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On the representation of the Picard modular function by θ constants I-II

Dedicated to Professor Kôtaro Oikawa on his 60th birthday

By

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§ 0. Introduction

In this paper the author shows the representation of the Picard modular function by θ constants, and characterizes this function as modular forms on the domain

$$D=\{(u,v)\in \mathbf{C}^2\colon 2\mathrm{Re}\, v+|u|^2<0\}=\{[\eta_0,\eta_1,\eta_2]\in \mathbf{P}^2(\mathbf{C})\colon {}^t\eta H\bar{\eta}<0\}$$

relative to a certain arithmetic discontinuous group,

$$\text{where } H=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We divide the paper in two parts. In Part I we discuss the former subject and in Part II we study the latter.

This modular function was constructed originally by [P], and was investigated by several mathematicians recently [D-M], [F], [H], [Sh] and [T]. This modular function is defined as the inverse mapping of the period mapping Φ for the family of the complex algebraic curves in (z, w)-space

$$C(\xi)\colon\ w^3=z(z-\xi_0)(z-\xi_1)(z-\xi_2),$$

where $\xi=[\xi_0,\xi_1,\xi_2]$ is a parameter on the domain

$$\Lambda=\{\xi\colon \xi_0\xi_1\xi_2(\xi_0-\xi_1)(\xi_1-\xi_2)(\xi_2-\xi_0)\neq 0\}$$

of $\mathbf{P}^2(\mathbf{C})$. As easily shown $C(\xi)$ can be regarded as a compact Riemann surface of genus 3. Then Φ is a multivalued analytic

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mapping from Λ to the Siegel upper half space $\mathfrak{S}_3$ of degree 3. The image of Φ is contained as an open dense subset in a nonsingular subvariety, which is biholomorphically equivalent to the domain D, of $\mathfrak{S}_3$. The inverse mapping of Φ is given as a single valued holomorphic mapping $[\xi_0, \xi_1, \xi_2]=[f_0, f_1, f_2]$ defined on D. The period mapping Φ induces a biholomorphic correspondence between the ξ-space $\boldsymbol{P}^2$ and the compactification $\widehat{D/\Gamma_1}$ of D/Γ_1, where Γ_1 is an arithmetic group defined afterwards. Hence $f_k/f_0(u, v)$ $(k=1, 2)$ are meromorphic automorphic functions relative to Γ_1 and they are the generators of the function field on $\widehat{D/\Gamma_1}$. As the conclusion of Part I we show the representation $f_k=\varphi_k(u, v)$ by θ constants of the Riemann theta function on $\mathfrak{S}_3$ (Proposition I-3). And this representation enables us to get an explicit Fourier expansion of the Picard modular function (Proposition I-4). As already shown in [Sh] the Picard modular function is a typical K3 modular function. So we wish to investigate it as a model case of the theory of the K3 modular functions. Hence we tried to obtain such a expansion of this modular function. In 1902 Alezais [A] has studied this representation but it contains essential faults, contrally our investigation is direct and complete. Our result can be considered as an extension of the classical Jacobi's representation $\lambda=\theta_2^4/\theta_3^4(\theta_i(z, \tau)$ indicates Jacobi's theta function and θ_i is the convention for $\theta_i(0, \tau))$ for the elliptic modular function $\lambda(\tau)$ to the special case of genus 3 (for the genus 2 case there is Rosenhein's representation, for the hyperelliptic case there is Thomae's representation).

We use several results of the theory of Riemann θ function. They are summed up as the Appendix at the end of this paper.

Next we give the summing up of Part II.

Putting $\Gamma(\boldsymbol{C})=\{g\in PGL(3, \boldsymbol{C}): {}^tgH\bar{g}=H\}$ we consider the following transformation groups acting on D:

$$\Gamma_1=\{g\in\Gamma(\boldsymbol{C})\cap PGL(3, \boldsymbol{Z}[\omega]): g\equiv I \ (\mathrm{mod.}\ (\sqrt{-3}))\},$$

where $\omega=\exp(-2\pi i/3)$.

$$\Gamma=\Gamma(\boldsymbol{C})\cap PGL(3, \boldsymbol{Z}[\omega]),$$
$$\Gamma_0=\Gamma(\boldsymbol{C})\cap PSL(3, \boldsymbol{Z}[\omega]),$$
$$\Gamma'=\Gamma_1\cap PSL(3, \boldsymbol{Z}[\omega]).$$

Where $PGL(3, *)$ $(PSL(3, *))$ indicates the group of projective transformations induced from $GL(3,*)\,(SL(3,*))$, respectively, and $g\equiv I$ (mod. $(\sqrt{-3})$) means that g represents the identity in $PGL(3, \boldsymbol{Z}[\omega]/(\sqrt{-3}))$.

In Part II we characterize the theta constants $\varphi_k(u, v)$ $(k=0, 1, 2)$ as modular forms on D relative to Γ' (II-§3 Proposition II-3). And we determine the structure and the generator system of $A(*)$ in terms of $\varphi_k(u, v)$, where $A(*)$ indicates the graded ring of modular forms on D with respect to $*=\Gamma'$ and Γ (II-§4 Proposition II-4 and II-5). There we use the transformation formula of theta functions as the main tool.

The results in Part II are also obtained by Holzapfel and Feustel [F], [H] independently in a different way.

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I. Representation by θ Constants

§ 1. Picard's Modular Function and His θ Representation

We consider an algebraic curve $C(\xi)$ in (z, w)-space $\boldsymbol{C}^2$ defined by

(1-1) $$C(\xi): w^3=z(z-\xi_0)(z-\xi_1)(z-\xi_2),$$

where the parameter $\xi=[\xi_0, \xi_1, \xi_2]$ is supposed to lie in the domain

$$\Lambda=\{\xi: \xi_0\xi_1\xi_2(\xi_0-\xi_1)(\xi_1-\xi_2)(\xi_2-\xi_0)\neq 0\}$$

of the complex projective space $P^2(C)$. Then we know that $C(\xi)$ can be considered as a compact Riemann surface of genus 3. Let us call $C(\xi)$ a Picard curve. And let us denote the totality of Picard curves by F.

Picard [P] constructed the period mapping for F and he showed that the inverse mapping defines a single valued automorphic function on a domain D which is biholomorphically equivalent to a complex 2 dimensional hyperball. Moreover he gave a representation of these functions by the Riemann θ function. Our aim is to get a representation by θ constant through a precise observation of Picard's result.

For this purpose we give a concrete description of the Picard's work in this section (the main part is owing to the private note of Wakabayashi [W]).

Let us fix the parameter $\Xi=[\xi_0, \xi_1, \xi_2]$ with $0<\xi_0<\xi_1<\xi_2$ and let us denote the corresponding Picard curve by C_0. In the sequel we construct a homology basis $\{A_i, B_i\}$ $(i=1, 2, 3)$ of C_0. We regard C_0 as a three sheeted covering surface over the z-sphere, and let π be the projection mapping from C_0 to the z-sphere. Then we get $Q_1=(z, w)=(0, 0)$, $Q_2=(z, w)=(\xi_0, 0)$, $Q_3=(z, w)=(\xi_1, 0)$, $Q_4=(z, w)=(\xi_2, 0)$, $Q_5=(z, w)=(\infty, \infty)$ as ramifying points. Put $\underline{Q}_i=\pi(Q_i)$ $(i=1, \dots, 5)$, and let t_0 be a fixed point on the z-plane with $\mathrm{Im}\, t_0<0$. Let γ_i be a line segment connecting t_0 and $\underline{Q}_i$ on the z-plane. Then we have three connected components σ_1, σ_2 and σ_3 of

$$\pi^{-1}(z\text{-sphere}-\bigcup_{i=1}^{5}\gamma_i),$$

and they are simply connected.

Let ρ be the automorphism of C_0 defined by $\rho(z, w)=(z, \omega w)$, where ω indicates $\exp(2\pi i/3)$. And the indices of σ are supposed to satisfy $\rho(\sigma_1)=\sigma_2$, $\rho(\sigma_2)=\sigma_3$. Let $\alpha^{(k)}(i,j)$ be the oriented arc from Q_i to Q_j on σ_k. Using above notations we define 1-cycles A_i, B_i on C_0 as follows:

$$(1\text{-}2)\qquad \begin{cases} A_1=\alpha^{(2)}(2, 3)+\alpha^{(3)}(3, 4)+\alpha^{(1)}(4, 2), \\ A_2=\alpha^{(3)}(2, 3)+\alpha^{(1)}(3, 2), \\ A_3=\alpha^{(3)}(2, 4)+\alpha^{(2)}(4, 3)+\alpha^{(1)}(3, 2), \\ B_1=\alpha^{(1)}(1, 3)+\alpha^{(3)}(3, 2)+\alpha^{(2)}(2, 1), \\ B_2=\alpha^{(3)}(2, 3)+\alpha^{(2)}(3, 2), \\ B_3=\alpha^{(2)}(1, 3)+\alpha^{(1)}(3, 2)+\alpha^{(3)}(2, 1). \end{cases}$$

They satisfy the relations $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$ $(1 \leqq i, j \leqq 3)$, so $\{A_i, B_j\}$ is a basis of $H_1(C_0, \boldsymbol{Z})$. Let us consider a path s on Λ from Ξ to a variable point ξ. Because F is a locally trivial topological fibre space over Λ, we can define a homology basis $\{A_i(\xi), B_i(\xi)\}$ of $H_1(C(\xi), \boldsymbol{Z})$ by the continuation of $\{A_i, B_i\}$ along s.

Let $\{\omega_i\}_{1 \leqq i \leqq 3}$ be a basis of Abelian differentials of first kind on $C(\xi)$ so that we have

$$\int_{A_j} \omega_i = \delta_{ij}.$$

Then we get a multivalued analytic mapping from Λ to the Siegel upper half space $\mathfrak{S}_3$ by the correspondence

$$\Omega(\xi) = (\Omega_{ij}(\xi)) = \left(\int_{B_j} \omega_i\right).$$

Remark 1-1. We get a basis of Abelian differentials of first kind $\varphi_1 = dz/w$, $\varphi_2 = dz/(w^2)$, $\varphi_3 = z dz/(w^2)$, but they do not coincide with $\{\omega_i\}$.

According to Picard the period matrix $\Omega(\xi)$ has a concrete description as follows:

$$(1\text{-}3) \qquad \Omega(\xi) = \begin{bmatrix} (u^2 + 2\omega^2 v)/(1-\omega) & \omega^2 u & (\omega u^2 - \omega^2 v)/(1-\omega) \\ \omega^2 u & -\omega^2 & u \\ (\omega u^2 - \omega^2 v)/(1-\omega) & u & (u^2 - 2v)/(\omega - \omega^2) \end{bmatrix},$$

where we use the notations

$$\eta_0 = \int_{A_1} \varphi_1, \quad \eta_1 = -\omega^2 \int_{B_1} \varphi_1, \quad \eta_2 = \int_{A_2} \varphi_1, \quad v = \eta_1/\eta_0, \quad u = \eta_2/\eta_0.$$

So we can regard the correspondence $\Phi: \xi \mapsto (u, v)$ as the period mapping for the family F.

We have the following properties about Φ (cf. [P], [T], [D-M], [H], [Sh]).

(1) The image of Φ is open-dense in the domain

$$D = \{(u, v) \in \boldsymbol{C}^2 : 2 \operatorname{Re} v + |u|^2 < 0\} = \{\eta \in \boldsymbol{P}^2 : {}^t\eta H \bar{\eta} < 0\},$$

where H indicates the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(2) Let Γ_1 be the monodromy group in Aut D, the group of

biholomorphic transformations of D, induced from $\pi_1(\Lambda, *)$. Then we have

$$\Gamma_1=\{g\in PGL(3, \boldsymbol{Z}[\omega]): {}^t gH\bar{g}=H,\ g\equiv id\ (\mathrm{mod.}\ (\sqrt{-3}))\},$$

where g acts on $\boldsymbol{P}^2$ from left and $\equiv$ means that g represents the identity in $PGL(3, \boldsymbol{Z}[\omega]/(\sqrt{-3}))$.

(3) Φ induces an injective mapping from Λ to D/Γ_1.

(4) If we take an affine coordinate $x=\xi_1/\xi_0$, $y=\xi_2/\xi_0$, then the system $\{\eta_i(x,y)\}$ $(i=0, 1, 2)$ gives a fundamental solution of the Appell's hypergeometric differential equation $F_1(1/3, 1/3, 1/3, 1; x, y)$.

(5) Φ extends to a biholomorphic correspondence between ξ-space $\boldsymbol{P}^2$ and the Satake-Baily-Borel compactification of D/Γ_1 (it is obtained by attatching 4 points corresponding to

$$P_0=[1, 0, 0],\ P_1=[0, 1, 0],\ P_2=[0, 0, 1],\ P_3=[1, 1, 1]$$

on the ξ-space.)

