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Asymptotic Behavior of Pseudo-Resolvents on Some Grothendieck Spaces

By

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Abstract

For a pseudo-resolvent $\{J_\lambda; \lambda \in \Omega \subset C\}$ of operators on a Grothendieck space X, it is proved that the strong convergence of λJ_λ as $\lambda \to 0$ [resp. $|\lambda| \to \infty$] is equivalent to that $\|\lambda J_\lambda\| = O(1)$ $(\lambda \to 0)$ [resp. $|\lambda| \to \infty$] and $\overline{R(\lambda J_\lambda^* - I^*)} = w^*-cl(R(\lambda J_\lambda^* - I^*))$ [resp. $\overline{R(J_\lambda^*)} = w^*-cl(R(J_\lambda^*))$]. If, in addition, X has the Dunford-Pettis property, then the strong convergence implies the uniform convergence. It is also shown that if a semigroup of class (E) on such a space is strongly Abel-ergodic at zero, then it must be uniformly continuous.

§ 1. Introduction

Let $\mathcal{B}(X)$ denote the set of all bounded linear operators on a Banach space X. A family $\{J_\lambda;\ \lambda \in \Omega\}$ of operators in $\mathcal{B}(X)$ is called a *pseudo-resolvent* on $\Omega \subset C$ if

$$J_\lambda - J_\mu = (\mu - \lambda) J_\lambda J_\mu\,, \qquad (\lambda,\ \mu \in \Omega)\,.$$

It is known that the ranges $R(J_\lambda)$, $R(\lambda J_\lambda - I)$, and the null spaces $N(J_\lambda)$, $N(\lambda J_\lambda - I)$ are independent of the parameter λ (cf. [6, p. 215]).

The strong convergence and the uniform convergence of λJ_λ as $\lambda \to 0$ or $|\lambda| \to \infty$ have been studied in Yosida [7] and Shaw [5], respectively. The results were obtained for general Banach spaces. In this note we investigate the strong convergence of λJ_λ on a Grothendieck space and the uniform convergence on a Grothendieck space with the Dunford-Pettis property.

A Banach space X is called a *Grothendieck space* if every w*-convergent sequence in the dual space X^* is weakly convergent. X is said to have the *Dunford-Pettis property* if $\langle x_n, x_n^* \rangle \to 0$ whenever $\{x_n\} \subset X$ tends weakly to 0 and $\{x_n^*\} \subset X^*$ tends weakly to 0. Examples of a Grothendieck space with the Dunford-Pettis property include L^∞, $B(S, \Sigma)$, $H^\infty(D)$, etc. (see [3].) On such a space, the weak convergence, the strong convergence, and the uniform con-

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vergence of λJ_λ are seen to be equivalent. This is similar to the recent results of Lotz [2, 3] and of Shaw [4] on the continuity and the ergodicity of operator semigroups and of cosine operator functions, repectively.

§ 2. Strong Ergodic Theorems

We shall denote by P [resp. Q] the mapping: $x \to \text{s-}\lim_{\lambda\to 0} \lambda J_\lambda x$ [resp. $\text{s-}\lim_{|\lambda|\to\infty} \lambda J_\lambda x$]. First, for the sake of convenience we state Yosida's theorem [7, pp. 217–218] in the following form:

Theorem 1. *If* $\|\lambda J_\lambda\| = O(1)$ $(\lambda \to 0)$ *[resp.* $|\lambda| \to \infty$*] and* $0 \in \bar{\Omega}$ *[resp.* Ω *is unbounded], then* P *[resp.* Q*] is a bounded linear projection with* $R(P) = N(\lambda J_\lambda - I)$ *and* $N(P) = \overline{R(\lambda J_\lambda - I)}$ *[resp.* $R(Q) = \overline{R(J_\lambda)}$ *and* $N(Q) = N(J_\lambda)$*]. Moreover,* $x \in D(P)$ *[resp.* $D(Q)$*] if and only if there is a sequence* λ_n *tending to* 0 *[resp.* ∞*] such that* $\text{w-}\lim_{n\to\infty} \lambda_n J_{\lambda_n} x$ *exists.*

