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# On Factor Decomposition of an Ergodic Groupoid

By

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Factor decomposition of cocycle regular representations of ergodic groupoids whose stabilizer groups are uniformly abelian is studied.

## Introduction

Given a measure groupoid $\Gamma$ and a torus-valued 2-cocycle $c$, a Hilbert algebra $\mathfrak{A}_c$ is defined in a definite way and the associated representation of $\mathfrak{A}_c$ on the $L^2$-completion of $\mathfrak{A}_c$ is called $c$-regular representation of $\Gamma$. In the study of $\mathfrak{A}_c$, one of the basic problems is the factor decomposition of $\mathfrak{A}_c$. Let us have a try at it in terms of decomposition of $\Gamma$. Since the ergodic decomposition of $\Gamma$ induces a central decomposition of $\mathfrak{A}_c$, the problem is reduced to the case when $\Gamma$ is ergodic. If, furthermore, $\Gamma$ is supposed to be a principal one, every $c$-regular representation of $\Gamma$ is known to generate a factor ([5]) and there remains no problem. In this paper, we deal with groupoids whose stabilizer groups are uniformly chosen (see the beginning of §1 for the precise meaning) and investigate the above mentioned problem.

Organization is as follows: §1 gathers facts needed in later sections. In § 2, we construct a measure space $S^*$ with an equivalence relation, from the information of $(\Gamma, c)$, and show that ergodic decomposition of $S^*$ is equivalent to the factor decomposition of $\mathfrak{A}_c$ in § 3. In § 4, we select a subgroup $\Sigma$ of the stabilizer of $\Gamma$ and ergodic quotient of $S^*$ is identified with a certain $\hat{\Sigma}$-principal homogeneous space. § 5 is only a matter of formulation and gives a factor decomposition of $(\Gamma, c)$, i. e., factor component of $\mathfrak{A}_c$ is realized as a cocycle regular representation of the quotient groupoid $\Gamma/\Sigma$. In § 6, an example is

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discussed.

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## § 1. Preliminaries

Let $\Gamma$ be an analytic groupoid (=Borel groupoid with its Borel structure analytic) and $G$ be a locally compact second countable abelian group. We assume the uniformity of stabilizer groups of $\Gamma$ in the following sense: For each $x \in \Gamma^{(0)}$, an isomorphism $\iota_x$ of $G$ onto $\Gamma_x^x (= \{\gamma \in \Gamma ; r(\gamma) = s(\gamma) = x\})$ is assigned and satisfies

1° for $\gamma \in \Gamma$ and $g \in G$, $\iota_{r(\gamma)}(g)\gamma = \gamma\iota_{s(\gamma)}(g)$

2 $G \times \Gamma \ni (g, \gamma) \mapsto \iota_{r(\gamma)}(g)\gamma$ is a Borel map.

For notational simplicity, we write $\iota_{r(\gamma)}(g)\gamma$ (resp. $\gamma\iota_{s(\gamma)}(g)$) as $g\gamma$ (resp. $\gamma g$). We assume that $\Gamma$ has a faithful $\sigma$-finite transverse function $\{\nu^x\}_{x \in \Gamma^{(0)}}$ and a transverse measure is specified by a pair $(\mu, \nu)$ where $\mu$ is a $\sigma$-finite measure in $\Gamma^{(0)}$. Then $m(d\gamma) = \int \mu(dy) \int \nu^y(d\gamma)$ is a $\sigma$-finite measure in $\Gamma$ and quasi-invariant under inversion $\gamma \mapsto \gamma^{-1}$. The measure groupoid $(\Gamma, m)$ is called ergodic if for each saturated Borel set $B$ of either $\mu(B) = 0$ or $\mu(\Gamma^{(0)} \setminus B) = 0$ holds. In the following, $(\Gamma, m)$ is supposed to be ergodic. For a normalized $\boldsymbol{T}$-valued Borel 2-cocycle $c$ of $\Gamma$, let $\mathfrak{A}_c$ be the set of functions $\xi$ on $\Gamma$ satisfying

(1) for each $y \in \Gamma^{(0)}$, $\xi|_{\Gamma^y}$ and $\xi^*|_{\Gamma^y}$ belong to $L^2(\Gamma^y, \nu^y)$

(2) $\xi$ and $\xi^*$ belong to $L^2(\Gamma, m)$

(3) $\sup_{y \in \Gamma^{(0)}} \int \nu^y(d\gamma)\, |\xi(\gamma)| < +\infty$ and $\sup_{y \in \Gamma^{(0)}} \int \nu^y(d\gamma)\, |\xi^*(\nu)| < +\infty$

where $\xi^*(\gamma) = \overline{\xi(\gamma^{-1}) c(\gamma^{-1}, \gamma)}$. We can define in $\mathfrak{A}_c$ a multiplication and an inner product as follows.

$$(4) \qquad (\xi_1 \xi_2)(\gamma) = \int \nu^{r(\gamma)}(d\gamma')\, \xi_1(\gamma')\, \xi_2(\gamma'^{-1}\gamma)\, c(\gamma', \gamma'^{-1}\gamma)$$

$$(5) \qquad (\xi_1 | \xi_2) = \int \mu(dy) \int \nu^y(d\gamma)\, \overline{\xi_1(\gamma)}\, \xi_2(\gamma).$$

Then, together with the above *-operation, $\mathfrak{A}_c$ becomes a right Hilbert algebra ([3], [5]). For $\varphi \in \mathfrak{A}_c$, we denote the right multiplication of $\varphi$ by $R(\varphi)$. Through the natural decomposition

$$L^2(\Gamma, m) \cong \int_{\Gamma^{(0)}}^{\oplus} \mu(dx)\ L^2(\Gamma^x, \nu^x),$$

$R(\varphi)$ is decomposed as $\int^{\oplus}\mu(dx)R^x(\varphi)$. Here $R^x(\varphi)$ is the right multiplication by $\varphi$ in $L^2(\Gamma^x, \nu^x)$. For later use, we give a characterization of the von Neumann algebra $\mathfrak{A}_c'$ (=von Neumann algebra generated by $R(\varphi)$, $\varphi\in\mathfrak{A}_c$). Let $\gamma\in\Gamma$ and define a unitary map $U_c(\gamma)$ of $L^2(\Gamma^{s(\gamma)}, \nu^{s(\gamma)})$ onto $L^2(\Gamma^{r(\gamma)}, \nu^{r(\gamma)})$ by

(6) $\qquad (U_c(\gamma)\xi)(\gamma')=c(\gamma, \gamma^{-1}\gamma')\xi(\gamma^{-1}\gamma')$.

**Lemma 1.1** ([5] *Th.* 4.1). *Let* $T=\int^{\oplus}\mu(dy)T^y$ *be a decomposable operator in* $\int^{\oplus}\mu(dy)L^2(\Gamma^y, \nu^y)$. *Then* $T$ *belongs to* $\mathfrak{A}_c'$ *if and only if there exists a suitable choice of measurable field of operators* $\{T^y\}_{y\in\Gamma^{(0)}}$ *such that* $U_c(\gamma)T^{s(\gamma)}=T^{r(\gamma)}U_c(\gamma)$ *for all* $\gamma\in\Gamma$.

