

Spectral Analysis in Krein Spaces

Dedicated to the memory of Professor Henry Abel Dye

By

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Abstract

For a bounded $\#$ -unitary, the existence of a Tomita's triangular matrix representation is equivalent to the existence of an invariant maximal nonnegative subspace due to Pontrjagin, Krein et al. In other words, if a bounded $\#$ -unitary u has such an invariant subspace, its spectral analysis can be reduced to the following three cases: (i) u is $\#$ -spectral; (ii) u is quasi- $\#$ -spectral; and (iii) u is represented in the form of a Tomita's triangular matrix.

Introduction

To solve the continuity problem on weights on an operator algebra, Tomita [15] introduced a new type of an involutive Banach algebra, called an observable algebra, as a representation of a full left Hilbert algebra, and showed that the continuity is equivalent to the semi-simplicity of the algebra. As the representation is of the form of an upper triangular 3×3 matrix, it can be interpreted as a representation on a Pontrjagin space of index 1. Taking this opportunity, he extended the representation to all bounded selfadjoint operators on a general Pontrjagin space so as to hold that the $(1, 1)$ and $(3, 3)$ elements act on neutral subspaces and the $(2, 2)$ element is identified with a selfadjoint operator on a Hilbert space. We will call such a representation a Tomita's triangular matrix. Recently, this result is generalized to a Krein space in [9, 12]. The main purpose of this paper is to show that this is a generalization of a so-called Pontrjagin's fundamental theorem [14]: each selfadjoint operator with respect to the indefinite inner product of a Pontrjagin space has an invariant maximal $\#$ -nonnegative subspace. Accordingly, the both

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study due to Tomita and Pontrjagin turns out to be the same, and the 4×4 matrix representation due to Langer [6] is also interpreted as a Tomita's triangular matrix. This fact allows us to find out a new treatment of operators in a Krein space.

Throughout this paper an indefinite inner product space is restricted to a Krein space. The adjoint operation: $x \rightarrow x^\sharp$ with respect to the indefinite inner product is called a \sharp -adjoint. The corresponding unitary, selfadjoint or projection operator is called \sharp -unitary, \sharp -selfadjoint or \sharp -projection, respectively. A \sharp -unitary operator and a \sharp -projection operator are automatically bounded in a Pontrjagin space, but they are not the case in a Krein space.

In this paper we will restrict our consideration mainly to a \sharp -unitary operator, because many of the results concerning \sharp -selfadjoint operators can be reduced to those for \sharp -unitaries by means of Cayley transformations. It is easy to see that the spectrum of a \sharp -unitary is symmetric (in the sense that $\lambda \leftrightarrow \bar{\lambda}^{-1}$) with respect to the unit circle, and may happen to be the whole complex plane. However the spectral analysis of operators has not been established yet. Before explaining our treatment of operators in a Krein space, we will introduce some terminologies used for a Krein space.

A subspace is called \sharp -positive (resp. neutral, \sharp -negative) if the values $\langle \xi, \hat{\xi} \rangle$ of the indefinite inner product are positive (resp. zero, negative) for all nonzero vectors ξ in the subspace. The \sharp -nonnegativity or \sharp -nonpositivity is defined similarly for a subspace. The same words are used for the corresponding projections and \sharp -projections. A \sharp -nonnegative subspace is called uniformly \sharp -positive if

$$\langle \xi, \hat{\xi} \rangle \geq \alpha \|\xi\|^2 \quad (\alpha > 0)$$

for all vectors ξ in the subspace. The uniform \sharp -negativity is also defined similarly. By virtue of Zorn's lemma the set of \sharp -nonnegative subspaces ordered by set inclusion has a maximal element called a *maximal \sharp -nonnegative subspace*. Of course it is closed.

Let x be a bounded \sharp -unitary or \sharp -selfadjoint operator. If it leaves a maximal \sharp -nonnegative subspace \mathfrak{M} invariant, then the subspace \mathfrak{M} is classified into one of the following three cases:

(i) \mathfrak{M} is uniformly \sharp -positive: In this case, x is called \sharp -spectral and turns out to be a unitary or a selfadjoint operator on a Hilbert

space by choosing a selfdual Hilbertian inner product of the Krein space suitably. Therefore, the spectral analysis is reduced to the case for a Hilbert space.

(ii) \mathfrak{M} is $\#$ -positive but not uniformly $\#$ -positive: In this case x is called quasi- $\#$ -spectral and is regarded as a unitary or a selfadjoint operator in a Hilbert space for some selfdual pre-Hilbertian inner product in a Krein space. The two structures of Krein spaces obtained by the completion are not connected continuously and hence the spectral structures are not preserved.

(iii) \mathfrak{M} is not $\#$ -positive: The operator x is represented in the form of a Tomita's triangular matrix. In this case the $(2, 2)$ element is $\#$ -spectral or quasi- $\#$ -spectral; our discussion is reduced to either the case (i) or (ii).

While the Pontrjagin's fundamental theorem assures the existence of such an invariant subspace in a Pontrjagin space, the same assertion in a Krein space is not yet proved in general and is known to be the Phillips' problem, [13]. A sufficient condition that the off diagonal components $x - JxJ$ is compact was obtained by Iohvidov, Krein and others, [4, 5]. In the following, this result will be referred as the Krein-Pontrjagin theorem.

In §1 we will recall the definition of a Krein space and the relationship between maximal $\#$ -nonnegative subspaces and angular operators. Using this correspondence, we will show that the set of $\#$ -positive maximal $\#$ -nonnegative subspaces corresponds bijectively to the set of positive selfadjoint $\#$ -unitaries. In §2 some conditions for a bounded $\#$ -unitary to be $\#$ -spectral will be given in order to compare with the corresponding more general results obtained in §3. The strong stability introduced will be needed in §5 as a sufficient condition for an operator algebra on a Krein space to be represented in the form of a Tomita's triangular matrix. In §3 some equivalent conditions for a bounded $\#$ -unitary to be quasi- $\#$ -spectral will be discussed with the aid of the preceding results. In §4 it will be shown that the problem of a Tomita's triangular matrix representation and the existence of an invariant maximal $\#$ -nonnegative subspace are equivalent. This assertion has two applications: One is a short proof to our previous theorems in [9, 12] by using the Krein-Pontrjagin theorem. The other is a new treatment for the operators explained

in the above. This second view is more significant. Finally, in §5, by using an important result due to Helton [3], we will show that a unital commutative involutive Banach algebra on a Krein space, containing a bounded #-unitary with a compact difference from a strongly stable #-unitary, has a Tomita's triangular matrix representation.