It follows from the last assertion that strong convergence and weak convergence are equivalent. Also, λJ_λ is strongly convergent whenever X is a reflexive space. Since $D(P)^\perp = \{N(\lambda J_\lambda - I) \oplus \overline{R(\lambda J_\lambda - I)}\}^\perp = N(\lambda J_\lambda - I)^\perp \cap N(\lambda J_\lambda^* - I^*)$, λJ_λ is strongly convergent as $\lambda \to 0$ if and only if $\|\lambda J_\lambda\| = O(1)$ $(\lambda \to 0)$, and $N(\lambda J_\lambda - I)$ separates $N(\lambda J_\lambda^* - I^*)$ (i.e. $N(\lambda J_\lambda - I)^\perp \cap N(\lambda J_\lambda^* - I^*) = \{0\}$).

The following theorem gives another characterization of the strong convergence of λJ_λ in the case that X is a Grothendieck space.

Theorem 2. *Let* X *be a Grothendieck space,* λJ_λ *is convergent in the strong operator topology as* $\lambda \to 0$ *[resp.* $|\lambda| \to \infty$*] if and only if* $\|\lambda J_\lambda\| = O(1)$ $(\lambda \to 0)$ *[resp.* $|\lambda| \to \infty$*] and* $\overline{R(\lambda J_\lambda^* - I^*)} = w^* - cl(R(\lambda J_\lambda^* - I^*))$ *[resp.* $\overline{R(J_\lambda^*)} = w^* - cl(R(J_\lambda^*))$*].*

Proof. We only prove the case "$\lambda \to 0$;" a similar argument works for the other case "$|\lambda| \to \infty$."

First, suppose that $P = \text{so-}\lim_{\lambda\to 0} \lambda J_\lambda$ exists. Then clearly one has $\|\lambda J_\lambda\| = O(1)$ $(\lambda \to 0)$, by the uniform boundedness principle. X being Grothendieck, it follows that $\text{w-}\lim_{n\to\infty} \lambda_n J_{\lambda_n}^* x^* = \text{w}^*\text{-}\lim_{n\to\infty} \lambda_n J_{\lambda_n}^* x^* = P^* x^*$ for any sequence $\{\lambda_n\} \to 0$ and any $x^* \in X^*$. Applying Theorem 1 to the pseudo-resolvent $\{J_\lambda^*\}$ we see that $P^* = \text{so-}\lim_{\lambda\to 0} \lambda J_\lambda^*$. Hence we have

$$\begin{aligned}\overline{R(\lambda J_\lambda^* - I^*)} &= N(P^*) = R(P)^\perp = N(\lambda J_\lambda - I)^\perp \\ &= [{}^\perp R(\lambda J_\lambda^* - I^*)]^\perp = w^* - cl(R(\lambda J_\lambda^* - I^*))\,.\end{aligned}$$

Conversely, if $||\lambda J_\lambda||=O(1)$ $(\lambda\to 0)$ and $\overline{R(\lambda J_\lambda^*-I^*)}=w^*\text{—}cl(R(\lambda J_\lambda^*-I^*)$ $(=N(\lambda J_\lambda-I)^\perp)$, then Theorem 1, applied to $\{J_\lambda\}$ and $\{J_\lambda^*\}$, implies that $D(P)=N(\lambda J_\lambda-I)\oplus\overline{R(\lambda J_\lambda-I)}$ and $\overline{R(\lambda J_\lambda^*-I^*)}\cap N(\lambda J_\lambda^*-I^*)=\{0\}$, so that $D(P)^\perp=\{0\}$. This shows that $D(P)=X$ because it is closed.

§ 3. Uniform Ergodic Theorems

For a pseudo-resolvent $\{J_\lambda\}$ on a general Banach space X, the uniform convergence of λJ_λ is characterized in the following theorem, which was proved in [5].

Theorem 3. (i) $\operatorname{uo-lim}_{\lambda\to 0}\lambda J_\lambda$ *exists if and only if* $||\lambda^2 J_\lambda||\to 0$ *as* $\lambda\to 0$ *and* $R(\lambda J_\lambda-I)$ *is closed.*

(ii) $\operatorname{uo-lim}_{|\lambda|\to\infty}\lambda J_\lambda=Q$ *exists if and only if* $||J_\lambda||\to 0$ *as* $|\lambda|\to\infty$ *and* $R(J_\lambda)$ *is closed, if and only if* $J_\lambda=Q(\lambda I-A)^{-1}$ *where* $Q^2=Q$, $A\in B(X)$ *and* $AQ=QA=A$.