Now let us relate the θ representation of Picard. For this purpose we apply Theorem A-5, A indicates the Appendix, for the Picard curve $X: w^3=z(z-1)(z-x)(z-y)$. Let us take a meromorphic function $f=z$ on X. It is a function of order 3 and we have $(f)=3Q_1-3Q_5$($(*)$ indicates the divisor defined by $*$). Let us consider the divisor $2Q_2+Q_3$ on X, then Q_2+Q_3 is a general divisor (see Appendix). In fact we have $(dz/w)=Q_1+Q_2+Q_3+Q_4$. If we take Q_5 as the initial point of the Abelian integral, we have

$$Ex=\prod_{k=1}^{3}\left\{\frac{\theta\left(2\int_{Q_5}^{Q_2}\omega+\int_{Q_5}^{Q_3}\omega-\int_{\alpha^{(k)}(5,1)}\omega-\Delta,\ \Omega(\xi)\right)}{\theta\left(2\int_{Q_5}^{Q_2}\omega+\int_{Q_5}^{Q_3}\omega-\Delta,\ \Omega(\xi)\right)}\right\}.$$

Owing to Remark A-5 the denominator and the numerator are different from zero. If we take the divisor Q_2+2Q_3 instead of $2Q_2+Q_3$, we have

$$Ex^2=\prod_{k=1}^{3}\left\{\frac{\theta\left(\int_{Q_5}^{Q_2}\omega+2\int_{Q_5}^{Q_3}\omega-\int_{\alpha^{(k)}(5,1)}\omega-\Delta,\ \Omega(\xi)\right)}{\theta\left(\int_{Q_5}^{Q_2}\omega+2\int_{Q_5}^{Q_3}\omega-\Delta,\ \Omega(\xi)\right)}\right\}.$$

By the similar argument we get

$$Ey=\prod_{k=1}^{3}\left\{\frac{\theta\left(2\int_{Q_5}^{Q_2}\omega+\int_{Q_5}^{Q_4}\omega-\int_{\alpha^{(k)}(5,1)}\omega-\Delta,\ \Omega(\xi)\right)}{\theta\left(2\int_{Q_5}^{Q_2}\omega+\int_{Q_5}^{Q_4}\omega-\Delta,\ \Omega(\xi)\right)}\right\},$$

$$Ey^2=\prod_{k=1}^{3}\left\{\frac{\theta\left(\int_{Q_5}^{Q_2}\omega+2\int_{Q_5}^{Q_4}\omega-\int_{\alpha^{(k)}(5,1)}\omega-\Delta,\ \Omega(\xi)\right)}{\theta\left(\int_{Q_5}^{Q_2}\omega+2\int_{Q_5}^{Q_4}\omega-\Delta,\ \Omega(\xi)\right)}\right\}.$$

By elliminating E from the above equalities we obtain

$$(1\text{-}4)\qquad x=\prod_{k=1}^{3}\left\{\frac{\theta\left(\int_{Q_5}^{Q_2}\omega+2\int_{Q_5}^{Q_3}\omega-\int_{\alpha^{(k)}(5,1)}\omega-\Delta,\ \Omega(\xi)\right)}{\theta\left(\int_{Q_5}^{Q_2}\omega+2\int_{Q_5}^{Q_3}\omega-\Delta,\ \Omega(\xi)\right)}\right\}$$

$$\times\prod_{k=1}^{3}\left\{\frac{\theta\left(2\int_{Q_5}^{Q_2}\omega+\int_{Q_5}^{Q_3}\omega-\Delta,\ \Omega(\xi)\right)}{\theta\left(2\int_{Q_5}^{Q_2}\omega+\int_{Q_5}^{Q_3}\omega-\int_{\alpha^{(k)}(5,1)}\omega-\Delta,\ \Omega(\xi)\right)}\right\},$$

$$y=\prod_{k=1}^{3}\left\{\frac{\theta\left(\int_{Q_5}^{Q_2}\omega+2\int_{Q_5}^{Q_4}\omega-\int_{\alpha^{(k)}(5,1)}\omega-\Delta,\ \Omega(\xi)\right)}{\theta\left(\int_{Q_5}^{Q_2}\omega+2\int_{Q_5}^{Q_4}\omega-\Delta,\ \Omega(\xi)\right)}\right\}$$

$$\times\prod_{k=1}^{3}\left\{\frac{\theta\left(2\int_{Q_5}^{Q_2}\omega+\int_{Q_5}^{Q_4}\omega-\Delta,\ \Omega(\xi)\right)}{\theta\left(2\int_{Q_5}^{Q_2}\omega+\int_{Q_5}^{Q_4}\omega-\int_{\alpha^{(k)}(5,1)}\omega-\Delta,\ \Omega(\xi)\right)}\right\}.$$

This is the θ representation of Picard.

§ 2. Observation of the Variables in the θ Representation

Now our aim is to express the parameters x and y in (1-4) as explicit functions of (u, v). The right hand sides are determined by the moduli variable $\Omega(\xi)$, the Riemann constant Δ and the Abelian integrals along certain arcs. The matrix $\Omega(\xi)$ in (1-4) is determined by the point (u, v) on D, in fact (1-3) gives the embedding of D in $\mathfrak{S}_3$. Hence it is determined by the periods on $C(\xi)$. The Riemann constant Δ (cf. (A-3)) is determined by the homology basis of $C(\xi)$ and the initial point P_0 of the Abelian integral as noted in Remark A-2. By Corollary to Theorem A-3, Δ is a half period if there exists an Abelian differential ω of first kind with $(\omega)=4P_0$. We already set $P_0=Q_5$ on $C(\xi)$. On the other hand if we consider $\varphi_2=dz/w^2$, we have $(\varphi_2)=4Q_5$. By Corollary 2 to Theorem A-3, the Riemann constant Δ in (1-4) is a half period on Jac $C(\xi)$. Namely we have $\Delta=\Omega(\xi)n_1+n_2$ for certain vectors n_1, n_2 of $(\boldsymbol{Z}/2)^3/\boldsymbol{Z}^3$. Here we note that n_1 and n_2 are independent of the parameter ξ, because

Δ is a continuous function of ξ (see (A-3)). We shall investigate Δ precisely in the next section.

So we determine the Abelian integrals of the form

$$\int_{\alpha^{(k)}(i,j)} \omega$$

in (1-4). At first we have the following:

Lemma 1. $\displaystyle\int_{\alpha^{(1)}(i,j)} \omega = \frac{1}{3}\int_{C-C'} \omega.$

Where we set $C = \alpha^{(1)}(i,j) - \alpha^{(2)}(i,j)$ *and* $C' = \alpha^{(3)}(i,j) - \alpha^{(1)}(i,j)$, *they are* 1-*cycles on* $C(\xi)$. *Hence the left hand side of the above equality is one thirds of a certain period.*

Proof. It is sufficient to show the equality for φ_i $(i=1,2,3)$. We have

$$\int_C \varphi_1 = \int_{\alpha^{(1)}(i,j)} \frac{dz}{w} - \int_{\alpha^{(1)}(i,j)} \frac{dz}{\omega w} = (1-\omega^2)\int_{\alpha^{(1)}(i,j)} \varphi_1.$$

Hence we obtain

$$\int_{\alpha^{(1)}(i,j)} \varphi_1 = \frac{1}{1-\omega^2}\int_C \varphi_1 = \frac{1-\omega}{3}\int_C \varphi_1 = \frac{1}{3}\int_{C-C'} \varphi_1.$$

The argument is almost same as for φ_2 and φ_3. q. e. d.

By the above consideration the θ variables in (1-4) can be represented by one sixths of the periods. In the sequel we find the exact values of them.

As for the cycles of type C we have:

Lemma 2.

$$\alpha^{(1)}(5,1) - \alpha^{(2)}(5,1) = A_1 - A_3,$$
$$\alpha^{(1)}(5,2) - \alpha^{(2)}(5,2) = B_1 + A_1 + A_2 - A_3,$$
$$\alpha^{(1)}(5,3) - \alpha^{(2)}(5,3) = B_1 + B_2 + A_1 - A_3,$$
$$\alpha^{(1)}(5,4) - \alpha^{(2)}(5,4) = B_1.$$

The automorphism ρ (in §1) of $C(\xi)$ acts on $H_1(C(\xi), \mathbf{Z})$. We denote this automorphism by the same notation. Then we have:

Lemma 3.

$$\rho^2 \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

By the above three Lemmas, we can describe the integral of the form

$$\int_{\alpha^{(k)}(i,j)} \omega$$

in (1-4) by the periods on $C(\xi)$. Namely we have:

Proposition I-1.

$$\int_{\alpha^{(1)}(5,2)} \omega = \frac{1}{3}\left\{ \Omega \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\},$$

$$\int_{\alpha^{(1)}(5,3)} \omega = \frac{1}{3}\left\{ \Omega \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\},$$

$$\int_{\alpha^{(1)}(5,4)} \omega = \frac{1}{3} \Omega \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \int_{\alpha^{(1)}(5,1)} \omega = \frac{1}{3} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix},$$

$$\int_{\alpha^{(2)}(5,1)} \omega = \frac{1}{3} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \int_{\alpha^{(3)}(5,1)} \omega = \frac{1}{3} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

Next let us observe the Riemann constant Δ. The corresponding half integral vectors n_1 and n_2 do not depend on the parameter ξ. So we consider the fixed Picard curve

$$X_1 : w^3 = z^4 - 1.$$

It is a three sheeted covering over the z-sphere. And we regard five ramifying points $Q_1 = (z, w) = (1, 0)$, $Q_2 = (z, w) = (i, 0)$, $Q_3 = (z, w)$

$=(-1,0)$, $Q_4=(z,w)=(-i,0)$, $Q_5=(z,w)=(\infty,\infty)$ are deformed from Q_i on C_0, respectively. And we suppose the deformed arc from line segment γ_i does not intersect the open unit disc $\{z: |z|<1\}$.

If we regard X_1 as a four sheeted covering over the w-sphere, we obtain 4 ramifying points $R_1=(z,w)=(\infty,\infty)=Q_5$, $R_2=(z,w)=(0,-\omega^2)$, $R_3=(z,w)=(0,-1)$, $R_4=(z,w)=(0,-\omega)$.

Here let us consider a deformation of Riemann surfaces:

$$(2\text{-}1)\qquad X(t):\exp(\pi i t)\cdot z^4 = \{w+\omega^2 t+(t-1)i\}\{w+\omega t+(1-t)i\}\{(1-t)w-1\}(w+1),$$

where t varies on the interval $[0,1]$. Then we obtain $X(1)=X_1$ and $X(0)=X_0$. As for the Fermat curve $X_0: z^4=w^4-1$ we have 4 ramifying points

$$R_1'=(z,w)=(0,1),\ R_2'=(z,w)=(0,i),\ R_3'=(z,w)=(0,-1),\ R_4'=(z,w)=(0,-i)$$

corresponding to R_1, R_2, R_3, R_4 respectively. In the next section we determine the Riemann constant Δ for X_0, and we get the one for X_1 by the deformation of Riemann surfaces (2-1).

§ 3. Determination of Δ and the Conclusion

For the simplicity we denote R_i' by R_i in the sequel. Let π be the projection $(z,w)\mapsto w$ and set $\underline{R}_i=\pi(R_i)$. Let γ_i be the line segment connecting $\underline{R}_i$ and ∞ on the w-sphere. We denote the connected component of $\pi^{-1}(\boldsymbol{P}-\bigcup_{j=1}^{4}\gamma_j)$ by Σ_i (i is an element of $\boldsymbol{Z}/4\boldsymbol{Z}$). Here we suppose that it holds $\tau\Sigma_i=\Sigma_{i+1}$ relative to the automorphism

$$\tau:\begin{cases} z'=iz \\ w'=w\ . \end{cases}$$

Let $\alpha^{(k)}(i,j)$ be the oriented arc on Σ_k from R_i to R_j. Using this notation we define the following 1-cycles on X_0:

$$(3\text{-}1)\qquad \begin{cases} A_1'=\alpha^{(2)}(4,1)+\alpha^{(4)}(1,4)+\alpha^{(3)}(1,2)+\alpha^{(1)}(2,1), \\ A_2'=\alpha^{(4)}(2,4)+\alpha^{(2)}(4,2), \\ A_3'=\alpha^{(1)}(2,3)+\alpha^{(3)}(2,3)+\alpha^{(2)}(3,2)+\alpha^{(4)}(3,2), \\ B_1'=\alpha^{(2)}(2,3)+\alpha^{(1)}(3,2), \\ B_2'=\alpha^{(1)}(1,4)+\alpha^{(4)}(4,1), \\ B_3'=\alpha^{(2)}(1,2)+\alpha^{(3)}(2,1). \end{cases}$$

Then we have $A_i' \cdot A_j' = B_i' \cdot B_j' = 0$ and $A_i' \cdot B_j' = \delta_{ij}$. Hence they constitute a basis of $H_1(X_0, \boldsymbol{Z})$. If we set

$$\varphi_1 = wz^{-3}dw, \quad \varphi_2 = z^{-3}dw, \quad \varphi_3 = z^{-2}dw,$$

we obtain

$$\left[\int_{A_j'} \varphi_i\right] = D$$

and

$$\Omega' := \left[\int_{B_j'} \varphi_i\right] = \frac{1}{2} D \begin{bmatrix} i & -1 & -1 \\ -1 & i & -1 \\ -1 & -1 & i \end{bmatrix},$$

for a certain diagonal matrix D. And put

$$Q_1 = (z, w) = (1, 0), \quad Q_2 = (z, w) = (i, 0),$$
$$Q_3 = (z, w) = (-1, 0), \quad Q_4 = (z, w) = (-i, 0).$$

These are the deformations of Q_1, Q_2, Q_3, Q_4 on X_1 relative to (2-1) respectively.

And we may assume Q_k is situated on Σ_k. By

$$\int_{Q_k}^{R_i} \varphi_j$$

we indicates the integral of φ_j along the arc from Q_k to R_i on Σ_k. And put

$$c_j = \int_{Q_1}^{R_1} \varphi_j.$$

By considering the automorphisms

$$\begin{cases} z' = z \\ w' = iw \end{cases}, \quad \begin{cases} z' = iz \\ w' = w \end{cases}$$

of X_0 we obtain the following table of

$$P_{ijk} = c_j^{-1} \int_{Q_k}^{R_i} \varphi_j:$$

Table 1

$\binom{i}{k}$	$\binom{1}{1}$	$\binom{2}{1}$	$\binom{3}{1}$	$\binom{4}{1}$	$\binom{1}{2}$	$\binom{2}{2}$	$\binom{3}{2}$	$\binom{4}{2}$	$\binom{1}{3}$	$\binom{2}{3}$	$\binom{3}{3}$	$\binom{4}{3}$	$\binom{1}{4}$	$\binom{2}{4}$	$\binom{3}{4}$	$\binom{4}{4}$
$j=1$	1	i	-1	$-i$	i	-1	$-i$	1	-1	$-i$	1	i	$-i$	1	i	-1
$j=2$	1	i	-1	$-i$	-1	$-i$	1	i	1	i	-1	$-i$	-1	$-i$	1	i
$j=3$	1	-1	1	-1	i	$-i$	i	$-i$	-1	1	-1	1	$-i$	i	$-i$	i

According to Table 1 we obtain the following:

Lemma 4.

$$\frac{1}{2}\int_{A_1'}\varphi\equiv 2\int_{Q_1}^{Q_4}\varphi,\quad \frac{1}{2}\int_{A_2'}\varphi\equiv\int_{R_1}^{Q_1}\varphi+\int_{R_3}^{Q_3}\varphi,$$

$$\frac{1}{2}\int_{A_3'}\varphi\equiv 2\int_{R_1}^{R_4}\varphi,\quad \frac{1}{2}\int_{B_1'-B_3'}\varphi\equiv\int_{Q_1}^{Q_3}\varphi+\int_{R_1}^{R_3}\varphi,$$

where we set $\varphi={}^t(\varphi_1,\varphi_2,\varphi_3)$ *and* $\equiv$ *indicates the equivalence modulo periods.*

Proof. Set

$$C'=\begin{bmatrix} c_1^{-1} & & 0 \\ & c_2^{-1} & \\ 0 & & c_3^{-1} \end{bmatrix},$$

using (3-1) we have

$$C'\int_{A_1'}\varphi=C'\left\{\left[\int_{Q_1}^{R_1}+\int_{Q_3}^{R_3}+\int_{Q_2}^{R_1}+\int_{Q_4}^{R_4}\right]-\left[\int_{Q_1}^{R_2}+\int_{Q_3}^{R_1}+\int_{Q_2}^{R_4}+\int_{Q_4}^{R_1}\right]\right\}\varphi,$$

where we set $\varphi={}^t(\varphi_1,\varphi_2,\varphi_3)$. By Table 1, we obtain:

the right hand side

$$=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\-i\\i\end{bmatrix}+\begin{bmatrix}i\\i\\-1\end{bmatrix}+\begin{bmatrix}i\\-1\\i\end{bmatrix}\right\}-\left\{\begin{bmatrix}-1\\i\\i\end{bmatrix}+\begin{bmatrix}-1\\-1\\1\end{bmatrix}+\begin{bmatrix}-i\\1\\i\end{bmatrix}+\begin{bmatrix}-i\\-i\\-1\end{bmatrix}\right\}$$

$$=\begin{bmatrix}4+4i\\0\\0\end{bmatrix}.$$

On the other hand we have

$$2C'\int_{Q_1}^{Q_4}\varphi \equiv C'\left[\int_{Q_1}^{R_1}+\int_{R_1}^{Q_4}+\int_{Q_1}^{R_3}+\int_{R_3}^{Q_4}\right]\varphi = \begin{bmatrix} 2+2i \\ 0 \\ 0 \end{bmatrix}.$$

Hence we have the first equality. By the similar computation we get the rests. q. e. d.