The contents of this paper are mostly contained in the lectures in [10, 11].

§ 1. Krein Subspace, Maximal #-Nonnegative Subspace and Angular Operator

A complex vector space \mathfrak{R} endowed with a (non degenerate) indefinite inner product $\langle \cdot, \cdot \rangle$ is called an indefinite inner product space. When a selfdual

$$\|\xi\| = \sup \{ |\langle \xi, \eta \rangle| : \|\eta\| \leq 1 \}$$

Hilbertian (resp. pre-Hilbertian) inner product $(\cdot | \cdot)$ is given on the indefinite inner product space, the space $\{\mathfrak{R}, \langle \cdot, \cdot \rangle\}$ is called a Krein (resp. pre-Krein) space, where $\|\xi\| = (\xi | \xi)^{1/2}$. By a Hilbertian (resp. pre-Hilbertian) inner product we mean an inner product with respect to which the space \mathfrak{R} is a Hilbert (resp. pre-Hilbert) space. The choice of such a selfdual Hilbertian inner product is uniquely determined up to bounded #-unitaries. That is, if v is a bounded #-unitary, then the inner product $(\cdot | \cdot)_v$ defined by

$$(\xi | \eta)_v = (v\xi | v\eta), \quad \xi, \eta \in \mathfrak{R}$$

is also a selfdual Hilbertian inner product, and each selfdual Hilbertian inner product is related to each other by such a relation. Therefore a selfdual Hilbertian inner product $(\cdot | \cdot)$ will be fixed as far as we do not specify it. The relation between the indefinite inner product and the selfdual Hilbertian inner product is given by a metric operator J :

$$\langle \xi, \eta \rangle = (J\xi | \eta), \quad \xi, \eta \in \mathfrak{R}.$$

We sometimes denote the Krein space by $\{\mathfrak{R}, J\}$. The metric operator J is decomposed into the difference $J^+ - J^-$ of two projections, whose ranges are denoted by $\mathfrak{R}^\pm = J^\pm \mathfrak{R}$.

In a Krein space there exists an adjoint operation: $x \rightarrow x^*$ with

respect to the Hilbertian inner product in addition to the $\#$ -adjoint: $x \rightarrow x^\#$. To distinguish these two adjoint operations we will use the following terminologies in addition to the traditional ones for a Hilbert space:

- $\#$ -unitary $\leftrightarrow D(x)$ and $R(x)$ are dense in \mathfrak{R} and $x^\# = x^{-1}$
- $\#$ -selfadjoint $\leftrightarrow D(x)$ is dense in \mathfrak{R} and $x^\# = x$
- $\#$ -projection $\leftrightarrow D(x)$ is dense in \mathfrak{R} and $x^\# = x = x^2$

where $R(x)$ and $D(x)$ denote the range and the domain of x , respectively. The relation between $x^\#$ and x^* is given by

$$x^\# = Jx^*J, \quad D(x^\#) = JD(x^*).$$

If \mathfrak{M} is a closed subspace, then the following three conditions are equivalent:

- (i) there exists a bounded $\#$ -projection to \mathfrak{M} ;
- (ii) $\{\mathfrak{M}, \langle \cdot, \cdot \rangle_{\mathfrak{M}}\}$ is a Krein space, where $\langle \cdot, \cdot \rangle_{\mathfrak{M}}$ is the restriction of $\langle \cdot, \cdot \rangle$ to \mathfrak{M} ; and
- (iii) $\mathfrak{M} + \mathfrak{M}^\perp = \mathfrak{R}$, where $\mathfrak{M}^\perp = \{\xi \in \mathfrak{R} : \langle \xi, \eta \rangle = 0, \eta \in \mathfrak{M}\}$.

In particular, for a $\#$ -positive closed subspace to have a bounded $\#$ -projection it is necessary and sufficient that it is uniformly $\#$ -positive. The following generalization is more or less known:

Theorem 1.1 ([5, 10]). *Let \mathfrak{M} be a closed subspace of a Krein space $\{\mathfrak{R}, \langle \cdot, \cdot \rangle\}$. The following three conditions are equivalent:*

- (i) *there exists a $\#$ -projection to \mathfrak{M} ;*
- (ii) *$\{\mathfrak{M}, \langle \cdot, \cdot \rangle_{\mathfrak{M}}\}$ is a pre-Krein space; and*
- (iii) *$\mathfrak{M} \cap \mathfrak{M}^\perp = \{0\}$.*

In particular, for a $\#$ -nonnegative closed subspace to have a $\#$ -projection if and only if it is $\#$ -positive.

Here we notice that the correspondence between a closed subspace \mathfrak{M} with $\mathfrak{M} \cap \mathfrak{M}^\perp = \{0\}$ and a $\#$ -projection e is given by the property that $D(e) = \mathfrak{M} + \mathfrak{M}^\perp$ and $e: \mathfrak{M} + \mathfrak{M}^\perp \rightarrow \mathfrak{M}$.

Now, we recall the correspondence between maximal $\#$ -nonnegative subspaces and angular operators. Every element in the unit ball

$$\{k \in \mathcal{L}(\mathfrak{R}^+, \mathfrak{R}^-) : \|k\| \leq 1\}$$

is called an angular operator. There exists a bijection from the set

of all maximal $\#$ -nonnegative subspaces \mathfrak{M} to the set of all angular operators k such that $\mathfrak{M}=G(k)$, where $G(k)$ is the graph $(\xi+k\xi: \xi \in \mathfrak{R}^+)$ of k . Under this correspondence, \mathfrak{M} is $\#$ -positive (resp. uniformly $\#$ -positive) if and only if $k^*k < 1$ (resp. $\|k\| < 1$). Here $h > 0$ means that $h \geq 0$ and $h\xi = 0$ implies $\xi = 0$. A linear transformation which maps \mathfrak{R}^+ to a uniformly $\#$ -positive maximal $\#$ -nonnegative subspace $\mathfrak{M}=G(k)$ is given by a bounded $\#$ -unitary $S_k = h_1 h_2^{-1}$, where

$$(1.1) \quad h_1 = \begin{pmatrix} 1 & k^* \\ k & 1 \end{pmatrix} \text{ and } h_2 = \begin{pmatrix} (1-k^*k)^{1/2} & 0 \\ 0 & (1-kk^*)^{1/2} \end{pmatrix}$$

Since $\|k\| < 1$, $S_k \geq 0$. A polar decomposition of a bounded $\#$ -unitary which transforms \mathfrak{R}^+ to $\mathfrak{M}=G(k)$ is expressed in the form

$$S_k \begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix},$$

where u^\pm are unitaries on \mathfrak{R}^\pm .

