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Involutive System of Effectively Hyperbolic Operators

By

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§ 1. Introduction

It is well known that for strictly hyperbolic operators one has energy estimate which is stable under a perturbation of lower order terms. Then it is clear that for a system of strictly hyperbolic operators the Cauchy problem is solved in C^∞ for any lower order terms. It is also known that the Cauchy problem for effectively hyperbolic operators is solved in C^∞ regardless of any lower order terms although energy estimate of those operators (measured in the usual Sobolev norms) essentially depends on lower order terms (see [4], [6], [8], [9]). Therefore in this paper we are interested in the same problem for a system of effectively hyperbolic operators. We shall show that involutive system of effectively hyperbolic operators (the sense will be clarified in the following) has the same property.

Let U be an open set in $\boldsymbol{R}^d$ with coordinates $x'=(x_1, \cdots, x_d)$. Denote by $(x', \xi')=(x_1, \cdots, x_d, \xi_1, \cdots, \xi_d)$ standard coordinates in the cotangent bundle T^*U. Let I be an open interval containing the origin and put $\Omega=I\times U$. We denote by $(x, \xi)=(x_0, x', \xi_0, \xi')$ standard coordinates in $T^*\Omega$ and

$$D_j=-i\partial/\partial x_j,\quad j=0, 1, \cdots, d,\quad D=(D_0, D'),\quad D'=(D_1, \cdots, D_d).$$

Let

$$(1.1)\qquad P^i(x, D)=-D_0^2+2A^i(x, D')D_0-B^i(x, D'),\ i=1, 2, \cdots, l$$

be differential operators in D_0 of order 2 with coefficients $A^i(x,D')$,

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$B^i(x, D')$ which are classical pseudodifferential operators of order 1, 2 respectively defined in a conic neighborhood of $(\hat{x}, \hat{\xi}') = (0, \hat{x}', \hat{\xi}') \in I \times (T^*U \backslash 0)$. We are concerned with the following microlocal Cauchy problem

$$(1.2) \qquad \begin{aligned} &P^i(x, D)u^i = \sum_{j=1}^{l} \sum_{k=0}^{1} C_{ijk}(x, D') D_0^{1-k} u^j + f^i, \\ &f^i = 0 \text{ in } x_0 < 0, \ 1 \leqq i \leqq l \end{aligned}$$

where $C_{ijk}(x, D')$ are classical pseudodifferential operators of order k defined near $(\hat{x}, \hat{\xi}')$. Sometimes we write (1.2) in a more concise form;

$$(1.2)' \qquad P(x, D)u = f, \quad f = 0 \text{ in } x_0 < 0$$

with $P(x, D) = \mathrm{diag}(p^1(x, D), \cdots, p^l(x, D)) + B_0(x, D')D_0 + B_1(x, D')$, $u = (u^1, \cdots, u^l)$, $f = (f^1, \cdots, f^l)$. Here $p^i(x, \xi)$ denotes the principal symbol of $P^i(x, D)$;

$$p^i(x, \xi) = -\xi_0^2 + 2a^i(x, \xi')\xi_0 - b^i(x, \xi') = -(\xi_0 - a^i(x, \xi'))^2 + q^i(x, \xi')$$

where $q^i(x, \xi) = a^i(x, \xi')^2 - b^i(x, \xi')$. We assume that $p^i(x, \cdot)$ are hyperbolic with respect to dx_0 near $(\hat{x}, \hat{\xi}')$ that is

$$(1.3) \qquad a^i(x, \xi') \text{ are real and } q^i(x, \xi') \geqq 0 \text{ near } (\hat{x}, \hat{\xi}').$$

Let $\rho = (\hat{x}, \hat{\xi}) = (\hat{x}, \hat{\xi}_0, \hat{\xi}')$ be a double characteristic for all $p^i(x, \xi)$ $(1 \leqq i \leqq l)$. Set $p(x, \xi) = \prod_{i=1}^{l} p^i(x, \xi)$ and introduce the localization $p_\rho(x, \xi)$, $p_\rho^i(x, \xi)$ of $p(x, \xi)$ and $p^i(x, \xi)$ at ρ;

$$p_\rho(x, \xi) = \lim_{s \to 0} s^{-2l} p(\rho + s(x, \xi)), \quad p_\rho^i(x, \xi) = \lim_{s \to 0} s^{-2} p^i(\rho + s(x, \xi)).$$

It is known that $p_\rho(x, \xi)$ and $p_\rho^i(x, \xi)$ are hyperbolic polynomials in $T_\rho(T^*\Omega)$ with respect to $H_{x_0} \in T_\rho(T^*\Omega)$ where H_{x_0} is the Hamilton field of x_0, defined by $\langle dx_0, Y \rangle = \sigma(Y, H_{x_0})$ for any $Y \in T_\rho(T^*\Omega)$ (see [3], [5]). Hence we can define the hyperbolic cone $\Gamma(p_\rho, H_{x_0})$ of p_ρ as the component of H_{x_0} in $\{X \in T_\rho(T^*\Omega) ; p_\rho(X) \neq 0\}$ and the propagation cone $C(p_\rho, H_{x_0})$ as

$$C(p_\rho, H_{x_0}) = \{X \in T_\rho(T^*\Omega) ; \sigma(X, Y) \geqq 0 \text{ for any } Y \in \Gamma(p_\rho, H_{x_0})\}.$$

Here σ is a natural 2 form on $T^*\Omega$ given in any standard coordinates (x, ξ) by

$$\sigma=\sum_{j=0}^{d} d\xi_j\wedge dx_j.$$

It is clear that $\Gamma(p_\rho, H_{x_0})=\bigcap_{i=1}^{l}\Gamma(p_\rho^i, H_{x_0})$. For $X\in T_\rho(T^*\Omega)$ we set $\langle X\rangle=\mathrm{span}(X)$ and for a subspace $V\subset T_\rho(T^*\Omega)$, V^σ denotes the σ orthogonal space of V.

Now we assume that

(1.4) there are l hyperplanes $H_j\subset T_p(T^*\Omega)$ which intersect involutively, that is, $\sigma(H_i^\sigma, H_j^\sigma)=0$ for any i, j, such that for $j=1,\cdots,l$

$$C(p_\rho, H_{x_0})\cap H_j=\{0\},\ H_j\supset \mathrm{KerHess}\ p^j(\rho)+\langle H_{x_0}\rangle$$

where KerHess $p^j(\rho)$ is the Kernel of the Hessian of $p^j(x,\xi)$ at ρ. Clearly (1.4) is invariant under a change of homogeneous symplectic coordinates preserving $x_0=$const. To see our hypothesis more intuitively we observe that

Lemma 1.1. *Let $r(x,\xi)$ be one of $p^i(x,\xi)$. Then the following five conditions are equivalent.*

a) $r(x,\xi)$ *is effectively hyperbolic at* ρ

b) $C(r_\rho, H_{x_0})\cap \mathrm{KerHess}\ r(\rho)=\{0\}$

c) *there is a hyperplane* $H\subset T_\rho(T^*\Omega)$ *such that*

$$H\cap C(r_\rho, H_{x_0})=\{0\},\ H\supset \mathrm{KerHess}\ r(\rho)+\langle H_{x_0}\rangle$$

d) $\Gamma(r_\rho, H_{x_0})\cap(\mathrm{KerHess}\ r(\rho))^\sigma\cap\langle H_{x_0}\rangle^\sigma\neq\emptyset$

e) $\Gamma(r_\rho, H_{x_0})\cap(\mathrm{KerHess}\ r(\rho))^\sigma\neq\emptyset$

We shall prove this lemma in §2. From this lemma it is clear that each $p^i(x,\xi)$ is effectively hyperbolic at ρ if (1.4) are verified, for $C(p_\rho^i, H_{x_0})\subset C(p_\rho, H_{x_0})$ and if $l=1$ (1.4) is equivalent to that $p^1(x,\xi)$ is effectively hyperbolic at ρ.

By $C^k(I, H^p)$ we denote the set of all k times continuously differentiable functions from I to the usual Sobolev space $H^p=H^p(\boldsymbol{R}^d)$ an dset $H^\infty=\bigcap_s H^s$. Main results in this paper were anounced in [11].

Theorem 1.1. *Suppose* (1.3) *and* (1.4). *Then there is a para-*

metrix at $(0, \hat{x}', \hat{\xi}')$ *of the Cauchy problem* (1.1) *with finite propagation speed of wave front sets. In particular there are a small interval* $\tilde{I}$ *containing the origin and a positive constant* β *such that*;

for any $f\in((C^0(\tilde{I}, H^p))^l$ *vanishing in* $x_0<0$ *with* $WF(f(t, \cdot))$ *contained in a sufficiently small conic neighborhood* Γ_1 *of* $(\hat{x}', \hat{\xi}')$ $(t\in\tilde{I})$ *there is a* $u\in(C^1(\tilde{I}, H^{p-1-\beta}))^l$ *vanishing in* $x_0<0$ *and satisfying*

$$Pu-f\in(C^0(\tilde{I}, H^{\infty}))^l, \quad WF(D_0^j u(t, \cdot))\subset\Gamma_2, \quad 0\leqq j\leqq 1, t\leqq\delta$$

with $\delta=\delta(\Gamma_1, \Gamma_2)>0$ *for any conic neighborhood* Γ_2 *of* $(\hat{x}', \hat{\xi}')$ *with* $\Gamma_1 \Subset\Gamma_2$.

For the definition and properties of parametrices with finite propagation speed of wave front sets, we refer to [12]. Next we study The propagation of wave front sets. Assume a variant of (1.4).

(1.4)′ There are l hyperplanes $H_j\subset T_\rho(T^*\Omega)$ intersecting involutively such that for $j=1, \cdots, l$
$$C(p_\rho, H_{x_0})\cap H_j=\{0\}, \quad H_j\supset \mathrm{KerHess}\ p^j(\rho).$$

Theorem 1.2. *Suppose* (1.3) *and* (1.4)′. *Let* $\varphi(x, \xi)$ *be real, homogeneous of degree* 0 *in* ξ, C^∞ *in a conic neighborhood of* ρ *such that*

$$\varphi(\rho)=0, \quad H_\varphi(\rho)\in\Gamma(p_\rho, H_{x_0})$$

and let ω *be a sufficiently small conic neighborhood of* ρ. *Then it follows from*

$$\omega\cap\{\varphi<0\}\cap WF(u)=\emptyset, \quad \rho\notin WF(Pu)$$

that

$$\rho\notin WF(u)$$

for any distribution $u\in(\mathscr{D}'(\Omega))^l$.

Of course if we drop the transversality condition;

there are l hyperplanes $H_j\subset T_\rho(T^*\Omega)$ such that

$$H_j\supset\mathrm{KerHess}\ p^j(\rho), \quad C(p_\rho, H_{x_0})\cap H_j=\{0\},$$

the situation becomes complicated. We give an example.

Example 1.1. Let $\rho=(0, 0, \cdots, 1)\in T^*\boldsymbol{R}^{d+1}\backslash 0$ and $p^i(x, \xi)$ be

$$p^1(x,\xi)=-(\xi_0-(1-r)^{1/2}x_0\xi_d)^2+(x_0-r^{-1}x_1)^2\xi_d^2,$$
$$p^2(x,\xi)=-\xi_0^2+(x_0-x_1)^2\xi_d^2+r\xi_1^2,\qquad 0<r<1.$$

Set $p(x,\xi)=p^1(x,\xi)p^2(x,\xi)$ and denote by $\Sigma(p^i)$ the doubly characteristic set of $p^i(x,\xi)$. Then noting $0<r<1$ there are hyperplanes H_i such that

$$H_i\supset T_\rho(\Sigma(p^i)),\qquad C(p^i_\rho,H_{x_0})\cap H_i=\{0\}.$$

In particular $p^i(x,\xi)$ are effectively hyperbolic at ρ. But for any choice of such H_1 we see that $C(p^2_\rho,H_{x_0})\cap H_1\neq\{0\}$ hence

$$C(p_\rho,H_{x_0})\cap H_1\neq\{0\}.$$

It is easy to see that (taking x_0 as a parameter)

$$x_1=rx_0,\ x_d=-2^{-1}(1-r)^{3/2}x_0^2,\ \xi_0=(1-r)^{1/2}x_0,\ \xi_1=-(1-r)^{1/2}x_0,\ \xi_d=1$$
$$(x_2,\cdots,x_{d-1})=\text{const.},\ \ (\xi_2,\cdots,\xi_{d-1})=\text{const.}$$

is a bicharacteristic of $p^2(x,\xi)$. We denote it by $\gamma=\gamma(x_0)$. Note that

$$\gamma\subset\Sigma(p^1).$$

Since $\gamma\subset\{(x,\xi)\,;p^2(x,\xi)=0\}$ we conclude that

$$\gamma\subset\{(x,\xi)\,;p(x,\xi)=dp(x,\xi)=d^2p(x,\xi)=0\}.$$

In § 2, we shall give a proof of Lemma 1. 1 which gives a geometric characterization of effective hyperbolicity (cf. [10]). From this we show the existence of l hypersurfaces which play an important role when deriving energy estimate. In § 3 we localize principal symbol $p^i(x,\xi)$ along l hypersurfaces and introduce a partition of unity associated with these surfaces. In § 4, we derive energy estimate for the terms which are squares of first order operators, and in §5, for the other term in an expression of $p^i(x,\xi)$ along the lines in [9] and [10]. In § 6 we shall estimate commutators which come from partition of unity. § 7 is devoted to give energy estimate for $\tilde{P}(x,D)$ blown up of $P(x,D)$, collecting estimates in §§ 4, 5 and 6. This shows the existence of parametrix in Theorem 1. 1. Finally in § 8 we estimate wave front sets applying energy estimate in § 7. This will be used to prove Theorem 1. 2 and also to show finiteness of propagation speed of wave front sets for a parametrix in Theorem 1. 1.

§ 2. Preliminaries

At first we prove Lemma 1.1 in a slightly more general form. For $X\in T^*_\rho(T^*\Omega)$ we denote by H_X the Hamilton vector of X, defined by $\langle X, Y\rangle=\sigma(Y, H_X)$ for any $Y\in T_\rho(T^*\Omega)$. Let $r(X)$ be a hyperbolic polynomial in $T_\rho(T^*\Omega)$ with respect to $H_\theta\in T_\rho(T^*\Omega)$, $\theta\in T^*_\rho(T^*\Omega)$. Denote by Σ the linearity space of r;

$$\Sigma=\{X\in T_\rho(T^*\Omega)\,;r(tX+Y)=r(Y) \text{ for any } t \text{ and } Y\}$$

(see [1], [2]). Then we have

Lemm 2.1. *Notations as above. Then the following four conditions are equivalent.*

a) $$C(r, H_\theta)\cap\Sigma=\{0\}$$

b) *there is a hyperplane* $H\subset T_\rho(T^*\Omega)$ *such that*

$$H\cap C(r, H_\theta)=\{0\},\ H\supset\Sigma+\langle H_\theta\rangle$$

c) $$\Gamma(r, H_\theta)\cap\Sigma^\sigma\cap\langle H_\theta\rangle^\sigma\neq\emptyset$$

d) $$\Gamma(r, H_\theta)\cap\Sigma^\sigma\neq\emptyset$$

Proof. At first we show a)$\Leftrightarrow$d). Assume $\Gamma(r, H_\theta)\cap\Sigma^\sigma=\emptyset$ then by the Hahn-Banach theorem there is $0\neq Y\in T_\rho(T^*\Omega)$ such that $\sigma(Y, X)\leqq 0$ for any $X\in\Gamma(r, H_\theta)$ and $\sigma(Y, X)\geqq 0$ for any $X\in\Sigma^\sigma$. These imply that $Y\in C(r, H_\theta)$ and $Y\in\Sigma$. This would give a contradiction to a) hence we have a)$\Rightarrow$d). Suppose $0\neq Y\in\Gamma(r, H_\theta)\cap\Sigma^\sigma$. Then it is clear that $\langle Y\rangle^\sigma\supset\Sigma$, $\langle Y\rangle^\sigma\cap C(r, H_\theta)=\{0\}$ because $\Gamma(r, H_\theta)$ is open. This implies obviously $C(r, H_\theta)\cap\Sigma=\{0\}$. Hence we have proved d)$\Rightarrow$a).

