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Gevrey Well-Posedness of an Abstract Cauchy Problem of Weakly Hyperbolic Type

By

Piero D'ANCONA*

Abstract

We consider here a general second order Cauchy problem of hyperbolic type, with coefficients Hölder continuous in the time variable and of Gevrey class in the space variables, in the abstract setting of Hilbert spaces. Some global and local existence results are proved to hold.

Introduction

Let $(H, |\cdot|)$ be a Hilbert space, $\mathscr{B}=(B_1, \ldots, B_n)$ an n-tuple of closed commuting operators on H, with common domain V dense in H; V with the norm $\|v\|_V = |v| + \sum_{i=1}^n |B_i v|$ is a Banach space. The triplet (V, H, V') forms then a *Hilbert triplet*, the duality between V, V' being the extension of the scalar product of H.

We will consider here the abstract Cauchy problem in H

$$u'' + (A(t) + M(t))u = f(t), \qquad t \in [0, T] \tag{1}$$

$$u(0) = u_0, \ u'(0) = u_1 \tag{2}$$

(A represents a second-order operator, M a first-order one) where

$$A \in C^\kappa([0, T]; \mathscr{L}(V, V')), \ 0 < \kappa \leq 2 \tag{3}$$

$$\langle A(t)v, w \rangle = \overline{\langle A(t)w, v \rangle} \tag{4}$$

$$M \in L^1(0, T; \mathscr{L}(V, H)) \tag{5}$$

$$f \in L^1(0, T; H). \tag{6}$$

(We recall that C^κ denotes the Hölder continuous functions of exponent κ if $0 < \kappa < 1$, and the C^1 functions whose first derivative is Hölder continuous of exponent $\kappa - 1$ if $1 < \kappa < 2$).

Problem (1, 2) is said to be *weakly hyperbolic* if

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* Department of Mathematics, University of Pisa, Via Buonarroti 2, 56100 Pisa, Italy.

$$\langle A(t)v, v\rangle \geq 0 \tag{7}$$

and *strictly hyperbolic* if, for some $\nu>0$,

$$\langle A(t)v, v\rangle \geq \nu^2 \|v\|_V^2. \tag{8}$$

Our aim is to extend to this abstract setting some well-known results about global and local solvability of hyperbolic second order equations in Gevrey classes (in particular, the results in Jannelli [7] and Nishitani [10]; but see also [3], [4], [11]). In one respect, this work can be considered as a generalization of the results obtained in [1] (see also [13]) for an analogous problem in the class of analytic functions.

The *abstract Gevrey spaces* of order $s\geq 1$, generated by $\mathscr{B}$, are defined as

$$X_r^s(\mathscr{B}) = \{v \in \bigcap_\alpha D(\mathscr{B}^\alpha) : \|v\|_{r,s} < \infty\}, \quad r>0$$

where, employing the notations

$$|\mathscr{B}^j v| = (\sum_{|\alpha|=j} |\mathscr{B}^\alpha v|^2)^{1/2},$$

$$\mathscr{B}^\alpha = B_1^{\alpha_1} \circ \ldots \circ B_n^{\alpha_n}, \quad \alpha = (\alpha_1, \ldots, \alpha_n),$$

the norms $\|\cdot\|_{r,s}$ are defined by

$$\|v\|_{r,s} = \sup_{j\geq 0} |\mathscr{B}^j v| \frac{r^j}{j!^s}.$$

Obviously $X_r^s(\mathscr{B})$ is a Banach space, and $\{X_r^s(\mathscr{B})\}_{r>0}$ for fixed s is a Banach scale. The space of *Gevrey vectors* of order s will be

$$X_{0+}^s(\mathscr{B}) = \bigcup_{r>0} X_r^s(\mathscr{B})$$

endowed with the (locally convex) inductive limit topology. For more details about Banach and Gevrey scales, see [2] (where, however, the norms defining the Gevrey spaces are slightly different from the ones introduced here) and [5].

Remark 1. To fix the ideas, think of the following *Ω-periodic realization*: let Ω be a bounded open non-void parallelepiped in $\mathbf{R}^n$; set

$H = \{\Omega - \textit{periodic functions } v,\ v|_\Omega \in L^2(\Omega)\}$, *with* $L^2(\Omega)$ *norm*
$V = \{\Omega - \textit{periodic functions } v,\ v|_\Omega \in H^1(\Omega)\}$, *with* $H^1(\Omega)$ *norm*

$$\mathscr{B} \equiv \nabla \equiv (\partial_{x_1}, \dots, \partial_{x_n}).$$

It is then easy to check that

$$X^s_{0+}(\mathscr{B}) \equiv \{v \in \gamma^{(s)}: v \text{ is } \Omega\text{-periodic}\},$$

where $\gamma^{(s)}$ are the usual *Gevrey classes*, defined by

$$v \in \gamma^{(s)} \Leftrightarrow \forall K \subset\subset \mathbf{R}^n, \ \exists C = C_K, \ A = A_K \text{ such that}$$
$$\sup_K |\partial^\alpha v| \leq CA^{|\alpha|}(|\alpha|!)^s, \ \alpha \in \mathbf{N}^n$$

(see for details [2], in particular Proposition 6).