Next, we will establish a correspondence from angular operators k with $k^*k < 1$ to positive selfadjoint $\#$ -unitaries. It is known that there exists a bijection from the set of angular operators k with $\|k\| < 1$ to the set of positive selfadjoint bounded $\#$ -unitaries u such that $u = S_k$. To extend this statement to more general angular operators k with $k^*k < 1$, we will introduce the following concept for $\#$ -unitaries:

Definition 1.2 ([10]). A $\#$ -unitary u is called *J-regular* if

- (i) $D(u) \cap \mathfrak{R}^+ + D(u) \cap \mathfrak{R}^-$ is a core for u , and
- (ii) $\mathfrak{M}^\pm = u\{D(u) \cap \mathfrak{R}^\pm\}$ are closed subspaces with $(\mathfrak{M}^+)^\perp = \mathfrak{M}^-$.

Here we notice that if we denote by p^\pm the projections onto \mathfrak{M}^\pm , then the condition $(\mathfrak{M}^+)^\perp = \mathfrak{M}^-$ is equivalent to $(p^+)^\# + p^- = 1$, although $\mathfrak{M}^+ + \mathfrak{M}^-$ may not coincides with \mathfrak{R} .

The following theorem was initially obtained in [10] under the assumption that \mathfrak{R} is separable.

Theorem 1.3 ([10]). *There exists a bijection from the set of angular operators k with $k^*k < 1$ to the set of J-regular, positive selfadjoint $\#$ -unitaries u such that u is the closure of S_k . In this case, $D(u) \cap \mathfrak{R}^+ = R((1-k^*k)^{1/2})$, $D(u) \cap \mathfrak{R}^- = R((1-kk^*)^{1/2})$ and $u\{D(u) \cap \mathfrak{R}^+\} = G(k)$.*

Proof. First we will show that if k is an angular operator with $k^*k < 1$, then the closure of S_k is a J -regular, positive selfadjoint $\#$ -unitary. Let h_1 and h_2 be the operators defined by (1.1). It is easy to see that $h_1 \geq 0$, $h_2 \geq 0$, $h_1 h_2 = h_2 h_1$, $h_2^{\#} = h_2$ and $h_1^{\#} h_1 = h_2^2$.

We begin by showing that S_k is positive and essentially selfadjoint. Since $D(S_k) = R(h_2)$, each ξ and η in $D(S_k)$ are of the forms $\xi = h_2 \xi'$ and $\eta = h_2 \eta'$ for some $\xi', \eta' \in \mathfrak{R}$. Hence

$$(S_k \xi | \eta) = (h_1 \xi' | h_2 \eta') = (h_2 \xi' | h_1 \eta') = (\xi | S_k \eta)$$

and so S_k is symmetric. In case of $\xi = \eta$,

$$(S_k \xi | \xi) = (h_1 h_2^{1/2} \xi' | h_2^{1/2} \xi') \geq 0;$$

hence S_k is positive.

To see the essential selfadjointness, we will use the Nelson's theorem by showing that any analytic vector for h_2^{-1} is also an analytic vector for S_k . Notice that

$$(1.2) \quad S_k^n \xi = h_1^n h_2^{-n} \xi$$

for any analytic vector ξ for h_2^{-1} . Indeed, this is verified by mathematical induction. The case $n=1$ is clear. Suppose (1.2) holds for $n=l$. Since

$$S_k^l \xi = h_1^l h_2^{-l} \xi = h_1^l h_2 h_2^{-(l+1)} \xi = h_2 h_1^l h_2^{-(l+1)} \xi \in D(S_k),$$

we find that

$$S_k^{l+1} \xi = S_k h_2 h_1^l h_2^{-(l+1)} \xi = h_1^{l+1} h_2^{-(l+1)} \xi.$$

Thus (1.2) holds. Therefore $\|S_k^n \xi\| = \|h_1^n h_2^{-n} \xi\| \leq \|h_1\|^n \|h_2^{-n} \xi\|$, and so

$$\sum_{n=0}^{\infty} \frac{\|S_k^n \xi\|}{n!} t^n \leq \sum_{n=0}^{\infty} \frac{\|h_2^{-n} \xi\|}{n!} (\|h_1\|t)^n.$$

Since ξ is analytic for h_2^{-1} , there exists a positive $t > 0$ for which the right hand side converges. Hence ξ is analytic for S_k , too. Since $D(S_k) = D(h_2^{-1})$ contains a total set of analytic vectors for h_2^{-1} , S_k is essentially selfadjoint.

Next, we will show that the closure S of S_k is $\#$ -unitary. It is clear that $JS_k J$ is essentially selfadjoint and its closure coincides with JSJ . Let $\xi \in D(JS_k J)$ and $\eta \in D(S_k)$. Since $h_2^{\#} = h_2$ and $h_2^2 = h_1^{\#} h_1$, we have

$$(1.3) \quad \begin{aligned} (JS_k J \xi | S_k \eta) &= (J h_1 h_2^{-1} J \xi | h_1 h_2^{-1} \eta) = (J h_2^{-1} J \xi | h_1^{\#} h_1 h_2^{-1} \eta) \\ &= (J h_2^{-1} J \xi | h_2 \eta) = (\xi | \eta). \end{aligned}$$

If $\xi \in D(JSJ)$ and $\eta \in D(S)$, then there exist two sequences $\{\xi_n\}_{n=1}^\infty \subset D(JS_k J)$ and $\{\eta_m\}_{m=1}^\infty \subset D(S_k)$ such that $\xi_n \rightarrow \xi$, $JS_k J \xi_n \rightarrow JSJ \xi$; and $\eta_m \rightarrow \eta$, $S_k \eta_m \rightarrow S \eta$. Hence, by (1.3), we have

$$\begin{aligned} (JSJ \xi | S \eta) &= \lim_{n, m \rightarrow \infty} (JS_k J \xi_n | S_k \eta_m) \\ &= \lim_{n, m \rightarrow \infty} (\xi_n | \eta_m) = (\xi | \eta). \end{aligned}$$

From this we find that the mapping: $\xi \in D(JSJ) \rightarrow (JSJ \xi | S \eta)$ is continuous. Since JSJ is closed, it follows that $S \eta \in D(JSJ)$ and $JSJS \eta = \eta$ for all $\eta \in D(S)$. Thus $S^{-1} \subset JSJ$. The selfadjointness of S and JSJ implies $S^{-1} = JSJ$. Therefore S is $\#$ -unitary.