Since c)$\Rightarrow$d) is obvious it suffices to show that a)$\Rightarrow$b)$\Rightarrow$c).

Proof of a)$\Rightarrow$b). When $H_\theta\in\Sigma+\Sigma^\sigma$ we write $H_\theta=X_1+X_2$ with $X_1\in\Sigma$ and $X_2\in\Sigma^\sigma$. Since $\Gamma(r, H_\theta)+\Sigma\subset\Gamma(r, H_\theta)$ and $\Gamma(r, H_\theta)\cap\Sigma=\emptyset$, it follows that $0\neq X_2\in\Gamma(r, H_\theta)$. It is clear that $\sigma(X_2, H_\theta)=0$ and hence $H_\theta\in\langle X_2\rangle^\sigma$. Noting that $X_2\in\Sigma^\sigma$, $X_2\in\Gamma(r, H_\theta)$ we get $\langle X_2\rangle^\sigma\supset\Sigma$ and $\langle X_2\rangle^\sigma\cap C(r, H_\theta)=\{0\}$, for $\Gamma(r, H_\theta)$ is open. Then $\langle X_2\rangle^\sigma$ is a desired hyperplane. Consider the case $H_\theta\notin\Sigma+\Sigma^\sigma$ and hence $(\Sigma+\Sigma^\sigma)\cap\langle H_\theta\rangle=\{0\}$. As proved above, a) implies that $\Gamma(r, H_\theta)\cap\Sigma^\sigma\neq$

$\emptyset$, and then we can take $0\neq Z\in\Gamma(r, H_\theta)\cap\Sigma^\sigma$. Note that

(2.1) $$\langle Z\rangle^\sigma\supset\Sigma,\ \langle Z\rangle^\sigma\cap C(r, H_\theta)=\{0\}.$$

Set $T=\langle Z\rangle^\sigma\cap(\Sigma+\Sigma^\sigma)$ hence

(2.2) $$T\supset\Sigma,\ T\cap C(r, H_\theta)=\{0\}.$$

We examine that $\dim T=\dim(\Sigma+\Sigma^\sigma)-1$. Indeed from $\Gamma(r, H_\theta)+\Sigma\subset\Gamma(r, H_\theta)$ it follows that

(2.3) $$C(r, H_\theta)\subset\Sigma^\sigma.$$

By (2.1) and (2.3) it follows that $\langle Z\rangle^\sigma\not\supset\Sigma+\Sigma^\sigma$ and this shows the desired assertion.

Take a subspace $V\subset T_\rho(T^*\Omega)$ so that $T_\rho(T^*\Omega)=(\Sigma+\Sigma^\sigma)\dot{+}V$ (direct sum) and write $H_\theta=Y_1+Y_2$, $Y_1\in\Sigma+\Sigma^\sigma$, $0\neq Y_2\in V$. Again we take a subspace $W\subset T_\rho(T^*\Omega)$ so that

$$V=\langle Y_2\rangle+W \text{ (direct sum)}.$$

Then the hyperplane $H=T+\langle H_\theta\rangle+W$ is the desired one. In fact we have $H\cap C(r, H_\theta)=\{0\}$ by (2.2) and (2.3). On the other hand it is obvious that $H\supset\Sigma+\langle H_\theta\rangle$.

Proof of b)$\Rightarrow$c). Take $0\neq Y\in T_\rho(T^*\Omega)$ so that $\langle Y\rangle=H^\sigma$. Then it is clear that $\langle Y\rangle\subset\Sigma^\sigma\cap\langle H_\theta\rangle^\sigma$. We show that Y or $-Y$ belongs to $\Gamma(r, H_\theta)$. If not we would have $\langle Y\rangle\cap\Gamma(r, H_\theta)=\emptyset$. Then by the Hahn-Banach theorem there is $0\neq Z\in T_\rho(T^*\Omega)$ such that $\sigma(Z, X)\leqq 0$ for any $X\in\Gamma(r, H_\theta)$, $\sigma(Z, X)\geqq 0$ for any $X\in\langle Y\rangle$. This shows that $Z\in C(r, H_\theta)$ and $X\in\langle Y\rangle^\sigma=H$ then we would have a contradiction to b). Thus we have proved b)$\Rightarrow$c).

Proof of Lemma 1.1. Noting that KerHess $r(\rho)$ is the linearity space of r_ρ, in view of Lemma 2.1, the statements b), c), d) and e) in the lemma are equivalent. On the other hand from Corollary 1.4.7 in Hörmander [3], it follows that a) and e) are equivalent.

Now we observe the hypothesis (1.4). It is clear from the proof of Lemma 2.1 that (1.4) implies that

(2.4) $$X_j \text{ or } -X_j\in\Gamma(p_\rho, H_{x_0})\cap(\text{KerHess } p^j(\rho))^\sigma\cap\langle H_{x_0}\rangle^\sigma$$

where $\langle X_j\rangle^\sigma=H_j$.

Lemma 2.2. *Suppose* (1.4). *Then there are* l *real functions* $f_i(x, \xi')$, *homogeneous of degree* 0 *in* ξ', C^∞ *in a conic neighborhood of* $\rho'=(\hat{x}, \hat{\xi}')$ *satisfying*

$$q^i(x, \xi')\geqq c_i f_i(x, \xi')^2|\xi'|^2 \quad \textit{near } \rho'$$

$$H_{f_i}(\rho')\in\Gamma(p_\rho, H_{x_0}),\ \{f_i, f_j\}(\rho')=0 \quad \textit{for any } i, j$$

with positive constant c_i *where* $\{\cdot, \cdot\}$ *is the Poisson bracket.*

Proof. Write $r(x, \xi)$ instead of $p^i(x, \xi)$ for arbitrarily fixed i. Recall that $r(x, \xi)$ has the form

$$r(x, \xi)=-(\xi_0-a(x, \xi'))^2+q(x, \xi').$$

Since $q(x, \xi')$ is non negative near ρ' the Morse lemma shows that there are functions $b_j(x, \xi')$ $(1\leqq j\leqq\nu)$, homogeneous of degree 1 in ξ', C^∞ in a conic neighborhood of ρ' such that

$$(2.5)\quad q(x, \xi')\geqq\sum_{j=1}^{\nu} b_j(x, \xi')^2 \text{ near } \rho',\ q_{\rho'}(x, \xi')=\sum_{j=1}^{\nu} db_j(x, \xi')^2.$$

Hence we have

$$(\text{KerHess } r(\rho))^\sigma=\text{span}(H_{\xi_0-a}(\rho), H_{b_j}(\rho');\ 1\leqq j\leqq\nu).$$

Take $0\neq X\in(\text{KerHess } r(\rho))^\sigma\cap\Gamma(p_\rho, H_{x_0})\cap\langle H_{x_0}\rangle^\sigma$ then we have with real constants α_i that

$$X=\sum_{j=1}^{\nu}\alpha_j H_{b_j}(\rho')+\alpha_0 H_{\xi_0-a}(\rho).$$

Since $X\in T_\rho\{x_0=0\}=\langle H_{x_0}\rangle^\sigma$ it follows that $\alpha_0=0$. Set

$$f(x, \xi')=\sum_{j=1}^{\nu}\alpha_j b_j(x, \xi')\,|\xi'|^{-1}$$

then it is clear that $H_f(\rho')=X\in\Gamma(p_\rho, H_{x_0})$. It is also clear from (2.5) that

$$q(x, \xi')\geqq cf(x, \xi')^2|\xi'|^2 \text{ near } \rho'$$

with a positive constant c. It remains to show that the last statement. We return to the original notation. Since we have chosen $f_i(x, \xi')$ so that $H_{f_i}(\rho')=X_i\in H_i^\sigma$ it follows that $\{f_i, f_j\}(\rho')=\sigma(H_{f_i}(\rho'), H_{f_j}(\rho'))=\sigma(X_i, X_j)=0$. This proves the lemma.

Remark 2.1. In Lemma 2.2, we can replace $f_i(x, \xi')$ by $e_i(x, \xi')$

$f_i(x, \xi')$ with $e_i(\rho')\neq 0$, homogeneous of degree 0 in ξ'.

Lemma 2.3. *Assume* (1.4). *Then we can choose a homogeneous symplectic coordinates* (x, ξ) *near* ρ *preserving* $x_0=const.$, *and a numbering of the indices i such that* $\rho=(0, e_p)$ *and* $f_i(x, \xi')$ *in Lemma* 2.2 *may take the following form*

$$f_i(x, \xi')=x_0-\psi_i(x', \xi') \quad (1\leqq i\leqq l),\ d\psi_i(\rho'')=dx_i \quad (1\leqq i\leqq p-1)$$

$$d\psi_i(\rho'')=\text{linear combination of } dx_j\ (1\leqq j\leqq p),\ (p\leqq i\leqq l)$$

where $p=\dim \operatorname{span}(dx_d, d\psi_j(\rho'')\ ;\ 1\leqq j\leqq l)$ *and* $\rho''=(0, e_p')\in T^*U\backslash 0$.

Proof. Note that we may assume that $\rho=(0, e_d)$. By a change of homogeneous symplectic coordinates near (x, ξ) preserving $x_0=$ const., we may assume that $a^1(x, \xi')=0$ and hence

$$p^1(x, \xi)=-\xi_0^2+q^1(x, \xi').$$

It is clear that $\Gamma(p_\rho^1, H_{x_0})\subset\{\xi_0<0\}$ and then the hypothesis

$$H_{f_i}(\rho')\in\Gamma(p_\rho, H_{x_0})\subset\Gamma(p_\rho^1, H_{x_0})$$

implies that $(\partial f_i/\partial x_0)(\rho')>0$. Thus we can write

$$f_i(x, \xi')=e_i(x, \xi')(x_0-\psi_i(x', \xi'))$$

with $e_i(\rho')>0$. Taking Remark 2.1 into account we may suppose that

$$f_i(x, \xi')=x_0-\psi_i(x', \xi').$$

We proceed to the next step. Note that $\{f_i, f_j\}(\rho')=\{\psi_i, \psi_j\}(\rho'')=0$ for any i, j. Renumbering f_i, if necessary, we may assume that

$$\operatorname{span}(dx_d, d\psi_i(\rho'')\ ;1\leqq i\leqq l)=\operatorname{span}(dx_d, d\psi_i(\rho'')\ ;1\leqq i\leqq p-1).$$

Set $\hat{\psi}_i(x', \xi')=d\psi_i(x', \xi')-(\partial\psi_i(\rho'')/\partial x_d)x_d$ and note that $\{\hat{\psi}_i, \hat{\psi}_j\}=\{\psi_i, \psi_j\}(\rho'')=0$. Put

$$X_i(\tilde{x}, \xi')=\hat{\psi}_i(\tilde{x}, \tilde{\xi}\xi_d^{-1})$$

with $\tilde{x}=(x_1, \cdots, x_{d-1})$, $\tilde{\xi}=(\xi_1, \cdots, \xi_{d-1})$. It is clear that $\{X_i\}_{i=1}^{p-1}$ form a partial homogeneous symplectic coordinates and dX_i $(1\leqq i\leqq p-1)$, dx_d are linearly independent at ρ''. Then we can extend $\{X_i\}_{i=1}^{p-1}$ to a full homogeneous symplectic coordinates $\{X_i, \Xi_i\}_{i=1}^{d}$ so that $\rho''=(0, e_d')$. We write (x', ξ') instead of (X', Ξ') and hence we have

$$d\psi_i(\rho'')=dx_i, \quad 1\leqq i\leqq p-1.$$

Since $d\psi_i(\rho'')$ $(p\leqq i\leqq l)$ are linear combinations of dx_i $(1\leqq i\leqq p-1)$ and dx_d, interchanging the coordinates x_p and x_d we get this lemma.

Proposition 2.1. *Assume* (1.4). *Then we have*

$$q^i(x,\xi')=\sum_{k=1}^{p-1} l_{ik}(x,\xi')^2+g_i(x,\xi')$$

where $l_{ik}(x,\xi')$, $g_i(x,\xi')$ *are homogeneous of degree* 1, 2 *respectively*, C^∞ *in a conic neighborhood of* ρ' *satisfying*

$$l_{ik}(\rho')=0, \quad g_i(\rho')=0, \quad (\partial^2 g_i/\partial\xi_s^2)(\rho')=0, \quad 1\leqq s\leqq p,$$
$$g_i(x,\xi')\geqq c_i\tilde{f}_i(x,\xi')^2|\xi'|^2 \quad \text{near } \rho'$$

with $\tilde{f}_i(x,\xi')=x_0-\tilde{\psi}_i(x,\xi')$ *such that* $d\tilde{\psi}_i(\rho')$ *are linear combinations of* dx_i $(1\leqq i\leqq p)$ *and* $H_{\tilde{f}_i}(\rho')\in\Gamma(p_\rho, H_{x_0})$.

Proof. We fix i and in what follows the index i will be omitted from notation. To simplify notation further we set $\delta_i(r)=0$ if $(\partial^2 r/\partial\xi_i^2)(\rho')=0$ and $\delta_i(r)=1$ otherwise for $r=r(x,\xi')$. We denote by $\Delta_k(r)\xi'$ the set of coordinates ξ_i with $1\leqq i\leqq k$ satisfying $\delta_i(r)=1$ and by $\Delta_k^c(r)\xi'$ the complement of $\Delta_k(r)\xi'$, that is, $\Delta_k^c(r)\xi'=\{\xi_1,\cdots,\xi_d\}\setminus(\Delta_k(r)\xi')$. We shall prove by induction on k $(k\leqq p-1)$ that we can express q as

$$(2.6)_k \qquad \begin{aligned} q(x,\xi')=&\sum_{j=1}^{k}\delta_j(q)e_j(x,\xi')(\xi_j-h_j(x,\Delta_k^c(q)\xi'))^2\\ &+r^k(x,\xi')g^k(x,\Delta_k^c(q)\xi')\end{aligned}$$

where $e_j(x,\xi')$, $r^k(x,\xi')$ are homogeneous of degree 0 in ξ' with $e_j(\rho')>0$, $r^k(\rho')>0$ and $g^k(x,\Delta_k^c(q)\xi')$ is homogeneous of degree 2 in $\Delta_k^c(q)\xi'$, non negative near ρ' such that

$$(2.7)_k \qquad (\partial^2 g^k/\partial\xi_s^2)(\rho')=0, \quad s=1,\cdots,k.$$

When $k=1$ and $\delta_1(q)=1$ Malgrange's preparation theorem gives that

$$\begin{aligned} q(x,\xi')&=e_1(x,\xi')\{(\xi_1-h_1(x,\Delta_1^c(q)\xi'))^2+g^1(x,\Delta_1^c(q)\xi')\}\\ &=\delta_1(q)e_1(x,\xi')(\xi_1-h_1(x,\Delta_1^c(q)\xi'))^2+e_1(x,\xi')g^1(x,\Delta_1^c(q)\xi').\end{aligned}$$

It is clear that $(\partial^2 g^1(x,\Delta_1^c(q)\xi')/\partial\xi_1^2)(\rho')=0$. If $k=1$ and $\delta_1(q)=0$, noting that $\Delta_1^c(q)\xi'=\xi'$, it suffices to put $g^1(x,\Delta_1^c(q)\xi')=q(x,\xi')$, $r^1(x,\xi')=1$. Now assume that $(2.6)_{k-1}$ and $(2.7)_{k-1}$ $(k-1\leqq p-2)$

are valid. If $\delta_k(q)=1$ Malgrange's preparation theorem again shows that

$$g^{k-1}(x, \Delta_{k-1}^c(q)\xi')=\tilde{e}(x, \Delta_{k-1}^c(q)\xi')\{(\xi_k-h_k(x, \Delta_k^c(q)\xi'))^2 + g^k(x, \Delta_k^c(q)\xi')\}.$$

With $e_k(x, \xi')=r^k(x, \xi')=r^{k-1}(x, \xi')\tilde{e}(x, \Delta_{k-1}^c(q)\xi')$ we have $(2.6)_k$. We examine $(2.7)_k$. It is obvious that $(\partial^2 g^k/\partial\xi_k^2)(\rho')=0$. For $1\leqq s\leqq k-1$ we have with some real constants a_s that

$$(\partial^2 g^{k-1}/\partial\xi_s^2)(\rho')=a_s^2+\tilde{e}(\rho')(\partial^2 g^k/\partial\xi_s^2)(\rho').$$

The inductive hypothesis gives that $(\partial^2 g^{k-1}/\partial\xi_s^2)(\rho')=0$, $1\leqq s\leqq k-1$, whereas from non negativity of g^k one has $(\partial^2 g^k/\partial\xi_s^2)(\rho')\geqq 0$ and hence $(\partial^2 g^k/\partial\xi_s^2)(\rho')=0$ for $s=1, \cdots, k-1$.