For $g\in G$ and $y\in\Gamma^{(0)}$ unitary operator $U_c^y(g)$ is defined to be $U_c(\iota_y(g))$. We have

(7) $\qquad U_c^y(g_1)U_c^y(g_2)=c_y(g_1, g_2)U_c^y(g_1g_2)$, for $g_1, g_2\in G$,

where $c_y$ is a 2-cocycle of $G$ given by $c_y(g_1, g_2)=c(\iota_y(g_1), \iota_y(g_2))$.

**Lemma 1.2** ([3] *Prop.* 15). *We can choose a sequence* $\{\xi_n\}_{n\geq 1}$ *in* $\mathfrak{A}_c$ *such that*

(*i*) $\{R(\xi_n)\}_{n\geq 1}$ *generates* $\mathfrak{A}_c'$,

(*ii*) *for each* $y\in\Gamma^{(0)}$, $\{R^y(\xi_n)\}_{n\geq 1}$ *generates* $U_c^y(G)'$.

Now we describe a factor decomposition of $U_c^y$. The following is implicitly contained in [2], [7]. Let $S(y)$ be a closed subgroup of $G$ defined by

(8) $\qquad S(y)=\{g\in G; c_y(g, g')=c_y(g', g)$ for all $g'\in G\}$,

and set

(9) $\qquad S^*(y)=\{b; b$ is a $\boldsymbol{T}$-valued Borel function on $S(y)$ satisfying $b(g_1)b(g_2)b(g_1g_2)^{-1}=c_y(g_1, g_2)$ for $g_1, g_2\in S(y)\}$.

Then $S^*(y)\neq\emptyset$ (symmetric cocycle is trivial) and, by point-wise multiplication, $S(y)^\wedge$ (=dual groups of $S(y)$) acts on $S^*(y)$ freely and transitively, so $S^*(y)$ inherits a standard Borel structure and an $S(y)^\wedge$-invariant measure $db$ from $S(y)^\wedge$. For $b\in S^*(y)$, consider a Borel function $\xi$ on $\Gamma^y$ such that

(10) $\quad \xi(g\gamma)=b(g)^{-1}c(g, \gamma)\xi(\gamma)$ for $g\in S(y)$, $\gamma\in\Gamma^y (c(g, \gamma)\equiv c(\iota_y(g), \gamma))$.

Then $|\xi(\gamma)|^2$ defines a Borel function on $S(y)\backslash\Gamma^y$. Given a Haar measure $ds$ of $S(y)$, we can define a measure $\oint_{S(y)\backslash\Gamma^y}\tilde{\nu}^y(d\gamma)$ in $S(y)\backslash\Gamma^y$ by the relation

$$(11)\qquad \int_{\Gamma^y}\nu^y(d\gamma)f(\gamma)=\oint_{S(y)\backslash\Gamma^y}\tilde{\nu}^y(d\gamma)\int_{S(y)}ds\ f(s\gamma)$$

where $f$ is a positive Borel function on $\Gamma^y$. We impose on $\xi$ the following condition

$$(12)\qquad \oint_{S(y)\backslash\Gamma^y}\tilde{\nu}^y(d\gamma)\,|\xi(\gamma)|^2<+\infty.$$

If $\mathfrak{H}_b$ is the set of all such $\xi$'s, it is a Hilbert space by the following inner product.

$$(13)\qquad (\xi_1|\xi_2)=\oint_{S(y)\backslash\Gamma^y}\tilde{\nu}^y(d\gamma)\overline{\xi_1(\gamma)}\xi_2(\gamma),\ \ \xi_1,\xi_2\in\mathfrak{H}_b.$$

Let $\xi\in L^2(\Gamma^y,\nu^y)\cap L^1(\Gamma^y,\nu^y)$, then

$$(14)\qquad \xi_b(\gamma)=\int_{S(y)}ds\ \ b(s)\ \ c(s,\gamma)^{-1}\ \ \xi(s\gamma)$$

defines an element in $\mathfrak{H}_b$. Taking various $\xi\in L^2\cap L^1$, $\{\xi_b\}_{b\in S^*(y)}$ determines a measurable field structure for $\{\mathfrak{H}_b\}_{b\in S^*(y)}$. The correspondence $\xi\mapsto\int_{S^*(y)}^{\oplus}db\ \xi_b$ gives rise to a unitary map from $L^2(\Gamma^y,\nu^y)$ to $\int_{S^*(y)}^{\oplus}db\ \mathfrak{H}_b$, and under this isomorphism $U_c^y(g)$ $(g\in G)$ is decomposed as $\int_{S^*(y)}^{\oplus}db\ U_c^b(g)$. Here $U_c^b(g)$ is a unitary operator in $\mathfrak{H}_b$ and defined by

$$(15)\qquad (U_c^b(g)\xi)(\gamma)=c(g,g^{-1}\gamma)\xi(g^{-1}\gamma),\ \ \xi\in\mathfrak{H}_b.$$

**Lemma 1.3** ([2], [7]).

(i) *For $b\in S^*(y)$, $U_c^b(G)''$ is a semifinite factor.*

(ii) *$U_c^y(g)\cong\int_{S^*(y)}^{\oplus}db\ U_c^b(g)$, $g\in G$, is a factor decomposition of $U_c^y$; $U_c^y(G)''\cap U_c^y(G)'$ is identified with $L^\infty(S^*(y))$.*

## § 2. Borel Structure of $S^*$

Let $S^*=\coprod_{y\in\Gamma^{(0)}}S^*(y)$ (disjoint union) and $p:S^*\to\Gamma^{(0)}$ be the canonical projection. In this section, we equip $S^*$ with a suitable Borel structure.

**Lemma 2.1.** *Let $Y$ be a Polish space and $X$ be an analytic Borel space. We assume that there are a Borel equivalence relation $R$ and an $R$-ergodic measure $\mu$ in $X$. Suppose that for each $x \in X$, a closed subset $F(x)$ of $Y$ is assigned and satisfies*

(i) $\tilde{F} \equiv \{(x, y); x \in X, y \in F(x)\}$ *is a Borel subset of* $X \times Y$.

(ii) *If* $x \overset{R}{\sim} x'$ $(x, x' \in X)$, $F(x) = F(x')$.