It remains to show that S is J -regular and that $D(S) \cap \mathfrak{R}^+ = R((1 - k^*k)^{1/2})$ and $D(S) \cap \mathfrak{R}^- = R((1 - kk^*)^{1/2})$. Since the domain of S_k is $R((1 - k^*k)^{1/2}) + R((1 - kk^*)^{1/2})$, we see that $R((1 - k^*k)^{1/2}) \subset D(S) \cap \mathfrak{R}^+$ and $R((1 - kk^*)^{1/2}) \subset D(S) \cap \mathfrak{R}^-$. Here we define \mathfrak{M}^\pm by setting $\mathfrak{M}^\pm = S\{D(S) \cap \mathfrak{R}^\pm\}$. Then

$$\begin{aligned} \mathfrak{M}^+ &= \{S\xi : \xi \in D(S) \cap \mathfrak{R}^+\} \supset \{S\xi : \xi \in R((1 - k^*k)^{1/2})\} \\ &= \{S_k \xi : \xi \in R((1 - k^*k)^{1/2})\} = G(k). \end{aligned}$$

Since \mathfrak{M}^+ is a $\#$ -positive subspace and $G(k)$ is a maximal $\#$ -non-negative subspace, we see that $\mathfrak{M}^+ = G(k)$ by maximality. Hence the above inclusion becomes the equality. Using the invertibility of S , we find that $D(S) \cap \mathfrak{R}^+ = R((1 - k^*k)^{1/2})$.

The similar discussion is applicable to \mathfrak{M}^- . Hence we have $\mathfrak{M}^- = G(k^*)$ and $D(S) \cap \mathfrak{R}^- = R((1 - kk^*)^{1/2})$. Since $(\mathfrak{M}^+)^\perp = G(k)^\perp = G(k^*) = \mathfrak{M}^-$, it follows that S is J -regular.

Finally, we will show the converse, namely, if u is a J -regular positive selfadjoint $\#$ -unitary, then there exists an angular operator k with $k^*k < 1$ such that u is the closure of S_k .

Since u is J -regular, $\mathfrak{M}^\pm = u\{D(u) \cap \mathfrak{R}^\pm\}$ satisfy $(\mathfrak{M}^+)^\perp = \mathfrak{M}^-$. Clearly \mathfrak{M}^+ is $\#$ -positive and \mathfrak{M}^- is $\#$ -negative. Since $(\mathfrak{M}^+)^\perp = \mathfrak{M}^-$, \mathfrak{M}^+ is a maximal $\#$ -nonnegative subspace. Therefore there exists an angular operator k with $k^*k < 1$ such that $\mathfrak{M}^+ = G(k)$. Let S denote the closure of $S_k = h_1 h_2^{-1}$. Since

$$u\{D(u) \cap \mathfrak{R}^\pm\} = \mathfrak{M}^\pm = S\{D(S) \cap \mathfrak{R}^\pm\},$$

there exist unitaries u^\pm on \mathfrak{R}^\pm which transform $D(u) \cap \mathfrak{R}^\pm$ to $D(S) \cap \mathfrak{R}^\pm$ bijectively such that

$$(1.4) \quad u = S \begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix}$$

on $\{D(u) \cap \mathfrak{R}^+\} + \{D(u) \cap \mathfrak{R}^-\}$. Since $\{D(u) \cap \mathfrak{R}^+\} + \{D(u) \cap \mathfrak{R}^-\}$ is a core for u and

$$\begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix} \begin{pmatrix} D(u) \cap \mathfrak{R}^+ \\ D(u) \cap \mathfrak{R}^- \end{pmatrix} = \begin{pmatrix} D(S) \cap \mathfrak{R}^+ \\ D(S) \cap \mathfrak{R}^- \end{pmatrix} = \begin{pmatrix} D((1-k^*k)^{-1/2}) \\ D((1-kk^*)^{-1/2}) \end{pmatrix}$$

is a core for S , the above (1.4) holds on $D(u)$. The uniqueness of the polar decomposition yields $u = S$. QED.

§ 2. #-Spectral Operator

A bounded #-unitary or a bounded #-selfadjoint operator x is said to be #-spectral if there exists a bounded #-unitary v such that vxv^* is unitary or selfadjoint, respectively, [9, 16]. In this case the operator x is unitary or selfadjoint with respect to the Hilbertian inner product $(\cdot | \cdot)_v$. Thus the spectral analysis of x is reduced to the analysis on a Hilbert space $\{\mathfrak{R}, (\cdot | \cdot)_v\}$. For a bounded #-unitary u the following four conditions are known to be equivalent [1, 5, 10]:

- (i) u is #-spectral;
- (ii) (Power bounded) $\sup \{\|u^n\| : n \in \mathbf{Z}\} < \infty$;
- (iii) there exists a (uniformly) #-positive maximal #-nonnegative bounded #-projection commuting with u ; and
- (iv) there exists an operator $T \in \mathcal{L}(\mathfrak{R})$ such that
 - a) $0 < T < 1$, $JTJ = 1 - T$, $\text{Sp}(T) \cap \{0, 1\} = \emptyset$
 - b) $Tu^*(1 - T) = (1 - T)u^*T$.

A bounded #-unitary satisfying the above conditions is also said to be stable. A stable bounded #-unitary u is said to be strongly stable if each bounded #-unitary in some norm neighbourhood of u is stable. For a bounded #-unitary u , it is strongly stable if and only if there exists a (uniformly) #-positive maximal #-nonnegative bounded #-projection e such that $\text{Sp}(u|e\mathfrak{R}) \cap \text{Sp}(u|(1-e)\mathfrak{R}) = \emptyset$, [5].

§ 3. Quasi-#-spectral Operator

In this § we will generalize a result in §2. On a Pontrjagin space, a #-unitary and a #-projection are bounded and the #-positivity

leads the uniform $\#$ -positivity. Hence the results in this § are proper to a Krein space and do not occur in a Pontrjagin space.