We turn to the case $\delta_k(q)=0$. In this case we put $r^k=r^{k-1}$, $g^k=g^{k-1}$. Noting that $\Delta_{k-1}^c(q)\xi'=\Delta_k^c(q)\xi'$, $q(x, \xi')$ takes the form $(2.6)_k$. It is also clear that $(\partial^2 g^{k-1}/\partial\xi_k^2)(\rho')=0$ since

$$(\partial^2 q/\partial\xi_k^2)(\rho')=a_k^2+r^{k-1}(\rho')(\partial^2 g^{k-1}/\partial\xi_k^2)(\rho').$$

Thus we have proved that $(2.6)_k$ and $(2.7)_k$ hold.

Next we solve

$$\delta_i(q)(\xi_i-h_i(x, \Delta_i^c(q)\xi'))=0, \quad i=1, \cdots, p-1. \tag{2.8}$$

Put $\{i; \delta_i(q)=1, 1\leqq i\leqq p-1\}=\{i_1<i_2<\cdots<i_s\}$ and $\xi_{(1)}=(\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_s})$. Denote by $\xi_{(2)}$ the complement of $\xi_{(1)}$ so that $\xi'=(\xi_{(1)}, \xi_{(2)})$. From (2.8) we have

$$\xi_{i_k}=h_{i_k}(x, \Delta_{i_k}^c(q)\xi')=h_{i_k}(x, \xi_{i_{k+1}}, \cdots, \xi_{i_s}, \xi_{(2)}).$$

Inductively we have $\xi_{i_k}=\tilde{h}_{i_k}(x, \xi_{(2)})$ and hence

$$\xi_{(1)}=H(x, \xi_{(2)}). \tag{2.9}$$

Thus $(2.6)_{p-1}$ and (2.9) give that

$$q(x, H(x, \xi_{(2)}), \xi_{(2)})=r^{p-1}(x, H(x, \xi_{(2)}), \xi_{(2)})g^{p-1}(x, \xi_{(2)}). \tag{2.10}$$

In view of Lemma 2.3 there is $f(x, \xi')=x_0-\psi(x', \xi')$ such that $H_f(\rho')\in\Gamma(p_\rho, H_{x_0})$ and

$$q(x, \xi_{(1)}, \xi_{(2)})\geqq cf(x, \xi_{(1)}, \xi_{(2)})^2|(\xi_{(1)}, \xi_{(2)})|^2 \quad \text{near } \rho' \tag{2.11}$$

with a positive constant c. Set

$$\tilde{f}(x, \xi')=f(x, H(x, \xi_{(2)}), \xi_{(2)})=x_0-\psi(x', H(x, \xi_{(2)}), \xi_{(2)}). \tag{2.12}$$

Since $r^{p-1}(\rho')>0$, $H(\rho')=0$ and $2|\xi_{(2)}|\geqq|\xi'|$ in a small conic neighborhood of ρ' it follows from (2.10) and (2.11) that

$$r^{p-1}(x, \xi')g^{p-1}(x, \xi_{(2)})\geqq\tilde{c}\ \tilde{f}(x, \xi')^2|\xi'|^2 \quad \text{near } \rho'$$

with a positive constant $\tilde{c}$. Finally we put

$$l_i(x, \xi')=\delta_i(q)\,(e_i(x, \xi'))^{1/2}(\xi_i-h_i(x, \Delta_i^c(q)\xi'))$$
$$g(x, \xi')=r^{p-1}(x, \xi')g^{p-1}(x, \xi_{(2)}).$$

Noting that $d\psi_i(\rho'')$ are linear combinations of dx_i $(1\leqq i\leqq p)$ and $d\tilde{\psi}_i(\rho')=d\psi_i(\rho'')$, $\tilde{\psi}_i(x, \xi')=\psi_i(x, H(x, \xi_{(2)}), \xi_{(2)})$, these l_i, g, $\tilde{f}$ are desired ones.

We rewrite the conditions $H_{\tilde{f}_i}(\rho')\in\Gamma(p_\rho, H_{x_0})$. This implies that $H_{\tilde{f}_i}(\rho')\in\Gamma(p_\rho^j, H_{x_0})$ for any j. Note that

$$p_\rho^j(x, \xi)=-dl_{j0}(x, \xi)^2+\sum_{k=1}^{p-1}dl_{jk}(x, \xi')^2+g_{j\rho'}(x, \xi')$$

with $l_{j0}(x, \xi)=\xi_0-a^j(x, \xi')$. It is clear that $H_{\tilde{f}_i}(\rho')\in\Gamma(p_\rho^j, H_{x_0})$ is equivalent to

$$dl_{j0}(H_{\tilde{f}_i}(\rho'))<0,\ dl_{j0}(H_{\tilde{f}_i}(\rho'))^2>\sum_{k=1}^{p-1}dl_{jk}(H_{\tilde{f}_i}(\rho'))^2+g_{j\rho'}(H_{\tilde{f}_i}(\rho')).$$

Remarking that $(\partial^2g_j/\partial\xi_s^2)(\rho')=0$ for $s=1, \cdots, p$ and $d\tilde{f}_i(\rho')$ are linear combinations of $dx_i(0\leqq i\leqq p)$, it follows that $g_{j\rho'}(H_{\tilde{f}_i}(\rho'))=0$. Thus we have

Lemma 2.4. *Let $\tilde{f}_i(x, \xi')$ be as in Proposition 2.1. Then*

$$\{l_{j0}, \tilde{f}_i\}(\rho)>0,\ \{l_{j0}, \tilde{f}_i\}^2(\rho)>\sum_{k=1}^{p-1}\{l_{jk}, \tilde{f}_i\}^2(\rho')$$

for any i, j.

§ 3. Localization

In this section, unless otherwise specified, we use the notation in [12]. In particular we use pseudodifferential calculus in § 4 in [12]. Recall

$$q^j(x, \xi')=\sum_{k=1}^{p-1}l_{jk}(x, \xi')^2+g_j(x, \xi'),\ l_{j0}(x, \xi')=\xi_0-a^j(x, \xi').$$

Following § 3 in [12] we introduce $y_j=y_j(x, \mu)$, $\eta_j=\eta_j(\xi, \mu)$;

$$y_0=\mu x_0,\ y_j=\mu\chi_0(x_j)\ \ (1\leqq j\leqq p),\qquad y_j=\mu^{1/2}\chi_1(\mu^{-1/2}x_j)x_j\ \ (p+1\leqq j\leqq d)$$
$$\eta_0=\mu^{-1}\xi_0,\ \eta_j=\mu^{-1/2}\chi_1(\mu^{-1/2}\xi_j\langle\xi'\rangle^{-1})\xi_j\ \ (p+1\leqq j\leqq d)$$
$$\eta_j=\mu^{-1}\chi_1(\mu^{-1}(\xi_j\langle\xi'\rangle^{-1}-\delta_{jp}))(\xi_j-\delta_{jp}\langle\xi'\rangle)+\mu^{-1}\delta_{jp}\langle\xi'\rangle\ \ (1\leqq j\leqq p)$$

where $0<\mu\leqq 1$ and δ_{ij} is Kronecker's delta. Here $\chi_0(s)$ is in $C^\infty(\boldsymbol{R})$, equal to s on $|s|\leqq 1$, $|\chi_0(s)|=2$ on $|s|\geqq 2$ and $0\leqq\chi_0^{(1)}(s)\leqq 1$ everywhere and $\chi_1(s)\in C_0^\infty(\boldsymbol{R})$ is equal to 1 on $|s|\leqq 1$ and has support in $|s|\leqq 2$ and $0\leqq\chi_1(s)\leqq 1$ everywhere. Let l, g, $\tilde{f}$ be one of l_{jk} $(0\leqq k\leqq p-1)$, g_j, $\tilde{f}_j$ $(1\leqq j\leqq l)$ respectively. Set

$$\hat{l}(x,\xi')=l(x,\xi)-\sum_{s=0}^{p-1}l^{(s)}(\rho)\xi_s,\quad l^{(s)}(\rho)=(\partial l/\partial\xi_s)(\rho)$$

and define $L(x,\xi,\mu)$ by

$$L(x,\xi,\mu)=\sum_{s=0}^{p-1}l^{(s)}(\rho)\xi_s+\tilde{L}(x,\xi',\mu) \tag{3.1}$$

where $\tilde{L}(x,\xi',\mu)=\hat{l}(y,\mu\eta')=\mu\hat{l}(y,\eta')$. By the definition we have

$$\hat{l}^{(s)}(\rho')=0\qquad\text{for } s=0,\cdots,p$$

and then it follows that (see the arguments preceding to Lemma 3.1 in [12])

$$\tilde{L}\in S(\mu\langle\xi'\rangle,dx_0^2+\tilde{G}'_\mu)+S(\mu^2\langle\xi'\rangle,dx_0^2+\tilde{G}_\mu). \tag{3.2}$$

Put

$$\tilde{G}(x,\xi',\mu)=\mu^2g(y,\eta')=g(y,\mu\eta').$$

Noting that $(\partial^2g/\partial\xi_s^2)(\rho')=0$, $s=0,\cdots,p$, the same argument to show (3.2) gives that

$$\tilde{G}\in S(\mu^2\langle\xi'\rangle^2,dx_0^2+\tilde{G}'_\mu)+S(\mu^3\langle\xi'\rangle^2,dx_0^2+\tilde{G}_\mu). \tag{3.3}$$

Next we define $F(x,\xi',\mu)$ by

$$F(x,\xi',\mu)=\mu^{-1}\tilde{f}(y,\mu\eta')=\mu^{-1}\tilde{f}(y,\eta').$$

It is easy to see that

$$F(x,\xi',\mu)=x_0+\sum_{s=1}^{p}\tilde{f}_{(s)}(\rho')\chi_0(x_s)+\tilde{F}(x,\xi',\mu), \tag{3.4}$$
$$\tilde{f}_{(s)}(\rho')=(\partial\tilde{f}/\partial x_s)(\rho')$$

with

$$\tilde{F}(x,\xi',\mu)\in S(\mu,dx_0^2+\tilde{G}_\mu). \tag{3.5}$$

We observe that when $|x_j|\leqq\mu^{1/2}$, $|\xi_j|\xi'|^{-1}-\delta_{jp}|\leqq\mu$ we have

$$L(x,\xi,\mu)=\mu l(M_\mu(x,\xi)),\qquad \tilde{G}(x,\xi',\mu)=\mu^2 g(M_\mu(x,\xi')),$$
$$F(x,\xi',\mu)=\mu^{-1}\tilde{f}(M_\mu(x,\xi'))$$

with $M_\mu(x,\xi)=(\mu x_0,\mu x'',\mu^{1/2}x''',\mu^{-1}\xi_0,\mu^{-1}\xi'',\mu^{-1/2}\xi''')$ where $x=(x_0,x'',x''')=(x_0,x_1,\cdots,x_p,x_{p+1},\cdots,x_d)$ and (ξ_0,ξ'',ξ''') is a corresponding partition of the coordinates ξ.

As in [12] we usually work with $S(m,G_\mu)/S_\mu^{-\infty}$ instead of $S(m,G_\mu)$. According to this remark "modulo $S_\mu^{-\infty}$" will not be indicated in inequalities and equalities in the sequel.

From Proposition 2.1 we have

$$\tilde{G}(x,\xi',\mu)\geqq cF(x,\xi',\mu)^2\langle\mu\xi'\rangle^2 \tag{3.6}$$

with a positive constant c independent of μ. Now we observe the Poisson bracket $\{L,F\}$. It is easy to see that

$$\{L,F\}=\{l,\tilde{f}\}(\rho)+\sum_{s=1}^{p-1}(\chi_0^{(1)}(x_s)-1)l^{(s)}(\rho)\tilde{f}_{(s)}(\rho')+r,\ r\in S(\mu^{1/2},dx_0^2+\tilde{G}_\mu).$$

To handle the second term on the right-hand side it is convenient to modify F, $\tilde{G}$ slightly. Set

$$b(x',\lambda)=\lambda\sum_{s=1}^{p-1}(1-\chi_0^{(1)}(x_s))$$

with a positive parameter λ. Note that $b(x',\lambda)\geqq 0$. Define $\varphi(x,\xi',\mu)$ by

$$\varphi(x,\xi',\mu)=(1+b)^{-1}(F+x_0 b).$$

It follows from (3.4) and (3.5) that

$$\varphi_{(\beta)}^{(\alpha)}\in S(\langle\xi'\rangle^{-|\alpha|},dx_0^2+\tilde{G}_\mu)\qquad\text{for}\quad |\alpha+\beta|\leqq 1. \tag{3.7}$$

Setting

$$D(L,\varphi)=(1+b)^{-1}(\{L,F\}+\{L,x_0\}b-r),$$
$$R(L,\varphi)=(1+b)^{-1}(\{L,b\}x_0+\{b,L\}\varphi+r)$$

we have

$$\{L,\varphi\}=D(L,\varphi)+R(L,\varphi). \tag{3.8}$$

Here we remark that (x_0 is regarded as a parameter) $\{L,b\}x_0$ is in $S(x_0,\tilde{G}_\mu)$ and hence we may assume that

$$\{L,b\}x_0\in S(\mu^{1/2},g_\mu)\qquad\text{when}\quad |x_0|\leqq\mu^{1/2}. \tag{3.9}$$

Put

$$G(x, \xi', \mu) = \tilde{G}(x, \xi', \mu) + b^2(x', \lambda)\langle\mu\xi'\rangle^2 x_0^2$$

then it is clear from (3.6) that

(3.10) $$G(x, \xi', \mu) \geqq c\varphi(x, \xi', \mu)^2\langle\mu\xi'\rangle^2$$

with a positive constant c independent of μ and λ. It is obvious that

(3.11) $$G(x, \xi', \mu) \in S(\mu^2\langle\xi'\rangle^2, dx_0^2 + \tilde{G}'_\mu) + S(\mu^3\langle\xi'\rangle^2, dx_0^2 + \tilde{G}_\mu).$$

Again we note that when $|x_j| \leqq \mu^{1/2}$, $|\xi_j|\xi'|^{-1} - \delta_{jp}| \leqq \mu$ we have

$$\varphi(x, \xi', \mu) = \mu^{-1}\hat{f}(M_\mu(x, \xi')), \quad G(x, \xi', \mu) = \mu^2 g(M_\mu(x, \xi')).$$

Let $L_{jk}(x, \xi, \mu)$, $G_j(x, \xi, \mu)$ be defined by preceding formula with l_{jk}, g_j. Set

$$\tilde{P}_j(x, D, \mu) = -L_{j0}^2(x, D, \mu) + \sum_{k=1}^{p-1} L_{jk}^2(x, D', \mu) + Q_j(x, D', \mu),$$

$$\tilde{P}(x, D, \mu) = \mathrm{diag}(\tilde{P}_1(x, D, \mu), \cdots, \tilde{P}_l(x, D, \mu))$$

where

(3.12) $$Q_j(x, D', \mu) = (G_j(x, D', \mu) + G_j^*(x, D', \mu))/2.$$

It is easy to see that

$\hat{P}(x, D, \mu) = \tilde{P}(x, D, \mu) + \tilde{B}_0(x, D', \mu)D_0 + \tilde{B}_1(x, D', \mu) \equiv P^\mu(x, D)$ near ρ' with $\tilde{B}_0(x, D', \mu) \in S(\mu, dx_0^2 + \tilde{G}_\mu)$, $\tilde{B}_1(x, D', \mu) \in S(\langle\mu\xi'\rangle, dx_0^2 + \tilde{G}_\mu)$ where $P^\mu(x, \xi) = \mu^2 P(y, \eta)$. Since $P^\mu(x, \xi) = \mu^2 P(M_\mu(x, \xi))$ when $|x_j| \leqq \mu^{1/2}$, $|\xi_j|\xi'|^{-1} - \delta_{jp}| \leqq \mu$, by Proposition A.3 in [12], to prove Theorem 1.1 it suffices to show that there is a parametrix with finite propagation speed of wave front sets of $\hat{P}$ (with some fixed positive μ) at $\rho' = (0, 0, e_p')$. Therefore in the following sections we shall study $\tilde{P}(x, D, \mu)$ and $\hat{P}(x, D, \mu)$.