*Then there exists a $\mu$-negligible saturated set $N \subset X$ such that $F(x) = F(x')$ for all $x, x' \in X \backslash N$.*

*Proof.* Let $\{U_i\}_{i \geq 1}$ be a countable open base for $Y$. Let $A_i$ be the image of $F_i \equiv \tilde{F} \cap (X \times U_i)$ under the projection $X \times Y \to X$, which is a saturated set by (*ii*). Since $A_i$ is an analytic set as the image of analytic set and every analytic set is absolutely measurable, the ergodicity implies that either $\mu(A_i) = 0$ or $\mu(X \backslash A_i) = 0$. Now let

$$N_i = \begin{cases} A_i & \text{if } \mu(A_i) = 0, \\ X \backslash A_i & \text{if } \mu(X \backslash A_i) = 0, \end{cases} \tag{16}$$

and set $N = \bigcup_{i \geq 1} N_i$. By the construction, if $x, x' \in X \backslash N$,

$$\{y \in Y; (x, y) \in F_i\} \neq \emptyset \Leftrightarrow \{y \in Y; (x', y) \in F_i\} \neq \emptyset. \tag{17}$$

Since $F(x)$ is closed, this implies that $F(x) = F(x')$.

**Lemma 2.2.**

(i) $\tilde{S} = \{(x, g) \in \Gamma^{(0)} \times G; g \in S(x)\}$ *is a Borel set of* $\Gamma^{(0)}$.

(ii) $S(s(\gamma)) = S(r(\gamma))$ *for* $\gamma \in \Gamma$.

*Proof.* (i): Take a countable dense subset $\{g_i\}_{i \geq 1}$ of $G$. Since

$$G \times G \ni (g, g') \mapsto c_x(g, g') / c_x(g', g)$$

is continuous ([6] Prop. 1.5), $\tilde{S} = \bigcap_{i \geq 1} \{(x, g); c_x(g, g_i) = c_x(g_i, g)\}$ is a Borel set.

(ii): This follows from

$$\frac{c_y(g, g')}{c_x(g', g)} = \frac{c(\gamma, gg')}{c(gg', \gamma)} \frac{c(g, \gamma)}{c(\gamma, g)} \frac{c(g', \gamma)}{c(\gamma, g')}, \tag{18}$$

which is an easy consequence of cocycle relation.

Due to above two lemmas, $S(x)$ is equal to a closed subgroup of

$G$, say $S$, for $\mu$-a.e. $x\in\Gamma^{(0)}$. So, for the purpose of factor decomposition, we may suppose that $S(x)=S$ for all $x\in\Gamma^{(0)}$ (inessential reduction).

To define a Borel structure in $S^*$, we need a special class of sections of $S^*\to\Gamma^{(0)}$. Let $L^\infty(S, \boldsymbol{T})$ be a subset of $L^\infty(S)$, consisting of $\boldsymbol{T}$-valued measurable function on $S$ and we give it weak* topology induced from $L^\infty(S)$. Then $L^\infty(S, \boldsymbol{T})$ is a Polish group by pointwise multiplication. Similarly $L^\infty(S\times S, \boldsymbol{T})$ is a Polish group. Define a continuous homomorphism $\delta; L^\infty(S, \boldsymbol{T})\to L^\infty(S\times S, \boldsymbol{T})$ by $(\delta b)(g, g')=b(g)b(g')b(gg')^{-1}$. Then $\delta^{-1}(1)=$ the inverse image of $\delta$ at unit of $L^\infty(S\times S, \boldsymbol{T})$ is a closed subgroup of $L^\infty(S, \boldsymbol{T})$ and naturally identified with the dual group of $S$. Set

(19) $C=\{[c]\in L^\infty(S\times S, \boldsymbol{T})\,; c$ is a symmetric Borel 2-cocycle of $S\}$.

Since symmetric cocycle is trivial and the image of $\delta$ is always symmetric, we have $C=\delta(L^\infty(S, \boldsymbol{T}))$. So $\delta$ induces a continuous isomorphism $\delta_*$ of $L^\infty(S, \boldsymbol{T})/\delta^{-1}(1)$ onto $C$, from which one sees that $C$ is a Borel subset of $L^\infty(S\times S, \boldsymbol{T})$ and $\delta_*$ is a Borel isomorphism (note that $L^\infty(S, \boldsymbol{T})/\delta^{-1}(1)$ is a Polish group). Since the natural projection $L^\infty(S, \boldsymbol{T})\to L^\infty(S, \boldsymbol{T})/\delta^{-1}(1)$ has a Borel section ([1] Th. 3.4.1), $\delta$ also has a Borel section on C. Due to the definition of $S$, $c_x(g, g')$ is a symmetric Borel cocycle on $S$, and we have a Borel map $\Gamma^{(0)}\ni x\mapsto[c_x]\in C$. As a conclusion, we can find a Borel map $\beta: \Gamma^{(0)}\ni x\mapsto\beta_x\in L^\infty(S, \boldsymbol{T})$ such that $\delta(\beta_x)=[c_x]$. Since in the class $\beta_x$ there is one and only one Borel function $b_x$ on $S$ satisfying $b_x(g)b_x(g')b_x(gg')^{-1}=c_x(g, g')$ (cf [2] p. 308), we have proved the following:

**Lemma 2.3.** *We can find a function $\Gamma^{(0)}\times S\ni(x, g)\mapsto b_x(g)\in\boldsymbol{T}$ such that*

(*i*) *for each $x\ni\Gamma^{(0)}$, $S\in g\mapsto b_x(g)$ is a Borel function on S and satisfies $b_x(g)b_x(g')b_x(gg')^{-1}=c_x(g, g')$, $g$, $g'\in S$.*

(*ii*) *$\Gamma^{(0)}\ni x\mapsto[b_x]\in L^\infty(S, \boldsymbol{T})$ is a Borel map.*

Now we define a Borel structure in $S^*$. Take a function $b_x(g)$ satisfying conditions in Lemma 2.3. We have a bijection $\Gamma^{(0)}\times\hat{S}\ni(x, \chi)\mapsto(x, b_x\chi)\in S^*$, which transfers the Borel structure of $\Gamma^{(0)}\times\hat{S}$ into $S^*$.

**Lemma 2.4.** *The above mentioned Borel structure of $S^*$ is independent of the choice of $b$.*

*Proof.* Let $b'_x(g)$ be another such function. By condition (i) in Lemma 2.3, we can find a function $\Gamma^{(0)} \ni x \mapsto \chi_x \in \hat{S}$ such that $b'_x(g) = \chi_x(g) b_x(g)$ and by (ii) in Lemma 2.3, $\Gamma^{(0)} \ni x \mapsto [\chi_x] \in \delta^{-1}(1) \cong \hat{S}$ is a Borel map. So $\Gamma^{(0)} \times S \ni (x, g) \mapsto \chi_x(g)$ is a Borel function and this implies that two Borel structures coincide.

