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A New Method Using the Circles of Curvature for Solving Equations in R^1

By

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Abstract

In this paper, we propose a numerical method using **the circles of curvature** for solving the equations in R^1, whose order of convergence is cubic. Some numerical examples are given, for which the method works well, while there is shown an example failed by means of **the Newton-Raphson's method**.

§1. A New Method for Solving Equations in R^1

Consider the equation

$$F(x) = 0 \tag{1}$$

in R^1 and let x_0 be an approximate solution for (1). As have been well known, the circle of curvature at the point $(x_0, y_0)=(x_0, F(x_0))$ on the curve $y=F(x)$ is given by

$$(x-x_0+\frac{y_0'(1+y_0'^2)}{y_0''})^2+(y-y_0-\frac{1+y_0'^2}{y_0''})^2=\frac{(1+y_0'^2)^3}{y_0''^2}, \tag{2}$$

provided that $F\in C^2$. Therefore, if we define the next approximation x_1 by the x-coordinate of the point at which the circle (2) intersects the x-axis, then we obtain the following iterative procedure for solving the equation (1):

$$(x_{n+1}-x_n+\frac{y_n'(1+y_n'^2)}{y_n''})^2=\frac{(1+y_n'^2)^3}{y_n''^2}-(y_n+\frac{1+y_n'^2}{y_n''})^2,$$

i.e.,

$$(x_{n+1}-x_n)^2+2B_n(x_{n+1}-x_n)+C_n=0 \tag{3}$$

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where $y_n=F(x_n)$, $y_n^{(i)}=F^{(i)}(x_n)$, $i=1,2$,

$$B_n = y_n'(1+y_n'^2)/y_n''$$

and

$$C_n = (y_n^2 y_n''+2y_n(1+y_n'^2))/y_n'' .$$

Substituting $(y_n/y_n')^2$ for $(x_{n+1}-x_n)^2$ in (2), we have

$$2B_n(x_{n+1}-x_n) = -C_n-y_n^2/y_n'^2 ,$$

i.e.,

$$x_{n+1} = x_n-(y_n^2 y_n''+2y_n y_n'^2)/(2y_n'^3) \qquad (n\geq 0) . \tag{4}$$

We shall call the procedure (4) as the first method of the curvature iteration.

On the one hand, if we solve the quadratic equation (3) with respect to $x_{n+1}-x_n$, then we have

$$x_{n+1}-x_n = -B_n+\text{sign}\,(B_n)(B_n^2-C_n)^{1/2} . \tag{5}$$

To avoid the loss of significant digits, we modify (5) as follows;

$$x_{n+1} = x_n-C_n/(B_n+\text{sign}\,(B_n)(B_n^2-C_n)^{1/2}) , \tag{6}$$

which we shall call the second method of the curvature iteration.

§2. Order of Convergence

To examine the order of convergence for (4), let α be a root of (1) and put

$$\varphi(x) = x-(y^2 y''+2yy'^2)/(2y'^3) .$$

Then, the procedure (4) may be written in the form of $x_{n+1}=\varphi(x_n)$, so that

$$\begin{aligned} x_{n+1}-\alpha &= \varphi(x_n)-\varphi(\alpha) \\ &= \varphi'(\alpha)(x_n-\alpha)+\frac{1}{2}\varphi''(\alpha)(x_n-\alpha)^2+\frac{1}{6}\varphi'''(\alpha)(x_n-\alpha)^3+\cdots \\ &= \frac{1}{6}\varphi'''(\alpha)(x_n-\alpha)^3+\cdots , \end{aligned}$$

since, as is easily seen, we have

$$\varphi'(\alpha) = \varphi''(\alpha) = 0 ,$$

and

$$\varphi'''(\alpha) = (2y'(\alpha)y''(\alpha)+3y''(\alpha)^2)/(2y'(\alpha)^2) ,$$

provided that $F\in C^5$. This implies that the order of convergence for (4) is cubic, if $F \in C^5$.

§3. Numerical Examples

Example 1. Table 3–1 shows the results of computation for both the Newton-Raphson method and the second method applied to the equation $\sin(2.1x-0.6)=0$ with the approximation, $x_0=1$. As we have observed, the second method converges, but the Newton-Raphson's method fails. The same situation occurs for the value x_0 in the interval [1, 2].

(Table 3–1)

Times of iteration	Newton Raphson	Curvature Iteration
1	−5.7149519920349	1.69681475779063
2	−5.6982652358711	1.78123734436566
3	−5.6982718813119	1.78171084762747
4	−5.6982721689230	
	....	
y	failure	(second formula) −8.7422783679D–08

* Here "1" is given as an initial approximation.

Example 2. In Table 3–2, we compare the methods (4) and (5) with the Newton-Raphson's method for the following equations.

1. $F(X) = X^3 - X^2 - 1 = 0$
2. $F(X) = X^4 - 3X^3 - X^2 + 2X + 3 = 0$
3. $F(X) = X^5 - 2X^4 - 4X^3 + X^2 + 5X + 3 = 0$
4. $F(X) = X^6 - 8X^4 - 4X^3 + 7X^2 + 13X + 6 = 0$
5. $F(X) = X^7 + X^6 - 8X^5 - 12X^4 + 3X^3 + 20X^2 + 19X + 6 = 0$

(Table 3–2)

Degree of Equation	Accuracy of the Last Root to Be Found			Times of Iteration		
	Newton	First Formula	Second Formula	Newton	First Formula	Second Formula
(*1) 3	E-016	E-016	E-016	5	4	4
(*1) 4	E-016	E-016	E-016	5	3	3
(*1) 5	E-014	E-015	E-015	4	4	4
(*1.5) 6	E-015	E-015	E-015	3	2	2
(*1.5) 7	E-015	E-014	E-015	3	2	2

(* Initial approximation)

As confirmed in Table 3–2, the number of iterations with which our method converges is fewer than the Newton-Raphson's method.