On the Picard curve X_1 we have the homology basis $\{A_i, B_i\}$ which is obtained by the continuation of (1-2). We can define the continuation of this basis relative to the deformation (2-1). Let us denote this shifted basis on X_0 by $\{A_i, B_i\}$ also. So we have two basis systems of $H_1(X_0, \boldsymbol{Z})$. They are changed each other by the following symplectic transformation:

Lemma 5.

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} B'_1 \\ B'_2 \\ B'_3 \\ A'_1 \\ A'_2 \\ A'_3 \end{pmatrix}.$$

Proof. It is sufficient to know the intersection multiplicities between these two systems. In fact we obtain the description:

$$A_1=\alpha^{(2)}(2, 3)+\alpha^{(3)}(3, 4)+\alpha^{(4)}(4, 2),$$
$$A_2=\alpha^{(2)}(2, 4)+\alpha^{(3)}(4, 2),$$
$$A_3=\alpha^{(2)}(2, 4)+\alpha^{(4)}(4, 3)+\alpha^{(3)}(3, 2),$$
$$B_1=\alpha^{(1)}(3, 2)+\alpha^{(3)}(2, 4)+\alpha^{(2)}(4, 3),$$
$$B_2=\alpha^{(2)}(3, 4)+\alpha^{(3)}(4, 3),$$
$$B_3=\alpha^{(1)}(4, 3)+\alpha^{(3)}(3, 2)+\alpha^{(2)}(2, 4).$$

So the above numbers are easily obtained by the geometric configuration. q. e. d.

Let $E_0, E_1, \dots, E_{63}$ be the totality of the linear equivalence classes of the divisor E on X_0 with $2E\equiv 0$. Put

$$d_i=\dim H^0(X_0, \mathcal{O}([E_i+2R_1])),$$

where [ ] indicates the corresponding line bundle. By Remark A-4, we have $d_i=0$ or 1. Set

$$\Omega_0=\int_{B_j}\omega_i \qquad 1\leqq i,\ j\leqq 3,$$

where $\omega={}^t(\omega_1, \omega_2, \omega_3)$ is a basis of Abelian differentials of first kind with

$$\int_{A_j}\omega_i=\delta_{ij}\,.$$

The Riemann constant Δ on X_0 relative to the homology basis $\{A_i, B_i\}$ and the initial point R_1 is also a half period, because we have $(\varphi_1)=(z^{-3}dw)=4R_1$. So Δ is represented by the form $\Omega_0\eta_1+\eta_2$ with certain elements η_1 and η_2 of $(\boldsymbol{Z}/2)^3/\boldsymbol{Z}^3$. Set

$$\int_{E_i}\omega=\Omega_0\eta_1^{(i)}+\eta_2^{(i)},$$

the notation is defined in Appendix, with vectors $\eta_1^{(i)}$ and $\eta_2^{(i)}$ of $(\boldsymbol{Z}/2)^3/\boldsymbol{Z}^3$. By the corollary to Theorem A-3, (η_1, η_2) is characterized by the condition:

$$4\,{}^t(\eta_1+\eta_1^{(i)})\,(\eta_2+\eta_2^{(i)})\equiv d_i \pmod 2 \tag{3-2}$$

for arbitrary index i.

Set $E_0=0$, $E_1=2Q_4-2Q_1$, $E_2=Q_1+Q_3-R_1-R_3$, $E_3=2R_4-2R_1$, $E_4=Q_3+R_3-Q_1-R_1$. Then we have $2E_i\equiv 0$ $(i=0,\dots,4)$ because of Lemma 4. Let us examine the condition (3-2) for these divisors.

Lemma 6. *We have $d_0=d_3=1$ and $d_1=d_2=d_4=0$.*

Proof. By the definition of d_i we obtain the required equalities. q. e. d.

Using Lemma 4 we have the following representation relative to the homology basis $\{A_i', B_i'\}$:

$$C'\int_{E_0}\varphi=0,\ C'\int_{E_1}\varphi=\frac{1}{2}\begin{bmatrix}1\\0\\0\end{bmatrix},\ C'\int_{E_2}\varphi=\frac{1}{2}\begin{bmatrix}0\\1\\0\end{bmatrix},$$

$$C'\int_{E_3}\varphi=\frac{1}{2}\begin{bmatrix}0\\0\\1\end{bmatrix},\ C'\int_{E_4}\varphi=\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

Using the transformation formula (A-7) we obtain:

Lemma 7.

$$ {}^t(\eta_1^{(1)}, \eta_2^{(1)}) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad {}^t(\eta_1^{(2)}, \eta_2^{(2)}) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, $$
$$ {}^t(\eta_1^{(3)}, \eta_2^{(3)}) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad {}^t(\eta_1^{(4)}, \eta_2^{(4)}) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}. $$

The condition (3-2) for $E_0, \dots, E_4$ induces only two possibilities:

$$ \Delta_1 = {}^t(\eta_1, \eta_2) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Delta_2 = {}^t(\eta_1, \eta_2) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}. $$

Next we examine the condition for the divisor E_5 with

$$ (*) \qquad \int_{E_5} \varphi = \frac{1}{2}\Omega\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. $$

Put $d_5 = \dim H^0(X_0, \mathcal{O}([E_5 + 2R_1]))$, then we have $\Delta = \Delta_1$ provided $d_5 = 1$. At first we determine E_5 and next we calculate d_5. By the definition of B_3' and Table 1 we have

$$ C'\int_{B_3'} \varphi = C'\left[\int_{Q_2}^{R_2} + \int_{Q_3}^{R_1} - \int_{Q_3}^{R_2} - \int_{Q_2}^{R_1}\right]\varphi = \begin{bmatrix} -2-2i \\ -2 \\ 2-2i \end{bmatrix}. $$

On the other hand we have

$$ C'\left[\int_{Q_2}^{R_2} + \int_{Q_3}^{R_1}\right]\varphi = \begin{bmatrix} -1-i \\ -2 \\ 1-i \end{bmatrix}, \quad C'\left[\int_{Q_1}^{R_1} + \int_{Q_3}^{R_3}\right]\varphi = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}. $$

Now we choose a point $R_1' = (a, b)$ on X_0 so that it holds

$$ \int_{Q_1}^{R_1'} \varphi_2 = \frac{1}{2}\int_{Q_1}^{R_1} \varphi_2 , $$

and put $R_3' = (-a, -b)$. Then it holds

$$ C'\left[\int_{Q_1}^{R_1'} + \int_{Q_3}^{R_3'}\right]\varphi = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. $$

Hence we have

$$ \frac{1}{2}\int_{B_3'} \varphi = \int_{E_5} \varphi = \frac{1}{2}\Omega'\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

for $E_5 = R_1' + R_3' + R_1 + R_2 - (Q_1 + Q_2 + 2Q_3)$.

Using the symplectic transformation in Lemma 5 we get the equality (*).

To obtain $d_5 = 1$ it is enough we construct a meromorphic function $f(\neq 0)$ with $(f) \geqq -(E_5 + 2R_1)$.

Let L_1, L_2 and L_3 be linear forms as the following:

(i) R_1' and R_3' belong to $L_1 = 0$,

(ii) $L_2 = 0$ is tangent to X_0 at R_1,

(iii) $L_3 = w$.

Then we have

$$(L_1)_{|X_0} = R_1' + R_2' + R_3' + R_4',$$

where $R_2' = (ia, ib)$, $R_4' = (-ia, -ib)$,

$$(L_2)_{|X_0} = 4R_1, \quad (L_3)_{|X_0} = Q_1 + Q_2 + Q_3 + Q_4.$$

Put $D = R_2' + R_4' + 2R_1 + Q_1 + Q_2 + 2Q_3 + R_3 + R_4$. If we find a cubic form F with

$$(**) \qquad (F)_{|X_0} \geqq D$$

then $f = F/(L_1 L_2 L_3)$ is a required function.

Let us regard 10 parameters of F as unknowns, then (**) induces 10 linear equations about these unknowns. We can calculate the determinant of the matrix of their coefficients and we see it vanishes. Hence we can find a solution F of (**). It shows $d_5 = 1$.

Hence Δ_1 is the Riemann constant on X_0 relative to the homology basis $\{A_i, B_i\}$ and the initial point $R_1 = (z, w) = (1, 0)$. By the shifting of Δ_1 along the deformation (2-1) we get the following.

Proposition I-2. *The Riemann constant Δ on the Picard curve X_1 relative to the homology basis $\{A_i, B_i\}$ and the initial point $Q_5 = R_1 = (z, w) = (\infty, \infty)$ of the Abelian integral is given by*

$$\Delta = \frac{1}{2}\Omega \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

By substituting the results Proposition I-1 and I-2 in the θ representation (1-4) we obtain the θ constant representation of the projective parameter ξ_i:

Proposition I-3.

$$[\xi_2, \xi_1, \xi_0] = \left[\theta^3\begin{bmatrix} 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 \end{bmatrix}, \ \theta^3\begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}, \ \theta^3\begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \right],$$

where we use the conventional notation $\theta\begin{bmatrix} a \\ b \end{bmatrix}$ *for* $\theta\begin{bmatrix} a \\ b \end{bmatrix}(0, \Omega)$.

Remark 3-1. The above θ constants are holomorphic on the whole domain D, because they are holomorphic on $\mathfrak{S}_3$. And we show that the theta constants in the right hand side have no common zero in Proposition 2 of II §3.

Henceforth we use the notation

$$\varphi_k(u, v) = \theta^3\begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix} (0, \Omega(u, v)),$$

$k=0, 1, 2$. If we write down $\varphi_k(u, v)$ explicitly using (1-3), we obtain the following Fourier expansion.

Proposition I-4.

$$\varphi_k(u, v) = \sum_{\nu=0}^{\infty} \{ \sum_{\nu=N(\mu)} (H_\mu^{(k)+}(u) + H_\mu^{(k)-}(u)) \} q^\nu,$$

where

$$H_\mu^{(k)\pm}(u) = \sum_\mu \exp\left\{\pi i \frac{\mu^2 u^2}{\sqrt{-3}}\right\} \theta\begin{bmatrix} \pm 1/6 \\ \pm 1/6 \end{bmatrix} (\mu u, -\omega^2) \cdot \exp\left\{\pm \frac{2}{3}\pi i k \ tr(\mu)\right\}$$

and $N(\mu) = \mu\bar{\mu}$, $tr(\mu) = \mu + \bar{\mu}$.

Remark 3-2. The coefficient $f_\mu^\pm(u) = H_\mu^{(k)\pm} \cdot \exp\left\{\pm \frac{2}{3}\pi i k \ tr(\mu)\right\}$ is a θ function of 1 variable satisfying the periodic property:

$$\text{(3-2)} \quad \begin{cases} f_\mu(u + (1-\omega^2)) = \exp\{2\pi i \nu \omega^2 (1+u)\} f_\mu(u), \\ f_\mu(u + (\omega - 1)) = \exp\{2\pi i \nu \omega (u+\omega)\} f_\mu(u) \\ f_\mu(u+3) = \exp\{-2\pi i \nu u \sqrt{-3} - 3\omega^2)\} f_\mu(u). \end{cases}$$

II. θ Constants $\varphi_k(u, v)$ as Modular Forms

§ 1. The Picard Modular Group

Let us begin with the situation of Part I §1. We considered a reference Picard curve C_0 corresponding to a fixed point Ξ on Λ. If we take an element δ of $\pi_1(\Lambda, \Xi)$, it is induced an automorphism δ^* of $H_1(C_0, \mathrm{Z})$ by the deformation of C_0 along δ. Let $N(\delta)$ be a matrix of δ^* relative to the basis $\{B_i, A_i\}$ given by $(1-2)$ in Part I. The transformation $N(\delta)$ preserves the intersection matrix of the system $\{B_i, A_i\}$, so it belongs to $Sp(3, \boldsymbol{Z})$. Namely $N(\delta)$ is a modular transformation of $\mathfrak{S}_3$. Here let us recall the definitions of η_j $(j=1, 2, 3)$, φ_1 and $\{B_i, A_i\}$ (see I§1), so we obtain the relation;

$$(1\text{-}1)\qquad \begin{cases} \int_{A_3} \varphi_1 = -\omega\eta_1, \\ \int_{B_3} \varphi_1 = -\eta_2 \\ \int_{B_2} \varphi_1 = -\omega\eta_3. \end{cases}$$

Therefore $N(\delta)$ induces an element $g(\delta)$ of $PGL(3, \boldsymbol{Z}[\omega])$, it acts on the domain

$$D = \{\eta \in \boldsymbol{P}^2 : \eta_1\bar{\eta}_2 + \bar{\eta}_1\eta_2 + \eta_3\bar{\eta}_3 < 0\}.$$

Set

$$G_1 = \{N(\delta) \in Sp(3\boldsymbol{Z}) : \ \delta \in \pi_1(\Lambda, \Xi)\},$$
$$\Gamma_1 = \{g(\delta) \in PGL(3, \boldsymbol{Z}[\omega]) : \ \delta \in \pi_1(\Lambda, \Xi)\}.$$

Let us write down the generator systems of G_1 and Γ_1. Let $(x_0, y_0) = (\xi_1/\xi_0, \xi_2/\xi_0)$ be the inhomogeneous coordinate of Ξ, then we have $1 < x_0 < y_0$. And set

$$L_x = \{(x, y) \in \boldsymbol{C}^2 : \ y = y_0\},$$
$$L_y = \{(x, y) \in \boldsymbol{C}^2 : \ x = x_0\}.$$

We define the following closed arcs δ_i $(i=1, \dots, 5)$ of $\pi_1(\Lambda, \Xi)$

δ_1; the loop goes around $x=1$ in the positive sense on L_x,
δ_2; the loop goes around $y=0$ in the positive sense on L_y,
δ_3; the loop goes around $x=y_0$ in the positive sense on L_x,

δ_4; the loop goes around $y=\infty$ in the positive sense on L_y,
δ_5; the loop goes around $x=0$ in the positive sense on L_x.

To make clear the way of construction we assume every δ_i is situated in the upper half plane of L_x (L_y respectively) except the lacet near the turning point. When we deform C_0 along δ_i, the branch locus ξ_j $(j=0, 1, 2)$ varies as Figure 1 on the z-plane.