Remark 2. Let $f: [0, T] \to D(\mathscr{B}^\infty) \equiv \cap_\alpha D(\mathscr{B}^\alpha)$ be a function such that $\mathscr{B}^\alpha f$ is (strongly) H-measurable for all α, and for a suitable $c(t) \in L^1(0, T)$ independent of α the following inequality holds

$$|\mathscr{B}^\alpha f(t)| \leq c(t) \frac{|\alpha|!^s}{r^{|\alpha|}}$$

with fixed s, r. Then f is $X^s_{r'}(\mathscr{B})$-measurable for every $r' < r$. This property, which can be proved in the same way as Lemma A. 1 of [1], allows us to assume merely the H-measurability in the following.

Definition. An operator P is said to have *order* m in the Banach scale $\{X^s_r(\mathscr{B})\}_{r>0}$ with constants (C, Λ) if P maps $D(\mathscr{B}^\infty)$ into itself, and for every $v \in X^s_{0+}(\mathscr{B})$, $j \geq 0$

$$|\mathscr{B}^j Pv| \leq C(j+m)!^s \sum_{h=0}^{j+m} \frac{|\mathscr{B}^h v|}{h!^s} \Lambda^{j+m-h}. \tag{9}$$

As in [2] it can be shown that an operator of order m maps continuously $X^s_r(\mathscr{B})$ into $X^s_{r-\delta}(\mathscr{B})$, for $0 < \delta < r < 1/\Lambda$. We state now the assumptions connecting the Gevrey scale $\{X^s_r(\mathscr{B})\}_{r>0}$ with the coefficients A, M, f:

Assumption 1. The functions $\mathscr{B}^\alpha A(\cdot)v$, $\mathscr{B}^\alpha M(\cdot)v$, $\mathscr{B}^\alpha f$ are H-measurable for any $v \in X^s_{0+}(\mathscr{B})$; moreover, there exist two constants C, Λ and two functions $\mu, \chi \in L^1(0, T)$ such that

i) $A(t)$ has order 2 with constants (C, Λ);

ii) $M(t)$ has order 1 with constants $(\mu(t), \Lambda)$;

iii) $\|f(t)\|_{1/\Lambda, s} \leq \chi(t)$.

Assumption 2. For every t, $A(t)$ *quasi-commutes* with $\mathscr{B}$, that is to say

$$(\sum_{|\alpha|=j}|[\mathscr{B}^{\alpha}, A(t)]v|^{2})^{1/2}\leq C(j+2)(\sum_{|\alpha|=j}\langle A(t)\mathscr{B}^{\alpha}v, \mathscr{B}^{\alpha}v\rangle)^{1/2}$$
$$+C(j+2)!^{s}\sum_{h=0}^{j}\frac{|\mathscr{B}^{h}v|A^{j+2-h}}{h!^{s}(h+1)^{\sigma}(h+2)^{\sigma}},$$

where

$$\sigma=s-1$$

Some remarks about these assumptions. As it will be proved in Section 3, in the concrete case in which $\mathscr{B}=\nabla$, any differential operator of order m with coefficients of Gevrey class in the space variables, uniformly in time, satisfies the above Definition of operator of order m in the scale. On the other hand, Assumption 2 is satisfied by selfadjoint second order operators such as

$$A(t)=-\sum_{h,k}^{1,n}\partial_{x_k}(a_{hk}(x,t)\partial_{x_h}).$$

In the case $s>1$, as no local existence result is available for Problem (1, 2), we will use the following additional assumption:

Assumption 3. There exists a countable basis of H made of common eigenvectors for the operators $B_1, \ldots, B_n$.

We point out that this is indeed the case in the Ω-periodic realization defined in Remark 1. We can now state our results (but see also Remark 1.2):

Theorem 1. i) (*Weak hyperbolicity*) *If* (3), (4), (5), (6), (7) *and Assumptions* 1, 2, 3 *are fulfilled, then Problem* (1, 2) *is globally solvable in* $X^s_{0+}(\mathscr{B})$ *for* $1\leq s<1+\kappa/2$, *in the sense that for every* u_0, $u_1\in X^s_{r_0}(\mathscr{B})$ *with* $r_0<1/A$, *there exists an unique solution* u *belonging to* $C^1([0,T]; X^s_{\bar{r}}(\mathscr{B}))$ *for some* $\bar{r}=\bar{r}(T)\in]0,r_0]$. *Moreover, if* $s=1+\kappa/2$, *then Problem* (1, 2) *is locally solvable in an analogous sense.*

ii) (*Strict hyperbolicity*) *Suppose that* $\kappa<1$. *If the same assumptions as in case* i) *hold but with* (7) *replaced by* (8), *then Problem* (1, 2) *is globally solvable in* $X^s_{0+}(\mathscr{B})$ *for* $1\leq s<1/(\kappa-1)$, *and locally solvable for* $s=1/(\kappa-1)$. *In particular, if* $A\in\mathrm{Lip}([0,T];\mathscr{L}(V,V'))$ *then Problem* (1, 2) *is globally solvable in* $X^s_{0+}(\mathscr{B})$ *for every* $s\geq 1$.

