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An Extension of Hodge Theory to Kähler Spaces with Isolated Singularities of Restricted Type

By

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Introduction

The present article is a continuation of the author's previous works [15] through [20], where degenerations of Hodge's spectral sequences have been observed on several non-compact Kähler manifolds. Here we shall be concerned with a problem related to a conjecture of Cheeger-Goreski-MacPherson on the coincidence of L^2 cohomology and intersection cohomology of projective varieties (cf. [C-G-M]). Let X be a compact complex space of pure dimension n equipped with a Kähler metric ds^2, let $\sum$ be the set of singular points of X and let $X_*:=X\backslash\sum$.

We denote by $H^r(X_*)$, $H^r_0(X_*)$ and $H^r_{(2)}(X_*)$, respectively the r-th de Rham cohomology of X_*, the r-th de Rham cohomology of X_* with compact support, and the L^2 de Rham cohomology of X_*, all with coefficients in $\mathbb{C}$. Correspondingly $H^{p,q}(X_*)$, $H^{p,q}_0(X_*)$ and $H^{p,q}_{(2)}(X_*)$ shall denote the Dolbeault cohomologies of type (p, q).

Our main result is stated as follows:

Theorem *If* $\dim\sum=0$, *then there exists a complete Kähler metric* ds^2_* *on* X_* *whose Kähler class is the same as that of* ds^2, *such that*

$$(1)\qquad H^r_{(2)}(X_*)\cong\begin{cases} H^r(X_*) & \text{if } r<n \\ Im(H^n_0(X_*)\to H^n(X_*)) & \text{if } r=n \\ H^r_0(X_*) & \textit{if } r>n \end{cases}$$

$$(2)\qquad H^{p,q}_{(2)}(X_*)\cong\begin{cases} H^{p,q}(X_*) & \text{if } p+q<n-1 \\ H^{p,q}_0(X_*) & \text{if } p+q>n+1\,, \end{cases}$$

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if the following condition $(*)$ *is satisfied.*

$(*)$ $\dim \sum = 0$ *and there exists a desingularization* $\pi\colon \tilde{X} \to X$ *by blowing up such that* $\pi^{-1}(\sum)$ *is a disjoint union of nonsingular divisors.*

Corollary 1. *Under the above situation, the multiplication L by the Kahler class of* ds^2 *induces the bijections*

$$L^k\colon H^{n-k}(X_*) \to H_0^{n+k}(X_*), \qquad k \neq 0. \tag{3}$$

Corollary 2. *Let X be as above, and let m be an odd integer. Then*

$$\dim_{\boldsymbol{C}} IH^m(X) \equiv 0 \qquad mod\, 2, \tag{4}$$

where IH^m *denotes the intersection cohomology in the sense of Goreski-MacPherson.*

In case $\dim X = 2$, M. Nagase [13] has established the relation (1) for the original Kähler metric ds^2 (see also [12]).

For higher dimensional varieties with isolated singularities, L. Saper [22] establishes (1) for certain complete Kähler metrics which are nonequivalent to ours, although he did it under a very restrictive assumption on the singularity as we do here. The publication of the present paper might be admitted because we have proved (1) by a completely different method.

§ 1. L^2 Estimates for the Exterior Derivative

Let $\varDelta \subset \boldsymbol{C}$ be the open unit disc centered at the origin, let $\varDelta_* = \varDelta \setminus \{0\}$, and let $(z_1, \cdots, z_{m+1})$ be the coordinate of $\boldsymbol{C}^{m+1}$. We put $z = (z_1, \cdots, z_{m+1})$, $|z| = \max |z_j|$ and $f(z) = z_{m+1}$. The modulus of the function f will be denoted by r. Let ds_e^2 be the euclidean metric on $\boldsymbol{C}^{m+1}$. We define a metric ds_f^2 on $\varDelta^m \times \varDelta_*$ by

$$ds_f^2 = \frac{ds_e^2}{(1-\ln r)\ln^2(2-\ln r)} + \frac{df d\bar{f}}{r^2(1-\ln r)^2 \ln^2(2-\ln r)}.$$

We shall first ask for estimates of the solutions to the equations $du = v$ on $\varDelta^m \times \varDelta_*$. The L^2 norms with respect to ds_e^2 and ds_f^2 will be denoted by $\|\ \|_e$ and $\|\ \|_f$, respectively.