Next following § 5 in [12] we introduce

$$\alpha_\varepsilon(\psi)(x, \xi', \mu) = \chi_3(\varepsilon n^{1/2}\psi(x, \xi', \mu)\langle\mu\xi'\rangle^{1/2})$$

$$J_\varepsilon(\psi)(x, \xi', \mu) = \varepsilon\{2\chi_2(\varepsilon\psi\langle\mu\xi'\rangle^{1/2}) - 1\}\psi + \langle\mu\xi'\rangle^{-1/2}$$

$$J_\varepsilon(r, \psi) = J_\varepsilon(\psi)^{-r}, \; I_\varepsilon(r, \psi) = \langle\mu\xi'\rangle^{n\varepsilon^*} J_\varepsilon(n\varepsilon + r, \psi)$$

$$m(\psi)(x, \xi', \mu) = \{\psi^2(x, \xi', \mu) + \langle\mu\xi'\rangle^{-1}\}^{1/2}$$

where $\varepsilon \in \{-1, 1\}$, $\varepsilon^* = \max(-\varepsilon, 0)$, $r \in \boldsymbol{R}$, $n \in \boldsymbol{R}^+$ and $\chi_2(s)$, $\chi_3(s) \in C^\infty(\boldsymbol{R})$ are the same ones in [12]. Note that

(3.13) $$c_1 m(\psi) \leqq J_\varepsilon(\psi) \leqq c_2 m(\psi)$$

with positive constants c_i independent of μ and

$$(3.14) \quad \sum_{\varepsilon} \alpha_\varepsilon(\psi) = 1, \ \alpha_\varepsilon(\psi)^{(\alpha)}_{(\beta)} \in S(\langle\xi'\rangle^{-|\alpha|} m(\psi)^{-|\alpha+\beta|}, g_\mu) \text{ for any } \alpha, \beta.$$

Remark that

$$(3.15) \quad \begin{aligned} &\partial J_\varepsilon(\psi)/\partial x_k = K_{\varepsilon k}(\psi)\,\partial\psi/\partial x_k, \\ &\partial J_\varepsilon(\psi)/\partial \xi_k = K^k_\varepsilon(\psi)\,\partial\psi/\partial\xi_k + S(\langle\xi'\rangle^{-1}\langle\mu\xi'\rangle^{-1/2}, g_\mu) \end{aligned}$$

where $K_{\varepsilon k}(\psi)$, $K^k_\varepsilon(\psi) \in S(1, g_\mu)$, $K_{\varepsilon k}(\psi) = K^k_\varepsilon(\psi) = \varepsilon$ on supp $\alpha_\varepsilon(\psi)$ when $n \geqq 16$ and for $|\alpha+\beta| \leqq 1$

$$(3.16) \quad I_\varepsilon(r, \psi)^{(\alpha)}_{(\beta)} \in S(\langle\xi'\rangle^{-|\alpha|}\langle\mu\xi'\rangle^{n\varepsilon^*} m(\psi)^{-n\varepsilon-r-|\alpha+\beta|}, g_\mu).$$

We define $\varphi_j(x, \xi', \mu)$ $(1 \leqq i \leqq l)$ according to the preceding formula from $\hat{f}_j(x, \xi')$. Let $S = (s(1), \cdots, s(l)) \in \{-1, 1\}^l$, and $R = (r_1, \cdots, r_l) \in \boldsymbol{R}^l$ we put

$$I_S(R, \varphi) = \prod_{j=1}^{l} I_{s(j)}(r_j, \varphi_j)(x, \xi', \mu), \ \alpha_S(\varphi) = \mathrm{Op}\alpha_{s(1)}(\varphi_1)\cdots\mathrm{Op}\alpha_{s(l)}(\varphi_l)$$
$$m(\varphi) = (m(\varphi_1), \cdots, m(\varphi_l)).$$

Note that $m(\varphi_j) \in \mathscr{M}$ for φ_j verifies (3.7) (see §4 in [12]). From (3.16) it follows that

$$(3.17) \quad \begin{aligned} &I_S(R, \varphi)^{(\alpha)}_{(\beta)} = \sum_{j=1}^{l} C_{\alpha\beta j}, \\ &C_{\alpha\beta j} \in S(\langle\xi'\rangle^{-|\alpha|}\langle\mu\xi'\rangle^{nS^*} m(\varphi)^{-nS-R-|\alpha+\beta|e_j}, g_\mu) \end{aligned}$$

for $|\alpha+\beta| \leqq 1$ where e_j is the unit vector in $\boldsymbol{R}^l$ with j-th component equal to 1 and $S^* = \sum_{j=1}^{l} s(j)^*$.

Denote by $[n, m]$ the set of integers $\{n, n+1, \cdots, m\}$ and let $K = \{i_1, \cdots, i_k\}$ be a subset of $[1, l] = I$ with $i_1 < i_2 < \cdots < i_k$. Then we shall write

$$\alpha_{S,K}(\varphi) = \mathrm{Op}\alpha_{s(i_1)}(\varphi_{i_1})\cdots\mathrm{Op}\alpha_{s(i_k)}(\varphi_{i_k}).$$

By $|K|$ we denote the number of elements of K. Let $K \subset L \subset I$. We set $\varepsilon_j(K, L) = 1$ if $j \in L \backslash K$ and $\varepsilon_j(K, L) = 0$ if $j \notin L \backslash K$ and set

$$\varepsilon(K, L) = (\varepsilon_1(K, L), \cdots, \varepsilon_l(K, L)).$$

If $L = I$ we write $\varepsilon(K)$ instead of $\varepsilon(K, I)$. For $Q = (q_1, \cdots, q_l) \in \boldsymbol{R}^l$ we put

$$\varepsilon(K, L) \circ Q = (\varepsilon_1(K, L) q_1, \cdots, \varepsilon_l(K, L) q_l)$$
$$|\varepsilon(K, L) \circ Q| = \varepsilon_1(K, L) q_1 + \cdots + \varepsilon_l(K, L) q_l$$

In what follows we shall write $I_S(R)$, $J_{s(j)}(r)$, α_S, $\alpha_{s(j)}$ instead of $I_S(R,\varphi)$, $J_{s(j)}(r,\varphi_j)$, $\alpha_S(\varphi)$, $\alpha_{s(j)}(\varphi_j)$ if there will be no confusion.

Lemma 3.1. *Let $S\in\{-1,1\}^l$, $L\subset I$ and $i\notin L$. Then we have*

$$[\alpha_{s(i)},\alpha_{S\circ L}]=\sum_K T_K\alpha_{S\circ K}$$

where the sum is taken over all $K\subset L$ with $|K|\leqq|L|-1$ and

$$T_K\in S(\langle\xi'\rangle^{-(|L|-|K|)}\langle\mu\xi'\rangle^{(|L|-|K|)+|\varepsilon(K,\hat{L})\circ Q|/2}m(\varphi)^{\varepsilon(K,\hat{L})\circ Q},g_\mu)$$

for any $Q\in\boldsymbol{R}^l$ with $\hat{L}=L\cup\{i\}$.

Lemma 3.2. *Let $S\in\{-1,1\}^l$ and $K\subset L\subset I$. Assume that*

$$T\in S(\langle\mu\xi'\rangle^{k+2nS^*+|\varepsilon(K,L)\circ Q|/2}m(\varphi)^{-2nS-R+\varepsilon(K,L)\circ Q},g_\mu)$$
$$(resp.\ \in S(\langle\mu\xi'\rangle^{k+nS^*+|\varepsilon(K,L)\circ Q|/2}m(\varphi)^{-nS-R+\varepsilon(K,L)\circ Q},g_\mu)).$$

Then for any $\tilde{S}\in\{-1,1\}^l$ with $\tilde{S}=S$ on $K\cup(I\backslash L)$ we have

$$T\in S(\langle\mu\xi'\rangle^{k+nS^*+n\tilde{S}^*+|\varepsilon(K,L)\circ U|/2}m(\varphi)^{-nS-n\tilde{S}-R+\varepsilon(K,L)\circ U},g_\mu)$$
$$(resp.\ \in S(\langle\mu\xi'\rangle^{k+n\tilde{S}^*+|\varepsilon(K,L)\circ U|/2}m(\varphi)^{-n\tilde{S}-R+\varepsilon(K,L)\circ U},g_\mu))$$

for any $U=(u_1,\cdots,u_l)\in\boldsymbol{R}^l$ satisfying $u_j=n(s(j)-\tilde{s}(j))+q_j$ when $j\in L\backslash K$. In particular if the hypothesis is verified for any $Q\in\boldsymbol{R}^l$ then the assertion holds for any $U\in\boldsymbol{R}^l$.

Proof. Note that

$$s(j)-\tilde{s}(j)=-2(s(j)^*-\tilde{s}(j)^*).$$

With this choice of u_j $(j\in L\backslash K)$ it follows that

$$\sum_{j\in L\backslash K}u_j=-2n\sum_{j\in L\backslash K}(s(j)^*-\tilde{s}(j)^*)+\sum_{j\in L\backslash K}q_j.$$

Recalling that $\varepsilon_j(K,L)=1$ if $j\in L\backslash K$ and $\varepsilon_j(K,L)=0$ if $j\notin L\backslash K$, these imply that

$$2nS^*+|\varepsilon(K,L)\circ Q|/2=nS^*+n\tilde{S}^*+|\varepsilon(K,L)\circ U|/2$$
$$(\text{resp. } nS^*+|\varepsilon(K,L)\circ Q|/2=n\tilde{S}^*+|\varepsilon(K,L)\circ U|/2)$$
$$-2nS-R+\varepsilon(K,L)\circ Q=-nS-n\tilde{S}-R+\varepsilon(K,L)\circ U$$
$$(\text{resp. } -nS-R+\varepsilon(K,L)\circ Q=-n\tilde{S}-R+\varepsilon(K,L)\circ U)$$

Clearly this proves the lemma.

Corollary 3.1. *Assume the same hypothesis as in Lemma 3.2. Then* *for any* $\tilde{S}\in\{-1, 1\}^{l}$ *with* $\tilde{S}=S$ *on* $K\cup(I\backslash L)$, *for any* $V\in\boldsymbol{R}^{l}$ *satisfying the same condition as that for* U *in Lemma* 3.2 *and for any* $R_i\in\boldsymbol{R}^{l}$ *with* $R_1+R_2=R-\varepsilon(K, L)\circ V$ (*resp.* $\tilde{R}$ *with* $\tilde{R}=R-\varepsilon(K, L)\circ V$) *we have*

$$T\equiv I_{\tilde{S}}^{*}(R_1)(B_{\tilde{S}}+C_{\tilde{S}})I_{\tilde{S}}(R_2) \qquad (resp.\ T\equiv(B_{\tilde{S}}+C_{\tilde{S}})I_{\tilde{S}}(\tilde{R}))$$

where $\sigma(B_{\tilde{S}})=\sigma(I_S(R_1))^{-1}\sigma(I_{\tilde{S}}(R_2))^{-1}\sigma(T)\in S(\langle\mu\xi'\rangle^{k+|\varepsilon(K,L)\circ V|/2}, g_\mu)$ *and* $C_{\tilde{S}}\in S(\mu\langle\mu\xi'\rangle^{k+|\varepsilon(K,L)\circ V|/2}, g)$ (*resp.* $\sigma(B_{\tilde{S}})=\sigma(I_{\tilde{S}}(\tilde{R}))^{-1}\sigma(T)$).

Lemma 3.3. *Assume that* T *satisfies the same hypothesis as in Lemma* 3.2 *for any* $Q\in\boldsymbol{R}^{l}$. *Then for any* R_i, $V\in\boldsymbol{R}^{l}$ *with* $R_1+R_2=R-\varepsilon(K, L)\circ V$ (*resp.* $\tilde{R}$ *with* $\tilde{R}=R-\varepsilon(K, L)\circ V$) *we have*

$$T\alpha_{S\circ K}\equiv\sum_{\tilde{S}} I_{\tilde{S}}^{*}(R_1)(B_{\tilde{S}}+C_{\tilde{S}})I_{\tilde{S}}(R_2)\alpha_{\tilde{S}\circ L}$$
$$(resp.\ T\alpha_{S\circ K}\equiv\sum_{\tilde{S}}(B_{\tilde{S}}+C_{\tilde{S}})I_{\tilde{S}}(\tilde{R})\alpha_{\tilde{S}\circ L})$$

where the sum is taken over all $\tilde{S}$ *with* $\tilde{S}=S$ *on* $K\cup(I\backslash L)$ *and* $\sigma(B_{\tilde{S}})=\sigma(I_S(R_1))^{-1}\sigma(I_{\tilde{S}}(R_2))^{-1}\sigma(T)\in S(\langle\mu\xi'\rangle^{k+|\varepsilon(K,L)\circ V|/2}, g_\mu)$, $C_{\tilde{S}}\in S(\mu\times\langle\mu\xi'\rangle^{k+|\varepsilon(K,L)\circ V|/2}, g)$ (*resp.* $\sigma(B_{\tilde{S}})=\sigma(I_{\tilde{S}}(\tilde{R}))^{-1}\sigma(T)$).

Remark 3.1. It is clear from the proof that we can write

$$TA\alpha_{S\circ K}\equiv\sum_{\tilde{S}} I_{\tilde{S}}^{*}(R_1)(B_{\tilde{S}}+C_{\tilde{S}})I_{\tilde{S}}(R_2)A\alpha_{\tilde{S}\circ L}$$

for any operator A.

Remark 3.2. In Lemma 3.3 and Corollary 3.1 if T verifies the same condition with the metric g then the same conclusion holds with $B_{\tilde{S}}\in S(\langle\mu\xi'\rangle^{k+|\varepsilon(K,L)\circ V|/2}, g)$ and $C_{\tilde{S}}=0$.

We denote by $\|\cdot\|$ L^2 norm in $L^2(\boldsymbol{R}^d)$. Let β, γ be operators from $H^{-\infty}=\bigcup_s H^s$ to $H^{-\infty}$ then we put

$$^{(\gamma)}|u|^2_{\beta,S,R,m}=\|\gamma\langle\mu D'\rangle^m I_S(R)\beta\alpha_S u\|^2$$
$$^{(\gamma)}|u|^2_{\beta,S,R\pm(t),m}=\sum_h {}^{(\gamma)}|u|^2_{\beta,S,R\pm h,m}$$

where the sum is taken over $h=(h_1,\cdots,h_l)\in(2^{-1}\boldsymbol{N})^{l}$ with $|h|=h_1+\cdots+h_l=t$. Also we set

$$^{(\gamma)}[u]^2_{\beta,R,m}=\sum_S {}^{(\gamma)}|u|^2_{\beta,S,R,m},\quad {}^{(\gamma)}[u]^2_{\beta,R+(t),m}=\sum_S {}^{(\gamma)}|u|^2_{\beta,S,R+(t),m}.$$

In these notations we drop β (resp. γ) when β (resp. γ) is the identity and drop both β and γ if $\beta=\gamma=$identity.