In the final section, Theorem 1 is applied to the study of the Cauchy problem

$$u_{tt}=\sum_{i,j}^{1,n} a_{ij}(x,t)u_{x_ix_j}+\sum_{j=1}^{n} b_j(x,t)u_{x_j}+c(x,t)u_t+d(x,t)u+f(x,t)$$
$$(x,t)\in\mathbf{R}^n\times[0,T] \qquad \text{(C)}$$
$$u(x,0)=\phi(x),\ u_t(x,0)=\psi(x)$$

under the following assumptions

$$\nu^2|\xi|^2\leq\sum a_{ij}\xi_i\xi_j\leq\nu_1^2|\xi|^2,\ \xi\in\mathbf{R}^n,\ \nu\geq 0;$$
$$a_{ij}=\overline{a_{ji}};$$
$$a_{ij}\in C^{\kappa}([0,T];\gamma^{(s)}),\ 0<\kappa\leq 2;$$
$$b_j,c,d,f\in L^1(0,T;\gamma^{(s)});$$
$$\phi,\psi\in\gamma^{(s)}.$$

If one assumes some suitable boundary conditions, namely periodicity in the x-variable, the preceding problem becomes an immediate consequence of Theorem 1. But it is also possible to re-obtain a theorem already proved by Jannelli [7] and Nishitani [10]:

Theorem 2. i) (*Weak hyperbolicity*) *Problem* (C) *is globally solvable in* $\gamma^{(s)}$ *if* $1<s<1+\kappa/2$, *in the sense that there exists an unique solution in* $W^{1,2}([0,T];\gamma^{(s)})$; *it is locally solvable for* $s=1+\kappa/2$.

ii) (*Strict hyperbolicity*) *Suppose that* $\nu>0$ *and* $\kappa<1$. *Then Problem* (*C*) *is globally solvable in* $\gamma^{(s)}$ *if* $1<s<1/(1-\kappa)$ *and locally solvable if* $s=1/(1-\kappa)$. *Moreover, if the coefficients of the second order term are Lipschitz continuous in the time variable, then Problem* (*C*) *is globally solvable in* $\gamma^{(s)}$ *for every* $s>1$.

Remark 3. As it was proved in [4], Theorem 2 and, a fortiori, Theorem 1, are optimal in the following sense: fixed κ and $s>1+\kappa/2$, there exist a nonnegative function $a(t)\in C^{\kappa}([0,T])$, and a pair of initial data $\phi,\psi\in\gamma^{(s)}$ such that for the Cauchy problem

$$u_{tt}=a(t)u_{xx} \qquad \text{on } \mathbf{R}\times[0,T]$$
$$u(x,0)=\phi(x)$$
$$u_t(x,0)=\psi(x)$$

even the local existence fails to hold; analogous counterexamples were found for the strictly hyperbolic case.

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§ 1. Sketch of the Proof of Theorem 1

We prove here some a-priori estimates of the "infinite order energy" of the solution (see [3], [1]). We will perform the proof only in the case $n=1$, namely, $\mathscr{B}=B$ (one single generating operator), adding a final remark for the general case. Note that, thanks to the substitution

$$\mathscr{B}_{\Lambda}=\mathscr{B}/\Lambda$$

and to the homogeneity of all the equations, we can suppose that $\Lambda=1$.

§ 1. 1 (Weak hyperbolicity). Suppose the assumptions of Theorem 1. i are fulfilled; moreover, assume $\kappa\leq 1$ (the case $1<\kappa\leq 2$ is considered successively). Let $\phi\in C_0^\infty(\mathbf{R})$, $0\leq\phi\leq 1$, $\phi(t)=1$ for $|t|\leq 1/2$, and $\phi=0$ for $|t|\geq 1$. Let $\{\delta_j\}$ be a decreasing sequence of positive numbers (to be chosen), and pose

$$\phi_j(t)=\phi\left(\frac{t}{\delta_j}\right)\left(\int_{\mathbf{R}}\phi\left(\frac{\tau}{\delta_j}\right)d\tau\right)^{-1}$$

Now if we extend $A(t)$ to all of $\mathbf{R}$ as

$$\begin{array}{ll} A(T) & \text{for } t>T \\ A(0) & \text{for } t<0 \end{array}$$

we can define the convolutions (performed in the H-norm)

$$A_j(t)=A*\phi_j(t).$$

From the κ-Hölder continuity of A it follows that, for some constant η depending on ϕ, A

$$\|A-A_j\|\leq\eta\delta_j^\kappa$$
$$\|A_j'\|\leq 2\eta\delta_j^{\kappa-1}$$

where the norms are in $L^\infty(0,T;\mathscr{L}(V,V'))$ (and the prime denotes a time derivative). Choosing $\delta_j=j^{-1}$ we obtain

$$\|A-A_j\|\leq\eta j^{-\kappa}$$
$$\|A_j'\|\leq 2\eta j^{1-\kappa}.$$

Suppose now u is a solution of (1). We define the *j-th order energy* E_j of u as

$$E_j^2 = \langle A_j B^{j-1}u, B^{j-1}u\rangle + j^2 |B^{j-1}u|^2 + j^{-\kappa}|B^j u|^2 + |B^{j-1}u'|^2.$$

Immediate consequences of the definition of E_j are

$$\|B^{j-1}u\|_V \le j^{\kappa/2}E_j,\quad \|B^{j-1}u'\|_V \le E_j + E_{j+1},\quad |B^{j-1}u| \le j^{-1}E_j$$
$$|B^j u| \le j^{\kappa/2}E_j,\quad |B^{j-1}u'| \le E_j,\quad |B^j u'| \le E_{j+1}$$

while from Assumption 1 it follows that

$$|B^{j-1}Mu| \le \mu(t) j!^s \sum_{h=0}^{j} \frac{|B^h u|}{h!^s} \le \mu(t) j!^s j^{\kappa/2} \sum_{h=1}^{j} \frac{E_h}{h!^s}$$
$$|B^{j-1}f| \le \chi(t)(j-1)!^s.$$