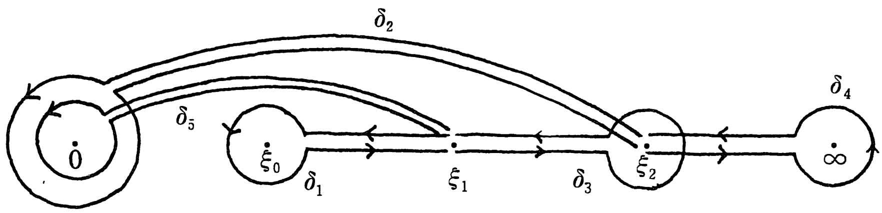


Figure 1

Then δ_i induces a monodromy transformation $g(\delta_i)$ as follows;

$$g(\delta_1)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&\omega\end{pmatrix},\qquad g(\delta_2)=\begin{pmatrix}-2\omega^2&\omega-1&0\\\omega-1&\omega^2&0\\0&0&1\end{pmatrix},$$

$$g(\delta_3)=\begin{pmatrix}1&0&0\\\omega-1&1&1-\omega^2\\\omega^2-\omega&0&\omega\end{pmatrix},\qquad g(\delta_4)=\begin{pmatrix}1&\omega-\omega^2&0\\0&1&0\\0&0&1\end{pmatrix},$$

$$g(\delta_5)=\begin{pmatrix}1&\omega-1&\omega-1\\0&1&0\\0&\omega-1&\omega\end{pmatrix},$$

these matrices are supposed to act to the system ${}^t(\eta_1, \eta_2, \eta_3)$ from left. We can examine the above transformation observing the deformation of C_0 along δ_i. The method is described also in the original paper of Picard (Reference [2] of Part I). So we omit the detail of an argument. And we choose the following generator system $\{\delta'_1, \dots, \delta'_5\}$ of $\pi_1(\Lambda, \Xi)$; $\delta'_1=\delta_1$, $\delta'_2=(\delta_1\delta_4\delta_2)^{-1}$, $\delta'_3=\delta_1\delta_3\delta_1$, $\delta'_4=\delta_4$, $\delta'_5=\delta_1^2\delta_5$, where the composition is supposed to perform from left to right. Then we obtain the corresponding transformations $g_i=g(\delta'_i)$ as follows;

$$
(1\text{-}2)\quad \begin{cases}
g_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, & g_2=\begin{pmatrix} 1 & 0 & 0 \\ \omega-\omega^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
g_3=\begin{pmatrix} 1 & 0 & 0 \\ \omega-1 & 1 & \omega-1 \\ 1-\omega^2 & 0 & 1 \end{pmatrix}, & g_4=\begin{pmatrix} 1 & \omega-\omega^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
g_5=\begin{pmatrix} 1 & \omega-1 & \omega-1 \\ 0 & 1 & 0 \\ 0 & 1-\omega^2 & 1 \end{pmatrix}. &
\end{cases}
$$

And we obtain the corresponding symplectic transformation $N_i=N(\delta_i')$ as follows;

$$
(1\text{-}3)\quad \begin{cases}
N_1=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & N_2=\begin{pmatrix} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
N_3=\begin{pmatrix} 1 & -2 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & N_4=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 \end{pmatrix} \\
N_5=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ -2 & -2 & -1 & 0 & 1 & 1 \end{pmatrix}, &
\end{cases}
$$

where N_i is supposed to act to the system ${}^t(B_1, B_2, B_3, A_1, A_2, A_3)$ from left. Naturally the systems (1-2) and (1-3) give generator systems of Γ_1 and G_1 respectively. The characterization of the group Γ_1 is obtained by several mathematicians independently. The result is already related in Part I §1 (2).

Now we set

$$(1\text{-}3)\quad \begin{cases} \Gamma=\{g\in PGL(3,\boldsymbol{Z}[\omega]):{}^tgH\bar{g}=H\}, \\ \Gamma_0=\{g\in SL(3,\boldsymbol{Z}[\omega]):{}^tgH\bar{g}=H\}, \\ \Gamma'=\{g\in SL(3,\boldsymbol{Z}[\omega]):{}^tgH\bar{g}\,H,\ g\equiv E \ \ (\text{mod. } \sqrt{3}i)\}, \end{cases}$$

where the notation $g\equiv E$ (mod. $\sqrt{3}i$) means that every entry of $cg-E$ belongs to the principal ideal $(\sqrt{3}i)$ of $\boldsymbol{Z}[\omega]$ for a certain complex number c. If we recall the relation (1-1), we obtain that Γ and Γ' induce subgroups of $Sp(3, \boldsymbol{Z})$. Let us denote them G and G' respectively.

Remark 1-1. (1) The following fact is already known; we have $\Gamma/\Gamma_1\cong S_4$ (the symmetric group) and the isomorphism $\rho: S_4\xrightarrow{\sim}\Gamma/\Gamma_1$ is given by

$$(1\text{-}4)\quad \begin{cases} \rho((12))=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \rho((123))=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \rho((12)(34))=\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \\ \rho((1234))=\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}. \end{cases}$$

(2) Clearly we have $[\Gamma_1;\Gamma']=3$ and g_1 represents the generator of Γ_1/Γ'.

(3) Γ'/Γ_0 is isomorphic to S_4.

Perhaps it is better to explain the above fact (1) by a geometric meaning. Let us consider the parameter space $\Lambda/S_4=T$ instead of Λ, where σ of S_4 acts on Λ as the projective linear transformation $L(\sigma)$ which induces the permutation of

$$\{P_0,\ P_1,\ P_2,\ P_3\}$$

by σ, where we use the notation

$$P_0=[1,0,0],\ P_1=[0,1,0],\ P_2=[0,0,1],\ P_3=[1,1,1].$$

In fact (1-4) gives the $L(\sigma)$ for a generator system of S_4. If we consider the monodromy transformation group induced from $\pi_1(T, *)$, it coincides with Γ. We denote an element g of Γ by

$$g=\begin{pmatrix} p_1(g) & q_1(g) & r_1(g) \\ p_2(g) & q_2(g) & r_2(g) \\ p_3(g) & q_3(g) & r_3(g) \end{pmatrix}, \tag{1-5}$$

and we denote an element N of $Sp(3, Z)$ by four $(3,3)$ blocks;

$$N=\begin{bmatrix} A(N) & B(N) \\ C(N) & D(N) \end{bmatrix}. \tag{1-6}$$

Suppose a discontinuous group H acting on D. If a holomorphic or meromorphic function $f(u,v)$ on D satisfies

$$f(g(u,v))(\det g)=\{p_1(g)+q_1(g)v+r_1(g)u\}^{3k}f(u,v) \tag{1-7}$$

for any point (u, v) of D and for any element g of H, we call $f(u, v)$ is a modular form or meromorphic modular form of weight k relative to H respectively. Here we note that it holds

$$\frac{\partial(u',v')}{\partial(u,v)}=\frac{\det g}{(p_1(g)+q_1(g)v+r_1(g)u)^3}, \tag{1-8}$$

where (u',v') indicates $g(u,v)$.

Let us denote the C-vector space of holomorphic modular forms of weight k relative to H by $A(H)_k$ and the graded ring $\bigoplus_{k=0}^{\infty} A(H)_k$ by $A(H)$.

§ 2. Modular Forms of Weight 1

In this section we show the following:

Proposition II-1. (1) $\dim_C(A(\Gamma_1)_1)=0$, $\dim_C(A(\Gamma_1)_2)=1$ *and* (2) $\dim_C(A(\Gamma')_1)=3$.

Let us introduce an affine coordinate of the ξ-space by $x=\xi_1/\xi_0$, $y=\xi_2/\xi_0$. According to the fact related in I§1(5) we may regard them as meromorphic modular functions on D relative to Γ_1. At first we note the following.

Lemma 2-1. (1) *The field of rational functions on the ξ-space $\boldsymbol{P}^2$ and the field of meromorphic modular functions on D relative to Γ_1 are isomorphic.*

(2) *We identify the above two fields and denote it by K_0. Then the K_0 vector space K_m of meromorphic modular forms of weight m on D relative to Γ_1 is isomorphic to K_0. And the isomorphism τ_m from K_0 to K_m is given by*

$$\tau_m(f)=f(x(u,v),\, y(u,v))\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^m. \tag{2-1}$$

Proof. (1) is obvious. (2) follows from an easy observation. q. e. d.

$\boldsymbol{P}^2-\Lambda$ is constituted of 6 lines. So we denote them as follows:

$H_0=\{\xi_0=0\}$, $H_1=\{\xi_1=0\}$, $H_2=\{\xi_2=0\}$, $H_3=\{\xi_1=\xi_2\}$,
$H_4=\{\xi_0=\xi_2\}$, $H_5=\{\xi_0=\xi_1\}$.

Here we obtain the following criterion for a meromorphic modular form $\tau_m(f)$ to be holomorphic.

Lemma 2-2. *$\tau_m(f)$ is a holomorphic modular form if and only if we have*

$$(f)\geqq\frac{7m}{3}H_0-\frac{2m}{3}\sum_{i=1}^{5}H_i \tag{2-2}$$

where (f) indicates the divisor on the ξ-space defined by f.

Proof. Let us consider the factor $\left[\frac{\partial(x,y)}{\partial(u,v)}\right]^m=A(u,v)$. The mapping $(x,y)=(x(u,v),\, y(u,v))$ is locally biholomorphic except the inverse image of H_i, and there it has a ramifying locus of order 3. Hence $A(u,v)$ has zeros of order 2 along the inverse image of H_i $(i\neq 0)$. Let $\alpha=(u,v)$ be a point on the inverse image of H_0, and set an affine coordinate $(x_1, y_1)=(1/x, y/x)$ which is valid on H_0. On the other hand let (u_1, v_1) be a local coordinate at α so that we have

$$x_1=u_1^3\times\text{unit function},\quad y_1=v_1\times\text{unit function}.$$

Then we have

$$\frac{\partial(x,y)}{\partial(u,\ v)}=\frac{\partial(x,\ y)}{\partial(x_1,\ y_1)}\ \frac{\partial(x_1,\ y_1)}{\partial(u_1,\ v_1)}\ \frac{\partial(u_1,\ v_1)}{\partial(u,\ v)}$$
$$=(u_1)^2(x_1)^{-3}\times\text{unit factor}.$$

Therefore $A(u,\ v)$ has a pole of order $7m$ along $u_1=0$. From the above observation we induce the required condition. q. e. d.

Proof of Proposition 1(1). It is a direct consequence of Lemma 2-2. q. e. d.

Next let us consider the modular forms relative to Γ'. According to Remark 1-1(2) the quatient space D/Γ' is a 3 sheeted ramified covering over D/Γ_1. So it defines a 3 sheeted ramified covering V of the ξ-space $\boldsymbol{P}^2$ as its compactification. The monodromy transformation g_1 shows that V has a ramifying locus along H_2 of degree 3. The situation is the same for every H_i. As easily shown V has 7 singular points over the points P_i $(i=0, 1, 2, 3)$ and $R_0=[0, 1, 1]$, $R_1=[1, 0, 1]$, $R_2=[1, 1, 0]$. We denote those singular points by the same notation as their projections (Figure 2).

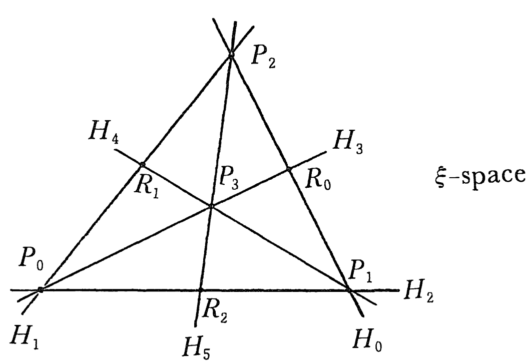


Figure 2

V has an affine representation

(2-3) $\qquad V: w^3=xy(x-1)(y-1)(x-y),$

so let V' be the inverse image of Λ relative to the covering mapping

$$\pi:(x,y,\ w)\rightarrow(x,y).$$

Lemma 2-3. (1) *The field of meromorphic functions on V and the field of meromorphic modular functions on D relative to Γ' are isomorphic.*

(2) *We identify the above two fields and denote it by K_0'. Then the K_0'-vector space K_m' of meromorphic modular forms of weight m on D relative to Γ' is isomorphic to K_0'. And the isomorphism $\tau_m': K_0' \to K_m'$ is given by*

$$\tau_m'(f) = f(P(u, v))\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^m \tag{2-4}$$

where $P(u, v)$ indicates the point on V corresponding to (u, v) of D.

Proof. (1) is obvious. So we show (2). Let f be a meromorphic function on V. Then $\tau_m'(f)$ determines a meromorphic function on D. And it has the automorphic factor $\left[\frac{\partial(u, v)}{\partial(u', v')}\right]^m$ for every point (u, v) of D corresponding to a point of V' and for every transformation $g(u, v) = (u', v')$ of Γ'. So it determines an element of K_m'. As easily shown it is an injective isomorphism. Next let us take an element $\phi(u, v)$ of K_m', then

$$\phi(u, v)\left[\frac{\partial(u, v)}{\partial(x, y)}\right]^m \tag{*}$$

defines a meromorphic function on V' because (x, y) is a local coordinate on V'. And it defines an algebraic function on the ξ-space $\boldsymbol{P}^2$, so it is single valued meromorphic on V. Therefore τ_m' maps (*) to $\phi(u, v)$, namely τ_m' is surjective. q. e. d.

Lemma 2-4. *Let f be a meromorphic function on V, then $\tau_m'(f)$ is a holomorphic modular form if and only if we have*

$$(f) \geqq 7mH_0 - 2m \sum_{i=1}^{5} H_i. \tag{2-5}$$

Proof. We can show the above condition by the same way as the proof of Lemma 2-2. q. e. d.

Next let us investigate the minimal nonsingular model $\tilde{V}$ of V. The singular point of V over R_i $(i=0, 1, 2)$ is a rational double singularity A_2. So we obtain two rational curves Θ_{i1} and Θ_{i2} as the exceptional divisor of its resolution, where we suppose Θ_{i1} intersects the proper image $\hat{H}_i$ of H_i and Θ_{i2} intersects $\hat{H}_{i+3}$ (that of H_{i+3}).

The singular point P_k ($k=0, 1, 2, 3$) of V is a simple elliptic singularity $\tilde{E}_6$. So we obtain an elliptic curve E_k as the exceptional divisor of its resolution. The resolution of the singularity A_2 and $\tilde{E}_6$ is wellknown, so we omitted the detailed discussion.