§ 4. Energy Estimate for A^2 and L_k^2

Let l_0, l_k be one of l_{j0}, l_{jk} $(1 \leqq k \leqq p-1)$ respectively. $L_0(x, \xi, \mu)$ and $L_k(x, \xi', \mu)$ will be defined by (3. 1). As mentioned in Introduction we proceed along the lines in [9] and [10] but we must be careful with negative terms in energy estimate which will be of the form $[u]_{\tilde{e}-\tilde{e}_\nu,1}$ where $\tilde{e}=2^{-1}e$, $\tilde{e}_\nu=2^{-1}e_\nu$, $e=(1, 1, \cdots, 1) \in \boldsymbol{R}^l$ and there we can take any $\nu \in I$. We put

$$A(x, \xi, \mu)=L_0(x, \xi_0-i\theta, \xi', \mu)=\xi_0-i\theta-a(x, \xi', \mu)$$

with a large positive parameter θ. We start with

$$\begin{aligned}(4.1)\quad & -2\mathrm{Im}(I_S(Q)A\alpha_S u, I_S(Q)\alpha_S u)=\partial_0 |u|^2_{S,Q,0}+2\theta |u|^2_{S,Q,0} \\ & +2\mathrm{Im}(aI_S(Q)\alpha_S u, I_S(Q)\alpha_S u)-2\mathrm{Im}([I_S(Q), A]\alpha_S u, I_S(Q)\alpha_S u).\end{aligned}$$

To simplify notation we write w_S instead of $\alpha_S u$ unless otherwise indicated. From (3. 1) and (3. 2) it follows that $a^*\text{-}a \in S(1, g_\mu)$ and hence the third term in the right-hand side of (4. 1) is estimated by

$$(c+c(a)\mu)\, |u|^2_{S,Q,0}$$

in view of Lemma 4. 6 in [12]. We observe $[I_S(Q), A]$. Since $A^{(\alpha)} \in S(\langle\xi'\rangle^{-1}, g_\mu)$, $A_{(\alpha)} \in S(\langle\xi'\rangle, g_\mu)$ for $|\alpha|=2$ it follows from (3. 17) that

$$\sigma([I_S(Q), A])=-i\{I_S(Q), A\}+r, \quad r \in S(\mu\langle\mu\xi'\rangle^{nS^*}m(\varphi)^{-nS-Q}, g).$$

Taking (3. 15) into account one has

$$\partial I_S(Q)/\partial x_k=-\sum_{j=1}^{l} (ns(j)+q_j)I_S(Q+e_j)K_{s(j)k}(\varphi_j)\,\partial\varphi_j/\partial x_k$$

$$\partial I_S(Q)/\partial \xi_k=-\sum_{j=1}^{l} (ns(j)+q_j)I_S(Q+e_j)K^k_{s(j)}(\varphi_j)\,\partial\varphi_j/\partial \xi_k+\sum_{j=1}^{l} r_j$$

with $r_j \in S(\langle\xi'\rangle^{-1}\langle\mu\xi'\rangle^{nS^*-1/2}m(\varphi)^{-nS-Q-e_j}, g_\mu)$. Recalling that

$$K_{s(j)k}(\varphi_j)=K^k_{s(j)}(\varphi_j)=s(j)$$

on $\mathrm{supp}\,\alpha_{s(j)}$ when $n \geqq 16$ we get

$$\begin{aligned}\sigma([I_S(Q), A])=&\, i\sum_{j=1}^{l} (n+q_j s(j))I_S(Q+e_j)\{\varphi_j, A\} \\ &+\sum_{j=1}^{l} (ns(j)+q_j)I_S(Q+e_j)B_j+R\end{aligned}$$

where $B_j \in S(1, g_\mu)$, $B_j = 0$ on supp$\alpha_{s(j)}$ when $n \geqq 16$ and R is in $S(\langle \mu\xi' \rangle^{nS^*} m(\varphi)^{-nS-Q}, g_\mu)$. Note that $R(A, \varphi_j) \in S(\mu^{1/2}, g_\mu) + S(m(\varphi_j), g_\mu)$ and substitute (3.8) into above expression to get

$$\begin{aligned}\sigma([I_S(Q), A]) = & -i\sum (n + q_j s(j)) I_S(Q+e_j) D(A, \varphi_j) \\ & + \sum (ns(j) + q_j) I_S(Q+e_j) B_j + \sum R_j + R\end{aligned} \tag{4.2}$$

with $R_j \in S(\mu^{1/2} \langle \mu\xi' \rangle^{nS^*} m(\varphi)^{-nS-Q-e_j}, g_\mu)$, $R \in S(\langle \mu\xi' \rangle^{nS^*} m(\varphi)^{-nS-Q}, g_\mu)$. Let us consider $I_S^*(Q)[I_S(Q), A]\alpha_S$. By Corollary 3.1 we have

$$I_S^*(Q) R_j \equiv I_S^*(Q+\tilde{e}_j) B_S I_S(Q+\tilde{e}_j), B_S \in S(\mu^{1/2}, g),$$

$$I_S^*(Q) R \equiv I_S^*(Q) B_S I_S(Q), B_S \in S(1, g)$$

and hence $(R_j w_S, I_S(Q) w_S)$, $(R w_S, I_S(Q) w_S)$ are estimated by

$$c(n, \lambda) \mu^{1/2} |u|^2_{S, Q+(1/2), 0}, \; c(n, \lambda) |u|^2_{S, Q, 0}$$

respectively. Observe $I_S^*(Q) \mathrm{Op}(I_S(Q+e_j) B_j) \alpha_S$. Put $M = [1, j-1]$, $L = [j+1, l]$ then

$$\alpha_S = [\alpha_{S \circ M}, \alpha_{s(j)}] \alpha_{S \circ L} + \alpha_{s(j)} \alpha_{S \circ J} \text{ with } J = M \cup L.$$

Applying Lemmas 3.1 and 3.3 one obtains

$$I_S^*(Q) \mathrm{Op}(I_S(Q+e_j) B_j) [\alpha_{S \circ M}, \alpha_{s(j)}] \equiv \sum_{\tilde{S}} I_S^*(Q+\tilde{e}_j) B_{\tilde{S}} I_{\tilde{S}}(Q+\tilde{e}_j) \alpha_{\tilde{S} \circ \hat{M}}$$

where $\tilde{S} = S$ on L, $B_{\tilde{S}} \in S(\mu, g)$ and $\hat{M} = M \cup \{j\}$. Note that the right-hand side multiplied by $\alpha_{S \circ L}$ to the right can be written as

$$\sum_{\tilde{S}} I_S^*(Q+\tilde{e}_j) B_{\tilde{S}} I_{\tilde{S}}(Q+\tilde{e}_j) \alpha_{\tilde{S}}.$$

We turn to $I_S^*(Q) \mathrm{Op}(I_S(Q+e_j) B_j) \alpha_{s(j)} \alpha_{S \circ J}$. Since $B_j = 0$ on supp$\alpha_{s(j)}$ when $n \geqq 16$ it follows from Lemma 4.8 in [12] that $\mathrm{Op}(I_S(Q+e_j) B_j) \alpha_{s(j)}$ belongs to

$$S(\langle \xi' \rangle^{-1} \langle \mu\xi' \rangle^{nS^*+1+|\varepsilon(\{j\}) \circ V|/2} m(\varphi)^{-nS-\tilde{e}-e_j+\varepsilon(\{j\}) \circ V}, g_\mu).$$

Then by Lemmas 3.1 and 3.3 again we obtain

$$I_S^*(Q) \mathrm{Op}(I_S(Q+e_j) B_j) \alpha_{s(j)} \equiv \sum I_S^*(Q+\tilde{e}_j) B_{\tilde{S}} I_{\tilde{S}}(Q+\tilde{e}_j) \alpha_{\tilde{S} \circ \{j\}}, B_{\tilde{S}} \in S(\mu, g).$$

The above argument shows that $|(\mathrm{Op}(I_S(Q+e_j) B_j) w_S, I_S(Q) w_S)|$ is estimated by

$$c(n, \lambda) [u]^2_{Q+(1/2), 0}.$$

Now we handle the term $\mathrm{Op}(I_S(Q+e_j) D(A, \varphi_j))$. Recall that

$$D(A, \varphi_j) = \{l_0, \tilde{f}_j\}(\rho) + b(x', \lambda) + \sum_{s=1}^{p-1} (\chi_0^{(1)}(x_s) - 1) l_0^{(s)}(\rho) \tilde{f}_{j(s)}(\rho').$$

Let $|l_i^{(s)}(\rho)\tilde{f}_{j(s)}(\rho')| \leqq \hat{c}$ $(0 \leqq i \leqq p-1)$ then one has

$$D(A, \varphi_j) \geqq \{l_0, \tilde{f}_j\}(\rho) + (\lambda - \hat{c}) \sum_s (1 - \chi_0^{(1)}(x_s)).$$

Then taking $\lambda \geqq \hat{c}$ we have $D(A, \varphi_j) \geqq \{l_0, \tilde{f}_j\}(\rho) > 0$ in view of Lemma 2.4. Set $\beta = \beta(A, \varphi_j) = D^{1/2}(A, \varphi_j) \in S_{1,0}^0$ then it follows that

$$I_S^*(Q)\,\mathrm{Op}(I_S(Q+e_j)D(A, \varphi_j)) \equiv I_S^*(Q+\tilde{e}_j)\beta^*(1+r)\beta I_S(Q+\tilde{e}_j)$$

with $r \in S(\mu, g)$. This shows that

$$\begin{aligned}(4.3)\qquad &\mathrm{Im}\, i \sum (n+s(j)q_j)\,\mathrm{Op}(I_S(Q+e_j)D(A, \varphi_j))w_S, I_S(Q)w_S) \\ &\geqq (1 - c(n, \lambda)\mu) \sum (n+s(j)q_j)\, {}^{(\beta)}|u|^2_{S, Q+\tilde{e}_j, 0}.\end{aligned}$$

Summing up we get an estimate of $-2\mathrm{Im}([I_S(Q), A]w_S, I_S(Q)w_S)$ from below by

$$\begin{aligned}&2(1 - c(n, \lambda)\mu) \sum (n+s(j)q_j)\, {}^{(\beta)}|u|^2_{S, Q+\tilde{e}_j, 0} \\ &\quad - c(n, \lambda)\mu^{1/2}[u]^2_{Q+(1/2), 0} - c(n, \lambda)\,|u|^2_{S, Q, 0}.\end{aligned}$$

Lemma 4.1.

$$\begin{aligned}&-2\mathrm{Im}(I_S(Q)Aw_S, I_S(Q)w_S) \geqq \partial_0 |u|^2_{S, Q, 0} + 2\theta(1 - c(n, \lambda)\theta^{-1})\,|u|^2_{S, Q, 0} \\ &\quad + 2(1 - c(n, \lambda)\mu) \sum (n+s(j)q_j)\, {}^{(\beta)}|u|^2_{S, Q+\tilde{e}_j, 0} - c(n, \lambda)\mu^{1/2}[u]^2_{Q+(1/2), 0}\end{aligned}$$

where $\beta = \beta(A, \varphi_j)$ *for any* $u \in C^\infty(I, H^\infty)$ *and* $|x_0| \leqq \mu^{1/2}$.

We shall estimate ${}^{(\beta)}|u|^2_{T, S, Q+\tilde{e}_j, 0}$ from above and below by $|u|^2_{T, S, Q, 0}$. Noting that $c(\lambda) \geqq D(A, \varphi_j) \geqq (1+\lambda)^{-1}\{l_0, \tilde{f}_j\}(\rho) \geqq (1+\lambda)^{-1}c$ with a positive constant c when $\lambda \geqq \hat{c}$ and

$$\beta^*\beta = \mathrm{Op}D(A, \varphi_j) + S_{1,0}^{-1},\ \|\langle D'\rangle^{-1/2} I_S(Q+\tilde{e}_j)Tw_S\| \leqq c(n)\mu^{1/2}|u|_{T, S, Q, 0}$$

it follows from the sharp Gårding inequality that

$$\begin{aligned}(4.4)\qquad &{}^{(\beta)}|u|^2_{T, S, Q+\tilde{e}_j, 0} \geqq c_1(\lambda)\,|u|^2_{T, S, Q+\tilde{e}_j, 0} - c(n, \lambda)\mu|u|^2_{T, S, Q, 0}, \\ &{}^{(\beta)}|u|^2_{T, S, Q+\tilde{e}_j, 0} \leqq c_2(\lambda)\,|u|^2_{T, S, Q+\tilde{e}_j, 0} + c(n, \lambda)\mu|u|^2_{T, S, Q, 0}\end{aligned}$$

with positive constants $c_i(\lambda)$.

Corollary 4.1. *Let* $|q_j| \leqq B$ *with a fixed* B. *Then we have*

$$-2\mathrm{Im} \sum_S (I_S(Q)Aw_S,\ I_S(Q)w_S) \geqq \partial_0[u]^2_{Q, 0} + \theta[u]^2_{Q, 0} + nc(\lambda)[u]^2_{Q+(1/2), 0}$$

with a positive constant $c(\lambda)$ *for* $(16 \leqq)\hat{n} \leqq n$, $0 < \mu \leqq \hat{\mu}(n, \lambda)$, $\hat{\theta}(n, \mu, \lambda) \leqq \theta$, $|x_0| \leqq \mu^{1/2}$ *and for any* $u \in C^\infty(I, H^\infty)$.

Taking $Q=\tilde{e}+\tilde{e}_i+\tilde{e}_k$ in Corollary 4.1 and noting

$$I_S^*(Q)I_S(Q)\equiv I_S^*(\tilde{e}+\tilde{e}_i+e_k)(1+r)I_S(\tilde{e}+\tilde{e}_i),\ r\in S(\mu,g)$$

it follows that

$$c_1(\lambda)[u]^2_{A,\tilde{e}+\tilde{e}_i,0}\geqq n\hat{\partial}_0[u]^2_{Q,0}+\theta n[u]^2_{Q,0}+n^2c_2(\lambda)[u]^2_{Q+(1/2),0}.$$

Summing up over i, $k\in I$ we get

$$c_1(\lambda)[u]^2_{A,\tilde{e}+(1/2),0}\geqq n\partial_0[u]^2_{\tilde{e}+(1),0}+n\theta[u]^2_{\tilde{e}+(1),0}+n^2c_2(\lambda)[u]^2_{\tilde{e}+(3/2),0}$$

for $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n,\lambda)$, $\hat{\theta}(n,\mu,\lambda)\leqq\theta$ with positive constants $c_i(\lambda)$. If we take $Q=\tilde{e}+\tilde{e}_i$ and sum over $i\in I$ we shall obtain a similar estimate. We summarize;

$$\begin{aligned}(4.5)_j\qquad & c_1(\lambda)[u]^2_{A,\tilde{e}+(j/2),0}\geqq n\partial_0[u]^2_{\tilde{e}+((j+1)/2),0}\\ &\quad+n\theta[u]^2_{\tilde{e}+((j+1)/2),0}+c_2(\lambda)n^2[u]^2_{\tilde{e}+(1+j/2),0}\end{aligned}$$

where $j=0$, 1. Next we observe

$$\begin{aligned}(4.6)\qquad & -2\mathrm{Im}(I_S(\tilde{e})A^2w_S, I_S(\tilde{e})Aw_S)=\partial_0|u|^2_{A,S,\tilde{e},0}+2\theta|u|^2_{A,S,\tilde{e},0}\\ &\quad+2\mathrm{Im}(aI_S(\tilde{e})Aw_S, I_S(\tilde{e})Aw_S)-2\mathrm{Im}([I_S(\tilde{e}),A]Aw_S, I_S(\tilde{e})Aw_S).\end{aligned}$$