Now, derivating E_j^2, applying the above inequalities and equation (1), and dividing by $2E_j$, we obtain

$$E_j' \le C_1\left[j^{-\kappa/2}E_{j+1} + jE_j + \mu(t) j!^s j^{\kappa/2} \sum_{h=1}^{j} \frac{E_h}{h!^s}\right] + \chi(t)(j-1)!^s + |[A, B^{j-1}]u|$$

where $C_1 = 2\eta + 1$. Applying now the commutator estimate of Assumption 2 and the inequalities $|B^h u| \le (h+1)^{-1}E_h$, $j^{\kappa/2} \le j+1 = (j+1)^s/(j+1)^\sigma$ we obtain

$$E_j' \le C_2\left[j^{-\kappa/2}E_{j+1} + \mu_1(t)(j+1)!^s \sum_{h=1}^{j} \frac{E_h}{h!^s (h+1)^\sigma}\right] + \chi(t)(j-1)!^s$$

where $\mu_1(t) = \mu(t) + 1$ and $C_2 = \max\{C, 2(C + C_1 + C\sqrt{C_1})\}$. We define now the Gevrey type *infinite order energy* of u as

$$\mathscr{E}(t) \equiv \mathscr{E}(\rho, u; t) \equiv \sum_{j\ge1} \frac{\rho^{j+1}}{(j+1)!^s} E_j$$

where $\rho \in AC([0, T])$ is a real function between 0 and 1 (to be chosen). Derivating $\mathscr{E}$ and using the above estimate we obtain

$$\mathscr{E}' \le C_2 \sum_{j\ge1} \frac{\rho^j}{j!^s} \frac{E_j}{(j+1)^\sigma}\left(\frac{(j+1)^\sigma}{(j-1)^{\kappa/2}} + \mu_1(t)\frac{\rho}{1-\rho} + \rho'\right) + \chi(t)$$

where we have applied a Fubini-like argument to derive the equality

$$\sum_{j\ge1} \rho^{j+1} \sum_{h=1}^{j} \frac{E_h}{h!^s (h+1)^\sigma} = \frac{\rho}{1-\rho} \sum_{j\ge1} \frac{\rho^j}{j!^s} \frac{E_j}{(j+1)^\sigma}.$$

We proceed now to the choice of ρ. For $\epsilon > 0$ small enough, the ordinary Cauchy problem

$$\rho' + \mu_1(t)\frac{\rho}{1-\rho} + \epsilon = 0, \qquad t \in [0, T]$$
$$\rho(0) = r_0, \qquad 0 < r_0 < 1$$

has a positive decreasing solution on $[0, T]$. In the case s is strictly less than $1 + \kappa/2$, for $j > j_\epsilon$ we have

$$\frac{(j+1)^\sigma}{(j-1)^{\kappa/2}} < \epsilon$$

and hence, by the above choice of ρ, the terms of the series are negative for $j > j_\epsilon$, so that

$$\mathscr{E}'(t) \le C(\eta, C, \mu, \kappa, s, r_0)\, \mathscr{E}(t) + \chi(t)$$

In the case $s = 1 + \kappa/2$, on the other hand, we choose an $\epsilon > (j+1)^{\kappa/2}/(j-1)^{\kappa/2}$ for all j; the resulting ρ will be a positive decreasing function on $[0, T^*]$ for some $T^*(\mu_1, r_0) > 0$. The terms of the series will be all less or equal to 0 on that interval, so that

$$\mathscr{E}'(t) \le \chi(t) \quad \text{on } [0, T^*].$$

In both cases, an application of Gromwall's Lemma yields the following fundamental a-priori estimate:

$$\mathscr{E}(t) \le C(\eta, C, \mu, \kappa, s, r_0)\, \mathscr{E}(0) + \|\chi\|_{L^1}$$

valid on $[0, T]$ if $s < 1 + \kappa/2$, and on $[0, T^*]$ with T^* depending on μ, r_0 if $s = 1 + \kappa/2$.

To deal with the case $1 < \kappa \le 2$, a different method is needed. We use the following lemma of real analysis (a proof can be found in [7]):

Let $\phi(t) \in C^\kappa([0, T])$ be a non-negative function, $1 \le \kappa \le 2$. Then

$$\|\phi^{1/\kappa}\|_{\mathrm{Lip}} \le c(\kappa)\|\phi\|_{C^\kappa}^{1/\kappa}.$$

Defining now the j-th order energy E_j of u as

$$E_j^2 = \langle AB^{j-1}u, B^{j-1}u\rangle + j^2|B^{j-1}u|^2 + j^{-\kappa}|B^j u|^2 + |B^{j-1}u'|^2$$

we proceed exactly as in the above proof, the only difference being that, when estimating E_j', the term $\langle A'B^{j-1}u, B^{j-1}u\rangle$ is estimated by $c(\kappa)jE_j$, by applying the Lemma to $\phi(t) = \langle A(t)v, v\rangle$, v a fixed element of V.

§ 1. 2 (Strict hyperbolicity). Suppose the assumptions of Theorem 1.ii are satisfied. The proof is similar to the weakly hyperbolic case, only the following modifications are needed:

a) δ_j is chosen as $j^{1/(\kappa-1)}(\nu^2/(4\eta))^{1/\kappa}$;

b) the j-th order energy is defined as

$$E_j^2 = \langle A_j B^{j-1}u, B^{j-1}u\rangle + j^2|B^{j-1}u|^2 + |B^{j-1}u'|^2.$$

The a-priori estimates obtained are the same as in the first part of the proof, exception made for the fact that the limit value for s is now $1/(1-\kappa)$.