Let $\hat{V}$ be the surface obtained after the performance of these resolutions. Then $\hat{H}_i$ ($0 \leqq i \leqq 5$) is an exceptional curve of first kind. So let us blow down $\hat{H}_{k+3}$, Θ_{k2} and Θ_{k1} ($k=0, 1, 2$) in this order (note that Θ_{ki} is a -2 curve). Let us denote the consequent surface by $\underline{V}$. Let us consider a complex line $l_t = l(t_1, t_2) = \{\xi \in \boldsymbol{P}^2 : t_1\xi_1 + t_2 + \xi_2 = (t_1+t_2)\xi_0\}$ on the ξ-space, so we get an elliptic curve $\pi^{-1}(l_t)$ for general value t. And moreover its invariant is always equal to 0, because $\pi^{-1}(l_t)$ is a 3 sheeted covering over $\boldsymbol{P}^1$ with three ramifying points of degree 3. Here we note that $\pi^{-1}(l_t)$ intersects E_0 except the case $l_t = \hat{H}_3$, $\hat{H}_4$ and $\hat{H}_5$. It is easy to show that every proper image of $\pi^{-1}(l_t)$ on $\underline{V}$ is an elliptic curve of the invariant 0. So we obtain a trivial fibration of elliptic curves on $\underline{V}$, therefore we know the following.

Lemma 2-5. *The minimal nonsingular model $\tilde{V}$ of V is isomorphic to $\boldsymbol{P} \times E$, where E is an elliptic curve of the invariant* 0.

Let us denote the image of E_i in $\tilde{V}$ by $\tilde{E}_i$ and that of $\hat{H}_i$ by $\tilde{H}_i$.

Remark 2-1. The elliptic curve $\tilde{E}_3$ is a three sheeted section of this fibration. The fibre $\tilde{E}_i$ ($i=0, 1, 2$) has one double contact with $\tilde{E}_3$ at $R_{ii} = \tilde{E}_i \cap \tilde{H}_i$. On the other hand $\tilde{H}_i$ ($i=0, 1, 2$) is an one sheeted section in $\tilde{V}$ (Figure 3).

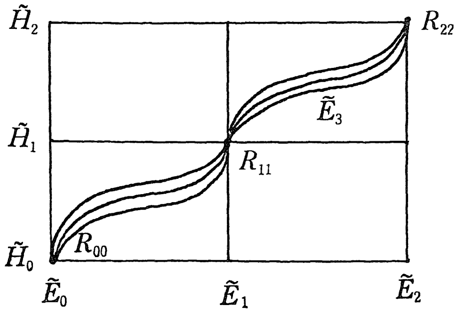


Figure 3

Lemma 2-6. *Suppose a meromorphic function f on V with*

$$(f)=\sum_{i=0}^{5} m_iH_i+D_1$$

where D_1 is an effective divisor with its support situated outside of H_i. And let $\tilde{f}$ be the corresponding function on $\tilde{V}$. Then it holds

$$\begin{aligned}(\tilde{f}) &= m_0H_0+m_1H_1+m_2H_2+\left(\frac{1}{3}(m_1+m_2+m_3)+\delta_0\right)E_0 \\ &+\left(\frac{1}{3}(m_0+m_2+m_4)+\delta_1\right)E_1+\left(\frac{1}{3}(m_0+m_1+m_5)+\delta_2\right)E_2 \\ &+\left(\frac{1}{3}(m_3+m_4+m_5)+\delta_3\right)E_3\end{aligned}$$

for certain nonnegative integers $\delta_0, \dots, \delta_3$.

Proof. If we follow the resolution process at P_i, the assertion is induced directly. q. e. d.

Lemma 2-7. *Suppose a meromorphic function F on $\tilde{V}$ with*

$$(F)=\sum_{i=0}^{2} m_iH_i+\sum_{j=0}^{3} n_jE_j+\tilde{D}$$

where $\tilde{D}$ is an effective divisor. Let $\bar{F}$ be the corresponding function on V. Then we have

$$(\bar{F})=\sum_{i=0}^{2} m_iH_i+\sum_{i=0}^{2}(3(n_3+n_i)+m_i+\varepsilon_i)H_{i+3}+\bar{D}$$

for certain nonnegative integers ε_i and the divisor $\bar{D}$ is effective and its support is situated outside of H_i $(0\leqq i\leqq 5)$.

Lemma 2-8. *Let l_i be the intersection multiplicity between $\tilde{D}$ and $\tilde{E}_i$ at R_{ii}. Then we have $\varepsilon_i=l_i$.*

Proofs of Lemma 2-7 *and Lemma* 2-8. By the fact related in Remark 2-1 we know that H_{i+3} is obtained after 3 times of blow up processes at R_{ii} (by the first and the second processes we get Θ_{i1} and Θ_{i2} respectively). The assertion follows from the observation of this process. q. e. d.

Proof of Proposition 1(2). Let A be the vector space of meromorphic functions on V satisfying the condition (2-5) for $m=1$. Then it is sufficient to know $\dim_{\mathbf{C}} A$. Let f be an element of A and let $\tilde{f}$ be the meromorphic function on $\tilde{V}$ which corresponds to f. By Lemma 2-6 we have

$$(2\text{-}6)\qquad (\tilde{f}) \geqq 7\tilde{H}_0 - 2\tilde{H}_1 - 2\tilde{H}_2 - 2\tilde{E}_3 - 2\tilde{E}_0 + \tilde{E}_1 + \tilde{E}_2.$$

Here we note the following linear equivalence relations for divisors on $\tilde{V}$:

(i) $\tilde{E}_0 \equiv \tilde{E}_1 \equiv \tilde{E}_2$, (ii) $3\tilde{H}_0 \equiv 3\tilde{H}_1 \equiv 3\tilde{H}_2$, (iii) $\tilde{E}_3 \equiv \tilde{E}_i + 3\tilde{H}_j$ $(0 \leqq i, j \leqq 2)$, (iv) $2\tilde{H}_i \equiv \tilde{H}_j + \tilde{H}_k$ $(\{i, j, k\} = \{0, 1, 2\})$. Hence (2-6) reduces to

$$(2\text{-}7)\qquad (G) \geqq -3\tilde{H}_0 - 2\tilde{E}_1,$$

for a certain meromorphic function G on $\tilde{V}$. Using the Riemann-Roch theorem and the Serre duality theorem we get

$$(2\text{-}8)\qquad \dim H^0(\tilde{V}, \mathcal{O}(3\tilde{H}_0 + 2\tilde{E}_1)) = 9.$$

So we consider the linear systems

$$L' := \{\delta \geqq 0 \mid \delta \equiv 3\tilde{H}_0 + 2\tilde{E}_0,\ \delta \cdot \tilde{E}_i|_{R_{ii}} \geqq 3\},$$

$$L = |3H_0 + 2E_0| = \{\delta \geqq 0 \mid \delta \equiv 3H_0 + 2E_0\} \quad (i = 0, 1, 2).$$

From (2-8) we have $\dim L = 8$. Let g be a meromorphic function with $(g) = \delta - (3\tilde{H}_0 + 2\tilde{E}_0)$. Then g is a linear combination of

$$1,\ 1/z,\ 1/z^2,\ \wp,\ \wp/z,\ \wp/z^2,\ \wp',\ \wp'/z,\ \wp'/z^2,$$

where z is an affine coordinate of $\boldsymbol{P}$ and $\wp$ is the Weierstrass $\wp$ function on the elliptic curve E.

So the contact condition $\delta \cdot E_i|_{R_{ii}} \geqq 3$ for δ of L imposes two linearly independent restrictions for each i. Hence we have

$$\dim L' = \dim A - 1 = 2.$$

q. e. d.

§ 3. The Possibility of Common Zeros of Theta Constants

Henceforth we use the following notations:

$$\alpha_k = \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{k}{3} & \frac{1}{6} & \frac{k}{3} \end{bmatrix} \quad (k = 0, 1, 2)$$

$$\theta_k(u, v) = \theta[\alpha_k](0, \Omega(u, v))$$
$$\varphi_k(u, v) = \{\theta_k(u, v)\}^3.$$

And let us denote the inverse of the period mapping Φ by

$$\Xi = [\xi_0(u,v), \xi_1(u,v), \xi_2(u,v)] = [\varphi_0(u,v), \varphi_1(u,v), \varphi_2(u,v)].$$

We discuss about the property of $\theta_k(u,v)$ in § 3 and § 4. For this purpose we cite the following transformation formula of theta constants (see [R-F] and [I]).

Theorem 3-1. *Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be an element of* $Sp(g, \boldsymbol{Z})$ *and suppose a characteristic* $\varepsilon = \begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}$ *of* $\boldsymbol{Q}^g \times \boldsymbol{Q}^g$. *Set*

$$M \circ \Omega = (A\Omega + B)(C\Omega + D)^{-1}$$

$$M \circ \varepsilon = \begin{bmatrix} D\varepsilon' - C\varepsilon'' + \frac{1}{2}\mathrm{dv}(C^tD) \\ -B\varepsilon' + D\varepsilon'' + \frac{1}{2}\mathrm{dv}(A^tB) \end{bmatrix},$$

where $\mathrm{dv}(*)$ *indicates the diagonal vector of* $*$. *Then we have*

$$\theta[M \circ \varepsilon](0, M \circ \Omega) = K(M, \varepsilon)\sqrt{\det(C\Omega + D)}\,\theta[\varepsilon](0, \Omega),$$

where $K(M, \varepsilon)$ *is a certain complex number with modulus* 1 *depending on* M *and* ε.

Remark 3-1. We shall relate the definition of the factor K and the branch of $\sqrt{\det(C\Omega + D)}$ aftarwards (see (4-5) and (4-6)).

Remark 3-2. We note here the following relation also:

$$\theta\begin{bmatrix} \varepsilon' + n' \\ \varepsilon'' + n'' \end{bmatrix}(z, \Omega) = \exp\{2\pi i \langle \varepsilon', n'' \rangle\}\, \theta\begin{bmatrix} \varepsilon' \\ \varepsilon'' \end{bmatrix}(z, \Omega)$$

for n', n'' of $\boldsymbol{Z}^g$, where $\langle\ ,\rangle$ indicates the Euclidean inner product.

Now let us investigate the possibility of common zeros of $\varphi_k(u, v)$. Suppose the parameter ξ is situated on $H_5 - \{P_2, P_3\}$, then the Picard curve degenerates to the curve

$$C' \colon w^3 = z(z - \xi_0)^2 (z - \xi_2).$$

And 1-cycles A_2 and B_2 vanish on C', moreover the system $\{A_1, A_3,$

$B_1, B_3\}$ becomes a canonical basis (see Part I (1-2)). In this case we have

$$\eta_3=\int_{A_2}\varphi=0,$$

so it holds $u=\eta_3/\eta_1=0$. Set a complex line $L=\{(u, v)\in D\,|\,u=0\}$. Then we have

$$\Phi(H_5-\{P_2, P_3\})=\Gamma_1\text{-orbit of } L,$$

namely

$$\Xi(L)=H_5-\{P_2, P_3\}.$$

Let us observe the behavior of θ_k on L. According to Part I Proposition 3 it holds

$$\theta_1(0,v)=\zeta_3\theta_2(0,v)$$

on L, where ζ_3 is a cubic root of 1 (we can show directly the equality $\theta_1(0, v)=\theta_2(0, v)$ also). And we have the following decomposition of theta constants:

Lemma 3-1.

$$\theta\begin{bmatrix}0 & g_2 & 0\\ h_1 & h_2 & h_3\end{bmatrix}\left(0,\begin{pmatrix}\tau_1 & 0 & \varepsilon\\ 0 & \tau_2 & 0\\ \varepsilon & 0 & \tau_3\end{pmatrix}\right)=\theta\begin{bmatrix}g_2\\ h_2\end{bmatrix}(0,\tau_2)\ \theta\begin{bmatrix}0 & 0\\ h_1 & h_2\end{bmatrix}\left(0,\begin{pmatrix}\tau_1 & \varepsilon\\ \varepsilon & \tau_3\end{pmatrix}\right).$$

Proof. The left hand term equals to

$$[\sum_{n_1\in\mathbf{Z}}\exp\{\pi i(n_2+g_2)^2\tau_2+2\pi i(n_2+g_2)h_2\}]$$
$$\times[\sum_{n_1,n_3\in\mathbf{Z}}\exp\{\pi in_1^2\tau_1+\pi in_3^2\tau_3+2\pi in_1n_3\varepsilon+2\pi i(n_1h_1+n_3h_3)\}].$$

Two parentheses are equal to the first and the second factors in the right hand term respectively. q. e. d.

If we recall the matrix $\Omega(u, v)$ in I (1-3), we have

$$\text{(3-1)}\qquad \theta[\alpha_k](0,\Omega(0,v))=\theta\begin{bmatrix}\frac{1}{6}\\ \frac{1}{6}\end{bmatrix}(0,\ -\omega^2)\,\theta\begin{bmatrix}0 & 0\\ \frac{k}{3} & \frac{k}{3}\end{bmatrix}(0,\ \Omega'(v))$$

by Lemma 3-1, where

$$\Omega'(v)=\begin{pmatrix} -\frac{2i}{\sqrt{3}}v & \frac{i}{\sqrt{3}}v \\ \frac{i}{\sqrt{3}}v & -\frac{2i}{\sqrt{3}}v \end{pmatrix}.$$

Let F and G_k be the first and the second factors in the right hand term of (3-1). It holds

$$F=\theta\left(\frac{1}{6}(1-\omega^2),\ -\omega^2\right)\times \text{unit}.$$

On the other hand $\theta(z, \tau)$ has only one zero represented by $(1+\tau)/2$ on the Jacobi variety $\boldsymbol{C}/\boldsymbol{Z}+\boldsymbol{Z}\tau$. Thus we have $F\neq 0$.

Moreover we have:

Lemma 3-2. *$\theta_0(u, v)$ has no zero on L.*

Proof. If we assume $\theta_0(0, v)=0$ for a certain value v, then it induces $G_0=\theta(0, \Omega'(v))=0$. According to Theorem A-2 there is a point P on C' with

$$(*)\qquad \Delta=\int_{P_0}^{P}\omega,$$

where P_0 is a certain fixed point and Δ is the Riemann constant determined by the homology basis $\{A_1, A_3, B_1, B_3\}$ and P_0 (see (A-3)). Here we note the condition $(*)$ is independent of P_0 and the choice of a homology basis. According to Theorem A-3 we have $D_0=P$, and D_0 corresponds to $(\eta', \eta'')=(0, 0)$. Now let us recall Theorem A-4, obviously we have

$$d(D_0)=\dim H^0(C', \mathcal{O}(D_0))=1$$

on the other hand we must have

$$d(D_0)\equiv 4{}^t\eta'\eta''\equiv 0 \pmod{2}.$$

This is a contradiction. q. e. d.

Proposition II-2. *There is no common zero of $\theta_k(u, v)$ $(k=0, 1, 2)$ on D.*

Proof. When $\Xi(u, v)$ belongs to Λ the assertion is already obtained by [5]. We can get his result by applying Remark A-5 for the representation I (1-4), because this representation (x,y) coincides with

$$(\xi_1/\xi_2,\ \xi_0/\xi_2)=(\varphi_1/\varphi_2,\ \varphi_0/\varphi_2).$$

So we suppose $\Xi(u,v)$ belongs to H_i $(0\leqq i\leqq 5)$. The case $i=5$ is already shown in Lemma 3-2. For general i set $\Xi(u,v)=[a_0, a_1, a_2]$. Then we can find a parameter $b=[b_0, b_1, b_2]$ on H_5 so that we have $C(a)\cong C(b)$, where $\cong$ indicates the biholomorphic equivalence relation.