As shown above the third term in the right-hand side of (4.6) is estimated by

$$(c+c(a)\mu)|u|^2_{A,S,\tilde{e},0}.$$

We turn to the last term in the right-hand side. Recall (4.2) with $Q=\tilde{e}$. It is clear that $(R_jAw_S, I_S(\tilde{e})Aw_S)$, $(RAw_S, I_S(\tilde{e})w_S)$ are estimated by

$$c(n)\mu^{1/2}[u]^2_{A,\tilde{e}+(1/2),0},\quad c(n)|u|^2_{A,S,\tilde{e},0}$$

respectively. We shall estimate $I_S^*(\tilde{e})\mathrm{Op}(I_S(\tilde{e}+e_j)B_j)A\alpha_S$. Using the same notation as in the proof of Lemma 4.1 we have

$$I_S^*(\tilde{e})\mathrm{Op}(I_S(\tilde{e}+e_j)B_j)(A[\alpha_{S\circ M},\alpha_{s(j)}]\alpha_{S\circ L}+A\alpha_{s(j)}\alpha_{S\circ J}).$$

From Lemma 3.1 we have

$$A[\alpha_{S\circ M},\alpha_{s(j)}]=\sum T_NA\alpha_{S\circ N}+\sum[A,T_N]\alpha_{S\circ N}.$$

It is clear that $I_S^*(\tilde{e})\mathrm{Op}(I_S(\tilde{e}+e_j)B_j)T_NA\alpha_{S\circ N}$ can be written as

$$\sum_{\tilde{S}} I_S^*(\tilde{e}+\tilde{e}_j)B_{\tilde{S}}I_{\tilde{S}}(\tilde{e}+\tilde{e}_j)A\alpha_{\tilde{S}\circ\hat{M}},\qquad B_{\tilde{S}}\in S(\mu,g)$$

where $\tilde{S}=S$ on L. Since

$$[A,T_N]\in S(\mu\langle\mu\xi'\rangle^{|\varepsilon(N,\hat{M})\circ Q'|/2+1/2}m(\varphi)^{\varepsilon(N,\hat{M})\circ Q},g_\mu),\qquad \hat{M}=M\cup\{j\}$$

for any $Q\in\boldsymbol{R}^l$, Lemma 3.3 shows that $I_S^*(\tilde{e})\mathrm{Op}(I_S(\tilde{e}+e_j)B_j)[A, T_N]$ $\alpha_{S\circ N}$ can be written as

$$\sum_{\tilde{S}} I_{\tilde{S}}^*(\tilde{e}+\tilde{e}_j)B_{\tilde{S}}I_{\tilde{S}}(\tilde{e}+\tilde{e}_j+e_j)\alpha_{\tilde{S}\circ\hat{M}},\qquad B_{\tilde{S}}\in S(\mu, g)$$

where $\tilde{S}=S$ on L. As for $I_S^*(\tilde{e})\,\mathrm{Op}\,(I_S(\tilde{e}+e_j)B_j)\,A\alpha_{s(j)}\alpha_{S\circ J}$ writing $A\alpha_{s(j)}\alpha_{S\circ J}=\alpha_{s(j)}\,A\alpha_{S\circ J}+[A, \alpha_{s(j)}]\alpha_{S\circ J}$ and noting that

$$[A, \alpha_{s(j)}]\in S(\langle\mu\xi'\rangle^{1/2+r/2}m(\varphi_j)^r, g_\mu)\qquad\text{for any } r\in\boldsymbol{R}$$

we shall obtain similar expressions. Combining above expressions we get

$$\begin{aligned}I_S^*(\tilde{e})\mathrm{Op}(I_S(\tilde{e}+e_j)B_j)A\alpha_S &\equiv \sum_{\tilde{S}} I_{\tilde{S}}^*(\tilde{e}+\tilde{e}_j)B_{\tilde{S}}I_{\tilde{S}}(\tilde{e}+\tilde{e}_j)A\alpha_{\tilde{S}}\\ &+\sum_{\tilde{S}} I_{\tilde{S}}^*(\tilde{e}+\tilde{e}_j)B_{\tilde{S}}I_{\tilde{S}}(\tilde{e}+3\tilde{e}_j)\alpha_{\tilde{S}},\qquad B_{\tilde{S}}\in S(\mu, g).\end{aligned}$$

For later use we state our arguments as a lemma.

Lemma 4.2. *For any i, k $(0\leqq i, k\leqq l)$ we have*

$$\begin{aligned}I_S^*(\tilde{e})[I_S(\tilde{e}), L_i]L_k\alpha_S &\equiv -i\sum_j (n+s(j)/2)I_S^*(\tilde{e}+\tilde{e}_j)\beta_j^*\gamma_jI_S(\tilde{e}+\tilde{e}_j)L_k\alpha_S\\ &+\sum_{\tilde{S},j} I_{\tilde{S}}^*(\tilde{e}+\tilde{e}_j)B_{j\tilde{S}}I_{\tilde{S}}(\tilde{e}+\tilde{e}_j)L_k\alpha_{\tilde{S}}+\sum_{\tilde{S},j} I_{\tilde{S}}^*(\tilde{e}+\tilde{e}_j)B_{j\tilde{S}}I_{\tilde{S}}(\tilde{e}+3\tilde{e}_j)\alpha_{\tilde{S}}\end{aligned}$$

with $B_{j\tilde{S}}\in S(\mu^{1/2}, g)$ where β_j, $\gamma_j\in S_{1,0}^0$, $\sigma(\beta_j)\sigma(\gamma_j)=D(L_i, \varphi_j)$.

Remark 4.1. Clearly the same argument shows that

$$[I_S(\tilde{e}), L_i]\alpha_S\equiv -i\sum_j (n+s(j)/2)\beta_j^*\gamma_jI_S(\tilde{e}+\tilde{e}_j)\alpha_S+\sum_{\tilde{S},j} B_{j\tilde{S}}I_{\tilde{S}}(\tilde{e}+\tilde{e}_j)\alpha_{\tilde{S}}$$

with $B_{j\tilde{S}}\in S(\mu^{1/2}, g)$ where β_j and γ_j are as in Lemma 4.2.

We return to estimate the last term in the right-hand side of (4.6). Since $\mathrm{Im}\, i\sum(n+s(j)/2)\,(\mathrm{Op}(I_S(\tilde{e}+\tilde{e}_j)\,D(A, \varphi_j))\,Aw_S, I_S(\tilde{e}+\tilde{e}_j)$ $Aw_S)$ is bounded from below by (see (4.3))

$$(1-c(n, \lambda)\mu)\sum(n+s(j)/2)^{(\beta)}|u|_{A,S,\tilde{e}+\tilde{e}_j,0}^2$$

we obtain the following estimate

$$\begin{aligned}&-2\mathrm{Im}(I_S(\tilde{e})A^2w_S, I_SAw_S)\geqq\partial_0|u|_{A,S,\tilde{e},0}^2+2\theta(1-c(n, \lambda)\theta^{-1})\,|u|_{A,S,\tilde{e},0}^2\\ (4.7)\qquad &+(1-c(n, \lambda)\mu)\sum(2n+s(j))^{(\beta)}|u|_{A,S,\tilde{e}+\tilde{e}_j,0}^2-c(n, \lambda)\mu^{1/2}[u]_{A,\tilde{e}+(1/2),0}^2\\ &-c(n, \lambda)\mu^{1/2}[u]_{\tilde{e}+(3/2),0}^2\end{aligned}$$

for any $u\in C^\infty(I, H^\infty)$, $|x_0|\leqq\mu^{1/2}$. We insert the estimate $(4.5)_j$ $(j=0, 1)$ into a part of the above estimate;

$$\delta n^{(\beta)}[u]^2_{A,\tilde{e}+(1/2),0}+2\theta\delta[u]^2_{A,\tilde{e},0}. \tag{4.8}$$

In view of (4.4) this is estimated from below by $\delta nc(\lambda)[u]^2_{A,\tilde{e}+(1/2),0}+\theta\delta[u]^2_{A,\tilde{e},0}$ when $\theta\geqq\hat{\theta}(n,\lambda)$. Substituting the estimates $(4.5)_j$ $(j=0,1)$ into this, (4.8) is estimated from below by

$$\begin{aligned}&\partial_0\{c_1(\delta,\lambda)n^2[u]^2_{\tilde{e}+(1),0}+c_2(\delta,\lambda)n\theta[u]^2_{\tilde{e}+(1/2),0}\}\\ &\qquad+c_3(n,\lambda)\{n\theta^2[u]^2_{\tilde{e}+(1/2),0}+n^2\theta[u]^2_{\tilde{e}+(1),0}+n^3[u]^2_{\tilde{e}+(3/2),0}\}.\end{aligned} \tag{4.9}$$

Inserting this into (4.7) it follows that for any sufficiently small $\delta>0$ we have

$$\begin{aligned}&-2\mathrm{Im}\sum_S(I_S(\tilde{e})A^2w_S,I_S(\tilde{e})Aw_S)\geqq\partial_0[u]^2_{A,\tilde{e},0}+\theta[u]^2_{A,\tilde{e},0}\\ &\qquad+(1-c(n,\lambda)\mu-\delta)\sum_{j,S}(2n+s(j))^{(\beta)}|u|^2_{A,S,\tilde{e}+\tilde{e}_j,0}+(4.9)\end{aligned} \tag{4.10}$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n,\lambda,\delta)$, $\hat{\theta}(n,\lambda,\delta,\mu)\leqq\theta$, $|x_0|\leqq\mu^{1/2}$, $u\in C^\infty(I,H^\infty)$ where $c_i(\delta,\lambda)$ are positive constants and $\beta=D^{1/2}(A,\varphi_j)$.

We turn to estimate L_k^2. We begin with

$$\begin{aligned}2\mathrm{Im}(I_S(\tilde{e})L_k^2w_S,I_S(\tilde{e})Aw_S)&=\partial_0|u|^2_{L_k,S,\tilde{e},0}+2\theta|u|^2_{L_k,S,\tilde{e},0}\\ &+\mathrm{Im}((a-a^*)I_S(\tilde{e})L_kw_S,I_S(\tilde{e})L_kw_S)\\ &+2\mathrm{Im}(I_S(\tilde{e})[A,L_k]w_S,I_S(\tilde{e})L_kw_S)\\ &+2\mathrm{Im}(I_S(\tilde{e})Aw_S,(L_k^*-L_k)I_S(\tilde{e})L_kw_S)\\ &+2\mathrm{Im}([A,I_S(\tilde{e})]L_kw_S,I_S(\tilde{e})L_kw_S)\\ &+2\mathrm{Im}([I_S(\tilde{e}),L_k]Aw_S,I_S(\tilde{e})L_kw_S)\\ &+2\mathrm{Im}(I_S(\tilde{e})Aw_S,[L_k,I_S(\tilde{e})]L_kw_S)\end{aligned} \tag{4.11}$$

The third term in the right-hand side of (4.11) is denoted by (I). Then it is clear that

$$|(I)|\leqq c(n)|u|^2_{L_k,S,\tilde{e},0}.$$

From (3.1) it is clear that $[A,L_k]\in S(\langle\mu\xi'\rangle,g_\mu)$. In view of Corollary 3.1 we have

$$I_S^*(\tilde{e})I_S(\tilde{e})[A,L_k]\equiv I_S^*(\tilde{e}-\tilde{e}_\nu)\langle\mu D'\rangle(B_S+r_S)I_S(\tilde{e}+\tilde{e}_\nu)$$

with $B_S\in S(1,g_\mu)$, $r_S\in S(\mu,g)$ for any $\nu\in I$. Then the fourth term in the right-hand side of (4.11), denoted by (II), has an estimate

$$|(II)|\leqq(c+c(n)\mu)|u|^2_{L_k,S,\tilde{e}+\tilde{e}_\nu,0}+(c+c(n)\mu)|u|^2_{S,\tilde{e}-\tilde{e}_\nu,1}.$$

Noting that $L_k^*-L_k\in S(1,g_\mu)$ the fifth term, denoted by (III), is estimated as follows

$$|(III)|\leqq(c+c(n)\mu)\{|u|^2_{A,S,\tilde e,0}+|u|^2_{L_k,S,\tilde e,0}\}.$$

Denote by (IV), (V), (VI) the sixth, seventh and the last term of (4.11) respectively. Applying Lemma 4.2 it follows that

$$(IV)\geqq\sum_j (2n+s(j))^{(\gamma_j)}|u|^2_{L_k,\tilde e+\tilde e_j,0}-c(n,\lambda)\mu^{1/2}[u]^2_{L_k,\tilde e+(1/2),0}$$
$$-c(n,\lambda)\mu^{1/2}[u]^2_{\tilde e+(3/2),0}\ \text{with}\ \gamma_j=D^{1/2}(A,\varphi_j).$$

Again by Lemma 4.2 we have

$$(V),(VI)\geqq-\sum_j(2n+s(j))\,|(\beta_{jk}I_S(\tilde e+\tilde e_j)Aw_S,\gamma_{jk}I_S(\tilde e+\tilde e_j)L_kw_S)|$$
$$(4.12)\qquad -c(n,\lambda)\mu^{1/2}[u]^2_{A,\tilde e+(1/2),0}-c(n,\lambda)\mu^{1/2}[u]^2_{L_k,\tilde e+(1/2),0}$$
$$-c(n,\lambda)\mu^{1/2}[u]^2_{\tilde e+(3/2),0}$$

with $\beta_{jk},\gamma_{jk}\in S^0_{1,0}$, $\sigma(\beta_{jk})\sigma(\gamma_{jk})=D(A,\varphi_j)$. We take $\gamma_{jk}=D^{1/2}(A,\varphi_j)$ and $\beta_{jk}=D(L_k,\varphi_j)D^{-1/2}(A,\varphi_j)$. Here we note that

$$2\sum_k\sum_j(2n+s(j))\,|(\beta_{jk}I_S(\tilde e+\tilde e_j)Aw_S,\gamma_{jk}I_S(\tilde e+\tilde e_j)L_kw_S)|$$
$$\leqq(1-\sigma)^{-1}\sum_j(2n+s(j))\sum_k{}^{(\beta_{jk})}|u|^2_{A,S,\tilde e+\tilde e_j,0}$$
$$+(1-\sigma)\sum_j(2n+s(j))\sum_k{}^{(\gamma_{jk})}|u|^2_{L_k,S,\tilde e+\tilde e_j,0}$$

for any $0<\sigma<1$. Summing up we obtain

$$2\mathrm{Im}(I_S(\tilde e)L_k^2w_S,I_S(\tilde e)Aw_S)\geqq\partial_0|u|^2_{L_k,S,\tilde e,0}+2\theta(1-c(n)\theta^{-1})$$
$$\times|u|^2_{L_k,S,\tilde e,0}+\sigma\sum_j(2n+s(j))\sum_k{}^{(\gamma_{jk})}|u|^2_{L_k,S,\tilde e+\tilde e_j,0}$$
$$(4.13)\qquad -(1-\sigma)^{-1}\sum_j(2n+s(j))\sum_k{}^{(\beta_{jk})}|u|^2_{A,S,\tilde e+\tilde e_j,0}$$
$$-c(n,\lambda)\mu^{1/2}\{[u]^2_{L_k,\tilde e+(1/2),0}+[u]^2_{A,\tilde e+(1/2),0}+[u]^2_{\tilde e+(3/2),0}\}$$
$$-c(n)|u|^2_{A,S,\tilde e,0}-(c+c(n)\mu^{1/2})\{|u|^2_{L_k,\tilde e+(1/2),0}+|u|^2_{S,\tilde e-\tilde e_\nu,1}\}$$

for any $\nu\in I$. Note that from (4.4) $\sigma\sum_j(2n+s(j))^{(\gamma_{jk})}|u|^2_{L_k,S,\tilde e+\tilde e_j,0}$ is estimated from below by

$$(4.14)\qquad c_1(\sigma,\lambda)n|u|^2_{L_k,S,\tilde e+(1/2),0}-c(\sigma,\lambda,n)|u|^2_{L_k,S,\tilde e,0}.$$

Let us consider

$$(1-\delta)^{(\beta)}|u|^2_{A,S,\tilde e+\tilde e_j,0}-(1-\sigma)^{-1}\sum_k{}^{(\beta_{jk})}|u|^2_{A,S,\tilde e+\tilde e_j,0}$$

where $\beta=D^{1/2}(A,\varphi_j)$. It turns out to consider

$$(1-\delta)\beta^*\beta-(1-\sigma)^{-1}\sum_k\beta^*_{jk}\beta_{jk}$$

$$=\mathrm{Op}((1-\delta)D(A,\varphi_j)-(1-\sigma)^{-1}\sum_k D(L_k,\varphi_j)^2 D(A,\varphi_j)^{-1})+S^{-1}_{1,0}.$$

Lemma 4.3. *For sufficiently small δ, $\sigma>0$ and for sufficiently large $\lambda>0$ we have*

$$(1-\delta)D(A,\varphi_j)-(1-\sigma)^{-1}\sum_k D(L_k,\varphi_j)^2 D(A,\varphi_j)^{-1}\geqq cD(A,\varphi_j)$$

with a positive constant c.