Remark 1. 1. To reproduce the proof in the general case in which $\mathscr{B}$ is composed of n operators $B_1, \ldots, B_n$, a number of minor modifications is needed. The j-th order energy must be defined as (in the weakly hyperbolic case)

$$E_j^2 = \sum_{|\alpha|=j-1} \langle A_j \mathscr{B}^\alpha u, \mathscr{B}^\alpha u\rangle + j^2|\mathscr{B}^{j-1}u|^2 + j^{-\kappa}|\mathscr{B}^j u|^2 + |\mathscr{B}^{j-1}u'|^2$$

and an analogous substitution is to be made in the strictly hyperbolic case. The computations are an almost word-by-word repetition of the above proof (replace everywhere B^j by $\mathscr{B}^j$, with the meaning defined in the Introduction).

§ 1. 3 (Conclusion of the proof). Using the a-priori estimates, it is not difficult to complete the proof of Theorem 1 by a standard Faedo-Galerkin argument. Assumption 3 ensures the existence of an increasing sequence of projections P_N in H, strongly converging to the identity map, commuting with B_j, $j=1, \ldots, n$, and with finite dimensional image V_N. Since for every v in V_N

$$||v||_{r,s} = \sup_{j\geq 0} \frac{|\mathscr{B}^j v|}{j!^s} r^j \leq |v| \sup_{j\geq 0} \Big(\sum_{h=1,n} ||B_h v||^2_{V_N}\Big)^{j/2} \frac{r^j}{j!^s} < \infty,$$

we have $V_N \subseteq X_r^s(\mathscr{B})$ for every s and r. Define now

$$A_N = P_N A, \qquad M_N = P_N M, \qquad f_N = P_N f.$$

It is easy to see that A_N, M_N, f_N verify Assumptions 1, 2 with the same constants as A, M, f. The Cauchy problem

$$u_N'' + (A_N(t) + M_N(t))u = f_N(t), \qquad t \in [0, T] \tag{10}$$

$$u(0) = P_N u_0, \; u'(0) = P_N u_1 \tag{11}$$

is globally solvable, since V_N is a finite-dimensional space; moreover, fixed the initial data

$$u_0, u_1 \in X^s_{r_0}(\mathscr{B}) \quad r_0 \in]0, 1[$$

the a-priori estimates of the preceding sections furnish a uniform bound for the energies $\mathscr{E}_N$ of the solution u_N of Problem (10, 11) (note that, if v is in V_N, then $|\mathscr{B}^\alpha v| = O(c^{|\alpha|})$). Naturally, the bound holds on $[0, T]$ or on $[0, T^*]$ according to which case is under consideration. By a compactness argument, the result follows.

***Remark* 1.2.** We point out that, with minor modifications in the proof, the same result can be proved for an equation of the type

$$u'' + (A(t) + M(t))u + Q(t)u' = f(t)$$

where $Q(t)$ is an operator of order 0 in the scale, with constants $(\mu(t), \Lambda)$ and $\mu \in L^1(0, T)$

§2. Estimates of Partial Differential Operators

We show in this section that concrete partial differential operators satisfy Assumptions 1, 2. To this end, we need the following Lemma, which is an easy consequence of Lemma A. 3 of [1]:

Lemma 2.1. *Let $K > 1$, and $\{x_\beta\}$ a sequence of non-negative real numbers, indicized by $\beta \in \mathbf{N}^n$. Then, for every integer j,*

$$\Big[\sum_{|\alpha|=j} \Big(\sum_{\substack{\beta\leq\alpha \\ |\beta|\leq l}} x_\beta\Big)^2\Big]^{1/2} \leq C(n, K) \sum_{r=0}^{l} K^{j-r} \Big(\sum_{|\beta|=r} x_\beta^2\Big)^{1/2}.$$

We begin by proving Assumption 2 for a self-adjoint second order operator.

Lemma 2.2. *Let Ω be an open non-void subset of $\mathbf{R}^n$, and a_{hk}, $h, k = 1, \dots, n$ continuous functions on $[0, T]$ with values in $\gamma^{(s)}$, such that*

$$\begin{aligned} &\text{i)} && a_{hk} = \overline{a_{kh}} \\ &\text{ii)} && \textstyle\sum a_{hk}\xi_h\xi_k \geq 0, \quad \xi \in \mathbf{R}^n \\ &\text{iii)} && |\partial^\alpha a_{hk}(x, t)| \leq M\Lambda_0^{|\alpha|}(|\alpha|!)^s \quad \text{on } \Omega\times[0, T] \end{aligned} \tag{12}$$

(where, as usual, $\partial^\alpha \equiv \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$) for some M, Λ_0 independent of α. Denote

by $A(t)$ *the operator*

$$-\sum_{h,k}^{1,n} \partial_{x_h}(a_{hk}(x,t)\,\partial_{x_k}).$$

Then, fixed an arbitrary $\Lambda > \Lambda_0$, *there exists a constant* $C = C(n, M, \Lambda_0, \Lambda)$ *such that for every* $v \in H^\infty(\Omega)$

$$\Big(\sum_{|\alpha|=j} \|[A(t), \partial^\alpha]v\|^2\Big)^{1/2} \le C(j+2)\Big(\sum_{|\alpha|=j} (A(t)\partial^\alpha v, \partial^\alpha v)\Big)^{1/2}$$

$$C(j+2)!^s \sum_{h=0}^{j}\Big(\sum_{|\beta|=h} \|\partial^\beta v\|^2\Big)^{1/2} \frac{\Lambda^{j+2-h}}{h!^s (h+1)^\sigma (h+2)^\sigma}$$

where $\sigma = s-1$, *and* $\|\cdot\|$, $(\ ,\ )$ *denote the norm and the scalar product in* $L^2(\Omega)$.