Lemma 1. *For any square integrable* $(m+1)$*-form* v *on* $(\varDelta^m \times \varDelta_*, ds_f^2)$ *satisfying* $dv = 0$, and supp $v \subset \varDelta^m \times \{z \in \varDelta_* \mid |z| < \frac{1}{2}\}$ *there exist a neighbourhood* $U \ni o$ *and a measurable m-form u on* $U \cap (\varDelta^m \times \varDelta_*)$, *square integrable with respect to* ds_f^2, *such that* $du = v$ *on* $U \cap (\varDelta^m \times \varDelta_*)$.

Proof. We expand v as

$$v = \sum_{\mu \in \mathbf{Z}} e^{i\mu\theta}(v'_\mu + d\theta \wedge v''_\mu),$$

where $\theta = \arg f$ and v'_μ, v''_μ are orthogonal to the forms $d\theta \wedge$. Then we have $dv'_\mu = 0$ and $i\mu v'_\mu - dv''_\mu = 0$ for all μ. Since $\deg v'_\mu = m+1$ and $\deg v''_\mu = m$, we have

$$(1) \qquad ||v'_\mu||_e \leq ||v'_\mu||_f$$

and

$$(2) \qquad ||v''_\mu||_e \leq ||d\theta \wedge v''_\mu||_f .$$

Moreover, it is clear that

$$(3) \qquad ||v''_\mu||_f \leq ||d\theta \wedge v''_\mu||_f .$$

From (2) and (3), the series $u' := \sum_{\mu \neq 0} i\mu^{-1} e^{i\mu\theta} v''_\mu$ converges with respect $||\ \ ||_f$ to and $||\ \ ||_e$ so that $||u'||_e < \infty$, $||u'||_f < \infty$ and

$$v'_0 + du' = v'_0 + d\theta \wedge v''_0 .$$

We put

$$v'_0 = \alpha + dr \wedge \beta ,$$

where α is orthogonal to the forms $dr \wedge \cdot$.

By assumption we have

$$\alpha = dr \wedge \beta = 0 \quad \text{on} \quad r^{-1}\left(\left[\frac{1}{2}, 1\right)\right).$$

Then, integrating $dr \wedge \beta$ along the gradient vector field of r, we obtain an m-form

$$\xi = \xi(r) := \int_{1/2}^{r} dt \wedge \beta$$

which is defined on $U \cap (\Delta^m \times \Delta_*)$ for some neighbourhood $U \ni 0$ and satisfies $\operatorname{supp} \xi \subset U \cap (\Delta^m \times \{z \in \Delta_* \mid |z| < \frac{1}{2}\})$.

Then we have

$$d\xi = (d_r + d')\xi = dr \wedge \beta + \int_{1/2}^{r} dt \wedge d'\beta$$

$$= dr \wedge \beta + \int_{1/2}^{r} d_r \alpha = dr \wedge \beta + \alpha ,$$

where d_r denotes the exterior derivative with respect to r and we put $d' = d - d_r$. The square integrability with respect to ds_f^2 follows immediately from the integral inequality

$$(4) \qquad \int_r^{1/2} s^{-1}(2-\ln s)^{-2} \ln^{-2}(2-\ln s) \left| \int_{1/2}^{s} g(t)dt \right|^2 ds$$
$$\leq \int_r^{1/2} s^{-1}(2-\ln s)^{-2} \left| \int_{1/2}^{s} g(t)dt \right|^2 ds$$
$$\leq 4 \int_r^{1/2} s\,|g(s)|^2 ds \qquad \text{for} \quad r \in \left(0, \frac{1}{2}\right),$$

which holds for any continuous function g: $\left(0, \frac{1}{2}\right] \to \boldsymbol{C}$.

Thus it only remains to solve the equation $du = d\theta \wedge v_0''$. But it is similar as above and left to the reader.

Lemma 1 shall be used to prove the vanishing of the middle L^2 cohomology around isolated singular points. As we have shown in earlier papers [18]$\sim$[20], the vanishing of higher L^2 cohomology groups follows from a very general argument by applying an estimate on complete Kähler manifolds due to Donnelly and Fefferman [7].

Lemma 2 *Let u be any compactly supported C^∞ r-form on a Kähler manifold (X, ds^2) of dimension n with a global potential function φ (i.e. $ds^2 = \partial\bar{\partial}\varphi$ on X). Then*

$$||u|| \leq 4 \sup \{ |\partial\varphi|_p \,|\, p \in \operatorname{supp} u \} \; (||du|| + ||d^*u||)$$

whenever $r \neq n$. Here d^* denotes the adjoint of d, and $||u||$ denotes the L^2-norm of u.

