_i^{\boldsymbol{I}} := \begin{cases} \sum_{I \in \boldsymbol{I}\,;\, I_* = i} y^{I'} \dfrac{\partial}{\partial y^I} & \text{if } (i) \in \boldsymbol{I}, \\ 0 & \text{if } (i) \notin \boldsymbol{I} \end{cases}$$

where $y^\phi := 1$ for convenience, and denote by $\mathfrak{g}_{\boldsymbol{I}}$ the Lie subalgebra of $\mathfrak{X}(\boldsymbol{R}^{\sharp \boldsymbol{I}})$ generated by $Q_i^{\boldsymbol{I}}$, $i \in \boldsymbol{E}$. Then it holds that

(A.2) $\begin{cases} \boldsymbol{R}^{\sharp G^I(\nu)} \textit{ can be regarded as a Lie group with a multiplication} \times \\ \textit{defined by means of the Campbell-Hausdorff formula, and if } \mathfrak{h}_I \\ \textit{denotes the right invariant Lie algebra of } \boldsymbol{R}^{\sharp G^I(\nu)}, \textit{ then } \mathfrak{g}_I \textit{ is} \\ \textit{isomorphic to } \mathfrak{h}_I \textit{ under the correspondence } Q_i^I \leftrightarrow R_i^I . \end{cases}$

Here

$$R_i^I := \begin{cases} \text{an element of } \mathfrak{h}_I \text{ such as } R_i^I(0) = \left(\dfrac{\partial}{\partial u^i}\right)_0 & \text{if } (i) \in \boldsymbol{I}, \\ 0 & \text{if } (i) \notin \boldsymbol{I}. \end{cases}$$

Since $Q_{[I]}^I = 0$ for $I \in \boldsymbol{E}(\infty) \setminus \boldsymbol{I}$ and $(\Pi_I^\nu)_* Q_i^{(\nu)} = Q_i^I$, where $\Pi_I^\nu \in \mathrm{Hom}(\boldsymbol{R}^{\sharp E(\nu)}, \boldsymbol{R}^{\sharp I})$ is a projection to $\boldsymbol{R}^{\sharp I}$, the following follows from (A.2):

$$R_{[I]}^I = 0 \quad \text{for } I \in \boldsymbol{E}(\infty) \setminus \boldsymbol{I}. \tag{A.3}$$

$$(P_I^\nu)_* R_{[I]}^{(\nu)} = R_{[I]}^I \quad \text{for any } I \in \boldsymbol{E}(\infty), \tag{A.4}$$

where $P_I^\nu \in \mathrm{Hom}(\boldsymbol{R}^{\sharp G(\nu)}, \boldsymbol{R}^{\sharp G^I(\nu)})$ is a projection to $\boldsymbol{R}^{\sharp G^I(\nu)}$. Let

$\mathfrak{b}_I :=$ the Lie subalgebra generated by R_i^I, $i \in \{1, \cdots, n\}$,
$\mathfrak{l}_I :=$ the Lie subalgebra generated by R_i^I, $i \in \{0, 1, \cdots, n\}$,
$\mathfrak{i}_I :=$ the ideal in $\mathfrak{l}_I$ generated by R_i^I, $i \in \{1, \cdots, n\}$.

Then it is easy to see that

$$\begin{cases} \text{(i)} & \mathfrak{l}_I = \mathfrak{b}_I, \\ \text{(ii)} & \mathfrak{b}_I = \mathfrak{l}_I = \mathfrak{i}_I \quad \text{if } (0) \notin \boldsymbol{I}, \\ \text{(iii)} & \dim \mathfrak{i}_I = \sharp \hat{G}^I(\nu) = \sharp G^I(\nu) - 1 \quad \text{if } (0) \in \boldsymbol{I}. \end{cases} \tag{A.5}$$

Next, recalling $U_t^{(\infty)} = (U_t^I)_{I \in G(\infty)}$ introduced in § 3, we define

$$U_t^I := (U_t^I)_{I \in G^I(\nu)}, \quad \hat{U}_t^I := (U_t^I)_{I \in \hat{G}^I(\nu)} .$$

By (A.1) and (3.6), it is easy to see that

$$U_t^I = \begin{cases} \hat{U}_t^I & \text{if } (0) \notin \boldsymbol{I} \\ \begin{bmatrix} t \\ U_t^I \end{bmatrix} & \text{if } (0) \in \boldsymbol{I}. \end{cases} \tag{A.6}$$

Since $U_t^I = P_I^\nu U_t^{(\nu)}$, by combining (A.4) and (3.2), U_t^I is the unique solution of

$$\begin{cases} dU_t = \sum_{i \in E} R_i^I(U_t) \circ dw_t^i \\ U_0 = 0 . \end{cases}$$

Generally, for each $u \in \boldsymbol{R}^{\sharp G^I(\nu)}$, $U_t^I \times u$ is the unique solution of the above SDE starting at u. So, from this fact, we see that for any $t, s\geq 0$

$$(1.\mathrm{A})\qquad P(U^I_{t+s}\in du)=\int_{\mathbb{R}^{\sharp G^I(\nu)}} P(U^I_t\in du\times(-v))P(U^I_s\in dv)\qquad u\in\mathbb{R}^{\sharp G^I(\nu)}.$$