## § 3. Factor Decomposition of $\mathfrak{A}_c$

Take and fix a Haar measure $ds$ in $S$ and a Haar measure $d\chi$ in $\hat{S}$ which are dually related, i.e., $\int ds \int d\chi\, f(\chi)\chi(s) = f(1)$ for $f \in C_c(S)$. $ds$ determines a Hilbert space $\mathfrak{H}_b$ for each $b \in S^*$ (see near (12)) and $d\chi$ is transformed to an $\hat{S}$-invariant measure $\lambda^x$ of $S^*(x)$ for each $x \in \Gamma^{(0)}$. Then $\{\lambda^x\}_{x \in \Gamma^{(0)}}$ forms a Borel field of measures and $\int_{\Gamma^{(0)}} \mu(dx)\lambda^x(db)$ defines a measure $\hat{\mu}$ in $S^*$. Since $ds$ and $d\chi$ are dually related, $\xi \mapsto \int_{S^*}^{\oplus} \hat{\mu}(db)\xi_b$ gives rise to a unitary map

$$(20) \qquad L^2(\Gamma, m) \rightleftarrows \int_{S^*}^{\oplus} \hat{\mu}(db)\mathfrak{H}_b.$$

Let $T \in \mathfrak{A}'_c \cap \mathfrak{A}''_c$. Then, due to Lemma 1.2 and Th. II. 3.1 in [4], we can find a measurable family of operators $\{T^y\}_{y \in \Gamma^{(0)}}$, $T^y \in \mathscr{B}(L^2(\Gamma^y, \nu^y))$ such that

$$(21) \qquad T = \int^{\oplus} \mu(dy) T^y \text{ and } T^y \in U_c^y(G)'' \cap U_c^y(G)'.$$

By Lemma 1.3, above isomorphism (20) transforms $\int^{\oplus} \mu(dy)\, T^y$ into a diagonalizable operator in $\int_{S^*}^{\oplus} \hat{\mu}(db)\, \mathfrak{H}_b$. Thus $\mathfrak{A}''_c \cap \mathfrak{A}'_c$ is identified with a closed subalgebra of $L^\infty(S^*, \hat{\mu})$. Conversely, let $F \in L^\infty(S^*, \hat{\mu})$. If we regard $F$ as $\int^{\oplus} \mu(dy)\, F^y$ with $F^y \in U_c^y(G)'' \cap U_c^y(G)'$, then $F^y$ commutes with $R^y(\xi_n)$ $(n \geq 1)$ and therefore $F$ commutes with $\mathfrak{A}'_c$. Thus $F \in \mathfrak{A}''_c$.

Let us seek the condition when $F$ belongs to $\mathfrak{A}'_c$. We begin with the construction of an action of $\Gamma$ on $S^*$. Let $\gamma \in \Gamma$ and $b \in S(r(\gamma))$ and define $b\gamma \in S^*(s(\gamma))$ by

(22) $(b\gamma)(g)=b(g)c(\gamma,g)/c(g,\gamma),\ g\in S.$

Then

(23) $b(\gamma_1\gamma_2)=(b\gamma_1)\gamma_2$, for $(\gamma_1,\gamma_2)\in\Gamma^{(2)}$, $b\in S^*(r(\gamma))$,

$\hat{\Gamma}\rightleftarrows\{(b,\gamma)\in S^*\times\Gamma\,;b\in S^*(r(\gamma))\}$ is a Borel set of $\Gamma\times S^*$, and $\hat{\Gamma}\ni(b,\gamma)\mapsto b\gamma\in S^*$ is a Borel map on this set. So we can makc $\hat{\Gamma}$ into a groupoid: The unit space of $\hat{\Gamma}$ is $S^*$ and for $(b,\gamma)\in\hat{\Gamma}$, $r(b,\gamma)=b$, $s(b,\gamma)=b\gamma$. $(b,\gamma)$ and $(b',\gamma')\in\hat{\Gamma}$ are composable if and only if $s(b,\gamma)=r(b',\gamma')$ and the composition $(b,\gamma)(b',\gamma')$ is given by '$(b,\gamma\gamma')$. Since $\hat{\Gamma}^b$ is identified with $\Gamma^{p(b)}$, $\nu^{p(b)}$ determines a measure $\hat{\nu}^b$ in $\hat{\Gamma}^b$ and $\{\hat{\nu}^b\}_{b\in S^*}$ forms a transverse function for $\hat{\Gamma}$. The pair $(\hat{\mu},\hat{\nu})$ gives a transverse measure of $\hat{\Gamma}$. We define a unitary map $U_c^b(\gamma)$ from $\mathfrak{H}_b$ onto $\mathfrak{H}_{b\gamma^{-1}}$ by

(24) $(U_c^b(\gamma)\xi)(\gamma')=c(\gamma,\gamma^{-1}\gamma')\xi(\gamma^{-1}\gamma'),\ \ \xi\in\mathfrak{H}_b.$

Now the following lemma is immediate.

**Lemma 3.1.** *According to decompositions*

$$L^2(\Gamma^x,\nu^x)\cong\int_{S^*(x)}^{\oplus}\lambda^x(db)\mathfrak{H}_b$$

*and*

$$L^2(\Gamma^y,\nu^y)\cong\int_{S^*(y)}^{\oplus}\lambda^y(db)\mathfrak{H}_b,\qquad U_c(\gamma):L^2(\Gamma^x,\nu^x)\to L^2(\Gamma^y,\nu^y)$$

*is decomposed to* $\int_{S^*(x)}^{\oplus}\lambda^x(db)U_c^b(\gamma)$.

**Lemma 3.2.** *Let* $L^\infty(S^*/\Gamma)=\{F\in L^\infty(S^*,\hat{\mu})\,;F(b)=F(b\gamma)$ *for* $\hat{m}$-*a.e.* $(b,\gamma)\in\hat{\Gamma}$, *where* $\hat{m}=\hat{\mu}\circ\hat{\nu}$ *is a* $\sigma$-*finite measure in* $\hat{\Gamma}$. *Then*

(i) $L^\infty(S^*/\Gamma)$ *is a weakly closed* $*$-*subalgebra of* $L^\infty(S^*,\hat{\mu})$.

(ii) *For each class* $F$ *in* $L^\infty(S^*/\Gamma)$, *we can find a Borel function* $f$ *on* $S^*$ *such that* $f$ *is a representative of* $F$ *and* $f(b)=f(b\gamma)$ *for all* $(b,\gamma)\in\hat{\Gamma}$.

*Proof.* (i) is immediate. (ii) follows from the proof of [3] Prop. II.8.

Combining these lemmas with Lemma 1.1, we obtain

**Corollary 3.3.** $F\in L^\infty(S^*,\hat{\mu})$ *belongs to* $\mathfrak{A}_c''\cap\mathfrak{A}_c'$ *if and only if* $F(b)=F(b\gamma)$ *for* $\hat{m}$-*a.e.* $(b,\gamma)\in\hat{\Gamma}$.

Let us introduce an equivalence relation $\sim$ in $S^*$ by $b\sim b' \Leftrightarrow b'=b\gamma$ for some $\gamma\in\Gamma$.

**Corollary 3.4.** *Ergodic decomposition of* $(S^*, \hat{\mu}, \sim)$ *gives a factor decomposition of* $\mathfrak{A}_c'$.