Theorem 3. *Let (X, ds^2) be a complete Kähler manifold of dimension n and $D \subset X$ an open subset. Suppose that there exists a proper C^∞ map φ: $D \to (c_0, \infty)$ for some $c_0 \in \boldsymbol{R} \cup \{-\infty\}$, such that*

1) *The eigenvalues of $i\partial\bar{\partial}\varphi$ are larger than a positive constant on D.*
2) $\sup_D |\partial\varphi| < \infty$.

Then, for any non-critical value c of φ and for any square integrable k-form (resp. (p, q)-form) v on $D_c := \{x \in D;\ \varphi(x) < c\}$ with $k > n$ (resp. $p+q > n$) and $dv = 0$ (resp. $\bar{\partial}v = 0$), there exists on $D_{c'}$, for any $c' < c$, a square integrable $(k-1)$-form *(resp. $(p, q-1)$-form) u such that $du=v$ (resp. $\bar{\partial}u=v$).*

Proof. See Theorem 1.1 in [18].

Remark. A metric of type ds_f^2 was first introduced by H. Grauert in [8] to show that every smooth (not necessarily compact) projective variety admits a complete Kahler metric. A remarkable property of ds_f^2 is that it admits a bounded potential function.

§ 2. Proof of Theorem

Let (X, ds^2) be a compact Kähler space of pure dimension n with isolated singular points $\sum$, and let $X_*=X\setminus\sum$. In virtue of Hironaka's desingularization theorem, there exists a Kähler manifold $\tilde{X}$ and a proper holomorphic map $\pi: \tilde{X}\to X$ which is a biholomorphism on $\pi^{-1}(X_*)$. One can take $\tilde{X}$ so that $E:=\pi^{-1}(\sum)$ is supported on a divisor of simple normal crossings and there exists an effective divisor E_* on $\tilde{X}$ supported on $|E|$ such that $[-E_*]$ is very ample (cf. [10]). Similarly as in [17], we have then a positive C^∞ function ψ on X_* such that

1) $\partial\bar{\partial}\ln\psi$ is extended smoothly along E as a metric on a neighbourhood $W\supset E$ and $-\ln\psi|W>1$.

2) $\ln\psi-\ln|s|^2$ is C^∞ on $\tilde{X}$, where s is a canonical section of $[E_*]$ and $|s|$ denotes the length of s with respect to some C^∞ metric of the bundle.

Let ρ be a nonnegative C^∞ function such that supp $\rho\Subset W$ and $\rho\equiv 1$ on a neighbourhood $U\supset E$. We put

$$ds_*^2 = Nds^2+\partial\bar{\partial}(\rho\ln^{-1}\ln^2\psi)\,.$$

Let $\pi: \tilde{X}\to X$ be as in the hypothesis of Theorem to be proved, and fix a positive constant N so that ds_*^2 is a complete Kähler metric on X_*. Since ds_*^2 is asymptotically equivalent to the metric of type ds_f^2 near each $p\in E$, we obtain the following.

Lemma 4. *Let $U\supset E$ be a neighbourhood and let u be a d-closed square integrable n-form on $U\setminus E$ satisfying* supp $u\cup E\Subset U$. *Then there exists a square integrable $(n-1)$-form u on $U\setminus E$ satisfying $du=u$.*

Proof is similar as in Lemma 1.

For any open set $V\subset X_*$, we denote by $L^k(V)$ (resp. $L^{p,q}(V)$) the set of square integrable k-forms (resp. (p, q)-forms) on V with respect to ds_*^2.