A bounded $\#$ -unitary (resp. bounded $\#$ -selfadjoint operator) x is said to be quasi- $\#$ -spectral if there exists a $\#$ -unitary v such that $xD(v^*v) = D(v^*v)$ (resp. $xD(v^*v) \subset D(v^*v)$), $vxv^*|_{D_0}$ is closable and the closure is unitary (resp. selfadjoint), where $D_0 = vD(v^*v)$. In this case, the pre-Hilbertian inner product $(\cdot | \cdot)_v$ defined by $(\xi | \eta)_v = (v\xi | v\eta)$ for $\xi, \eta \in D(v^*v)$ is selfdual, and the operator x is unitary (resp. selfadjoint) in the Hilbert space \mathfrak{R}_v , the completion of \mathfrak{R} with respect to $(\cdot | \cdot)_v$. As the restriction of the indefinite inner product to $D(v^*v)$ is continuous in norm $\|\xi\|_v = (\xi | \xi)_v^{1/2}$, it is naturally extended to \mathfrak{R}_v and makes \mathfrak{R}_v a Krein space. In this case, both \mathfrak{R} and \mathfrak{R}_v contain $D(v^*v)$ simultaneously as a dense subspace, and the identity mapping on $D(v^*v)$ is considered to be the mapping from a dense subspace of \mathfrak{R} to \mathfrak{R}_v . However this mapping is not continuous and hence the spectral structure of x in \mathfrak{R} does not necessarily preserved in \mathfrak{R}_v .

Now we are ready to relate the study of Tomita with that of Pontrjagin, Krein, Langer et al. The meaning of the following theorem will become clear if we compare it with the results stated in §2. Theorem 1.3 will be utilized in the following proof.

Theorem 3.1 ([11]). *Let u be a bounded $\#$ -unitary. The following five conditions are equivalent:*

- (i) u is quasi- $\#$ -spectral;
- (ii) there exists a $\#$ -unitary v such that $uD(v^*v) = D(v^*v)$ and $\sup \{ \|\{u|D(v^*v)\}^n\|_v : n \in \mathbf{Z} \} < \infty$;
- (iii) there exists a $\#$ -positive maximal $\#$ -nonnegative $\#$ -projection e commuting with u in the sense that $ueu^* = e$;
- (iv) there exists an operator $T \in \mathcal{L}(\mathfrak{R})$ such that
 - a) $0 < T < 1$, $JTJ = 1 - T$,
 - b) $Tu^*(1 - T) = (1 - T)u^*T$; and
- (v) there exists a $\#$ -unitary v such that $uD(v) = D(v)$, vvu^* is closable and the closure is unitary.

From condition (v) it is immediate that $JD(v) = R(v^*)$ and $u^*R(v^*) = R(v^*)$. Some examples of such quasi- $\#$ -spectral operators have been given in [9, 12].

Proof. The equivalence between (i) and (iv) is proved in [9]. The equivalence of (i) and (ii) is immediate from §2.

(i)→(v): Suppose that u is quasi-#-spectral. Then there exists a #-unitary v such that $uD(v^*v) = D(v^*v)$, $vu v^*|_{D_0}$ is closable and the closure w is unitary, where $D_0 = vD(v^*v)$. Considering the polar decomposition, we may assume that v is positive selfadjoint. Since $D(v^2)$ is a core for v , for any $\xi \in D(v)$ there exists a sequence $\{\xi_n\}_{n=1}^\infty \subset D(v^2)$ such that

$$\xi_n \longrightarrow \xi \quad \text{and} \quad v\xi_n \longrightarrow v\xi.$$

Since u and w are bounded, we have

$$u\xi_n \longrightarrow u\xi \quad \text{and} \quad vu\xi_n = wv\xi_n \longrightarrow wv\xi.$$

Since v is closed and $u\xi_n \in D(v)$, it follows that $u\xi \in D(v)$ and $vu\xi = wv\xi$. Therefore $uD(v) \subset D(v)$. Similarly, the boundedness of $u^\#$ and $w^\#$ implies

$$u^\#\xi_n \longrightarrow u^\#\xi \quad \text{and} \quad vu^\#\xi_n = w^\#v\xi_n \longrightarrow w^\#v\xi.$$

Hence $u^\#\xi \in D(v)$ and $vu^\#\xi = w^\#v\xi$. Thus $u^\#D(v) \subset D(v)$. Consequently, $uD(v) = D(v)$. Moreover, since $vu v^*|_{D_0} \subset vu v^* \subset w$, we see that $vu v^*$ is closable and the closure is w , which is unitary. Thus (v) is proved.

(v)→(iii): We may assume that v is positive selfadjoint as in the above proof. We begin by showing that the sum $\mathfrak{M}^+ + \mathfrak{M}^-$ of the closures \mathfrak{M}^\pm of $v^\#\{R(v) \cap \mathfrak{R}^\pm\}$ is dense in \mathfrak{R} .

Let p_n be the spectral projection of v corresponding to the interval $[n^{-1}, n]$ for each $n \in \mathbb{N}$. It is easy to see that the union D_1 of all $p_n\mathfrak{R}$ is a core for $v^\# (=v^{-1})$ as well as v . Indeed, if $\eta \in D(v^\#)$, then $p_n\eta \rightarrow \eta$. Since $\|v^\#\eta\|^2 = \|v^\#p_n\eta\|^2 + \|v^\#(1-p_n)\eta\|^2$, $\{\|v^\#p_n\eta\|\}_{n=1}^\infty$ is a bounded increasing sequence and hence a Cauchy sequence. Since $\|v^\#p_n\eta - v^\#p_m\eta\|^2 = \|v^\#p_n\eta\|^2 - \|v^\#p_m\eta\|^2$ for $n \geq m$, it follows that $\{v^\#p_n\eta\}_{n=1}^\infty$ is a Cauchy sequence. Thus the closedness of $v^\#$ implies that $v^\#p_n\eta \rightarrow v^\#\eta$.

Since $JvJ = v^{-1}$, p_n is a #-projection commuting with J . Hence J maps D_1 onto itself. Therefore D_1 is of the form $D_1 \cap \mathfrak{R}^+ + D_1 \cap \mathfrak{R}^-$. If $\xi \in D(v)$, then $\eta = v\xi \in D(v^\#)$. The above discussion tells us that

$$p_n\eta \longrightarrow \eta \quad \text{and} \quad v^\#p_n\eta \longrightarrow v^\#\eta.$$

Since $J^\#p_n\eta \in D_1 \cap \mathfrak{R}^\pm$ and $D_1 \cap \mathfrak{R}^\pm \subset R(v) \cap \mathfrak{R}^\pm$, we see that $v^\#p_n\eta \in v^\#\{R(v) \cap \mathfrak{R}^+\} + v^\#\{R(v) \cap \mathfrak{R}^-\}$, namely, $v^\#p_n\eta \in \mathfrak{M}^+ + \mathfrak{M}^-$. Since $D(v)$ is dense in \mathfrak{R} , so is $\mathfrak{M}^+ + \mathfrak{M}^-$.