Let σ be the isomorphism from $C(a)$ to $C(b)$. And let $\{\gamma_i\}$ and $\{\gamma_i'\}$ $(1\leqq i\leqq 6)$ be the homology basis on $C(a)$ and $C(b)$ corresponding to the point (u,v) and a point (u',v') on L respectively. So we can find an element M of $Sp(3,\mathbf{Z})$ with $(\sigma\gamma_i)=M(\gamma_i')$. Then it holds

$$\theta[\alpha_k](0, \Omega(u, v))=\theta[\alpha_k](0, M\circ\Omega(u', v')).$$

On the other hand we have

$$\theta[\alpha_k](0, M\circ\Omega(u', v'))=unit\times\theta[\alpha_k](0, \Omega(u', v'))$$

for a point (u',v') of L from Theorem 3-1. Hence the problem is reduced to Lemma 3-2. q. e. d.

Remark 3-3. Using the above result we can easily show that $\theta_j(u,v)$ and $\theta_k(u,v)$ has no common zero on D for any pair (j,k) with $j\neq k$.

§ 4. Characterization of $\varphi_k(u,v)$ as Modular Forms

Now let us observe the automorphic factor of $\theta_k(u, v)$ relative to the transformation $g_1,\dots, g_5$. We have:

Lemma 4-1. $\theta[\alpha_k](0, N_j\circ\Omega)=\rho_j\theta[N_j\circ\alpha_k](0, N_j\circ\Omega)$
$(1\leqq j\leqq 5, k=0,\ 1.2)$,

where $\rho_1=\exp\left(\frac{1}{3}\pi i\right)$ *and* $\rho_2=\rho_3=\rho_4=\rho_5=1$.

Proof. By an elementary calculation we have $N_j\circ\alpha_k=\alpha_k+n_{jk}$ as follows:

$$n_{1k}=\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\end{bmatrix},\ n_{2k}=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix},\ n_{3k}=\begin{bmatrix}0 & 0 & 0\\ -1 & 0 & 1\end{bmatrix},$$

$$n_{4k}=\begin{bmatrix}k-1 & 0 & k-1\\ 0 & 0 & 0\end{bmatrix},\ n_{5k}=\begin{bmatrix}0 & k & k-1\\ 0 & 0 & 0\end{bmatrix}.$$

Considering Remark 3-2 we have the assertion. q. e. d.

Combining Lemma 4-1 and Theorem 3-1 we deduce that

$$\theta[\alpha_k](0, N_j\circ\Omega)=\rho_j K(N_j, \alpha_k)\sqrt{det\{C(N_j)\Omega+D(N_j)\}}\,\theta[\alpha_k](0, \Omega). \tag{4-1}$$

Namely it indicates the required automorphic factor. According to [R-F] we have the method to determine it.

We choose a generator system of $Sp(g,\boldsymbol{Z})$ as the following:

$$\begin{aligned}&{}^{\pm}B_i:=I\pm E_{i,g+i}\,(1\leqq i\leqq g),\ {}^{\pm}C_i:={}^t({}^{\pm}B_i),\\&{}^{\pm}A_{ij}:=I\pm(E_{i,j}-E_{g+j,g+i})\quad(1\leqq i,\ j\leqq g,\ i\neq j),\\&D_i:=I-2(E_{i,i}+E_{g+i,g+i})\quad(1\leqq i\leqq g),\end{aligned}$$

where $E_{i,j}$ indicates the matrix (m_{kl})

$$\text{with } m_{kl}=\begin{cases}1 & (i,\ j)=(k,\ l)\\ 0 & \text{otherwise}\end{cases}.$$

We denote this system by S, and the element M of S will be called of type B, type C, type A or type D neglecting the signature and the subscript.

For an element M of S we define the following:

$$\gamma(M):=\begin{cases}1 & \text{for } M \text{ of type } A,\ B \text{ and } C.\\ -\mathrm{i} & \text{for } M \text{ of type } D,\end{cases} \tag{4-2}$$

$$\sqrt{det\{C(M)\Omega+D(M)\}}$$
$$:=\begin{cases}1 & \text{for } M \text{ of type } A \text{ and } B\\ i & \text{for } M \text{ of type } D\\ \text{the branch in the right half plane} & \\ \{z\in\boldsymbol{C}:Re\ z>0\} & \text{for } M \text{ of type } C\end{cases} \tag{4-3}$$

Remark 4-1. For the type C case the determination of (4-3) is given more precisely: namely it belongs to the upper half plane for ${}^{+}C$ type, and it belongs to the lower half plane for ${}^{-}C$ type.

For an element $M=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$ of $Sp(g,\boldsymbol{Z})$ and a characteristic $\varepsilon=\begin{bmatrix}\varepsilon'\\ \varepsilon''\end{bmatrix}$ we set

$$\varphi(M,\varepsilon):=-{}^t\varepsilon'{}^tDB\varepsilon'+2{}^t\varepsilon''{}^tCB\varepsilon'-{}^t\varepsilon''{}^tCA\varepsilon''+{}^t(D\varepsilon'-C\varepsilon'')\,\mathrm{dv}(A{}^tB).$$

And for elements M_1, M_2 of $Sp(g,\boldsymbol{Z})$ we set

$$\begin{bmatrix}m_1\\ m_2\end{bmatrix}:=(M_2M_1)\circ\begin{bmatrix}0\\ 0\end{bmatrix}-M_2\circ M_1\circ\begin{bmatrix}0\\ 0\end{bmatrix},$$

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} := 2(M_2M_1)\circ\begin{bmatrix}0\\0\end{bmatrix},$$
$$[M_2, M_1] := (-1)\langle\lambda_1, m_2\rangle.$$

When M' and M'' belong to S we define

$$\gamma(M'M'') := [M', M'']\gamma(M')\gamma(M'')\exp\left\{\pi i\varphi\ M', M''\circ\begin{bmatrix}0\\0\end{bmatrix}\right\}.$$

For a general element M of $Sp(g, \boldsymbol{Z})$ let

$$M = M_r \cdots M_1 \tag{4-4}$$

be a decomposition by the elements of S. Then we can define

$$\gamma(M) = [M_r, M_{r-1}\cdots M_1]\gamma(M_r)\gamma(M_{r-1}\cdots M_1)$$
$$\times\exp\left\{\pi i\varphi\left(M_r, M_{r-1}\cdots M_1\circ\begin{bmatrix}0\\0\end{bmatrix}\right)\right\}$$

by the induction relative to r. If we set $S_k = M_k\cdots M_1$ $(S_0 = I)$, we have

$$C(M)\Omega + D(M) = [(C(S_r)\Omega + D(S_r))(C(S_{r-1})\Omega + D(S_{r-1}))^{-1}]$$
$$\times\cdots\times[(C(S_2)\Omega + D(S_2))(C(S_1)\Omega + D(S_1))^{-1}][C(S_1)\Omega + D(S_1)]$$
$$= \prod_{k=1}^{r}\{C(M_k)(S_{k-1}\circ\Omega) + D(M_k)\}.$$

So we define

$$\sqrt{det\{C(M)\Omega + D(M)\}} := \prod_{k=1}^{r}\sqrt{det\{C(M_k)(S_{k-1}\circ\Omega) + D(M_k)\}}. \tag{4-5}$$

Remark 4-2. The values of $\gamma(M)$ and $\sqrt{det\{C(M)\Omega + D(M)\}}$ are dependent of the decomposition (4-4).

According to [R-F] the unit factor in Theorem 3-1 is given by

$$K(M, \varepsilon) = \gamma(M)\exp\{\pi i\varphi(M, \varepsilon)\}. \tag{4-6}$$

Remark 4-3. If every M_k is of type A or of type B for the decomposition (4-4), we have $K(M, \varepsilon) = 1$ and $\sqrt{\det\{C(M)\Omega + D(M)\}} = 1$.

To determine $K(N_j, \alpha_k)$ and $\sqrt{\det\{C(N_j)\Omega + D(N_j)\}}$ we use the following decomposition:

$$\begin{cases} N_1 = {}^+C_2D_2{}^-B_2, \\ N_2 = {}^-A_{13}{}^+B_1{}^+B_3{}^+A_{13}{}^+B_3, \\ N_3 = ({}^-A_{12})^2({}^+B_1)^{3+}B_3{}^-B_2{}^+A_{12}{}^+A_{32}{}^+B_2{}^-B_1{}^-A_{12}, \\ N_4 = {}^-C_1{}^-C_3{}^-A_{13}{}^-C_1{}^+A_{13}, \\ N_5 = {}^-A_{13}{}^-A_{23}{}^-C_2{}^+A_{21}{}^-C_1({}^+C_2)^{2+}A_{13}{}^-A_{23}{}^-C_2. \end{cases} \tag{4-7}$$

Then we have:

Lemma 4-2. $K(N_1, \alpha_k) = -1, \quad K(N_2, \alpha_k) = K(N_3, \alpha_k) = 1,$

$$K(N_4, \alpha_k) = \exp\left(\frac{2\pi i}{3}k^2\right) \text{ and } K(N_5, \alpha_k) = \exp\left(\frac{\pi i}{3}k(k+1)\right).$$

Proof. Following the definition of $\gamma(M)$ we obtain

$$\gamma(N_1) = \gamma({}^+C_2D_2{}^-B_2) = \exp\left(-\frac{3}{4}\pi i\right).$$

Next we consider $\gamma(N_j)$ $(j=2, 3, 4, 5)$. Suppose both M_1 and M_2 of system S are of type A or of type B. Then it holds $C(M_2M_1) = 0$. It induces $2(M_2M_1)\circ\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\\varepsilon''\end{bmatrix}$ for certain ε''. Hence it follows $[M_2, M_1] = 1$. And if we suppose both M_1 and M_2 are of type A or of type C. Then we have $B(M_1) = B(M_2) = B(M_2M_1) = 0$. So it induces

$$(M_2M_1)\circ\begin{bmatrix}0\\0\end{bmatrix} - M_2\circ M_1\circ\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}\varepsilon'\\0\end{bmatrix}$$

for some ε'. Hence it follows $[M_2, M_1] = 1$ also. And for the former case we have

$$\varphi\left(M_2, M_1\circ\begin{bmatrix}0\\0\end{bmatrix}\right) = \varphi\left(M_2, \begin{bmatrix}0\\\varepsilon''\end{bmatrix}\right)$$
$$= -{}^t\varepsilon''{}^tC(M_2)A(M_2)\varepsilon'' - ({}^t(C(M_2)\varepsilon'')\,\mathrm{dv}(A(M_2){}^tB(M_2))) = 0,$$

because it holds $C(M_2) = 0$. By a similar argument we obtain

$$\varphi\left(M_2, M_1\circ\begin{bmatrix}0\\0\end{bmatrix}\right) = 0$$

for the latter case also.

Recalling the decomposition (4-7) we have $\gamma(N_j) = 1$ $(j=2, 3, 4, 5)$. By the definition of $\varphi(M, \varepsilon)$ we can calculate $\varphi(N_j, \alpha_k)$, namely we have:

$$\varphi(N_1, \alpha_k) = -\frac{1}{4},$$
$$\varphi(N_2, \alpha_k) = \varphi(N_3, \alpha_k) = 0,$$
$$\varphi(N_4, \alpha_k) = \frac{2}{3}k^2,$$
$$\varphi(N_5, \alpha_k) = \frac{1}{3}k(k+1).$$

Now the assertion is clear, because we have (4-6). q. e. d.

Now we obtain:

Lemma 4-3.

$$\{\theta[\alpha_k](0,\ M\circ\Omega)\}^3=[\sqrt{\det\{C(M)\Omega+D(M)\}}]^3\{\theta[\alpha_k](0,\Omega)\}^3$$

for any element M of G'.

Proof. It is sufficient to prove the equality for the generator system $\{N_1N_2N_1^{-1},\ N_2,\dots,N_5\}$ of G'. Recalling (4-1), Lemma 4-1 and Lemma 4-2 we get the required equality. q. e. d.

Remark 4-4. We can check easily the following:

(1) $\det\{C(N_1)\Omega(u,v)+D(N_1)\}=\omega^2$ and
$p_1(g_1)+q_1(g_1)v+r_1(g_1)u=1$,

(2) $\det\{C(N_j)\Omega(u,v)+D(N_j)\}=\{p_1(g_k)+q_1(g_k)v+r_1(g_k)u\}^2$.

Moreover we can determine the square root of
$\det\{C(N_j)\Omega(u,v)+D(N_j)\}$:

Lemma 4-4. *Set* $T_j=\sqrt{\det\{C(N_j)\Omega(u,v)+D(N_j)\}}$ *then we have*

$$T_j=\begin{cases}\omega & \text{for } j=1,\\ 1 & \text{for } j=2,3,\\ 1+(\omega-\omega^2)v & \text{for } j=4,\\ 1+(\omega-1)v+(\omega-1)u & \text{for } j=5.\end{cases}$$

Especially it holds

$$T_j^{-3}=\{p_1(g_j)+q_1(g_j)v+r_1(g_j)u\}^{-3}.$$

And also it holds

$$T_j=p_1(g_j)+q_1(g_j)v+r_1(g_j)u$$

for $j=2,3,4,5$.

Proof. For the case $j=1$ we have

$$\sqrt{\det\{C(^-B_2)\Omega(0,v)+D(^-B_2)\}}=1,$$

$$\sqrt{\det\{C(D_2)(^-B_2\circ\Omega(0,v))+D(D_2)\}}=\exp\left(\frac{1}{2}\pi i\right),$$

$$\sqrt{\det\{C(^+C_2)((D_2{}^-B_2)\circ\Omega(0,v))+D(^+C_2)\}}=\exp\left(\frac{1}{6}\pi i\right).$$

Then by the definition (4-5) we have $T_1=\omega$.

For the case $j=2$, 3 the assertion is obvious. The arguments are same for both cases $j=4$ and $j=5$, so we discuss only the latter. Let us denote the decomposition of N_5 in (4-7) by $N_5=M_{10}\cdots M_1$, and set

$$L_k=\sqrt{\det\{(C(S_k)\Omega+D(S_k))(C(S_{k-1})\Omega+D(S_{k-1}))^{-1}\}}$$
$$=\sqrt{\det\{C(M_k)(S_{k-1}\circ\Omega)+D(M_k)\}}.$$

The ambiguity of T_j comes only from the signature of the square root.

Therefore it is sufficient to show the equality for $\Omega(0, v)$ (v is a real negative variable). From Remark 4-3 we get $L_{10}=L_9=L_7=L_3=L_2=1$. If we set

$$\theta_k=\lim_{v\to-\infty} Arg\, L_k\,,$$

we obtain

$$\theta_1=-\frac{1}{6}\pi,\ \theta_4=\frac{1}{4}\pi,\ \theta_5=0,\ \theta_6=-\frac{1}{4}\pi,\ \theta_8=0.$$

On the other hand if we set

$$\beta(u, v)=p_1(g_5)+q_1(g_5)v+r_1(g_5)u,$$

we can check directly

$$\lim_{v\to-\infty} Arg\ \beta(0, v)=-\frac{1}{6}\pi.$$

Thus we get the assertion for the case $j=5$. q. e. d.

