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On Operator Inequalities due to Ando-Kittaneh-Kosaki

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Abstract

Operator norm inequalities due to Ando-Kittaneh-Kosaki for positive operators A, B and a non-negative operator monotone function f on $[0, \infty)$ are discussed: Main inequality is $\|f(A)-f(B)\| \leqq \|f(|A-B|)\|$. It is shown that the equality holds for invertible A, B and non-linear f if and only if $A=B$ and $f(0)=0$. Similarly, from the Kittaneh-Kosaki inequality, we show that $\|f(A)-f(B)\|=f'(t)\|A-B\|$ for $A, B \geqq t>0$ and nonlinear f if and only if $A=B$.

§ 1. Introduction

From the viewpoint of the Schatten p-norm, Kittaneh and Kosaki [2] showed some inequalities for the operator norm. Recently, T. Ando [1] showed two comparison theorems for unitarily invariant norms of positive semi-definite matrices making use of Ky Fan norm technique, and summed up the interesting inequalities related to operator monotone functions.

A real function f is called operator monotone (on $[0, \infty)$) if $A \leqq B$ implies $f(A) \leqq f(B)$ for (bounded linear) positive operators A, B on a Hilbert space. In the below, we assume an operator monotone function is non-negative. Then, a main inequality of the Ando-Kittaneh-Kosaki is as follows:

(a) $$\|f(A)-f(B)\| \leqq \|f(|A-B|)\| .$$

On the other hand, Kittaneh and Kosaki discussed an equation:

(b) For $A, B \geqq t>0$, $2t\|A-B\| = \|A^2-B^2\|$ if and only if $A=B$.

Note that (b) is the equality case for $f(t)=t^{1/2}$ in the following inequality by them:

(c) $$\|f(A)-f(B)\| \leqq f'(t)\|A-B\| \qquad \text{for} \quad A, B \geqq t>0 .$$

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In this note, by examining such inequalities, we shall consider the equality conditions for non-linear f. It is shown that equality in (a) holds for invertible A, B if and only if $A=B$. The condition $A=B$ is also equivalent to the equality in the Kittaneh-Kosaki inequality (c). In addition, as an application of the inequality (a), we shall give an improvement of [2; Theorem 3.4]:

$$\|\log(A+t)-\log(B+t)\|<\log(2\|A-B\|/t) \qquad \text{for} \quad 0<t<\|A-B\|.$$

§ 2. Ando-Kittaneh-Kosaki Inequalities

First, we shall consider the equality condition

$$\text{(d)} \qquad \|f(A)-f(B)\| = \|f(|A-B|)\|$$

for the inequality (a) which is stated in [1; Theorem 1], [2; Theorem 2.3]. Since the equation (d) always holds for linear f, we assume that f is non-linear. Then, it is natural to expect that (d) implies $A=B$. But, even in a commutative case, a counter-example is given: The equality (d) holds for $f(t)=t^{1/2}$, $A=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $B=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$. In this example, it should be noted that A is not invertible. As a matter of fact, we have the following:

Theorem 1. *If f is a non-linear non-negative operator monotone function on $[0, \infty)$, then $\|f(A)-f(B)\|=\|f(|A-B|)\|$ for positive invertible operators A and B if and only if $A=B$ and $f(0)=0$.*

Proof. Since a non-linear operator monotone function is strictly concave, for any $t>0$ and positive invertible X there is $\varepsilon=\varepsilon(t, X)>0$ such that $f(t+X)-f(X)+\varepsilon\leqq f(t)-f(0)$, hence

$$\|f(t+X)-f(X)\|<f(t)-f(0).$$

Therefore, if either $\|A-B\|>0$ or $f(0)>0$, then

$$a = \|f(\|A-B\|+A)-f(A)\|<f(\|A-B\|), \quad \text{and}$$
$$b = \|f(\|A-B\|+B)-f(B)\|<f(\|A-B\|).$$

Putting $c=\max\{a, b\}$, we have

$$f(A)-f(B) = f(A-B+B)-f(B)\leqq f(\|A-B\|+B)-f(B)\leqq c,$$

and similarly $f(B)-f(A)\leqq c$. Since $|f(A)-f(B)|\leqq c$, it follows that

$$\|f(A)-f(B)\|\leqq c<f(\|A-B\|)=\|f(|A-B|)\|.$$

Thus the equality shows that $||A-B||=0$ and $f(0)=0$. The converse is clear.