Since v is $\#$ -unitary, $v^*\{R(v) \cap \mathfrak{R}^+\}$ is $\#$ -positive and $v^*\{R(v) \cap \mathfrak{R}^-\}$ is $\#$ -negative. Moreover $v^*\{R(v) \cap \mathfrak{R}^+\} \subset \{v^*\{R(v) \cap \mathfrak{R}^-\}\}^\perp$. Thus \mathfrak{M}^+ and \mathfrak{M}^- are $\#$ -nonnegative and $\#$ -nonpositive closed subspaces satisfying $\mathfrak{M}^\pm \subset (\mathfrak{M}^\mp)^\perp$. Furthermore \mathfrak{M}^+ is $\#$ -positive and \mathfrak{M}^- is $\#$ -negative, for $\mathfrak{M}^\pm \cap (\mathfrak{M}^\pm)^\perp = \{0\}$ follows from the density of $\mathfrak{M}^+ + \mathfrak{M}^-$. Next, we will show that $\mathfrak{M}^+ = (\mathfrak{M}^-)^\perp$. Since \mathfrak{M}^+ is $\#$ -positive and closed, $J^+\mathfrak{M}^+$ is also closed. The maximal $\#$ -nonnegativity of \mathfrak{M}^+ is equivalent to $J^+\mathfrak{M}^+ = \mathfrak{R}^+$. Let p_n be the above spectral projection of v corresponding to the interval $[n^{-1}, n]$. Since p_n commutes with v and J , $\{p_n\mathfrak{R}, J_n\}$ is a Krein space and the restriction of v to $p_n\mathfrak{R}$ is a bounded $\#$ -unitary whose spectrum contained in $[n^{-1}, n]$, where J_n is the restriction of J to $p_n\mathfrak{R}$. Hence $v^*p_n\mathfrak{R}^+$ is uniformly $\#$ -positive in $\{p_n\mathfrak{R}, J_n\}$, and so $J^+v^*p_n\mathfrak{R}^+ = J^+p_n\mathfrak{R}^+ = p_n\mathfrak{R}^+$. Since the union of $p_n\mathfrak{R}^+$ is contained in $J^+\mathfrak{M}^+$ and dense in \mathfrak{R}^+ , it follows that $J^+\mathfrak{M}^+ = \mathfrak{R}^+$. Thus $\mathfrak{M}^+ = (\mathfrak{M}^-)^\perp$. Furthermore, it is u -invariant. Really, the boundedness of u implies

$$\begin{aligned} u\mathfrak{M}^+ &\subset \{uv^*\{R(v) \cap \mathfrak{R}^+\}\}^- = \{v^*(vuv^*)\{R(v) \cap \mathfrak{R}^+\}\}^- \\ &\subset \{v^*\{R(v) \cap \mathfrak{R}^+\}\}^- = \mathfrak{M}^+. \end{aligned}$$

Thus the $\#$ -projection $e: \xi \in \mathfrak{M}^+ + \mathfrak{M}^- \rightarrow \xi^+ \in \mathfrak{M}^+$ commutes with u , where $\xi = \xi^+ + \xi^-$ with $\xi^\pm \in \mathfrak{M}^\pm$.

(iii) \rightarrow (i): Let e be an invariant maximal $\#$ -nonnegative $\#$ -projection. Since the uniform $\#$ -positivity is equivalent to the $\#$ -spectrality, we may assume that e is a $\#$ -positive $\#$ -projection that is not uniformly $\#$ -positive. Let k be the angular operator with $G(k) = e\mathfrak{R}$. Since e is $\#$ -positive, k satisfies $k^*k < 1$. By virtue of Theorem 1.3 the closure v of S_k is a J -regular positive selfadjoint $\#$ -unitary. The J -regularity of v yields that $D(v) \cap \mathfrak{R}^+ + D(v) \cap \mathfrak{R}^-$ is a core for v , that $\mathfrak{M}^\pm = v\{D(v) \cap \mathfrak{R}^\pm\}$ are closed subspaces with $\mathfrak{M}^- = (\mathfrak{M}^+)^\perp$ and that $\mathfrak{M}^+ = G(k)$.

Since $ueu^* = e$, v^*uv transforms $D(v) \cap \mathfrak{R}^\pm$ onto itself bijectively. Denote by w_2 the restriction of v^*uv to D_2 , where $D_2 = D(v) \cap \mathfrak{R}^+ + D(v) \cap \mathfrak{R}^-$. Since, for $\xi^\pm \in D(v) \cap \mathfrak{R}^\pm$,

$$\begin{aligned} \|w_2(\xi^+ + \xi^-)\|^2 &= \langle w_2\xi^+, w_2\xi^+ \rangle - \langle w_2\xi^-, w_2\xi^- \rangle \\ &= \langle \xi^+, \xi^+ \rangle - \langle \xi^-, \xi^- \rangle = \|\xi^+ + \xi^-\|^2, \end{aligned}$$

it follows that w_2 is bounded. Since D_2 is a core for v , it is dense in \mathfrak{R} and hence the closure w of w_2 is a unitary.

Next we will show that $wD(v) = D(v)$ and $wR(v) = R(v)$. If $\xi \in D(v)$, then there exists a sequence $\{\xi_n\}_{n=1}^\infty \subset D_2$ such that $\xi_n \rightarrow \xi$ and

$v\xi_n \rightarrow v\xi$. The boundedness of w and u implies

$$v^*uw\xi_n = w\xi_n \longrightarrow w\xi, \quad uw\xi_n \longrightarrow uw\xi,$$

and hence $w\xi \in D(v^*)$ and $w\xi = v^*uw\xi$. Hence $w\xi \in R(v^*) = D(v)$, in other words, $wD(v) \subset D(v)$. Similarly, the boundedness of $w^* = w^\#$ and $u^\#$ implies

$$v^*u^*v\xi_n = w^*\xi_n \longrightarrow w^*\xi, \quad u^*v\xi_n \longrightarrow u^*v\xi,$$

and hence $w^{-1}D(v) \subset D(v)$, for $w^* = w^{-1}$. Thus $wD(v) = D(v)$, which is equivalent to $wR(v) = R(v)$. Really $JD(v) = R(v)$ and $w^{-1} = w^* = JwJ$ imply $wR(v) = R(v)$.