Proof. Fixed α, and denoting by $e_1, \ldots, e_n$ the canonical base of $\mathbf{R}^n$, it is readily seen that

$$[A, \partial^\alpha]u = I_\alpha + II_\alpha + III_\alpha$$

where

$$I_\alpha = \sum_{h,k} \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial^{\alpha+e_h-\beta} a_{hk}\, \partial^{\beta+e_k} u$$

$$II_\alpha = \sum_{h,k} \sum_{\substack{\beta<\alpha \\ |\beta|<|\alpha|-2}} \binom{\alpha}{\beta} \partial^{\alpha-\beta} a_{hk}\, \partial^{\beta+e_h+e_k} u$$

$$III_\alpha = \sum_{h,k} \sum_{\substack{\beta<\alpha \\ |\beta|=|\alpha|-1}} \binom{\alpha}{\beta} \partial^{\alpha-\beta} a_{hk}\, \partial^{\beta+e_h+e_k} u$$

and we will estimate the three terms separately. Using (12. iii) and the fact that $|\alpha+e_h-\beta| = j+1-|\beta|$, $\binom{\alpha}{\beta} \le \binom{|\alpha|}{|\beta|}$, we have in a few passages

$$\Big(\sum_{|\alpha|=j} \|I_\alpha\|^2\Big)^{1/2} \le M\Big[\sum_{|\alpha|=j}\Big(\sum_{h,k}\sum_{\beta<\alpha} \binom{j}{|\beta|}(j+1-|\beta|)!^s \Lambda_0^{j+1-|\beta|} \|\partial^{\beta+e_k} u\|\Big)^2\Big]^{1/2}.$$

Now apply Lemma 2. 1 with $l = j-1$, $K = \Lambda/\Lambda_0 > 1$ and

$$x_\beta = \sum_{h,k} \binom{j}{|\beta|}(j+1-|\beta|)!^s \Lambda_0^{j+1-|\beta|} \|\partial^{\beta+e_k}\|$$

to obtain

$$\Big(\sum_{|\alpha|=j} \|I_\alpha\|^2\Big)^{1/2} \le C \sum_{\nu=1}^{j} \binom{j}{\nu-1}(j+2-\nu)!^s \Lambda^{j+2-\nu}\Big(\sum_{|\beta|=\nu} \|\partial^\beta u\|^2\Big)^{1/2}.$$

The terms II_α yield an analogous inequality, with $\binom{j}{\nu-2}$ instead of $\binom{j}{\nu-1}$. Sum up, observing that

$$\binom{j}{\nu-1}+\binom{j}{\nu-2}=\binom{j+1}{\nu-1}$$

and that

$$\binom{j+1}{\nu-1}(j+2-\nu)!^s\leq\binom{j+1}{\nu-1}^{-\sigma}\frac{(j+2)!^s}{\nu!^s}\leq\frac{\text{const.}}{(\nu+1)^\sigma(\nu+2)^\sigma}\frac{(j+2)!^s}{\nu!^s}$$

when $\nu\leq j$.

Finally, to estimate the terms III_α, apply the following Lemma due to O. Oleinik (see [12], Lemma 4; see also [6]):

Let $(a_{hk}(x))$ be a hermitian non-negative matrix of functions in $W^{2,\infty}(\Omega)$. Then for every $n\times n$ symmetric matrix (ξ_{hk}), for $j=1,\dots,n$

$$\Big(\sum_{h,k}\partial_{x_j}a_{hk}(x)\xi_{hk}\Big)^2\leq C_1(n)C_2(a_{hk})\sum_{h,k,q}a_{hk}(x)\xi_{hq}\overline{\xi_{kq}}$$

where C_2 is the $W^{2,\infty}$ norm of the a_{hk}.

With this Lemma in mind, it is not difficult to see that (taking $\xi_{hk}=\partial^\gamma u$ with $\gamma=\alpha-e_j-e_h-e_k$)

$$\Big(\sum_{|\alpha|=j}||III_\alpha||^2\Big)^{1/2}\leq Cj\Big[\sum_{|\alpha|=j}(A(t)\partial^\alpha u,\partial^\alpha u)\Big]^{1/2}.$$

Lemma 2.3. *With the same notations of Lemma 2.2, let*

$$P=\sum_{|\gamma|\leq m}a_\gamma(x,t)\partial^\gamma$$

be a partial differential operator on Ω, with measurable coefficients, infinitely differentiable in the x-variable, and such that, for a $\mu\in L^1(0,T)$ and a $\Lambda_0>0$

$$|\partial^\alpha a_\gamma|\leq\mu(t)\Lambda_0^{|\alpha|}(|\alpha|!)^s. \tag{13}$$

Then, for any $\Lambda>\Lambda_0$, there exists a constant $C=C(n,\Lambda,\Lambda_0)$ such that for any v in $H^\infty(\Omega)$

$$\Big(\sum_{|\alpha|=j}||\partial^\alpha Pv||^2\Big)^{1/2}\leq C\mu(t)(j+m)!^s\sum_{h=0}^{j+m}\frac{\Lambda^{j+m-h}}{h!^s}\Big(\sum_{|\beta|=h}||\partial^\beta v||^2\Big)^{1/2}.$$