Definition. Let V be as above. We put

$$H^k_{(2)}(V) := \{f \in L^k(V);\ df = 0\} / \{g \in L^k(V);\ \exists u \in L^{k-1}(V) \text{ s.t. } g = du\}$$
$$H^{p,q}_{(2)}(V) := \{f \in L^{p,q}(V);\ \bar\partial f = 0\} / \{g \in L^{p,q}(V);\ \exists u \in L^{p,q-1}(V) \text{ s.t. } g = \bar\partial u\}.$$

Then we have the following exact sequences:

$$(4) \qquad \lim_K H^{k-1}_{(2)}(X_* \backslash K) \to H^k_0(X_*) \to H^k_{(2)}(X_*) \to \lim_K H^k_{(2)}(X_* \backslash K)$$
$$(5) \qquad \lim_K H^{p,q-1}_{(2)}(X_* \backslash K) \to H^{p,q}_0(X_*) \to H^{p,q}_{(2)}(X_*) \to \lim_K H^{p,q}_{(2)}(X_* \backslash K),$$

where K runs through the compact subsets of X_*. As a consequence we obtain

Lemma 5. *If* $\lim_K H^{k-1}_{(2)}(X_* \backslash K) = \lim_K H^k_{(2)}(X_* \backslash K) = 0$ (*resp.* $\lim_K H^{p,q-1}_{(2)}(X_* \backslash K) = \lim_K H^{p,q}_{(2)}(X_* \backslash K) = 0$), *then*

$$H^k_0(X_*) \simeq H^k_{(2)}(X_*) \ (\textit{resp. } H^{p,q}_0(X_*) \simeq H^{p,q}_{(2)}(X_*)).$$

Since the function $\varphi = \ln^{-1} \ln^2 \psi$ satisfies that

$$\partial\bar\partial\varphi \geq \partial\varphi\bar\partial\varphi$$

on a neighbourhood of E, combining Theorem 3 with Lemma 5 we obtain the isomorphisms

$$H^k_0(X_*) \simeq H^k_{(2)}(X_*)$$

and

$$H^{p,q}_0(X_*) \simeq H^{p,q}_{(2)}(X_*)$$

for $k,\ p+q > n+1$.

By Poincaré and Serre's duality, taking the finiteness of $\dim H^{p,q}_0(X_*)$ $(p+q > n+1)$ and $\dim H^{p,q}(X_*)$ $(p+q > n-1)$ into account (cf. [2]), we have

$$H^k(X_*) \simeq H^k_{(2)}(X_*)$$

and

$$H^{p,q}(X_*) \simeq H^{p,q}_{(2)}(X_*)$$

if $k,\ p+q < n-1$.

The proof of Theorem will be finished if we show the following.

Proposition 6.

$$\lim_K H^n_{(2)}(X_* \backslash K) = 0.$$

In fact, from the exact sequence (4) one has

$$H^{n+1}_{(2)}(X_*)\cong H^{n+1}_0(X_*)\,,$$

hence by the Poincaré duality

$$H^{n-1}_{(2)}(X_*)\cong H^{n-1}(X_*)\,.$$

As for the n-th L^2 cohomology, the map

$$H^n_0(X_*)\to H^n_{(2)}(X_*)$$

is surjective, therefore the map

$$H^n_{(2)}(X_*)\to H^n(X_*)$$

is injective. Hence we have

$$H^n_{(2)}(X_*)\cong \mathrm{Im}\,(H^n_0(X_*)\to H^n(X_*))\,.$$

Proof of Proposition 6: The L^2 vanishing shall be reduced to a vanishing theorem which has nothing to do with L^2 conditions. We note that the proof we give below does not use the assumption that Sing $|E|=\phi$. Therefore, the following is valid for any compact complex space X with isolated singularities and any desingularization $\pi\colon \tilde{X}\to X$ by blowing up.

Lemma. *If $\tilde{X}\backslash K$ is homotopically equivalent to E, the homomorphism*

$$H^n(\tilde{X}\backslash K)\to H^n(X_*\backslash K)$$

is a zero map.

Proof. It suffices to show that the map

$$\iota\colon\ H^n_0(\tilde{X}\backslash K)\to H^n(\tilde{X}\backslash K)$$

is surjective. Since $\dim H^n_0(\tilde{X}\backslash K)=\dim H^n(\tilde{X}\backslash K)$, the surjectivity will follow from the injectivity of ι. Since we may assume that $\tilde{X}\backslash K$ is an arbitrarily small neighbourhood of E, we may assume that $\tilde{X}\backslash K$ is biholomorphically equivalent to an open subset U of a nonsingular projective variety Y such that the image $\hat{Y}$ of Y under the blow down along E is projective algebraic. (Artin's theorem, cf. [4]). Let $Z\subset Y$ be a nonsingular divisor which does not intersect with E and defines an ample divisor on $\hat{Y}$. Shrinking U if necessary, we may assume that $Z\cap U=\phi$. Then, applying the Morse theory as in Andreotti-Frankel [1], we have

$$H^k(Y\backslash Z)\cong H(U) \qquad \text{for} \quad k>n$$

and that the restriction map

$$H^n(Y\backslash Z) \to H^n(U)$$

is surjective.