On the other hand, by Proposition (3.8), $\hat{U}^I_1(\in\mathbb{D}^\infty(\mathbb{R}^{\sharp\hat{G}^I(\nu)}))$ is non-degenerate in the Malliavin sense, and generally so is $\hat{U}^I_t$ for $t>0$. Hence, for each $t>0$, $\hat{U}^I_t$ has a smooth density p_t with respect to the Lebesgue measure on $\mathbb{R}^{\sharp\hat{G}^I(\nu)}$:

$$P(\hat{U}^I_t\in d\hat{u})=p_t(\hat{u})d\hat{u}\,,\qquad \hat{u}\in\mathbb{R}^{\sharp\hat{G}^I(\nu)}.$$

Thus, combining this with (A.6), we have

$$(\mathrm{A}.8)\qquad P(U^I_t\in du)=\begin{cases} p_t(u)du & \text{if } (0)\notin\mathcal{I}\,,\\ \delta_t(du^0)p_t(\hat{u})d\hat{u} & \text{if } (0)\in\mathcal{I}\,,\end{cases}$$

where $u=\begin{bmatrix} u^0\\ \hat{u}\end{bmatrix}$, $u^0\in\mathbb{R}^1$, $\hat{u}\in\mathbb{R}^{\sharp\hat{G}^I(\nu)}$, for $u\in\mathbb{R}^{\sharp G^I(\nu)}$.

We further view the case when $(0)\notin\mathcal{I}$. In this case, from (A.7) and (A.8), we easily see that for any $t, s>0$ and $u\in\mathbb{R}^{\sharp G^I(\nu)}$,

$$(\mathrm{A}.9)\qquad p_{t+s}(u)=\int_{\mathbb{R}^{\sharp G^I(\nu)}} p_t(u\times(-v))p_s(v)\,|\det\partial_u(u\times(-v))|\,dv\,.$$

On the other hand, from (A.5) (i), (ii), the support of $P(U^I_t\in\cdot)$ coincides with $\mathbb{R}^{\sharp G^I(\nu)}$ for $t>0$ (cf. [7], [14]). Hence, from this and (A.9), it follows immediately that $p_t(u)>0$ for any $t>0$ and $u\in\mathbb{R}^{\sharp G^I(\nu)}$.

Next we view the case when $(0)\in\mathcal{I}$. To this end, define a smooth function f: $\mathbb{R}^1\times\mathbb{R}^1\times\mathbb{R}^{\sharp\hat{G}^I(\nu)}\times\mathbb{R}^{\sharp\hat{G}^I(\nu)}\to\mathbb{R}^{\sharp\hat{G}^I(\nu)}$ such that the following holds: For $u^0, v^0\in\mathbb{R}^1$ and $\hat{u}, \hat{v}\in\mathbb{R}^{\sharp\hat{G}^I(\nu)}$

$$(\mathrm{A}.10)\qquad \begin{bmatrix} u^0\\ \hat{u}\end{bmatrix}\times\begin{bmatrix} v^0\\ \hat{v}\end{bmatrix}=\begin{bmatrix} u^0+v^0\\ f(u^0, v^0, \hat{u}, \hat{v})\end{bmatrix}.$$

Then it is easily seen that

(i) for fixed $u^0, v^0\in\mathbb{R}^1$ and $\hat{u}\in\mathbb{R}^{\sharp\hat{G}^I(\nu)}$, $f(u^0, v^0, \hat{u}, \cdot)$ is diffeomorphic,

(ii) for fixed $u^0, v^0\in\mathbb{R}^1$ and $\hat{v}\in\mathbb{R}^{\sharp\hat{G}^I(\nu)}$, $f(u^0, v^0, \cdot, \hat{v})$ is also diffeomorphic.

Further, putting (A.7), (A.8) and (A.10) together, we see that for any $t, s>0$ and $\hat{u}\in\mathbb{R}^{\sharp\hat{G}^I(\nu)}$

$$(\mathrm{A}.11)\ p_{t+s}(\hat{u})=\int_{\mathbb{R}^{\sharp\hat{G}^I(\nu)}} p_t(f(t+s, -s, \hat{u}, -\hat{v}))p_s(\hat{v})\,|\det\partial_{\hat{u}}f(t+s, -s, \hat{u}, -\hat{v})|\,d\hat{v}.$$

On the other hand, if we assume that

(A.12) $$[\mathfrak{b}_I, \mathring{\mathfrak{i}}_I](u) \subset \mathfrak{b}_I(u) \quad \text{for every } u \in \boldsymbol{R}^{\sharp G^I(\nu)},$$

then, by applying Theorem 6.1 in Kunita [7], this together with (A.5) (i), (iii) implies that the support of $P(\hat{U}_t^I \in \cdot)$ coincides with $\boldsymbol{R}^{\sharp \hat{G}^I(\nu)}$. Hence, under the assumption (A.12), it follows from this and (A.11) that $p_t(\hat{u})>0$ for any $t>0$ and $\hat{u} \in \boldsymbol{R}^{\sharp \hat{G}^I(\nu)}$.

$\boldsymbol{E}^0(\nu) := \{I \in \boldsymbol{E}(\nu);\ \alpha(I)=0\}$ is in the case when $(0) \notin I$; $\boldsymbol{E}((2))$ and $\boldsymbol{E}((3))$ are in the case when $(0) \in I$, and by virtue of (A.3), in these cases, the assumption (A.12) is satisfied. Therefore, from the above, we can state the following:

Proposition A.13. *For each* $t>0$, $(U_t^I)_{I \in \boldsymbol{E}^0(\nu) \cap G(\nu)}$, $\nu \geq 1$, $(U_t^I)_{I \in \hat{G}((2))}$ *and* $(U_t^I)_{I \in \hat{G}((3))}$ *have positive smooth densities.*

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