Proof. It is obviously sufficient to prove the Lemma in the case P is composed of one single term of degree m. The proof is similar to the estimates of terms I_α and II_α in the preceding Lemma; after having applied (13), just observe that

$$\binom{j}{\nu-m}(j+m-\nu)!^s=\binom{j}{\nu-m}^{-\sigma}\frac{j!^s}{(\nu-m)!^s}\leq\frac{(j+m)!^s}{\nu!^s}.$$

§ 3. Applications

We will apply in this section the abstract theory developed so far to the study of the Cauchy problem

$$u_{tt}=\sum_{i,j}^{1,n} a_{ij}(x,t)u_{x_ix_j}+\sum_{j=1}^{n} b_j(x,t)u_{x_j}+c(x,t)u_t+d(x,t)u+f(x,t)$$
$$(x,t)\in\mathbf{R}^n\times[0,T] \tag{14}$$
$$u(x,0)=\phi(x),\ u_t(x,0)=\psi(x),\ x\in\mathbf{R}^n$$

under the following assumptions

$$\begin{aligned}&\nu^2|\xi|^2\le\sum a_{ij}\xi_i\xi_j\le\nu_1^2|\xi|^2,\ \xi\in\mathbf{R}^n,\ \nu\ge 0;\\ &a_{ij}=\overline{a_{ji}};\\ &a_{ij}\in C^\kappa([0,T];\gamma^{(s)}),\ 0<\kappa\le 2;\\ &b_j,c,d,f\in L^1(0,T;\gamma^{(s)});\\ &\phi,\psi\in\gamma^{(s)}.\end{aligned} \tag{15}$$

The following result is proved:

Theorem 3.1. (Jannelli [7], Nishitani [10]) i) (*Weak hyperbolicity*) *Problem* (14) *is globally solvable in* $\gamma^{(s)}$ *if* $1<s<1+\kappa/2$, *in the sense that there exists an unique solution in* $W^{1,2}([0,T];\gamma^{(s)})$; *it is locally solvable for* $s=1+\kappa/2$.

ii) (*Strict hyperbolicity*) *Suppose that* $\nu>0$ *and* $\kappa<1$. *Then Problem* (14) *is globally solvable in* $\gamma^{(s)}$ *if* $1<s<1/(1-\kappa)$ *and locally solvable if* $s=1/(1-\kappa)$. *Moreover, if the coefficients of the second order term are Lipschitz continuous in the time variable, then Problem* (14) *is globally solvable in* $\gamma^{(s)}$ *for every* $s>1$.

Proof. As the proofs of the various cases are analogous, we will limit ourself to the weakly hyperbolic case, $s<1+\kappa/2$. Define in a natural way $A(t)$ as in Lemma 2.2, and

$$M=-\sum_j b_j(x,t)\partial_{x_j}+\sum_{k,j}(\partial_{x_k}a_{kj})\partial_{x_j}-d(x,t)$$
$$Q=-c(x,t).$$

With these definitions, Problem (14) looks formally like the problem considered in Remark 1.2. We will solve it by reduction of an analogous Ω-periodic problem. First of all, suppose we are in the following situation: let Ω be a bounded open non-void parallelepiped in $\mathbf{R}^n$, and suppose that

$\phi, \psi, a_{ij}, b_j, c, d, f$ are Ω-periodic in the space variables;

then global existence and uniqueness are an immediate consequence of Theorem 1. In fact, choose V, H and $\mathscr{B}$ as in the Ω-periodic realization of Remark 1; assumptions (3)-(7) follow from (15), while Assumptions 1, 2 are the abstract counterpart of Lemmas 2. 3, 2. 2.

Suppose now

f and the initial data vanish for $|x-x_0|>r$

(no assumption of periodicity about the coefficients). Owing to the finite speed of propagation (see the Appendix), which incidentally guarantees the uniqueness for Problem (14) under no restrictive assumptions, we can reduce this case to the periodic one in the following way: choose a $\chi(x)$ in $\gamma^{(s)}$ with compact support such that $\chi=1$ on $|x-x_0|\leq r+\nu_1 T$, and multiply all the coefficients of Problem (14) by χ. Then fix an open bounded parallelepiped Ω containing supp χ, and extend data and coefficients to the outside of Ω by Ω-periodicity. The problem thus obtained admits a solution $\tilde{u}$, with the property that $\tilde{u}|_\Omega(x,t)=0$ if $\chi(x)\neq 1$. If now we define a function u as equal to $\tilde{u}$ for $x\in\Omega$ and 0 outside, we obtain a solution of the original problem.

To conclude the proof, it is sufficient to choose a locally finite partition of unity $\{\chi_\alpha\}$ in $\gamma^{(s)}$, solve the "localized" problems obtained by multiplying f and the data by each χ_α, and finally sum up the solutions thus obtained.

Remark 3. 1. For a different proof of the same result, see [10] (where Problem (14) was for the first time considered in its full generality, with an additional assumption of continuity with respect to time of the lower order terms).