Now we can state

Proposition II-3. (1) *The system* $\{\varphi_k(u, v)\}$ $(k=0, 1, 2)$ *gives a basis of* $A(\Gamma')_1$.

(2) *We have* $\theta_k(g_1(u, v))=\theta_k(u, v)$ $(k=0, 1, 2)$, *especially it holds* $\varphi_k(g_1(u, v))=\varphi_k(u, v)$.

Proof. (1) Combining Lemma 4-3 and Lemma 4-4 we get

$$(*)\qquad \varphi_k(g(u, v))=\{p_1(g)+q_1(g)v+r_1(g)u\}^3\varphi_k(u, v)$$

for $g=g_j$ $(j=2, 3, 4, 5)$. Because $\{g_1g_2g_1^{-1}, g_2, \ldots, g_5\}$ is a generator system of Γ', the equality (*) holds for any g of Γ'. Recalling the relation (1-8) and the fact $\det(g)=1$ for any g of Γ' it is clear that $\varphi_k(u, v)$ belongs to $A(\Gamma')_1$. According to I Proposition 3 the system $\{\varphi_0, \varphi_1, \varphi_2\}$ is linearly independent. Combining this fact with Proposition II-1 we get the assertion.

(2) Recalling (4-1) we know that it is sufficient to determine ρ_1, $K(N_1, \alpha_k)$ and $\sqrt{\det C(N_1)\Omega+D(N_1)}$. Lemma 4-1, Lemma 4-2 and Lemma 4-4 give these values. q. e. d.

Remark 4-5. (1) We can deduce the fact that $\varphi_k(g_1(u, v))=\varphi_k(u, v)$ from the fact that $g_1^3=id$ and that the automorphic factor (4-1) of $\theta[\alpha_k]$ for N_1 is a complex number of modulus 1. (2) We note that

$$\varphi_k(g_1(u, v))\frac{\partial(u', v')}{\partial(u, v)}=\omega\varphi_k(u, v) \tag{4-8}$$

for $k=0, 1, 2$, because we have

$$\frac{\partial(g_1(u, v))}{\partial(u, v)}=\omega$$

from Lemma 4-4 and (1-8).

§ 5. The Generator System of the Graded Ring of Modular Forms

If we set $x=\frac{\xi_1}{\xi_0}$, $y=\frac{\xi_2}{\xi_0}$, $w=\frac{\xi_3}{\xi_0}$ in the affine representation (2-3) of D/Γ', we get a projective representation

$$V_1: \xi_3^3\xi_0^2=\xi_1\xi_2(\xi_0-\xi_1)(\xi_1-\xi_2)(\xi_2-\xi_0). \tag{5-1}$$

We may regard w as a meromorphic function on D/Γ'. Therefore it may be considered as a meromorphic modular function on D relative to Γ'. Combining (5-1) and I Proposition 3 we obtain

$$w=\varphi_0^{-2}\{\varphi_0\varphi_1\varphi_2(\varphi_0-\varphi_1)(\varphi_1-\varphi_2)(\varphi_2-\varphi_0)\}^{(1/3)}.$$

And set

$$\zeta=\{\varphi_0\varphi_1\varphi_2(\varphi_0-\varphi_1)(\varphi_1-\varphi_2)(\varphi_2-\varphi_0)\}^{(1/3)}, \tag{5-2}$$

then we have $\zeta=w\varphi_0^2$. Hence ζ is a single valued meromorphic modular form of weight 2 relative to Γ'. On the other hand it is clear that ζ has no pole on D because of its definition. Therefore ζ belongs to $A(\Gamma')_2$.

Lemma 5-1. *$A(\Gamma')$ is generated by $\varphi_0, \varphi_1, \varphi_2, \zeta$.*

Proof. Let σ be an element of $A(\Gamma')_m$, and suppose that φ_1 is not a factor of σ. Then $\sigma\varphi_1^{-m}$ can be regarded as a meromorphic function on V_1. Hence we can describe it as

$$\sigma\varphi_1^{-m}=\frac{P(\varphi_0, \varphi_1, \varphi_2, \zeta)}{Q(\varphi_0, \varphi_1, \varphi_2, \zeta)},$$

where P and Q are polynomials of $\varphi_0, \varphi_1, \varphi_2$ and ζ without common factor. At first we suppose that the divisor (σ) and $m(\varphi_1)$ on V have no common component. Then we have $m(\varphi_1) \leqq (Q(\varphi_0, \varphi_1, \varphi_2, \zeta))$. According to Remark 3-3 it holds $(\varphi_1)=H_1$, and φ_1 is irreducible. Therefore we must have $m(\varphi_1)=(Q(\varphi_0, \varphi_1, \varphi_2, \zeta))$. Namely we have

$$(\sigma)=(P(\varphi_0, \varphi_1, \varphi_2, \zeta)).$$

It implicates that σ belongs to $\boldsymbol{C}[\varphi_0, \varphi_1, \varphi_2, \zeta]$. For general element σ we can choose a certain linear form l of φ_0, φ_1 and φ_2 so that (σ) and (l) has no common component. Then we can proceed the argument by the same way as the above. q. e. d.

Remark 5-1. From (4-8) we may suppose that

$$\zeta(g_1(u, v))=w(g_1(u, v))\varphi_0^2(g_1(u, v))=\omega\zeta(u, v).$$

Proposition II-4. *We have*

$$A(\Gamma')=\boldsymbol{C}[\varphi_0, \varphi_1, \varphi_2, \zeta]/(\zeta^3-\varphi_0\varphi_1\varphi_2(\varphi_0-\varphi_1)(\varphi_1-\varphi_2)(\varphi_2-\varphi_0)).$$

Proof. It is sufficient only to discuss that $A(\Gamma')$ is not a proper quatient ring of the right hand term (saying R). But it is clear because R is the graded ring of homogeneous polynomials on V_1. q. e. d.

Here we note that two Picard curves $C(\xi)$ and $C(\xi')$ are biholomorphically equivalent if and only if ξ and ξ' belongs to a same orbit of the action $\rho(S_4)$ of (1-4) (see [N]). Hence if we consider $\hat{T}=\boldsymbol{P}/\rho(S_4)$, it is the parameter space which determines the biholomorphic equivalence class of $\{C(\xi)\}$. From Remark 1-1 we have

$$T=\Lambda/\rho(S_4)\cong(D/\Gamma_1)/(\Gamma/\Gamma_1)=D/\Gamma,$$

hence it holds

$$\hat{T} \cong (\widehat{D/\Gamma}). \tag{5-3}$$

$(\widehat{D/\Gamma})$ has 4 boundary points which correspond to P_i ($i=0, 1, 2, 3$). And these 4 points belong to a same orbit of the action $\rho(S_4)$. Hence $(\widehat{D/\Gamma})$ has only one boundary point.

Here we use the following notations:

$$\Lambda^a = \boldsymbol{C}^3 - \{(\xi_0, \xi_1, \xi_2) \in \boldsymbol{C}^3 : \xi_0\xi_1\xi_2 \prod_{i<j} (\xi_i - \xi_j) = 0\}$$

Γ_1^a: the group of affine transformations on the vector space

$$V = \boldsymbol{C}\eta_0 \oplus \boldsymbol{C}\eta_1 \oplus \boldsymbol{C}\eta_2$$

induced from the element of $\pi_1(\Lambda, *)$ $(=\pi_1(\Lambda^a, *))$. For an element l of $\pi_1(\Lambda, *)$ we denote $g(l)$ the element of Γ_1^a above mentioned. And we say $g(l)$ an affine monodromy induced from l. Here we note we already have

$$\Gamma_1^a = \langle g_1, \dots, g_5 \rangle. \tag{5-4}$$

Γ^a: the group of affine monodromies induced from the element of $\pi_1(\Lambda^a/S_4, *)$, where we define the action of S_4 on Λ^a as follows:

$$\begin{cases} \rho((12))\,{}^t(\xi_0, \xi_1, \xi_2) = {}^t(\xi_1, \xi_0, \xi_2), \\ \rho((13))\,{}^t(\xi_0, \xi_1, \xi_2) = {}^t(\xi_0, \xi_2, \xi_1), \\ \rho((14))\,{}^t(\xi_0, \xi_1, \xi_2) = {}^t(-\xi_0, \xi_1 - \xi_0, \xi_2 - \xi_0). \end{cases} \tag{5-5}$$

$\hat{A}_k(\Gamma_1^a)$: the vector space of holomorphic automorphic forms of weight k of Neben type. Namely a holomorphic function $f(v, u)$ on D belongs to $\hat{A}_k(\Gamma_1^a)$ when it holds

$$f(g(v, u))\,|\det g|^k (a_1 + b_1 v + c_1 u)^{-3k} = f(v, u)$$

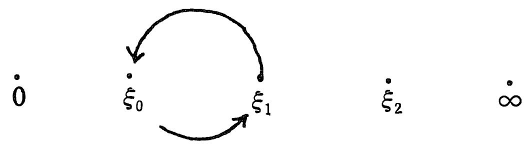
for every element $g = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ of Γ_1^a.

And we set $\hat{A}(\Gamma_1^a) = \bigoplus_{k=0}^{\infty} \hat{A}_k(\Gamma_1^a)$. By the similar way we define

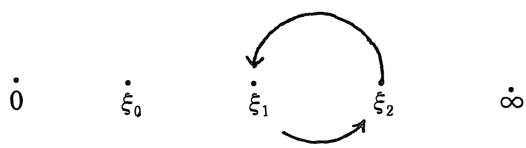
$$\hat{A}(\Gamma^a) = \bigoplus_{k=0}^{\infty} \hat{A}_k(\Gamma^a).$$

Let us fix a point $\xi = {}^t(\xi_0, \xi_1, \xi_2)$ of Λ^a with $0 < \xi_0 < \xi_1 < \xi_2$. And let $l(12)$, $l(13)$ and $l(14)$ be the loops on Λ^a/S_4 defined by the arcs on Λ^a start from ξ and go to $\sigma(12)\xi$, $\sigma(13)\xi$ and $\sigma(14)\xi$, respectively.

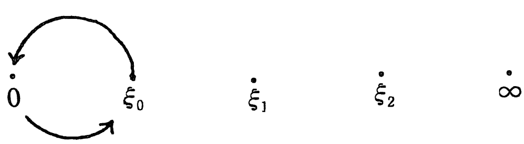
We may suppose $l\,(12)$, $l\,(13)$ and $l\,(14)$ induce the permutations of $\{0,\ \xi_0,\ \xi_1,\ \xi_2,\ \infty\}$ as Figure 4.



deformation by $l\,(12)$



deformation by $l\,(13)$



deformation by $l\,(14)$

Figure 4

We have

Fact 1.

$$
(5\text{-}6)\qquad
\begin{cases}
g_{12}=g(l\,(12))\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-\omega^2\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix},\\[2ex]
g_{13}=g(l\,(13))\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}1&0&0\\ \omega&1&1\\ \omega^2&0&-\omega^2\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix},\\[2ex]
g_{14}=g(l\,(14))\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}1&-\omega&1\\0&1&0\\0&-\omega^2&-\omega^2\end{pmatrix}\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}.
\end{cases}
$$

By considering

$$
\begin{pmatrix}\varphi_2(g(l\,(\tau))\,(u,v))\\ \varphi_1(g(l\,(\tau))\,(u,v))\\ \varphi_0(g(l\,(\tau))\,(u,v))\end{pmatrix}du'{}_\wedge dv'=\det(g(l\,(\tau)))\,A(\tau)\begin{pmatrix}\varphi_2\\ \varphi_1\\ \varphi_0\end{pmatrix}du_\wedge dv
$$

for every element τ of S_4, where $(u', v') = g(l(\tau))(u, v)$, we obtain a representation $\rho: \tau \mapsto A(\tau)$ of S_4 on ξ-space C^3.

Fact 2. The representation ρ is given by (1-4).

Fact 3. The affine monodromy group Γ^a is generated by the system $\{g_1, \ldots, g_5, g_{12}, g_{13}, g_{14}\}$. And Γ_0 is generated by the system $\{g_1 g_2 g_1^{-1}, g_2, \ldots, g_5, -g_{12}^3, -g_{13}^3, -g_{14}^3\}$.

Fact 4. We have $C[\xi_0, \xi_1, \xi_2]^{\rho_4(S_4)} = C[G_2, G_3, G_4]$.

Proof of the Facts.

We obtain the first part of Fact 3 from (5-4) and Fact 1. And the second part is induced from $\Gamma_0/\Gamma' \cong S_4$.

So let us examine Fact 2. By the representation theory there are only two different representations, up to conjugate representations, of S_4 to $GL(3, C)$ (cf. Serre's book). So we have only two possibilities for ρ, namely one is ρ of (1-4) and another (saying ρ') is given by $\mathrm{sgn}(\tau)\rho(\tau)$ for τ of S_4. If we set $\tau = (12)$ then it is easy to check

$$A((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next let us consider Fact 4. If we set

$$\begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{bmatrix}, \tag{5-7}$$

we get an equivalent representation ρ_4:

$$\rho_4((12)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_4(123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\rho_4((12)(34)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \rho_4((1234)) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The representation ρ_4 induces the permutation group of 4 points

$$\hat{P}_0=[1,1,1],\ \hat{P}_1=[1,-1,-1],\ \hat{P}_2=[-1,1,-1],\ \hat{P}_3=[-1,-1,1].$$

By the classical invariant theory the invariant subring of $\boldsymbol{C}[\zeta_0,\zeta_1,\zeta_2]$ under the action ρ_4 is given as:

$$\boldsymbol{C}[\zeta_0,\zeta_1,\zeta_2]^{\rho_4(S_4)}=\boldsymbol{C}[\hat{G}_2,\hat{G}_3,\hat{G}_4],$$

where

$$\hat{G}_2=\zeta_0^2+\zeta_1^2+\zeta_2^2,\qquad \hat{G}_3=\zeta_0\zeta_1\zeta_2,$$

$$\hat{G}_4=\zeta_0\zeta_1+\zeta_1\zeta_2+\zeta_2\zeta_0,\qquad \varDelta=(\zeta_0^2-\zeta_1^2)(\zeta_1^2-\zeta_2^2)(\zeta_2^2-\zeta_0^2).$$

And if we consider the representation ρ', the transformation (5-7) induces

$$\rho_5(\tau)=(\operatorname{sgn}\tau)\,\rho_4(\tau)\qquad \text{for } \tau\in S_4.$$

By the same argument we have

$$\boldsymbol{C}[\zeta_0,\zeta_1,\zeta_2]^{\rho_5(S_4)}=\boldsymbol{C}[\hat{G}_2,\hat{G}_4,\hat{G}_3^2,\hat{G}_3\varDelta].$$

So we can conclude ρ is given by (1-4).