Proof. To simplify notation we put $m(x')=\sum_{s=1}^{p-1}(1-\chi_0^{(1)}(x_s))$ in this proof. Then we have

$$|D(L_k,\varphi_j)|\leqq(1+b)^{-1}(|\{l_k,\tilde{f}_j\}(\rho')|+\hat{c}m(x')),$$
$$D(A,\varphi_j)\geqq(1+b)^{-1}(\{l_0,\tilde{f}_j\}(\rho)+(\lambda-\hat{c})m(x')).$$

Hence it follows that

$$(1+b)^2D(A,\varphi_j)^2\geqq\{l_0,\tilde{f}_j\}^2(\rho)+(\lambda-\hat{c})^2m(x')^2\geqq(1-\kappa)\sum_k\{l_k,\tilde{f}_j\}^2(\rho')+(\lambda-\hat{c})^2m(x')^2$$

with some $\kappa>0$ by Lemma 2.4. On the other hand it is clear that

$$(1+b)^2\sum_k D^2(L_k,\varphi_j)\leqq(1+\tilde{\kappa})\sum_k\{l_k,\tilde{f}_j\}^2(\rho')+c(\tilde{\kappa})m(x')^2$$

for any $\tilde{\kappa}>0$. Then taking $\tilde{\kappa}>0$ sufficiently small and λ sufficiently large it follows that

$$\sum_k D^2(L_k,\varphi_j)\leqq(1-\kappa/2)D^2(A,\varphi_j).$$

This proves the lemma.

In what follows we fix λ, δ, σ so that Lemma 4.3 holds. Then the sharp Gårding inequality gives that

$$(4.15)\qquad \begin{aligned}&(1-\delta)^{(\beta)}|u|^2_{A,S,\bar{e}+\bar{e}_j,0}-(1-\sigma)^{-1}\sum_k{}^{(\beta_{jk})}|u|^2_{A,S,\bar{e}+\bar{e}_j,0}\\&\geqq c|u|^2_{A,S,\bar{e}+\bar{e}_j,0}-c(n)\mu^{1/2}|u|^2_{A,S,\bar{e},0}\end{aligned}$$

with a positive constant c. Now (4.10), (4.13), (4.14) and (4.15) give

Proposition 4.1. *For any $\nu\in I$ we have*

$$2\mathrm{Im}\sum_S(I_S(\bar{e})(-A^2+\sum_k L_k^2)w_S,I_S(\bar{e})Aw_S)\geqq\partial_0\{[u]^2_{A,\bar{e},0}+\sum_k[u]^2_{L_k,\bar{e},0}$$

$$+c_1 n\theta[u]^2_{\bar{e}+(1/2),0}+c_2 n^2[u]^2_{\bar{e}+(1),0}\}+c_3\theta\{[u]^2_{A,\bar{e},0}+\sum_k [u]^2_{L_k,\bar{e},0}\}$$
$$+c_3 n\{[u]^2_{A,\bar{e}+(1/2),0}+\sum_k [u]^2_{L_k,\bar{e}+(1/2),0}\}+c_3\{n\theta^2[u]^2_{\bar{e}+(1/2),0}$$
$$+n^2\theta[u]^2_{\bar{e}+(1),0}+n^3[u]^2_{\bar{e}+(3/2),0}\}-(c+c(n)\mu^{1/2})[u]^2_{\bar{e}-\bar{e}_\nu,1}$$

for $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\hat{\theta}(n,\mu)\leqq\theta$, $|x_0|\leqq\mu^{1/2}$, $u\in C^\infty(I,H^\infty)$ *with positive constants* c_i, c *independent of* n, μ *and* θ.

§ 5. Energy Estimate for Q

As noted in § 4 we proceed along the lines in [9] and [10]. Let Q be one of Q_j (see (3.12)) then by (3.11) we have

$$\sigma(Q)=\sigma(G)+S(\langle\mu\xi'\rangle,g_\mu),\quad G=G_j. \tag{5.1}$$

In what follows we use the following notation;

$$|u;Q|_{S,R+\langle h\rangle}=\mathrm{Re}\sum_j (QJ_{s(j)}(h_j)I_S(R)w_S,\ J_{s(j)}(h_j)I_S(R)w_S)$$
$$|u;Q|_{S,R+(t)}=\sum_k |u;Q|_{S,R+\langle k\rangle}$$

where $R\in\boldsymbol{R}^l$, $h\in(2^{-1}\boldsymbol{N})^l$ and the sum is taken over $k=(k_1,\cdots,k_l)\in(2^{-1}\boldsymbol{N})^l$ with $|k|=t$. We put

$$[u;Q]_{R+\langle h\rangle}=\sum_S |u:Q|_{S,R+\langle h\rangle},\ [u;Q]_{R+(t)}=\sum_S |u;Q|_{S,R+(t)}$$
$$\|\nabla Q u\|^2=\sum_{j=1}^d\{\|\langle D'\rangle^{-1}Q_{(j)}u\|^2+\|Q^{(j)}u\|^2\}.$$

To simplify notation we set

$$E_S(i,\nu)=|u;Q|_{S,\bar{e}+\langle\bar{e}_i\rangle}+|u|^2_{S,\bar{e}-\bar{e}_\nu,1}+|u|^2_{S,\bar{e}+(3/2),0}+c(n,\mu)|u|^2_{S,\bar{e}+(1/2),0}$$
$$E_S^\sharp(\nu)=|u|^2_{S,\bar{e}-\bar{e}_\nu,1}+|u|^2_{S,\bar{e}+(3/2),0}+c(n,\mu)|u|^2_{S,\bar{e}+(1/2),0}$$
$$e_S(\nu)=|u;Q|_{S,\bar{e}}+|u|^2_{S,\bar{e}-e_\nu,1}+|u|^2_{S,\bar{e}+(1),0}+c(n,\mu)|u|^2_{S,\bar{e}+(1/2),0}$$
$$e_S^\sharp(\nu)=|u|^2_{S,\bar{e}-e_\nu,1}+|u|^2_{S,\bar{e}+(1),0}+c(n,\mu)|u|^2_{S,\bar{e}+(1/2),0}.$$

Also we write J_ε instead of $J_\varepsilon(1/2)$ unless otherwise indicated.

Noting that $Q^*=Q$ we start with the following identity

$$\begin{aligned}&2\mathrm{Im}(I_S(\bar{e})Qw_S,I_S(\bar{e})Aw_S)=\partial_0|u;Q|_{S,\bar{e}}+2\theta|u;Q|^2_{S,\bar{e}}\\&-|u;\partial_0 Q|_{S,\bar{e}}+2\mathrm{Im}([I_S(\bar{e}),Q]w_S,I_S(\bar{e})Aw_S)\\&-2\mathrm{Im}(QI_S(\bar{e})w_S,aI_S(\bar{e})w_S)+2\mathrm{Im}(QI_S(\bar{e})w_S,[I_S(\bar{e}),A]w_S).\end{aligned}\tag{5.2}$$

We shall estimate first the third term in the right-hand side of (5.2) denoted by (I). Note that for $T\in S(\langle\mu\xi'\rangle,g_\mu)$ we have

(5.3) $$|u\,;T|_{S,\tilde{e}+\langle e_j\rangle}\leqq(c+c(n)\mu)E_S^\sharp(\nu)$$

for any $\nu\in I$. In fact Corollary 3.1 shows that $I_S^*(\tilde{e})J_{s(j)}^*TJ_{s(j)}I_S(\tilde{e})$ is written as

$$I_S^*(\tilde{e}+e_j+\tilde{e}_\nu)(B_S+C_S)\langle\mu D'\rangle I_S(\tilde{e}-\tilde{e}_\nu)\quad\text{for any }\nu\in I$$

where $B_S\in S(1,g_\mu)$, $C_S\in S(\mu,g)$ and B_S.
This proves (5.3) and in particular

(5.4) $$\|\langle\mu D'\rangle^{1/2}J_{s(j)}I_S(\tilde{e})w_S\|^2\leqq(c+c(n)\mu)E_S^\sharp(\nu).$$

Lemma 5.1.

$$\|\nabla QJ_{s(j)}I_S(\tilde{e})w_S\|^2\leqq\mu(c+c(n)\mu)E_S(j,\nu)$$

for any $\nu\in I$.

Proof. Noting (5.1), (5.3) and (5.4) it suffices to show the lemma with $Q=G$. By (3.11) it is clear that

$$Q_{(j)}^*-Q_{(j)}\in S(\mu^{1/2}\langle\mu\xi'\rangle,g_\mu),\,Q^{(j)*}-Q^{(j)}\in S(\mu^{1/2},g_\mu).$$

Hence we have $Q_{(j)}^*\langle D'\rangle^{-2}Q_{(j)}=Q_{(j)}\langle D'\rangle^{-2}Q_{(j)}+S(\mu\langle\mu\xi'\rangle,g_\mu)=\mathrm{Op}\,|\langle\xi'\rangle^{-1}Q_{(j)}|^2+r_1$ with $r_1\in S(\mu\langle\mu\xi'\rangle,g_\mu)$. Similarly we get $Q^{(j)*}Q^{(j)}=\mathrm{Op}\,|Q^{(j)}|^2+r_2$ with $r_2\in S(\mu\langle\mu\xi'\rangle,g_\mu)$. Melin's inequality gives that (since $\sum\{|Q_{(j)}\langle\xi'\rangle^{-1}|^2+|Q^{(j)}|^2\}\leqq c\mu Q$)

(5.5) $$c\mu\mathrm{Re}(Qv,v)+\nu\|\langle\mu D'\rangle^{1/2}v\|^2+c(\nu,\mu)\|v\|^2\geqq\|\nabla Qv\|^2-\sum_i|(r_iv,v)|$$

for any $\nu>0$. Taking $\nu=\mu$ and $v=J_{s(j)}I_S(\tilde{e})w_S$, (5.3), (5.4) and (5.5) prove this lemma.

Remark 5.1. The same argument shows that

$$\|\nabla QI_S(\tilde{e})w_S\|^2\leqq\mu(c+c(n)\mu)e_S(\nu)$$

for any $\nu\in I$.

We slightly sharpen Lemma 5.1.

Lemma 5.2. *For any* $0\leqq j\leqq p$ *and* i, $\nu\in I$ *we have*

$$\|\langle\mu D'\rangle^{-1}Q_{(j)}J_{s(i)}I_S(\tilde{e})w_S\|^2\leqq(c+c(n)\mu^{1/2})E_S(i,\nu).$$

Proof. As noted in the proof of Lemma 5.1 we may assume that $Q=G$. (3.11) means that for $0\leqq j\leqq p$

$$Q_{(j)}\in S(\mu^2\langle\xi'\rangle^2, dx_0^2+\tilde{G}'_\mu)+S(\mu^3\langle\xi'\rangle^2, dx_0^2+\tilde{G}_\mu)$$

and hence $Q^*_{(j)}-Q_{(j)}\in S(\langle\mu\xi'\rangle, g_\mu)$. This shows that

$$Q^*_{(j)}\langle\mu D'\rangle^{-2}Q_{(j)}=\mathrm{Op}\,|\langle\mu\xi'\rangle^{-1}Q_{(j)}|^2+r, \qquad r\in S(\langle\mu\xi'\rangle, g_\mu).$$

Since $|\langle\mu\xi'\rangle^{-1}Q_{(j)}|^2\leqq cQ$, using Melin's inequality the rest of the proof goes along the same lines as that in Lemma 5. 1.

We turn to estimate of (I). Writing

$$I_S(\tilde{e})\equiv J^*_{s(\nu)}(1+r)I_S(\tilde{e}-\tilde{e}_\nu), \qquad r\in S(\mu, g)$$

we are led to consider

$$J_{s(\nu)}\partial_0 Q I_S(\tilde{e})=[J_{s(\nu)}, \partial_0 Q]I_S(\tilde{e})+\partial_0 Q J_{s(\nu)}I_S(\tilde{e}).$$

In view of (3. 11) and (3. 16) it follows that $[J_{s(\nu)}, \partial_0 Q]$ is in $S(\mu^{1/2}\langle\mu\xi'\rangle m(\varphi_\nu)^{-3/2}, g_\mu)$ and hence $I^*_S(\tilde{e}-\tilde{e}_\nu)(1+r^*)[J_{s(\nu)}, \partial_0 Q]I_S(\tilde{e})$ is written

$$I^*_S(\tilde{e}-\tilde{e}_\nu)\langle\mu D'\rangle B_S I_S(\tilde{e}+3\tilde{e}_\nu), \qquad B_S\in S(\mu^{1/2}, g).$$

This expression gives an estimate of $([J_{s(\nu)}, \partial_0 Q]I_S(\tilde{e})w_S, (1+r)\times I_S(\tilde{e}-\tilde{e}_\nu)w_S)$ by $c(n)\mu^{1/2}E^\sharp_S(\nu)$ for any $\nu\in I$. Applying Lemma 5. 2 we get

$$|(I)|\leqq(c+c(n)\mu^{1/2})E_S(\nu,\nu) \tag{5. 6}$$

for any $\nu\in I$.

We next estimate the fourth term in the right-hand side of (5. 1) denoted by (II). We write

$$[I_S(\tilde{e}), Q]\equiv -i\sum_j(I_S(\tilde{e})^{(j)}Q_{(j)}-I_S(\tilde{e})_{(j)}Q^{(j)})+r.$$

In view of (3. 11), (3. 17) and (5. 1) we see that

$$r=\sum_j r_j, \qquad r_j\in S(\mu\langle\mu\xi'\rangle^{nS^*+1/2}m(\varphi)^{-nS-\tilde{e}-e_j}, g_\mu)$$

hence $r\in S(\mu\langle\mu\xi'\rangle^{nS^*+1}m(\varphi)^{-nS-\tilde{e}}, g_\mu)$. Corollary 3. 1 shows that

$$I^*_S(\tilde{e})r\equiv\sum I^*_S(\tilde{e}+\tilde{e}_\nu)B_S\langle\mu D'\rangle I_S(\tilde{e}-\tilde{e}_\nu), \qquad B_S\in S(\mu, g)$$

for any $\nu\in I$. This gives an estimate of $(rw_S, I_S(\tilde{e})Aw_S)$ by

$$c(n)\mu\{|u|^2_{S,\tilde{e}-\tilde{e}_\nu,1}+|u|^2_{A,S,\tilde{e}+(1/2),0}\}.$$

Again by Corollary 3. 1 and (3. 17) we have an expression of $I^*_S(\tilde{e})I_S(\tilde{e})_{(j)}$ as

$$\sum_k I^*_S(\tilde{e}+\tilde{e}_k)B_{kjS}J_{s(k)}I_S(\tilde{e})$$

with $B_{kjS}=B^1_{kjS}+C_{kjS}$ where $B^1_{kjS}\in S(1,g_\mu)$, $C_{kjS}\in S(\mu,g)$. Here we note that $[J_{s(k)}I_S(\tilde{e}),\ Q^{(j)}]$ is written as

$$\sum_i R_{ikj}I_S(\tilde{e}+\tilde{e}_k+e_i),\qquad R_{ikj}\in S(\mu,g).$$

Hence $I^*_S(\tilde{e})I_S(\tilde{e})_{(j)}Q^{(j)}$ can be expressed

$$\sum_k I^*_S(\tilde{e}+\tilde{e}_k)B_{kjS}Q^{(j)}J_{s(k)}I_S(\tilde{e})+\sum_{i,k}I^*_S(\tilde{e}+\tilde{e}_k)B_{ik}I_S(\tilde{e}+\tilde{e}_k+e_i)$$

with $B_{ik}\in S(\mu,g)$. For $I^*_S(\tilde{e})I_S(\tilde{e})^{(j)}Q_{(j)}$, one has a similar expression. Now we have an estimate of (II) by

$$c(n)\{\mu^{1/2}|u|^2_{A,S,\bar{e}+(1/2),0}+\mu^{-1/2}\sum_k\|\nabla Q J_{s(k)}I_S(\tilde{e})\alpha_S u\|^2+\mu E^{\#}_S(\nu)\}$$

for any $\nu\in I$. Applying Lemma 5.1 we have

$$|(II)|\leqq\mu^{1/2}c(n)\{\sum_{j,S}E_S(j,\nu)+|u|^2_{A,S,\bar{e}+(1/2),0}\}$$

for any $\nu\in I$.