## § 4. Ergodic Decomposition of $S^*$

In this section we assume that $S$ is a discrete subgroup of $G$. In that case, we can go further into the ergodic decomposition of $S^*$. We begin with the selection of certain subgroup of $S$. For $g\in G$, $C_g$ is, by definition, the set of all Borel functions $f$ on $\Gamma^{(0)}$ such that

(25) $\quad f(r(\gamma))=f(s(\gamma))c(g,\gamma)/c(\gamma,g)$ for all $\gamma\in\Gamma$.

We identify two $\mu$-a.e. equal functions in $C_g$.

**Lemma 4.1.** *For* $f\in C_g$, $|f(x)|$ *is constant for* $\mu$*-a.e.* $x\in\Gamma^{(0)}$ *and two functions in* $C_g$ *is proportional (up to* $\mu$*-negligible set).*

*Proof.* An immediate consequence of the ergodicity of $\mu$.

**Lemma 4.2.** *Set* $\Sigma=\{g\in G; C_g\neq\{0\}\}$. *Then*

(i) *there exists a* $\mu$*-negligible saturated Borel set* $N$ *such that*

$$\Sigma\subset S(x) \text{ for } x\in\Gamma^{(0)}\setminus N.$$

(ii) $\Sigma$ *is a subgroup of* $G$.

*Proof.* (i) Take a countable dense set $\{g_i\}_{i\geq 1}$ of $\Sigma$ and let $f_i$ be a non-trivial function in $C_{g_i}$ $(i\geq 1)$. Then the saturated Borel set $N=\{x\in\Gamma^{(0)}; f_i(x)=0$ for some $i\geq 1\}$ is $\mu$-negligible, and we have $\{g_i\}_{i\geq 1}\subset S(x)$ for $x\in\Gamma^{(0)}\setminus N$. Since $S(x)$ is closed, this proves (i). (ii) Let $g_1$, $g_2\in\Sigma$ and take non-trivial $f_i\in C_{g_i}$ $(i=1,2)$. Set $f(x)=f_1(x)f_2(x)^{-1}c_x(g_1,g_2^{-1})^{-1}c_x(g_2,g_2^{-1})$ ($f_2(x)^{-1}$ is defined to be zero if $f_2(x)=0$). Due to the cocycle relation, one sees that $f\in C_{g_1g_2^{-1}}$, which implies $g_1g_2^{-1}\in\Sigma$. Thus $\Sigma$ is a subgroup of $G$.

**Definition 4.3.**
$\Omega \rightleftarrows \{\omega\,;\omega$ is a function from $\Sigma$ into $L^\infty(\Gamma^{(0)}, \boldsymbol{T})$ such that $\omega_g \in C_g$ and $\omega_{g_1}(x)\omega_{g_2}(x) = c_x(g_1, g_2)\omega_{g_1g_2}(x)$ for $\mu$-a. e. $x \in \Gamma^{(0)}$ (coboundary condition)$\}$.

Let us check that $\Omega$ is not void. Due to Lemma 4.1, we can find a function on $\Sigma \times \Gamma^{(0)}$, $(g, x) \mapsto f_g(x)$ with the property, $f_g \in C_g$ and $|f_g(x)| = 1$ for $\mu$-a. e. $x \in \Gamma^{(0)}$. For $g_1, g_2 \in \Sigma$,

$$x \mapsto f_{g_1}(x) f_{g_2}(x) f_{g_1g_2}(x)^{-1} c_x(g_1, g_2)^{-1}$$

is a $\mu$-measurable function and by (18) it is constant on canonical equivalence class. So by the ergodicity of $\mu$, there is a unique $a(g_1, g_2) \in \boldsymbol{T}$ such that

$$f_{g_1}(x) f_{g_2}(x) = f_{g_1g_2}(x) c_x(g_1, g_2) a(g_1, g_2) \text{ for } \mu\text{-a. e. } x \in \Gamma^{(0)}, \tag{26}$$

and from this relation $a(g_1, g_2)$ is a symmetric cocycle on $\Sigma$. So we can find a function $b: \Sigma \to \boldsymbol{T}$ such that $a(g_1, g_2) = b(g_1g_2) b(g_1)^{-1} b(g_2)^{-1}$. Replacing $f_g$ with $b(g) f_g$, we may assume that $a(g_1, g_2) \equiv 1$. In other words, $g \mapsto [f_g] \in L^\infty(\Gamma^{(0)}, \boldsymbol{T})$ is in $\Omega$.

Let $\omega, \omega' \in \Omega$. By equivariance condition, there is a uniquely defined $\boldsymbol{T}$-valued function $\chi$ on $\Sigma$ satisfying $\omega'_g = \chi(g)\omega_g$, $g \in \Sigma$. Then, by coboundary condition, $\chi$ is a character on $\Sigma$. Conversely for each $\chi \in \hat{\Sigma}$ and each $\omega \in \Omega$, $g \mapsto \chi(g)\omega_g$ defines an element $\chi\omega$ in $\Omega$. $\hat{\Sigma} \times \Omega \ni (\chi, \omega) \mapsto \chi\omega \in \Omega$ is an action of $\hat{\Sigma}$ on $\Omega$, and with respect to which, $\Omega$ is a $\hat{\Sigma}$-principal homogeneous space. Since $\hat{\Sigma}$ is a compact abelian group, $\Omega$ has a unique $\hat{\Sigma}$-invariant probability measure $d\omega$. In the rest of this section, we show that $L^\infty(S^*/\Gamma)$ is isomorphic to $L^\infty(\Omega, d\omega)$.

Let $\mathscr{C}$ be the set of measurable functions $\varphi$ on $\Gamma^{(0)} \times \Sigma$ such that

$$\varphi(r(\gamma), g) = \varphi(s(\gamma), g) c(\gamma, g)/c(g, \gamma) \quad \text{for } g \in \Sigma \text{ and } \gamma \in \Gamma, \tag{27}$$

$$\{g \in \Sigma\,; x \mapsto \varphi(x, g) \text{ is not trivial in } L^\infty(\Gamma^{(0)}, \mu)\} \text{ is finite,} \tag{28}$$

and give it $*$-algebra structure by

$$(\varphi_1\varphi_2)(x, g) = \sum_{g' \in \Sigma} \varphi_1(x, g') \varphi_2(x, g'^{-1}g) c_x(g', g'^{-1}g) \tag{29}$$

$$\varphi^*(x, g) = \overline{\varphi(x, g^{-1}) c_x(g, g^{-1})}. \tag{30}$$

Further, inner product in $\mathscr{C}$ is introduced as

(31) $\quad (\varphi_1|\varphi_2)=\sum_{g\in\Sigma}\overline{\varphi_1(x,g)}\varphi_2(x,g).$

Here we must comment on the meaning of the right hand side. Since $x\mapsto\overline{\varphi_1(x,g)}\varphi_2(x,g)$ is constant on canonical equivalence class, it is constant, say $a_g$, for $\mu$-a.e. $x\in\Gamma^{(0)}$. The summation in (31) is, then, defined to be $\sum_{g\in\Sigma}a_g$. With all these structures, $\mathscr{C}$ becomes a commutative Hilbert algebra. A representation $\Phi$ of $\mathscr{C}$ on $L^2(\Gamma^{(0)},\mu)\otimes\ell^2(S)$ is defined by

(32) $\quad (\Phi(\varphi)\xi)(x,g)=\sum_{g'\in\Sigma}\varphi(x,g')\xi(x,g'^{-1}g)c_x(g',g'^{-1}g),\ x\in\Gamma^{(0)}, g\in S.$

**Lemma 4.4.** *$\Phi$ is extended continuously to an isomorphism of $\mathscr{C}''$ onto $\Phi(\mathscr{C})''$.*

*Proof.* See the argument before Theorem 1 in [9].