Furthermore, we will show that $uR(v^2) = R(v^2)$. If $\eta \in R(v^2)$, then $\eta = v^2\xi$ for some $\xi \in D(v^2)$. Since $uv^2\xi = vuv\xi \in vR(v)$, it follows that $uR(v^2) \subset R(v^2)$. Since $u^*v^2\xi = vw^*v\xi \in vR(v)$ as well, it follows that $u^*R(v^2) \subset R(v^2)$. Thus $uR(v^2) = R(v^2)$.

Finally, we set $D_0 = v^*D((v^*)^2) = v^{-1}R(v^2)$. Since $R(v^2) \subset R(v)$, we have $D_0 \subset v^{-1}R(v) = D(v)$. Since $w = v^*uw$ on $D(v)$ as shown in the above and D_0 is dense in \mathfrak{R} , the closure of $v^*uw|_{D_0}$ coincides with w . Hence it is unitary. QED.

Problem 3.2. Improve condition (ii) so as to be described by the words in \mathfrak{R} without using \mathfrak{R}_v . For example, can we weaken the inequality into the following form: for any $\xi, \eta \in D(v)$

$$\sup \{ |(u^n\xi|\eta)| : n \in \mathbf{Z} \} < \infty ?$$

§ 4. Tomita's Triangular Matrix

Let x be a bounded $\#$ -unitary or $\#$ -selfadjoint operator which has an invariant maximal $\#$ -nonnegative subspace. If x is neither $\#$ -spectral nor quasi- $\#$ -spectral, it will be represented in the form of a Tomita's triangular matrix.

If a neutral projection p with $pp^\# = 0$ is invariant under x and its $\#$ -adjoint $x^\#$ ($xp = pxp$ and $x^\#p = px^\#p$), then x and the metric operator J are represented in the forms

$$(4.1) \quad x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \quad J = \begin{pmatrix} 0 & 0 & J_{13} \\ 0 & J_{22} & 0 \\ J_{31} & 0 & 0 \end{pmatrix}$$

where $p_1 = p$, $p_2 = 1 - p - p^\sharp$, $p_3 = p^\sharp$, $x_{ij} = p_i x | p_j \mathfrak{R}$ and $J_{ij} = p_i J | p_j \mathfrak{R}$. The former triangular matrix is called a Tomita's triangular matrix for x if the $(2, 2)$ component x_{22} is quasi- \sharp -spectral in a Krein space $\{p_2 \mathfrak{R}, J_{22}\}$. If x is \sharp -unitary or \sharp -selfadjoint, then so is x_{22} .

The following theorem is an immediate consequence from Theorem 3. 1.

Theorem 4. 1 ([11]). *Let u be a bounded \sharp -unitary. The following two conditions are equivalent:*

- (i) u has an invariant maximal \sharp -nonnegative subspace; and
- (ii) u has a Tomita's triangular matrix representation.

Proof. (i) \rightarrow (ii): Let p be an invariant maximal \sharp -nonnegative projection ($pu = up$ by maximality). Then $1 - p^\sharp$ is an invariant maximal \sharp -nonpositive projection. Put $p_1 = p \wedge (1 - p^\sharp)$, $p_2 = 1 - p_1 - p_1^\sharp$ and $p_3 = p_1^\sharp$. Since p commutes with u , $p_1 \mathfrak{R}$ is invariant under u and u^\sharp . Since p_1 is neutral, u is represented in the form (4. 1). Put $p^+ = p - p_1$ and $p^- = 1 - p^\sharp - p_1$. Then p^+ is \sharp -positive, p^- is \sharp -negative and they satisfy $(p^+)^\sharp + p^- = p_2$. Since $p_2^\sharp = p_2$, we see that $p^\pm \leq p_2$ and p^+ is maximal \sharp -nonnegative on $\{p_2 \mathfrak{R}, J_{22}\}$. Since

$$\begin{aligned} p_2 u p^+ &= p_2 u (p - p_1) = p_2 (p u p - p_1 u p_1) \\ &= p_2 (p u p) (p - p_1) = p_2 p u (p - p_1) \\ &= p_2 (p - p_1) u (p - p_1) + p_2 p_1 u (p - p_1) \\ &= p^+ u p^+, \end{aligned}$$

if we set $u_{22} = p_2 u | p_2 \mathfrak{R}$, then u_{22} is a bounded \sharp -unitary on $\{p_2 \mathfrak{R}, J_{22}\}$ which satisfies $u_{22} p^+ = p^+ u_{22} p^+$. Therefore a maximal \sharp -nonnegative projection p^+ is \sharp -positive and invariant under u_{22} . Hence, u_{22} is quasi- \sharp -spectral by Theorem 3. 1. Thus the above matrix is a Tomita's triangular matrix.

(ii) \rightarrow (i): Let p_1 , p_2 and p_3 be projections used in the construction of a Tomita's triangular matrix. Since the bounded \sharp -unitary u_{22} is quasi- \sharp -spectral, there exists a maximal \sharp -nonnegative projection p^+ invariant under u_{22} . It is easy to see that $p_1 + p^+$ is an invariant maximal \sharp -nonnegative projection. QED.

In the above proof, if we define q_j , $j = 1, 2, 3, 4$ and u_{ij} by setting $q_1 = p^+$, $q_2 = p_1$, $q_3 = p_3$, $q_4 = p^-$ and $u_{ij} = q_i u | q_j \mathfrak{R}$, then u is represented in

the form of 4×4 matrix (u_{ij}) . This is nothing but a generalization of Langer's matrix [6] to a Krein space.

A sufficient condition for (i) in Theorem 4.1 to hold is known as the Krein-Pontrjagin theorem [4, 14]: If u is a bounded $\#$ -unitary with $u-JuJ$ compact, then u has an invariant maximal $\#$ -nonnegative subspace. Combining this with Theorem 4.1, we have a new proof for the following theorem.

Theorem 4.2 ([10, 12]). *If x is a bounded $\#$ -unitary (or a bounded $\#$ -selfadjoint operator) with $x-JxJ$ compact, then x has a Tomita's triangular matrix representation.*

This was proved in [10, 12] independently of the Krein-Pontrjagin theorem. Conversely the last theorem is deduced from Theorem 4.2. It is desirable to extend Theorem 4.2 to an unbounded $\#$ -selfadjoint operator. But we have not yet succeeded. For a Pontrjagin space we know the following:

Theorem 4.3 ([12]). *If h is a $\#$ -selfadjoint operator in a Pontrjagin space, then there exists a selfdual Hilbertian inner product for which h is represented by a Tomita's triangular matrix such that $D(h) = p_1\mathfrak{R} + D(h_{22}) + p_3\mathfrak{R}$.*

If we use the similar assertion for $\#$ -selfadjoint operator as Theorem 4.1, then this theorem is a restatement of a Pontrjagin's fundamental theorem [14].