Now, we apply Theorem 1 and the inequality (a) to typical operator monotone functions. The following inequalities are due to Ando [1]:

Corollary 1.1. *The following inequalities hold for positive operators A and B, and the equality for invertible A, B holds only when $A=B$:*

(i) $||A^p-B^p|| \leqq || \, |A-B|^p||$ *for* $0<p<1$, *and*

(ii) $||\log(A+1)-\log(B+1)|| \leqq ||\log(|A-B|+1)||$.

The inequality (ii) in the above leads us an improvement of [2; Theorem 3.4]. From the viewpoint of this note, the following inequality shows that the equality condition itself is not reasonable in their theorem.

Corollary 1.2. *For positive operators A, B with $0<t<||A-B||$ for some constant t,* $||\log(A+t)-\log(B+t)||<\log(2||A-B||/t)$.

Proof. Since $||C-D||>1$ for $C=A/t$ and $D=B/t$, Corollary 1.1. (ii) implies that

$$\begin{aligned} ||\log(A+t)-\log(B+t)|| &= ||\log(C+1)-\log(D+1)|| \\ &\leqq ||\log(|C-D|+1)|| = \log(||C-D||+1) \\ &< \log(2||C-D||) = \log(2||A-B||/t)\,. \end{aligned}$$

§ 3. Inverse Inequalities

Symmetrically, we shall discuss an inverse inequality for the inverse function of operator monotone one, cf. [1; Theorem 3]:

Corollary 1.3. *If a continuous increasing unbounded function g on $[0, \infty)$ with $g(0)=0$ has the inverse function f which is operator monotone, then $||g(A)-g(B)|| \geqq ||g(|A-B|)||$ for positive operators A and B. Moreover, the equality for invertible A, B holds for nonlinear g if and only if $A=B$.*

Proof. Applying the inequality (a) for $g(A)$ and $g(B)$, we have

$$f(||g(A)-g(B)||) \geqq ||f(g(A))-f(g(B))|| = ||A-B||\,.$$

It follows from monotonity of g that

$$||g(A)-g(B)|| = g(f(||g(A)-g(B)||)) \geqq g(||A-B||) = ||g(|A-B|)||\,.$$

The second statement follows from Theorem 1.

Like Corollary 1.1, we can get the operator norm version of [1; Corollary 4] (cf. [1; Lemma 5]):

Corollary 1.4. *The following inequalities hold for positive operators A and B, and the equality for invertible A, B holds only when* $A=B$:

(i) $\|e^A-e^B\|\geqq\|e^{|A-B|}-1\|$,

(ii) $\|A^p-B^p\|\geqq\||A-B|^p\|$ *for* $p\geqq 1$, *and*

(iii) $\|A^p\log(A+1)-B^p\log(B+1)\|\geqq\||A-B|^p\log(|A-B|+1)\|$ *for* $p\geqq 1$.

§ 4. Estimation by Derivative

As a generalization of the van Hemmen-Ando theorem [3; Proposition 4.1], Kittaneh and Kosaki established the following inequality [2; Theorem 3.1]: *Let f be a non-negative continuous operator monotone function on* $[0, \infty)$, *and A, B positive operators with* $0\leqq a\leqq A$, $0\leqq b\leqq B$. *Then, for every operator X,* $\|f(A)X-Xf(B)\|\leqq C(a, b)\|AX-XB\|$ *where* $C(a, b)=f'(a)$ *when* $a=b$, $=(f(a)-f(b))/(a-b)$ *otherwise.* In this section, we shall consider the equality condition in (*c*), that is, the case $X=1$ and $a=b$ in the above. We note that (b) is a special case of this: Let $f(t)=t^{1/2}$. Since A^2, $B^2\geqq c^2$, we have that $\|A-B\|\leqq(2t)^{-1}\|A^2-B^2\|$ by (c). In this case, the equality is eqivalent to $A=B$. More generally:

Theorem 2. *Let f be a non-negative non-linear operator monotone function on* $(0, \infty)$, *and A, B positive operators with* $A, B\geqq c>0$ *for some scalar c. Then,* $\|f(A)-f(B)\|=f'(c)\|A-B\|$ *if and only if* $A=B$.