To relate $\Phi(\mathscr{C})''$ with $L^\infty(S^*/\Gamma)$, we use a partial Fourier transform. Let $\xi$ be a support-finite function on $S$ and $y\in\Gamma^{(0)}$. Set

(33) $\quad \hat{\xi}(b)=\sum_{g\in S}\xi(g)b(g)\ \text{ for } b\in S^*(y).$

Then $\hat{\xi}$ is in $L^2(S^*(y),\lambda^y)$ and $\xi\mapsto\hat{\xi}$ is extended to a unitary map of $\ell^2(S)$ to $L^2(S^*(y),\lambda^y)$, which is also denoted by ^ (this is essentially Fourier transform of $S$). Now let $\{\xi_x\}_{x\in\Gamma^{(0)}}$ be a family of vectors in $\ell^2(S)$. Then it can be easily checked that $\{\xi_x\}_{x\in\Gamma^{(0)}}$ is $\mu$-measurable if only if $\{\hat{\xi}_x\}_{x\in\Gamma^{(0)}}$ is $\mu$-measurable. Then a unitary map $V$ of $L^2(\Gamma^{(0)},\mu)\otimes\ell^2(S)$ to $L^2(S^*,\hat{\mu})=\int^\oplus\mu(dx)L^2(S^*(x),\lambda^x)$ is defined by

(34) $\quad V\xi=\int^\oplus\mu(dx)\hat{\xi}_x \quad \text{if} \quad \xi=\int^\oplus\mu(dx)\xi_x \text{ with } \xi_x\in\ell^2(S).$

**Lemma 4.5.**

$$V^*L^\infty(S^*/\Gamma)V=\Phi(\mathscr{C})''.$$

*Proof.* Let $\varphi\in\mathscr{C}$. A direct computation shows that $V\Phi(\varphi)V^*$ is a multiplication operator by

(35) $\quad \hat{\varphi}(b)\overset{\leftarrow}{=}\sum_{g\in\Sigma}\varphi(p(b),g)b(g),\ b\in S^*,$

and $\hat{\varphi}$ is constant on equivalence class of $\sim$ (due to (27)). Thus $V\Phi(\mathscr{C})''V^*\subset L^\infty(S^*/\Gamma)$. Conversely let $F\in L^\infty(S^*/\Gamma)$. According to

Lemma 3.2, we may assume that $F$ is represented by a $\hat{\mu}$-measurable function $f$ on $S^*$ satisfying

(36) $\qquad f(b\gamma)=f(b)$ for all $(b,\gamma)\in\hat{\Gamma}$.

Let $\xi\in L^2(\Gamma^{(0)},\mu)\otimes\ell^2(S)$ be $S$-support finite; there is a finite set $F$ of $S$ such that if $g\notin F$, $\xi(x,g)=0$ for $\mu$-a.e. $x\in\Gamma^{(0)}$. By a direct computation, we have

(37) $$(V^*FV\xi)(x,g)=\sum_{g'\in S}\varphi(x,g')\xi(x,g'^{-1}g)c_x(g',g'^{-1}g),$$

$(x,g)\in\Gamma^{(0)}\times S$, where a measurable function $\varphi$ on $\Gamma^{(0)}\times S$ is defined by

(38) $$\varphi(x,g)=\int_{S^*(x)}\lambda^x(db)f(b)b(g)^{-1}.$$

Due to (35), $\varphi$ satisfies

(39) $$\varphi(r(\gamma),g)=\varphi(s(\gamma),g)c(\gamma,g)/c(g,\gamma)$$

and then, by the definition of $\Sigma$, $\varphi$ vanishes outside of $\Sigma$. Since such a function is approximated by elements in $\mathscr{C}$ (cf. arguments in the proof of [9] Theorem 1), we deduce that $V^*FV\in\Phi(\mathscr{C})''$.

Now we relate $\mathscr{C}''$ with $L^\infty(\Omega)$. This is also achieved by Fourier transform. Let $\varphi\in\mathscr{C}$ and define a function $W\varphi$ on $\Omega$ by

(40) $$W\varphi(\omega)=\sum_{g\in\Sigma}\varphi(x,g)\omega_g(x)$$

where $\varphi(x,g)\omega_g(x)$ is constant for $\mu$-*a.e.* $x$ and the summation is taken over these constants. Since $\varphi$ has an $S$-finite support, the summation in (40) is finite and $W\varphi$ is in $L^\infty(\Omega)\subset L^2(\Omega)$. $\varphi\mapsto W\varphi$ is extended to a unitary map from $L^2(\mathscr{C})$ to $L^2(\Omega)$, which is also denoted by $W$.

**Lemma 4.6.**

$$W^*L^\infty(\Omega)W=\mathscr{C}''.$$

*Proof.* By a direct computation we have

(41) $$(W(\varphi_1\varphi_2))(\omega)=(W\varphi_1)(\omega)(W\varphi_2)(\omega),$$

$\omega\in\Omega$, $\varphi_1,\varphi_2\in\mathscr{C}$, and we can apply Stone-Weierstrass theorem to obtain the assertion.

**Definition 4.7.** Set $D=\{W\varphi;\varphi\in\mathscr{C}\}$. $D$ is a dense $*$-subalgebra of $L^\infty(\Omega)$. We define a linear map $\tau$ of $D$ into $L^\infty(S^*, \hat{\mu})$ by

$$\tau(\psi)(b) = \sum_{g\in\Sigma}\int_\Omega d\omega\psi(\omega)b(g)\omega_g(p(b))^{-1},\ b\in S^*. \tag{42}$$

Here we give comments on the integration in (41). Take any $\omega^0\in\Omega$ and a representative of $\omega^0$ by a function $\omega_g^0(x)$. Then $\int_\Omega d\omega\psi(\omega)\omega_g^0(x)^{-1}$ is defined to be $\int_{\hat{\Sigma}} d\chi\psi(\chi\omega)\chi(g)^{-1}\omega_g^0(x)^{-1}$. Another choice of $\omega^0$ and $\omega_g^0(x)$ gives the same integration for $\mu$-$a.e.\,x$. Thus as an element of $L^\infty(\Gamma^{(0)}, \mu)$, $x\mapsto\int_\Omega d\omega\psi(\omega)\omega_g(x)$ is well-defined. Furthermore if we express $\psi$ as $W\varphi$ ($\varphi\in\mathscr{C}$), then $\varphi(x, g) = \int_\Omega d\omega\psi(\omega)\overline{\omega_g(x)}$ for $g\in\Sigma$ and for $\mu$-$a.e.\,x\in\Gamma^{(0)}$. So the summation in (41) is essentially finite and the right hand side of (42) gives a well-defined element in $L^\infty(S^*)$.