Finally let us consider Fact 1. We investigate the case $\tau=(12)$. Set $\xi'={}^t(\xi_0',\xi_1',\xi_2')=\sigma((12))\xi={}^t(\xi_1,\xi_0,\xi_2)$. And we identify

$$C(\xi):\ w^3=z(z-\xi_0)(z-\xi_1)(z-\xi_2)$$

and

$$C(\xi'):\ w'^3=z'(z'-\xi_0')(z'-\xi_1')(z'-\xi_2')$$

by $z'=z$ and $w'=w$. Next we deform $C(\xi)$ to $C(\xi')$ along the arc $l(12)$ from ξ to ξ'. By this procedure the branch points $\{\xi_0,\xi_1,\xi_2\}$ moves like Figure 5.

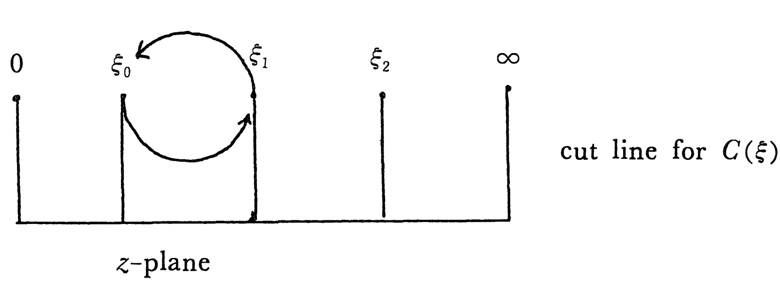


Figure 5

As a consequence we get $C(\xi')$ like Figure 6

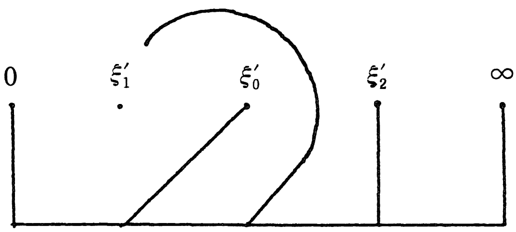


Figure 6

Let us draw the homology basis $\{A_i, B_i\}$ on Figure 5 and deform them to Figure 6. And change the cut line with the original one on $C(\xi')$.

Then we can check A_1, A_3, B_1 and B_3 are invariant under this deformation. And A_2 is deformed to A_2' of Figure 7 (the encircled number means the branch).

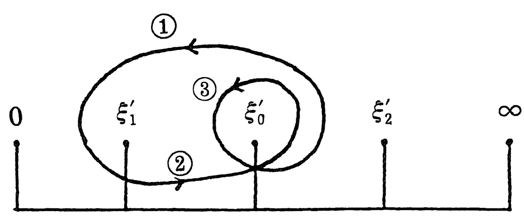


Figure 7

Hence we have

$$\eta_2' = \int_{A_2'} \varphi = -\omega^2 \int_{A_2} \varphi = -\omega^2 \eta_2.$$

As for $\tau = (13)$ and (14) we can proceed the similar trick. q. e. d.

Using the above facts we obtain the characterization of monodromy invariant subrings:

Proposition II-5. (1) $\hat{A}(\Gamma_1^a) = \boldsymbol{C}[\varphi_0, \varphi_1, \varphi_2]$,

(2) $\hat{A}(\Gamma^a) = \boldsymbol{C}[G_2, G_3, G_4]$,

where

$$G_2=(\xi_0-\xi_1-\xi_2)^2+(-\xi_0+\xi_1-\xi_2)^2+(-\xi_0-\xi_1+\xi_2)^2.$$
$$G_3=(\xi_0-\xi_1-\xi_2)(-\xi_0+\xi_1-\xi_2)(-\xi_0-\xi_1+\xi_2)$$
$$G_4=(\xi_0-\xi_1-\xi_2)^2(-\xi_0+\xi_1-\xi_2)^2+(-\xi_0+\xi_1-\xi_2)^2(-\xi_0-\xi_1+\xi_2)^2 \\ +(-\xi_0-\xi_1+\xi_2)^2(\xi_0-\xi_1-\xi_2)^2.$$

Proof. (1) Already we know φ_i belongs to $A(\Gamma')$ $(i=0,1,2)$, so it is sufficient to examine $\varphi_i(g_1(u,v))=\varphi_i(u,v)$. But it is obtained in Remark 4-8. Conversely let ϕ be an element of $\hat{A}(\Gamma')$, especially it belongs to $A(\Gamma')$. Using Proposition 4 we can write in the form

$$\phi=\sum_{i=0}^{2} P_i(\varphi_0,\varphi_1,\varphi_2)\zeta^i.$$

But we have $\zeta(g_1(u,v))=\omega\zeta(u,v)$ by Remark 5-1. Therefore P_1 and P_2 must be 0.

(2) it is the direct consequence of Fact 2, Fact 3 and Fact 4.

Combining the definition (5-3) of T and the above argument we obtain that $\hat{T}$ is a twisted projective space given by

$$\hat{T}=\operatorname{Proj}\boldsymbol{C}[G_2,G_3,G_4].$$

Hence we have

Proposition II-6. *The field of modular functions on D relative to Γ is given by* $\boldsymbol{C}(G_4/G_2^2,\ G_3^2/G_2^3)$.

Appendix

Here we cite up the theorems concerning the θ functions which we used in the preceeding sections, for precise arguments see [M], [R-F].

The θ function is defined by

$$\theta(z,\Omega)=\sum_{n\in\boldsymbol{Z}^g}\exp(\pi i\,{}^t n\Omega n+2\pi i\,{}^t nz)$$

for a θ variable z of $\boldsymbol{C}^g$ and a moduli variable Ω of the Siegel upper half space $\mathfrak{S}_g$. It is holomorphic on $\boldsymbol{C}^g\times\mathfrak{S}_g$ and satisfies the periodic relation

$$\text{(A-1)}\quad \begin{cases}\theta(z+e_j,\Omega)=\theta(z,\Omega),\\ \theta(z+\Omega e_j,\Omega)=\exp(-\pi i\Omega_{jj}-2\pi iz_j)\cdot\theta(z,\Omega),\end{cases}$$

where we use the notations

$$e_j={}^t(0,\dots,0,1,0,\dots,0),$$
$$\Omega=(\Omega_{ij})_{1\leq i,j\leq g}j,$$
$$z={}^t(z_1,\dots,z_g).$$

For rational g vectors a, b of $\boldsymbol{Q}^g$ the θ function with a characteristic $\begin{bmatrix} a \\ b \end{bmatrix}$ is defined by

$$\text{(A-2)} \quad \theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\Omega)=\exp\{\pi i{}^t a\Omega a+2\pi i{}^t a(z+b)\}\cdot\theta((z+\Omega a+b),\Omega)$$
$$=\sum_{n\in \boldsymbol{Z}^g}\exp\{\pi i{}^t(n+a)\Omega(n+a)+2\pi i{}^t(n+a)(z+b)\}.$$

Remark A-1. For half integral vectors η', η'' of $(\boldsymbol{Z}/2)^g$ the function $\theta\begin{bmatrix} \eta' \\ \eta'' \end{bmatrix}(z,\Omega)$ of z is even (odd) if $4{}^t\eta'\eta''$ is even (odd), respectively.

Next let us consider a compact Riemann surface X of genus g. Let $\{A_1,\dots,A_g,B_1,\dots,B_g\}$ be a basis of $H_1(X,\boldsymbol{Z})$ and let $\omega_1,\dots,\omega_g$ be a basis of Abelian differentials of first kind with the properties;

$$A_i\cdot A_j=B_i\cdot B_j=0,$$
$$A_i\cdot B_j=\delta_{ij},$$
$$\int_{A_j}\omega_i=\delta_{ij}.$$

And set

$$\Omega=(\Omega_{ij})=\left(\int_{B_j}\omega_i\right)_{1\leq i,j\leq g}.$$

For arbitrary points P and Q of X we denote the column vector of Abelian integrals

$${}^t\left(\int_P^Q\omega_1,\dots,\int_P^Q\omega_g\right)$$

by

$$\int_P^Q\omega.$$

And for a divisor $D=\sum_{i=1}^m P_i-\sum_{i=1}^m Q_i$ of degree 0 we use the notation

$$\sum_{i=1}^m\int_{Q_i}^{P_i}\omega=\int_D\omega \text{ or } I(D).$$

The Jacobi variety $\boldsymbol{C}^g/(\Omega\boldsymbol{Z}^g+\boldsymbol{Z}^g)$ of X shall be denoted by Jac X. The point P_0 is supposed to be fixed as the common terminal of A_i and B_i on X.

Theorem A-1. *We suppose z is a fixed point on* Jac X. *The multivalued function*

$$\theta(z+\int_{P_0}^{P}\omega,\ \Omega)$$

of P on X has g zeros $Q_1, \ldots, Q_g$ provided not to be constantly zero. And we have the Jacobi inverse relation between z and the divisor $\sum_{i=1}^{g} Q_i$:

$$z=\Delta-\sum_{i=1}^{g}\int_{P_0}^{Q_i}\omega,$$

where Δ is a constant defined by

$$\text{(A-3)}\qquad \Delta=-\frac{1}{2}\begin{pmatrix}\Omega_{11}\\ \vdots\\ \Omega_{gg}\end{pmatrix}-\sum_{k=1}^{g}\int_{A_k}\left(\int_{P_0}^{P}\omega\right)\omega_k.$$

Let us call Δ the Riemann constant.

Remark A-2. The Riemann constant Δ is determined by the homology basis $\{A_i, B_i\}$ and the terminal point P_0.

Theorem A-2. *We have $\theta(z, \Omega)=0$ if and only if there is an effective divisor $P_1+\cdots+P_{g-1}$ with the property*

$$z=\Delta-\sum_{i=1}^{g-1}\int_{P_0}^{P_i}\omega.$$

Here we use the notation

$$\Sigma=\{\text{a divisor } D \text{ on } X: 2D\equiv K\},$$

where $\equiv$ indicates the linear equivalence and K is a canonical divisor.

Remark A-3. If we fix an element D_1 of Σ, then we have $\Sigma=\{D_1+E: 2E\equiv 0\}$. Moreover we know that Σ has 2^{2g} elements because of Abel's theorem.

Theorem A-3. *We have $\Delta=I(D_0-(g-1)P_0)$ for a certain divisor D_0 of Σ.*

For a divisor D of Σ let us consider a pair of half integral vectors (η', η'') with the property

$$\Omega\eta'+\eta''\equiv I(D_0-D).$$

It defines a bijective correspondence between Σ and $(\boldsymbol{Z}/2)^{2g}/\boldsymbol{Z}^{2g}$.

Corollary 1. *We have* $\theta(\Omega\eta'+\eta'', \Omega)=0$ *for a pair of half integral vectors* (η', η'') *if and only if the corresponding divisor* D *of* Σ *is effective.*

Corollary 2. *The Riemann constant* Δ *is a half period on* Jac X *if and only if we have* $(2g-2)P_0\equiv K$.

Theorem A-4. *Set* $\mathrm{d}(D)=\dim H^0(X, \mathcal{O}([D]))$. *Then the divisor* D_0 *is characterized as an element of* Σ *with the condition*

$$\mathrm{d}(D_0+E)\equiv 4{}^t\eta'\eta'' \pmod{2}$$

for every $D=D_0+E$ *of* Σ, *where* (η', η'') *is a pair of half integral vectors corresponding to* D.

Remark A-4. If X is a non hyper-elliptic Riemann surface of genus 3, we have $\mathrm{d}(D)=0$ or 1 for every D of Σ.

Theorem A-5. *Let* f *be a meromorphic function on* X, *and let*

$$\sum_{i=1}^{m} a_i-\sum_{i=1}^{m} b_i$$

be the divisor defined by f. *Let us take paths from* P_0 *to* a_i *and* b_i *so that we have*

$$\sum_{i=1}^{m}\int_{P_0}^{a_i}\omega=\sum_{i=1}^{m}\int_{P_0}^{b_i}\omega.$$

For an effective divisor $P_1+\cdots+P_g$ *we have*

$$\text{(A-4)}\qquad f(P_1)\cdots f(P_g)=\frac{1}{E}\prod_{k=1}\left\{\frac{\theta\left(\sum_i\int_{P_0}^{P_i}\omega-\int_{P_0}^{a_k}\omega-\Delta, \Omega\right)}{\theta\left(\sum_i\int_{P_0}^{P_i}\omega-\int_{P_0}^{b_k}\omega-\Delta, \Omega\right)}\right\}$$

where the equality indicates as meromorphic functions on the g *times symmetric product of* X, E *is a constant independent of* $P_1, \ldots, P_g$ *and the integrals from* P_0 *to* P_i *take the same paths in the numerator and the denominator.*

An effective divisor D of degree $g-1$ on X is said to be *general* if we have $\dim H^0(X, \mathcal{O}([K-D]))=1$. For a general divisor D of degree $g-1$ let ω be an Abelian differential of first kind such that $(\omega)-D$ is effective. In this case $D'=(\omega)-D$ is said to be *the complement* of D.

Remark A-5. In the situation of Theorem A-5 let us suppose the following conditions:

(i) $D=P_2+\cdots+P_g$ is general and let $Q_2+\cdots+Q_g$ be its complement,

(ii) $P_1,\ldots,P_g$ are different from a_k, b_k $(1\leqq k\leqq m)$,

(iii) P_1 is different from $Q_2,\ldots,Q_g$.

Then both of the numerator and the denominator in (A-4) are different from zero.

Suppose a basis $\{A_i', B_i'\}$ of $H_1(X, \mathbf{Z})$ obtained by a symplectic transformation $M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ from the basis $\{A_i, B_i\}$;

$$ {}^t(B_1',\ldots, B_g', A_1',\ldots, A_g')=\begin{bmatrix} A & B \\ C & D \end{bmatrix} {}^t(B_1,\ldots, B_g, A_1,\ldots, A_g), $$

where we have ${}^tAC={}^tCA$, ${}^tBD={}^tDB$ and ${}^tDA-{}^tBC=E_g$ from the symplectic condition. Let $\omega'={}^t(\omega_1',\ldots,\omega_g')$ be a basis of Abelian differentials of first kind with the property

$$ \int_{A_j'} \omega_i'=\delta_{ij}, $$

and put

$$ \Omega'=(\Omega_{ij}')=\int_{B_j'} \omega_i'. $$

Then we have

$$ \text{(A-5)} \qquad \Omega'=(A\Omega+B)(C\Omega+D)^{-1}. $$

And also we have

$$ \text{(A-6)} \qquad z'={}^t(C\Omega+D)^{-1}z $$

for $z=\int_{P_0}^{P}\omega$ and $z'=\int_{P_0}^{P}\omega'$.

For a divisor D of degree zero let us put

$$ \int_D \omega=\Omega\zeta_1+\zeta_2 \text{ and } \int_D \omega'=\Omega'\zeta_1'+\zeta_2', $$

then we have

$$\text{(A-7)} \qquad \begin{cases} \zeta_1' = D\zeta_1 - C\zeta_2 \\ \zeta_2' = -B\zeta_1 + A\zeta_2 . \end{cases}$$

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