Proof. Suppose $\|f(A)-f(B)\|=f'(c)\|A-B\|$. Here we use the integral representation of f: $f(x)=\alpha+\beta x+\int_0^\infty (t:x)\,\mathrm{dm}(t)$ where $t:x$ means the parallel sum $tx/(t+x)$, $\alpha=f(0)$, $\beta=\lim_{t\to\infty}f(t)/t$ and $d\mu(t)=\{t/(1+t)\}dm(t)$ is a positive Radon measure. Notice that the support of m is non-trivial since f is non-affine.

Putting $X=(t+A)^{-1}(A-B)(t+B)^{-1}=t^{-2}(t:A-t:B)$, we have

$$\|f(A)-f(B)\|=\|\beta(A-B)+\int(t:A-t:B)dm(t)\|$$

$$\leqq\beta\|A-B\|+\int t^2\|X\|dm(t)$$

$$(*)\qquad\leqq\beta\|A-B\|+\int t^2\|(t+A)^{-1}\|\,\|A-B\|\,\|(t+B)^{-1}\|dm(t)$$

$$\leqq\left\{\beta+\int t^2(t+a)^{-1}(t+b)^{-1}dm(t)\right\}\|A-B\|$$

$$(**)\qquad \leqq \left\{\beta+\int t^2(t+c)^{-2}dm(t)\right\}||A-B||$$
$$=f'(c)||A-B|| = ||f(A)-f(B)||\,,$$

where $a=\min\sigma(A)$ and $b=\min\sigma(B)$. Therefore, on the support of m, we have two equations up to null sets:

$$(1)\qquad ||X||=||(t+A)^{-1}||\,||A-B||\,||(t+B)^{-1}|| \qquad \text{by } (*),$$

$$(2)\qquad (t+a)^{-1}(t+b)^{-1}||A-B|| = (t+c)^{-2}||A-B|| \qquad \text{by } (**).$$

Here suppose $A\neq B$ to the contrary. Then (2) implies $a=b=c$. We may assume that there exists a state ω with $\omega(X)=\omega(|X|)=||X||$ since the condition is symmetric for A, B and $X=X^*$. Noting that $\omega(|YZ|)\leqq||Y||\omega(|Z|)$, it follows from (1) that the following:

$$||X|| = \omega(|X|)\leqq||(t+A)^{-1}||\,||A-B||\omega((t+B)^{-1})$$
$$\leqq||(t+A)^{-1}||\,||A-B||\,||(t+B)^{-1}||\,,$$

imply $\omega((t+B)^{-1})=||(t+B)^{-1}||=1/(t+b)$. Since $t{:}B=t(1-t(t+B)^{-1})$, we have $\omega(t{:}\ B)=t{:}\ b$. Similarly, since

$$||X|| = \omega(|X|)\leqq||(t+B)^{-1}||\,||A-B||\omega((t+A)^{-1})$$
$$\leqq||(t+B)^{-1}||\,||A-B||\,||(t+A)^{-1}||$$

by the self-adjointness of X, we have $\omega(t{:}A)=t{:}a$. Therefore,

$$t^2\omega(X) = \omega(t{:}A-t{:}B) = \omega(t{:}A)-\omega(t{:}B) = t{:}a-t{:}b = 0\,,$$

which implies $X=0$, hence $A=B$. This is a contradiction, that is, the equality implies $A=B$. The converse is clear.

Remark. It is essential that $f'(c)$ dominates $C(a,b)$ and $f'(t)$ on the spectra of A and B. Indeed, the equation $||f(A)-f(B)||=C(a,b)||A-B||$ does not always imply $A=B$. For example, let $f(t)=t^{1/2}$, $A=1$ and $B=1\oplus\varepsilon$ for $0<\varepsilon<1$. Then, $||A-B||=1-\varepsilon$ and $||f(A)-f(B)||=1-\varepsilon^{1/2}$. Since $C(a,b)=(1-\varepsilon^{1/2})/(1-\varepsilon)$ for $a=1$ and $b=\varepsilon$, we have the equation $||f(A)-f(B)||=C(a,b)||A-B||$ although $A\neq B$.

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