**Theorem 4.8.** *$\tau$ is extended to a normal *-isomorphism of $L^\infty(\Omega)$ into $L^\infty(S^*)$ and $\tau(L^\infty(\Omega)) = L^\infty(S^*/\Gamma)$.*

*Proof.* A computation shows that

$$\tau(W\varphi) = V\Phi(\varphi)V^*, \text{ for } \varphi\in\mathscr{C}.$$

Now the assertion follows from Lemma 4.5～4.7.

**Corollary 4.9.** *$\mathfrak{A}_c''$ is a factor if and only if $\Sigma = \{e\}$.*

## §5. Factor Decomposition of $(\Gamma, c)$

In this section $S$ is continued to be assumed discrete, and we work out a factor decomposition of $\mathfrak{A}_c$ in groupoid level, using the results of §4. To simplify the construction, we adopt another point of view for the description of cocycle regular representation.

**Definition 5.1.** Let $\Gamma$ be a Borel groupoid and let $B = \{B_\gamma\}_{\gamma\in\Gamma}$ be a Borel field of 1-dimentional Hilbert spaces over $\Gamma$ and suppose that

(i) multiplication $B_{\gamma_1}\otimes B_{\gamma_2}\simeq B_{\gamma_1\gamma_2}$ ($(\gamma_1, \gamma_2)\in\Gamma^{(2)}$) is given. It is associative and Borel in the following sense: Let $\xi_1$, $\xi_2$, $\xi$ be a Borel section of $B$, then $(\gamma_1, \gamma_2)\mapsto(\xi_1(\gamma_1)\xi_2(\gamma_2)\,|\,\xi(\gamma_1\gamma_2))$ is Borel,

(ii) anti-unitary involution $*: B_\gamma\to B_{\gamma^{-1}}$ is given. It transforms Borel sections to Borel sections and satisfies

$$(\xi_1\xi_2)^*=\xi_2^*\xi_1^*, \quad \xi_1\in B_{\gamma_1}, \quad \xi_2\in B_{\gamma_2}, \ (\gamma_1,\gamma_2)\in\Gamma^{(2)},$$

(iii) $(\xi_1\xi_2|\xi)=(\xi_2|\xi_1^*\xi), \quad \xi_1\in B_{\gamma_1}, \quad \xi_2\in B_{\gamma_2}, \quad \xi\in B_{\gamma_1\gamma_2},$

(iv) if $\xi\in B_x$ $(x\in\Gamma^{(0)})$ and $\xi^2=\xi\neq 0$, then $(\xi|\xi)=1$.

We call such $B$ a groupoid ring.

Take $0\neq\xi\in B_x$ $(x\in\Gamma^{(0)})$. Since $B_x\otimes B_x\cong B_x$, $\xi^2=z\xi$ with $0\neq z\in\boldsymbol{C}$. Replacing $\xi$ by $z^{-1}\xi$, we may assume that $\xi^2=\xi$. Then by (iv), $(\xi|\xi)=1$. Let $\eta\in B_\gamma$ with $s(\gamma)=x$. By $B_\gamma\otimes B_x\cong B_\gamma$, we can find $\eta'\in B_\gamma$ such that $\eta=\eta'\xi$ and then $\eta\xi=\eta'\xi^2=\eta'\xi=\xi$, *i.e.*, $\xi$ is a right unit for $B_\gamma$. Furthermore, as $\eta^*\eta\in B_x$ and $(\eta^*\eta|\xi)=(\eta|\eta\xi)=(\eta|\eta)$, we conclude that $\eta^*\eta=(\eta|\eta)\xi$. As a corollary of this fact, $\{\xi\in B;(\xi|\xi)=1\}$ is closed under multiplication. Above arguments also show that there is a Borel section $\sigma:\Gamma\to B$ such that

(43) $$(\sigma(\gamma)|\sigma(\gamma))=1 \text{ for } \gamma\in\Gamma,$$

(44) $$\sigma(x)^2=\sigma(x) \text{ if } x\in\Gamma^{(0)}.$$

Associated with $\sigma$, we define a $\boldsymbol{T}$-valued Borel 2-cocycle $c$ of $\Gamma$ by

(45) $$\sigma(\gamma_1)\sigma(\gamma_2)=c(\gamma_1,\gamma_2)\sigma(\gamma_1\gamma_2), \ (\gamma_1,\gamma_2)\in\Gamma^{(2)}.$$

Due to (44), $c$ is normalized, i.e., $c(\gamma,s(\gamma))=c(r(\gamma),\gamma)=1$ for $\gamma\in\Gamma$. If we change $\sigma$ (under the condition that it satisfies (43) and (44)), $c$ is changed to a cohomologous one. In this way, groupoid ring $B$ determines an element in the Borel 2-cohomology group $H^2(\Gamma,\boldsymbol{T})$. Conversely, for any normalized Borel cocycle $c$, a groupoid ring structure is defined in the trivial bundle $B=\Gamma\times\boldsymbol{C}$ by

(46) $$(\gamma,z)(\gamma',z')=(\gamma\gamma',\ zz'c(\gamma,\gamma'))$$

(47) $$(\gamma,z)^*=(\gamma^{-1},\ \overline{zc(\gamma^{-1},\gamma)})$$

(48) $$((\gamma,z)|(\gamma,z))=|z|^2.$$

If we change $c$ by a coboundary, the groupoid ring obtained in this way is changed to an isomorphic one. So we have proved

**Proposition 5.2.** *There is a 1-1 correspondence between isomorphism class of groupoid ring over $\Gamma$ and $\boldsymbol{T}$-valued cohomology class of $\Gamma$.*

Now we can rewrite various objects related with a cocycle $c$, in terms of the corresponding groupoid ring $B$. For example $\mathfrak{A}_c$ is realized as a set $\mathfrak{A}$ of Borel sections of $B$. Its Hilbert algebra structure is described as

(49) $(\xi_1\xi_2)(\gamma)=\int\nu^{r(\gamma)}(d\gamma')\xi_1(\gamma')\xi_2(\gamma'^{-1}\gamma),\ \xi_1,\xi_2\in\mathfrak{A},\ \gamma\in\Gamma.$

(50) $\xi^*(\gamma)=\xi(\gamma^{-1})^*,\ \xi\in\mathfrak{A},\ \gamma\in\Gamma.$

(51) $(\xi_1\,|\,\xi_2)=\int\mu(dy)\int\nu^y(d\gamma)\,(\xi_1(\gamma)\,|\,\xi_2(\gamma)),$