As a last application, we will consider the following Cauchy problem of super-kowalewskian type:

$$\begin{aligned} &u_{tt}+c(t)\Delta^2 u+\sum_{i,j}^{1,n} m_{ij}(t)u_{x_i x_j}=f(x,t) \qquad (x,t)\in\mathbf{R}^n\times[0,T] \\ &u(x,0)=\phi(x), \qquad u_t(x,0)=\psi(x). \end{aligned} \tag{16}$$

The assumptions will be the following: let Ω be an open bounded non-void parallelepiped in $\mathbf{R}^n$, and denote with $\gamma_{\text{per}}^{(s)}$ the class of Ω-periodic Gevrey functions of index $s>0$; then

$$\begin{aligned}&c(t)\in C^{\kappa}([0,T]), \qquad 0<\kappa\leq 2\\ &c(t)\geq\nu\geq 0\\ &f(x,t)\in L^1(0,T;\gamma_{\text{per}}^{(s)})\\ &m_{ij}(t)\in L^1(0,T).\end{aligned}\tag{17}$$

Theorem 3.2. i) (*Weak hyperbolicity*) *Under assumptions* (17), *Problem* (16) *is globally solvable in* $\gamma_{\text{per}}^{(s)}$ *for* $1\leq 2s<1+\kappa/2$, *and locally solvable if* $2s=1+\kappa/2$.

ii) (*Strict hyperbolicity*) *Assume* $\nu>0$, $\kappa<1$. *Then Problem* (16) *under assumptions* (17) *is globally solvable in* $\gamma_{\text{per}}^{(s)}$ *for* $1\leq 2s<1/(1-\kappa)$, *and locally solvable if* $2s=1/(1-\kappa)$.

Proof. In the notations of Theorem 1, choose $\mathscr{B}=\Delta$ (one single operator), H and V as in the Ω-periodic realization of Remark 1. A simple application of the Theorem on elliptic iterates (see [9], vol. III, chapter 8, Theorem 1.2) shows that $X_{0+}^{s}(\mathscr{B})=\gamma_{\text{per}}^{(s/2)}$. The operator $A(t)=c(t)\Delta^2$ is evidently of second order in the scale, while $M(t)=\sum m_{ij}\partial_{x_i}\partial_{x_j}$ is of first order (use Fourier series development). A simple application of Theorem 1 allows us to conclude the proof.

Appendix

By sake of completeness, we furnish here a proof of the "finite speed of propagation" property, for a weakly hyperbolic equation of type (14).

Lemma. *Let* B_r *be an open ball in* $\mathbf{R}^n$ *with center* x_0 *and radius* r. *Suppose* a_{ij}, b_j, c, d *are functions satisfying* (15) *on* $B_r\times[0,T]$, $T>0$, *and let* $u\in C^1([0,T];\ C^2(B_r))$ *be a solution of*

$$u_{tt}=\sum_{ij}a_{ij}u_{x_i}u_{x_j}+\sum_j b_ju_{x_j}+cu_t+du$$

on $B_r\times[0,T]$. *If*

$$u(x,0)=u_t(x,0)=0 \text{ on } B_r$$

then

$$u=0 \text{ on } \Gamma(\nu_1, B_r) \equiv \{(x,t): t\in[0,T],\ |x-x_0|\leq r-\nu_1 t\}$$

(the cone with base B_r *and slope* $1/\nu_1$*).*

Proof. Let r' be an arbitrary element of $]0, r[$ and denote by ω the open ball with center x_0 and radius r'. We will prove that $u=0$ on $\Gamma(\nu_1, \omega)$, whence the result will follow. Let Ω be a bounded open parallelepiped in $\mathbf{R}^n$ containing $\overline{B_r}$. Let χ be a Gevrey function with support in B_r and equal to 1 in ω. Denote by a tilde multiplication by χ and extension by Ω-periodicity. Then we will have

$$\begin{aligned} &\tilde{u}_{tt}-(\sum \tilde{a}_{ij}\tilde{u}_{x_i x_j}+\sum \tilde{b}_j\tilde{u}_{x_j}+\tilde{c}\tilde{u}_t+\tilde{d}\tilde{u})=g(x,t) \\ &\tilde{u}(x,0)=\tilde{u}_t(x,0)=0 \end{aligned} \tag{A. 1}$$

for some function g vanishing on $\omega\times[0,T]$. We now approximate this Ω-periodic problem with a sequence of strictly hyperbolic ones, satisfying assumptions (15) (with constants converging to the constants appearing in (15)) and coefficients Lipschitz continuous in time (the second order terms) or continuous (the lower order terms): this is easily achieved by convolving with a sequence of Friedrichs mollifiers, then adding $\epsilon^2\delta_{ij}$ to the second order term. In particular, we have

$$\sum a^{\epsilon}_{ij}\xi_i\xi_j\leq(\nu_1+\epsilon)^2|\xi|^2.$$

As initial data we choose the null ones. From the first part of Theorem 3. 1 it follows that these approximating periodic problems are uniquely solvable, and their solutions u^{ϵ} satisfy an uniform estimate in $C^1([0,T];\gamma^{(s)})$. We can thus suppose that they converge uniformly to a function, which is, of course, a solution of (A. 1); hence they converge uniformly to $\tilde{u}$. But now observe that

$$u^{\epsilon}=0 \text{ on } \Gamma(\nu_1+\epsilon, \omega)$$

as the finite speed of propagation holds for strictly hyperbolic equations, so that

$$\tilde{u}=0 \text{ on } \Gamma(\nu_1, \omega).$$

To conclude, observe that on $\omega\times[0,T]$

$$\tilde{u}(x,t)=\chi(x)u(x,t)=u(x,t).$$

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