Sublemma. *The restriction map*

$$H^n(Y) \to H^n(U)$$

is surjective.

Proof. Note that the Green operator commutes with the complex exterior derivatives ∂ and $\bar{\partial}$. Since the Gysin map $H^{n-1}(Z)\to H^{n+1}(Y)$ is of type (1, 1), and Y is Kählerian the above property of the Green operator implies the following. Let v be any d-cosed C^∞ n-form on $Y\backslash Z$ with logarithmic poles along Z. Let $v=v_{0,n}+\cdots+v_{n,0}$ be the decomposition into different types. Suppose that $v_{p,q}=0$ for $p\leq k$ for some integer k. Then there exists a C^∞ d-closed n-form v' on $Y\backslash Z$, with logarithmic poles along Z, such that $v'_{k,n-k}|U=0$, $v'_{p,q}=0$ for $p\leq k$, and $\mathrm{res}_Z(v-v')_{k,n-k}=0$. Here $v'_{p,q}$ and $(v-v')_{p,q}$ denote the (p, q) components of v' and $v-v'$, respectively, and res_Z denotes the residue along Z. Combining this fact with the surjectivity of $H^n(Y\backslash Z)\to H^n(U)$, the surjectivity of $H^n(Y)\to H^n(U)$ follows immediately.

Now we proceed to prove the injectivity of ι.

Let v be a C^∞ compactly supported d-closed n-form on U, and suppose that there exists a C^∞ $(n-1)$-form u on U with $du=v$. We shall show that the harmonic representative v_h of v as a cohomology class on Y is zero. By Sublemma, it will then follow that v represents zero in $H^n_0(U)$. Let $v_h=v_p+Lv_n$ be the decomposition into the primitive and nonprimitive parts. Then

$$\begin{aligned}
&\int_Y v_h\wedge\bar{*}v_h \\
&=\int_Y v\wedge\bar{*}v_h \\
&=\int_U v\wedge\bar{*}v_p+\int_U v\wedge\bar{*}Lv_n \\
&=\int_U v\wedge{}_p\bar{*}v+\int_U v\wedge Lw\,.
\end{aligned}$$

Here w is some C^∞ d-closed $(n-2)$-form on Y.

Since $Lw=\frac{i}{2}\partial\bar{\partial}\ln\psi\wedge w$ on $U\backslash E$ and v is d-exact on U, we have

$$\int_U v\wedge Lw=0\,.$$

Since $v_p\perp Lv_n$, we have

$$\int_Y v_h\wedge\bar{*}v_h=\int_Y v_p\wedge\bar{*}v_p\,.$$

Thus $Lv_u=0$, which impies that v_p is d-exact on U. Note that

$$\int_U v\wedge\overline{v_p}$$
$$=(-1)^{n(n+1)/2}\int_U v\wedge\overline{Cv_p}\,,$$

where C denotes the Weil's operator. Since $v_p|_U$ is d-exact, $Cv_p|_U$ must be d-exact, too, since C is compatible with the canonical spectral sequence which abuts to $H^*(E)$ on the varieties with normal crossings.

Therefore,

$$\int_U v\wedge\overline{Cv_p}=0\,.$$

Thus

$$\int_Y v_h\wedge\bar{*}v_h=0$$

which impiles that $v_h=0$, and the proof of Lemma is completed.

Now we shall finish the proof of Proposition 6. By the Lemma, the image of the homomorphism

$$\alpha\colon \varinjlim_K H^n_{(2)}(X_*\backslash K)\to\varinjlim_K H^n(X_*\backslash K)$$

is zero. The injectivity of α follows immediately from Lemma 4. Therefore $\varinjlim_K H^n_{(2)}(X_*\backslash K)=0$. Q.E.D.

Corollaries 1 and 2 are straightforward applications of our theorem.

Remark. In case X is projective algebraic, the corollaries have been obtained by Navaro Aznar [14] and Morihiko Saito [21] independently by different methods. As for the basic results in this direction, see also [5] and [6].

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Added in proof. The author must apologize to the reader that our result is not so satisfactory. He promises to give a complete result, i.e. one without any restriction on the singularity, in a forthcoming article.