$\xi_1,\xi_2\in\mathfrak{A}$. $C_g$, $g\in G$ (see §4) is also described by making use of B. Let $B^g$ be a Borel line bundle over $\Gamma^{(0)}$ defined by $B^g=\bigcup_{x\in\Gamma^{(0)}}B_{\iota_x(g)}$. Each $\gamma\in\Gamma$ gives rise to a linear map of $B^g_{s(\gamma)}$ into $B^g_{r(\gamma)}$, $B^g_{s(\gamma)}\ni\eta\mapsto\gamma\eta\equiv\xi\eta\xi^*\in B^g_{r(\gamma)}$, where $\xi\in B_\gamma$ is a unit vector, and $B^g$ becomes a $\Gamma$-bundle. Now $C_g$ is identified with the set of Borel section $\xi$ of $B^g$ such that

(52) $\xi(r(\gamma))=\gamma\xi(s(\gamma)),\quad \gamma\in\Gamma,$

and then $\Omega$ consists of sequences $\{\omega_g\}_{g\in\Sigma}$ ($\Sigma=\{g\in G\,;C_g\neq 0\}$) such that

(53) $\omega_g\in C_g,\ \|\omega_g(x)\|=1,$

and

(54) $\omega_{g_1}(x)\omega_{g_2}(x)=\omega_{g_1g_2}(x)$ for $\mu$-a. e. $x\in\Gamma^{(0)}$.

Consider the quotient space $\Gamma/\Sigma$. It has a structure of analytic Borel groupoid induced from $\Gamma$. Let $\omega\in\Omega$ and define an action of $\Sigma$ on $B$ by

(55) $g\xi=\omega_g(r(\gamma))\xi\in B_{g\gamma},\ \xi\in B_\gamma.$

Taking quotient, we have a groupoid ring $B^\omega$ over $\Gamma/\Sigma$ (cf. comments in Definition 4. 7). Let $\mathfrak{A}_\omega$ be the Hilbert algebra associated with $B^\omega$. Note that each section of $B^\omega$ is identified with a $\Sigma$-equivariant section of B. Now we specify the Borel field structure for the collection of Hilbert algebras $\{\mathfrak{A}_\omega\}_{\omega\in\Omega}$. Let $\xi$ be a section in $\mathfrak{A}$ and suppose that $\{g\in\Sigma\,;$ there exists $\gamma\in\Gamma$ such that $\xi(\gamma)\neq 0$ and $\xi(g\gamma)\neq 0\}$ is finite. Then we can form a family of vectors $\{\xi_\omega\}_{\omega\in\Omega}\in\{\mathfrak{A}_\omega\}_{\omega\in\Omega}$ by

(56) $\xi_\omega(\gamma)=\sum_{g\in\Sigma}\omega_g(r(\gamma))\xi(g^{-1}\gamma),\quad \gamma\in\Gamma.$

Such families for various $\xi$, give a Borel field structure and, from the results in § 4, we obtain

**Theorem 5. 3.**

(*i*) *For each* $\omega\in\Omega$, $\mathfrak{A}''_\omega$ *is a factor.*

*(ii)* $\mathfrak{A}''\cong\int_{\Omega}^{\oplus}d\omega\mathfrak{A}''_{\omega}$ *is a central factor decomposition of* $\mathfrak{A}''$ ($\mathfrak{A}''\cap\mathfrak{A}'$ *corresponds to* $L^{\infty}(\Omega, d\omega)$).

## § 6. Example

Let $\lambda_i$ $(i=1, 2)$ be a real number satisfying $1°$ $\lambda_i\notin\boldsymbol{Q}\pi$ and $2°$ there are integers $m_1$, $m_2$, $m$ such that $\lambda_1 m_1+\lambda_2 m_2=2\pi m$ with $(m_1, m_2, m)=1$ (relatively prime), and define an action $\varphi$ of $\boldsymbol{Z}^2$ on $\boldsymbol{T}$ by

$$\varphi_{(n_1,n_2)}(z)=e^{i(\lambda_1 n_1+\lambda_2 n_2)}z, \qquad z\in\boldsymbol{T}. \tag{57}$$

We construct a groupoid $\Gamma=\boldsymbol{Z}^2\times\boldsymbol{T}$ by semi-direct product using $\varphi$, which is ergodic if we give a Haar measure to $\boldsymbol{T}$. Note that

$$s(n_1, n_2, z)=z, \qquad r(n_1, n_2, z)=\varphi_{(n_1,n_2)}(z),$$

and

$$(n_1, n_2, z)(n_1', n_2', z')=(n_1+n_1', n_2+n_2', z').$$

The stabilizer of $\Gamma$ at $z$ is $\{(m_1 n, m_2 n, z)\in\Gamma; n\in\boldsymbol{Z}\}$ and we can define isomorphism $\iota_z:\boldsymbol{Z}\to\Gamma^z$ by $\iota_z(n)=(m_1 n, m_2 n, z)$. $\{\iota_z\}_{z\in\boldsymbol{T}}$ satisfies the conditions $1°$, $2°$ in §1. For $\alpha\in\boldsymbol{R}$, let $c$ be a cocycle of $\Gamma$ given by

$$c(n_1, n_2, z; n_1', n_2', z')=e^{i\,\alpha/2(n_1 n_2'-n_1' n_2)}. \tag{58}$$

Let us find out $\Sigma$. For $n\in\boldsymbol{Z}$ $(=G)$, the condition (25) is expressed in this example as

$$f(\mathrm{e}^{i(\lambda_1 n_1+\lambda_2 n_2)}z)=\mathrm{e}^{-i\alpha n(m_1 n_2-m_2 n_1)}f(z), \tag{59}$$

for $n_1, n_2\in\boldsymbol{Z}$ and for *a. e.* $z\in\boldsymbol{T}$. Using Fourier expansion $f(z)=\sum_{k\in\boldsymbol{Z}}f_k z^k$, we have $f\not\equiv 0$ if and only if the condition

(*) $\exists \mathrm{k}\in\boldsymbol{Z}$ such that $\mathrm{k}\lambda_1-\alpha m_2 n\in 2\pi\boldsymbol{Z}$,
$k\lambda_2+\alpha m_1 n\in 2\pi\boldsymbol{Z}$,

is satisfied.

*Case* $\alpha\notin\boldsymbol{Q}\lambda_1+\boldsymbol{Q}\pi$.

There is no $n\in\boldsymbol{Z}$ satisfying (*), i.e., $\Sigma=0$, and $c$-regular representation generates a factor.

*Case* $\alpha\in\boldsymbol{Q}\lambda_1+\boldsymbol{Q}\pi$.

We can choose integers $a, b, c$ such that $\alpha a=\lambda_1 b+2\pi c$ with $(a, b, c)=1$.

Then some computations show that $n \in \boldsymbol{Z}$ satisfies (*) if and only if it is an integer multiple of $a/(m_2, bm+cm_1, a)$, i. e., $\Sigma = a/(m_2, bm+cm_1, a)\boldsymbol{Z}$. In this case, $c$-regular representation has a factor decomposition parametrized by $\hat{\Sigma} \cong \boldsymbol{T}$.

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