In general the strong convergence of λJ_λ is weaker than the uniform convergence. But it is to be shown that these two kinds of convergence coincide in the class of Grothendieck spaces with the Dunford-Pettis property. To prove this we need the following lemma of Lotz [3].

Lemma 4. *Let* $\{V_n\}$ *be a sequence of operators on a Banach space* X *with the Dunford-Pettis property. Suppose that* $\text{w-lim}\, V_n x_n=0$ *for every bounded sequence* $\{x_n\}$ *in* X *and* $\text{w-lim}\, V_n^* x_n^*=0$ *for every bounded sequence* $\{x_n^*\}$ *in* X^*. *Then* $||V_n^2||\to 0$. *In particular,* V_n-I *and* V_n+I *are invertible for large* n.

Theorem 5. *Let* $\{J_\lambda\}$ *be a pseudo-resolvent on a Grothendieck space* X *with the Dunford-Pettis property. The following statements are equivalent*:

(1) $||\lambda J_\lambda||=O(1)$ $(\lambda\to 0)$ *and for each* $x\in X$ *there is a sequence* $\lambda_n\to 0$ *such that* $\text{w-lim}_{n\to\infty}\lambda_n J_{\lambda_n}x$ *exists.*

(2) $P:=\operatorname{so-lim}_{\lambda\to 0}\lambda J_\lambda$ *exists.*

(3) $||\lambda J_\lambda-P||\to 0$ *as* $\lambda\to 0$.

(4) $||\lambda^2 J_\lambda||\to 0$ *as* $\lambda\to 0$, *and* $\overline{R(\lambda J_\lambda-I)}$ *is closed.*

(5) $||\lambda J_\lambda||=O(1)$ $(\lambda\to 0)$ *and* $R(\lambda J_\lambda^*-I^*)=w^*-cl(R(\lambda J_\lambda^*-I^*))$.

Proof. "(1)⇔(2)", "(2)⇔(5)", and "(3)⇔(4)" are contained in Theorem 1, Theorem 2, and Theorem 3 (i), respectively. Thus it remains to show that (2) implies (3).

Suppose (2) holds. Then Theorem 1 implies that $X=R(P)\oplus N(P)$ and

$R(P)=N(\lambda J_\lambda - I)$ for all $\lambda\in\Omega$. So, in order to prove that $\|\lambda J_\lambda - P\|\to 0$, it is no loss of generality to assume that $P=0$.

Let $V_n=n^{-1}J_{1/n}$. Then s-lim $V_n x=Px=0$ for all $x\in X$ so that $\{V_n^* x_n^*\}$ converges weakly* and hence weakly to zero for any bounded sequence $\{x_n^*\}$ in X^*. In particular, $\{n^{-1}J_{1/n}^* x^*\}$ converges weakly to zero for all $x^*\in X^*$. Now Theorem 1 applies to $\{J_\lambda^*\}$ to yield that $\{V_n^* x^*\}$ converges strongly to zero for all $x^*\in X^*$. Hence $\{V_n x_n\}$ converges weakly to zero for any bounded sequence $\{x_n\}$ in X. It follows from Lemma 4 that $V_n - I$ is invertible for large n.

Finally, it follows from the estimate

$$\begin{aligned}\|\lambda J_\lambda\| &\leq \|\lambda J_\lambda(n^{-1}J_{1/n}-I)\|\ \|(V_n-I)^{-1}\| \\ &= \left\|\left(\frac{1}{n}-\lambda\right)^{-1}\left[\lambda^2 J_\lambda - \frac{\lambda}{n}J_{1/n}\right]\right\|\ \|(V_n-I)^{-1}\|\end{aligned}$$

that $\|\lambda J_\lambda\|\to 0$ as $\lambda\to 0$. This proves the theorem.

If we let V_n be $nJ_n - I$, then a similar argument as above, together with Theorems 1, 2, and 3 (ii), will give the following uniform ergodic theorem for the case "$|\lambda|\to\infty$". This is a slight extension of a result of Lotz [2] which treated the case $Q=I$ and did not include conditions (4) and (5).