Problem 4.4. Is a bounded $\#$ -unitary (or $\#$ -selfadjoint) operator x is quasi- $\#$ -spectral if x has no nonzero invariant neutral subspaces?

If this is true then the Phillips' problem is affirmative via Theorem 4.1.

§ 5. Tomita's Triangular Representaion of Commutative Lorentz Algebras

Let $\mathcal{L}(\mathfrak{R})$ be the set of all bounded operators on a Krein space

$\{\mathfrak{R}, \langle, \rangle\}$. It is a Banach algebra with respect to the operator norm induced from a selfdual Hilbertian inner product. A Banach subalgebra of $\mathcal{L}(\mathfrak{R})$ closed under the involution: $x \rightarrow x^\#$ is called a *Lorentz algebra*. It should be noted that the definition does not depend on the choice of a selfdual Hilbertian inner product.

As easily seen from the discussion in §§1~4, if the Lorentz algebra \mathcal{A} leaves a uniformly $\#$ -positive maximal $\#$ -nonnegative subspace invariant, then \mathcal{A} turns out to be a C^* -algebra commuting with a metric operator with respect to some selfdual Hilbertian inner product. If \mathcal{A} leaves a $\#$ -positive (but not uniformly $\#$ -positive) maximal $\#$ -nonnegative subspace invariant, then, by choosing a selfdual pre-Hilbertian inner product on a dense subspace of \mathfrak{R} , \mathcal{A} is represented by a dense $*$ -subalgebra of a C^* -algebra commuting with a metric operator in the Hilbert space constructed by the completion. However, this representation is not continuous. If \mathcal{A} leaves a (not $\#$ -positive) maximal $\#$ -nonnegative subspace invariant, \mathcal{A} is represented in the form of a Tomita's triangular matrix.

Therefore there gives rise to an interesting problem: When has a Lorentz algebra an invariant maximal $\#$ -nonnegative subspace? For instance, $\mathcal{L}(\mathfrak{R})$ does not have any such invariant subspaces. From the preceding discussion, the Lorentz algebra which we can treat seems to be limited to subalgebras of

$$(5.1) \quad \mathcal{L}(\mathfrak{R}^+) \oplus \mathcal{L}(\mathfrak{R}^-) + \mathcal{L}\mathcal{C}(\mathfrak{R}),$$

although the latter does not have any invariant maximal $\#$ -nonnegative subspace. In the following we will give some examples of Lorentz algebras which have an invariant maximal $\#$ -nonnegative subspace.

The following theorem is a generalization of the Phillips-Naimark-Langer's theorem [13, 8, 7] on a Pontrjagin space to that on a Krein space. The proof will be omitted, for it is an immediate consequence of a Helton's theorem [3].

Theorem 5.1 ([11]). *Let \mathcal{A} be a unital commutative Lorentz subalgebra of $\mathcal{L}(\mathfrak{R}^+) \oplus \mathcal{L}(\mathfrak{R}^-) + \mathcal{L}\mathcal{C}(\mathfrak{R})$. If \mathcal{A} has a $\#$ -unitary u with*

$$u - \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in \mathcal{L}\mathcal{C}(\mathfrak{R}), \quad Sp(c) \cap Sp(d) = \emptyset,$$

then \mathcal{A} has an invariant maximal $\#$ -nonnegative subspace.

This theorem is restated as follows:

Theorem 5.2 ([11]). *The Lorentz algebra \mathcal{A} in Theorem 5.1 is of the form*

$$\left(\sum_{j=1}^n \oplus \mathcal{A}_{1j}\right) \oplus \left(\sum_{j=1}^m \oplus \mathcal{A}_{2j}\right) \oplus \mathcal{A}_3,$$

where

$$\mathcal{A}_{1j} \simeq \left\{ \left(\begin{array}{cccc} \mu_j(x) & & & \\ & \ddots & & \\ & & \mu_j(x) & \\ \dots & & & \dots \\ & & \overline{\mu_j(x)} & \\ & 0 & & * \\ & & 0 & \ddots \\ & & & & \overline{\mu_j(x)} \end{array} \right) : x \in \mathcal{A} \right\}$$

μ_j is a non real character ($\mu_j(x^\#) \neq \overline{\mu_j(x)}$);

$$\mathcal{A}_{2j} \simeq \left\{ \left(\begin{array}{ccc} \lambda_j(x) & & * \\ & \ddots & \\ & & \lambda_j(x) \end{array} \right) : x \in \mathcal{A} \right\}$$

λ_j is a real character ($\lambda_j(x^\#) = \overline{\lambda_j(x)}$); and

$\mathcal{A}_3 \simeq C(\Omega^+) \oplus C(\Omega^-)$, Ω^\pm are compact.

Applying this theorem to a single operator, we have immediately the following:

Corollary 5.3. *If \mathcal{A} is a Lorentz algebra in Theorem 5.1, then the (2, 2) element of a Tomita's triangular matrix for a $\#$ -unitary or a $\#$ -selfadjoint operator in \mathcal{A} is $\#$ -spectral.*

Since the Helton's theorem treated a non commutative set of operators, we can prove the similar assertions for slightly more general non commutative Lorentz algebras. For example,

Theorem 5.4. *Let u and v be strongly continuous one parameter bounded $\#$ -unitary groups which satisfy the commutation relation:*

$$u(s)v(t) = e^{ist}v(t)u(s) \quad s, t \in \mathbf{R}.$$

If $u(s)$ and $v(t)$ belong to $\mathcal{L}(\mathbb{R}^+) \oplus \mathcal{L}(\mathbb{R}^-) + \mathcal{L}\mathcal{C}(\mathbb{R})$ for all s and t , and if the Lorentz algebra \mathcal{A} generated by them contains a $\#$ -unitary w with

$$w - \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in \mathcal{L}\mathcal{C}(\mathbb{R}), \quad Sp(c) \cap Sp(d) = \phi,$$

then \mathcal{A} has an invariant maximal $\#$ -nonnegative subspace.

Remark. We can obtain examples of Lorentz algebras whose off diagonal components are not necessarily compact by using the above Helton's theorem as well as crossed product.

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