Theorem 6. *Let X be a Grothendieck space with the Dunford-Pettis property. The following statements are equivalent*:

(1) $\|\lambda J_\lambda\|=O(1)$ $(|\lambda|\to\infty)$, *and for each* $x\in X$ *there is a sequence* $\{\lambda_n\}$, $|\lambda_n|\to\infty$, *such that* $\text{w-}\lim_{n\to\infty}\lambda_n J_{\lambda_n}x$ *exists.*

(2) $Q:=so\text{-}\lim_{|\lambda|\to\infty}\lambda J_\lambda$ *exists.*

(3) $\|\lambda J_\lambda - Q\|\to 0$ *as* $|\lambda|\to\infty$.

(4) $\|J_\lambda\|\to 0$ *as* $|\lambda|\to\infty$, *and* $R(J_\lambda)$ *is closed.*

(5) $\|\lambda J_\lambda\|=O(1)$ $(|\lambda|\to\infty)$ *and* $\overline{R(J_\lambda^*)}=w^*-cl(R(J_\lambda^*))$.

(6) $J_\lambda=Q(\lambda I-A)^{-1}$ *for some* $Q, A\in\mathcal{B}(X)$ *satisfying* $Q^2=Q, AQ=QA=A$.

§ 4. Uniform Continuity of Semigroups and Cosine Functions

Lotz [2] has proved that every semigroup of class (A) in the sense of [1, p. 342] on a Grothendieck space with the Dunford-Pettis property is uniformly continuous. In what follows we shall apply Theorem 6 and a theorem of Hille [1, Theorem 18.8.3] to deduce a slight generalization.

A semigroup $\{T(t); t>0\}$ of type w_0 is said to be of class (E) if

(a) $T(\cdot)$ is strongly continuous on $(0, \infty)$;

(b) $X_0 := \{x; \int_0^1 ||T(t)x||dt < \infty\}$ is dense in X;

(c) the linear operator $R(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)xdt$ is defined on X_0 for each $\lambda > w_0$, (see [1, p. 509]). It is known that $\{R(\lambda); \lambda > w_0\}$ is a pseudo-resolvent (cf. [1, p. 510]).

$T(\cdot)$ is said to be *strongly* (resp. *uniformly*) *Abel-ergodic* to Q at zero if $\lambda R(\lambda)$ converges to Q in the strong (resp. uniform) operator topology as λ tends to infinity. Theorem 18.8.3 of [1] asserts that if $T(\cdot)$ is of class (E) and is uniformly Abel-ergodic to Q at zero, then $T(t) = Q\ exp(tA)$ with $Q^2 = Q$, $A \in \mathcal{B}(X)$ and $AQ = QA = A$. We combine this with Theorem 6 to formulate the following result.

Corollary 7. *Let $T(\cdot)$ be a semigroup of class (E) on a Grothendieck space with the Dunford-Pettis property. If $T(\cdot)$ is strongly Abel-ergodic to Q at zero, then $T(t) = Q\ exp(tA)$ where $Q^2 = Q$, $A \in \mathcal{B}(X)$ and $AQ = QA = A$.*

In particular, if $T(\cdot)$ is a semigroup of class (A), then $T(\cdot)$ belongs to the class (E) and it is strongly Abel-ergodic to I at zero. Corollary 7 shows that $T(t) = exp(tA)$ with $A \in \mathcal{B}(X)$ and so is uniformly continuous.

We close this section with another application of Theorems 1 & 6. Let A be the generator of a strongly continuous cosine operator function $C(\cdot)$ on a Grothendieck space with the Dunford-Pettis property. Then $\overline{D(A)} = X$, and there are constants $M > 0$ and $w > 0$ such that $\lambda^2 \in \rho(A)$ and $||\lambda(\lambda^2 - A)^{-1}|| \leq M/(\lambda - w)$ for all $\lambda > w$. With $J_\lambda := (\lambda - A)^{-1}$, Theorems 1 & 6 show that A is bounded and hence $C(\cdot)$ is uniformly continuous (cf. [6]). This proves the following corollary which has appeared in [4] with a different proof.

Corollary 8. *Every strongly continuous cosine operator function on a Grothendieck space with the Dunford-Pettis property is uniformly continuous.*

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