We turn to the fourth term in the right-hand side of (5.2), denoted by (III). We start with the following lemma;

Lemma 5.3. *Let $R\in S(1,g)$. Then we have*

$$|(RQu_1,u_2)|\leqq c\sum_{i=1}^2\{\mathrm{Re}(Qu_i,u_i)+\mu^{-1}\|\nabla Qu_i\|^2+\|\langle\mu D'\rangle^{1/2}u_i\|^2+c(\mu)\|u_i\|^2\}$$

with a positive constant c independent of μ.

Proof. Set

$$X_\pm=X_\pm(R,\lambda)=\lambda\pm(R+R^*)/2,\qquad Y_\pm=iX_\pm(-iR,\lambda)$$

with a large positive constant λ. Note that we can take λ so that $(X^{1/2}_\pm)^*=X^{1/2}_\pm$, $X^{1/2}_\pm\in S(1,g)$. The same argument as in §5 in [9] shows that

$$\begin{aligned}(5.7)\qquad 2\lambda\mathrm{Re}(Qu,u)\geqq&|(RQu,u)|-|([Q,R]u,u)|\\&-\textstyle\sum^{\pm}\{|([Q,X^{1/2}_\pm]u,X^{1/2}_\pm u)|+|([Q,Y^{1/2}_\pm]u,Y^{1/2}_\pm u)|\}\\&-c\|\langle\mu D'\rangle^{1/2}u\|^2-c(\lambda,\mu)\|u\|^2.\end{aligned}$$

Since $Q^{(\alpha)}\in S(\mu,g_\mu)$, $Q_{(\alpha)}\in S(\mu\langle\xi'\rangle^2,g_\mu)$ for $|\alpha|=2$ we have

$$\begin{aligned}[X^{1/2}_\pm,Q]&\equiv-i\sum\{(X^{1/2}_\pm)^{(j)}Q_{(j)}-(X^{1/2}_\pm)_{(j)}Q^{(j)}\}+r_1,\\ [R,Q]&\equiv-i\sum\{R^{(j)}Q_{(j)}-R_{(j)}Q^{(j)}\}+r_2\end{aligned}$$

with $r_i\in S(\langle\mu\xi'\rangle,g)$. Then one obtains that

$$2\lambda \mathrm{Re}(Qu, u) \geqq |(QRu, u)| - \mu^{-1}\|\nabla Qu\|^2 - c(\lambda)\|\langle\mu D'\rangle^{1/2}u\|^2 - c(\lambda, \mu)\|u\|^2.$$

Applying this inequality to $u_1 \pm u_2$, $u_1 \pm iu_2$ we get the lemma.

Remark 5.2. When $R \in S(\mu^\delta, g)$ we obtain

$$|(RQu_1, u_2)| \leqq c\sum_{i=1}^{2} \{\mu^\delta \mathrm{Re}(Qu_i, u_i) + \mu^{\delta-1}\|\nabla Qu_i\|^2 + \mu^\delta\|\langle\mu D'\rangle^{1/2}u_i\|^2 + c(\mu)\|u_i\|^2\}$$

because $\mu^{-\delta}R \in S(1, g)$.

Corollary 5.1. *Let* $R \in S(\mu^\delta, g)$. *Then we have*

$$|u\,; RQ|_{S, \tilde{e}+\langle e_j\rangle} \leqq \mu^\delta(c + c(n)\mu)E_S(j, \nu),$$

$$|u\,; RQ|_{S, \tilde{e}} \leqq \mu^\delta(c + c(n)\mu)e_S(\nu)$$

for any $\nu \in I$.

Consider the fifth term, denoted by (III), which is equal to

$$\mathrm{Im}((aQ - Qa)I_S(\tilde{e})w_S, I_S(\tilde{e})w_S) + \mathrm{Im}((a^* - a)QI_S(\tilde{e})w_S, I_S(\tilde{e})w_S) = |u : i[a, Q]|_{S, \tilde{e}} + |u\,; i(a^* - a)Q|_{S, \tilde{e}}.$$

Taking the fact $a^* - a \in S(1, g)$ into account and applying Corollary 5.1 with u replaced by $I_S(\tilde{e})w_S$, $|u\,; i(a^* - a)Q|_{S, \tilde{e}}$ is estimated by

$$(c + c(n)\mu)e_S(\nu) \tag{5.8}$$

for any $\nu \in I$. To estimate $|u\,; i[a, Q]|_{S, \tilde{e}}$ we observe $i[a, Q]$;

$$i[a, Q] = \sum_j \{a^{(j)}Q_{(j)} - a_{(j)}Q^{(j)}\} + r = K + r, \qquad r \in S(\langle\mu\xi'\rangle, g_\mu).$$

Similar argument as to prove (5.3) gives an estimate of $|u\,; r|_{S, \tilde{e}}$ by

$$(c + c(n)\mu)e_S^\sharp(\nu)$$

for any $\nu \in I$. Writing $I_S^*(\tilde{e})KI_S(\tilde{e}) \equiv I_S^*(\tilde{e} - e_\nu)(1 + r)J_{s(\nu)}KI_S(\tilde{e})$ we shall show that the right-hand side of the above is written

$$\begin{aligned} &I_S^*(\tilde{e} + \tilde{e}_\nu)B_{1S}I_S(\tilde{e} + 2\tilde{e}_\nu) + I_S^*(\tilde{e} + 3\tilde{e}_\nu)(B_{2S} + C_{2S})I_S(\tilde{e} - \tilde{e}_\nu) \\ &+ I_S^*(\tilde{e} + \tilde{e}_\nu)\{(B_{3S} + C_{3S})\langle D'\rangle^{-1}Q_{(j)} + (B_{4S} + C_{4S})Q^{(j)}\}J_{s(\nu)}I_S(\tilde{e}) \\ &+ I_S^*(\tilde{e} - \tilde{e}_\nu)(1 + r)KJ_{s(\nu)}I_S(\tilde{e}) \end{aligned} \tag{5.9}$$

for any $\nu \in I$ where $B_{1S} \in S(\mu, g)$, $B_{iS} \in S(1, g_\mu)$, $C_{iS} \in S(\mu, g)$. Since arguments for $a_{(j)}Q^{(j)}$ is just parallel to that for $a^{(j)}Q_{(j)}$ it will suffice to show (5.9) with K replaced by $a^{(j)}Q_{(j)}$. Write

$$J_{s(\nu)}a^{(j)}Q_{(j)}=[J_{s(\nu)},\ a^{(j)}]Q_{(j)}+a^{(j)}[J_{s(\nu)},Q_{(j)}]+a^{(j)}Q_{(j)}J_{s(\nu)}.$$

Since $[J_{s(\nu)},\ Q_{(j)}]\in S(\langle\mu\xi'\rangle m(\varphi_\nu)^{-3/2},\ g_\mu)$ and $[J_{s(\nu)},\ a^{(j)}]\equiv(B_3+C_3)\langle D'\rangle^{-1}J_{s(\nu)}$ with $B_3\in S(m(\varphi_\nu)^{-1},g_\mu)$, $C_3\in S(\mu m(\varphi_\nu)^{-1},g)$ then one has

$$[J_{s(\nu)},a^{(j)}]Q_{(j)}\equiv B_1+(B_3+C_3)\langle D'\rangle^{-1}Q_{(j)}J_{s(\nu)}$$

with $B_1\in S(\mu m(\varphi_\nu)^{-5/2},g)$. Setting $B_2=a^{(j)}[J_{s(\nu)},Q_{(j)}]$ which is in $S(\langle\mu\xi'\rangle m(\varphi_\nu)^{-3/2},g_\mu)$ one can write $J_{s(\nu)}a^{(j)}Q_{(j)}$ as

$$B_1+B_2+((B_3+C_3)\langle D'\rangle^{-1}+a^{(j)})Q_{(j)}J_{s(\nu)}. \tag{5.10}$$

Here note that B_2 is independent of μ. In view of Corollary 3.1 (5.9) follows from (5.10). Hence $|u;K|_{S,\tilde e}$ is estimated by

$$|(\langle\mu D'\rangle^{-1}(1+r)KJ_{s(\nu)}I_S(\tilde e)w_S,\langle\mu D'\rangle I_S(\tilde e-\tilde e_\nu)w_S)|+(c+c(n)\mu)E_S(\nu,\nu).$$

Since $a_{(j)}\in S(\mu^{1/2}\langle\xi'\rangle,g_\mu)$ $(1\leqq j\leqq d)$, $a^{(j)}\in S(\mu^{1/2},g_\mu)$ $(p+1\leqq j)$ it is easy to see that with $T=a_{(j)}Q^{(j)}$ $(1\leqq j\leqq d)$ or $T=a^{(j)}Q_{(j)}$ $(p+1\leqq j)$ $|(\langle\mu D'\rangle^{-1}(1+r)TJ_{s(\nu)}I_S(\tilde e)w_S,\ \langle\mu D'\rangle I_S(\tilde e-\tilde e_\nu)w_S)|$ is estimated by $(c+c(n)\mu^{1/2})E_S(\nu,\nu)$ from Lemma 5.1. In the case $T=a^{(j)}Q_{(j)}$ $(1\leqq j\leqq p)$ we apply Lemma 5.2 to conclude that this is bounded by $(c+c(n)\mu^{1/2})\times E_S(\nu,\nu)$ also. Combining these estimates we see that $|u;K|_{S,\tilde e}$ is estimated by $(c+c(n)\mu^{1/2})\{E_S(\nu,\nu)+e_S(\nu)\}$. Hence we have

$$|(III)|\leqq(c+c(n)\mu^{1/2})\{E_S(\nu,\nu)+e_S(\nu)\}.$$

We turn to the last term in the right-hand side of (5.2), denoted by (IV). Remark 4.1 gives that

$$[I_S(\tilde e),A]\alpha_S\equiv-i\sum_j(n+s(j)/2)J^*_{s(j)}\beta^*_j\gamma_jJ_{s(j)}I_S(\tilde e)\alpha_S$$
$$+\sum_{j,\bar S}J^*_{s(j)}B_{j\bar S}J_{\bar s(j)}I_{\bar S}(\tilde e)\alpha_{\bar S}$$

with $B_{j\bar S}\in S(\mu,g)$. Here we take $\gamma_j=1$ and hence $\beta_j=D(A,\varphi_j)$. Hence (IV) turns to

$$\mathrm{Re}\sum_j(2n+s(j))(\beta_jJ_{s(j)}QI_S(\tilde e)w_S,J_{s(j)}I_S(\tilde e)w_S) \tag{5.12}$$
$$+\sum_{j,\bar S}(B^*_{j\bar S}J_{s(j)}QI_S(\tilde e)w_S,\ J_{\bar s(j)}I_{\bar S}(\tilde e)w_{\bar S}).$$

We want to commute $J_{s(j)}$ through Q. First we observe that

$$[J_{s(j)},Q]\equiv-i\sum\{M^k\langle D'\rangle^{-1}Q_{(k)}J_{s(j)}-M_kQ^{(k)}J_{s(j)}\}+r \tag{5.13}$$

with M^k, $M_k\in S(m(\varphi_j)^{-1},g)$, $r\in S(\mu^{1/2}\langle\mu\xi'\rangle m(\varphi_j)^{-1/2},g)$. Indeed we can write $J_{s(j)}^{(k)} \equiv \tilde{M}^k J_{s(j)}$, $J_{s(j)(k)} \equiv \tilde{M}_k J_{s(j)}$ with $\tilde{M}^k \in S(\langle\xi'\rangle^{-1} m(\varphi_j)^{-1}, g)$, $\tilde{M}_k \in S(m(\varphi_j)^{-1}, g)$ then (5.13) follows from the fact $[Q_{(k)}, \tilde{M}^k]$, $[Q_{(k)}, \tilde{M}_k] \in S(\mu\langle\xi'\rangle^{1/2} m(\varphi_j)^{-1}, g)$. In view of (5.13) the first term in (5.12), up to a constant factor, is equal to a sum of

$$(5.14) \qquad |u\,;\beta_j T Q^{(k)}|_{S,\bar{e}+\langle\bar{e}_j\rangle}, \qquad |u\,;\beta_j T\langle D'\rangle^{-1} Q_{(k)}|_{S,\bar{e}+\langle\bar{e}_j\rangle},$$
$$(\beta_j r I_S(\bar{e}) w_S, J_{s(j)} I_S(\bar{e}) w_S)$$

where $T \in S(m(\varphi_j)^{-1}, g)$, $r \in S(\mu^{1/2}\langle\mu\xi'\rangle m(\varphi_j)^{-1/2}, g)$. Writing

$$I_S^*(\bar{e}) J_{s(j)}^* \beta_j r I_S(\bar{e}) \equiv I_S^*(\bar{e}-\bar{e}_\nu)\langle\mu D'\rangle B_S I_S(\bar{e}+\bar{e}_\nu+2\bar{e}_j)$$

with $B_S \in S(\mu^{1/2}, g)$ the last term in (5.14) is estimated by

$$c(n)\,\mu^{1/2} E_S^\sharp(\nu).$$

Noting that $(\beta_j T)^* J_{s(j)} I_S(\bar{e}) \equiv B_S I_S(\bar{e}+3\bar{e}_j)$ with $B_S \in S(1, g)$ it is clear that the first two terms in (5.14) multiplied by $c(n)$ are both estimated by $\mu^{-1/2}\|\nabla Q J_{s(j)} I_S(\bar{e}) w_S\|^2 + c(n)\,\mu^{1/2}|u|^2_{S,\bar{e}+(3/2),0}$ and hence by $c(n)\,\mu^{1/2} E_S(j,\nu)$. Since the same argument can be applicable to estimate $(B_{j\bar{S}}^*[J_{s(j)}, Q] I_S(\bar{e}) w_S, J_{\bar{s}(j)} I_{\bar{S}}(\bar{e}) w_{\bar{S}})$ because

$$I_{\bar{S}}^*(\bar{e}) J_{\bar{s}(j)}^* B_{j\bar{S}}^* r I_S(\bar{e}) \equiv I_{\bar{S}}^*(\bar{e}+\bar{e}_\nu+2\bar{e}_j)\langle\mu D'\rangle \tilde{B}_S I_S(\bar{e}-\bar{e}_\nu),$$
$$T^* B_{j\bar{S}} J_{\bar{s}(j)} I_{\bar{S}}(\bar{e}) \equiv \tilde{B}_{\bar{S}} I_{\bar{S}}(\bar{e}+3\bar{e}_j)$$

with $\tilde{B}_S \in S(\mu^{3/2}, g)$, $\tilde{B}_{\bar{S}} \in S(\mu, g)$ then we have an estimate of (IV) from below by

$$(5.15) \qquad \sum_j (2n+s(j))\,|u\,;\beta_j Q|_{S,\bar{e}+\langle\bar{e}_j\rangle}$$
$$-\sum_{j,\bar{S}} |(B_{j\bar{S}}^* Q J_{s(j)} I_S(\bar{e}) w_S, J_{\bar{s}(j)} I_{\bar{S}}(\bar{e}) w_{\bar{S}})| - c(n)\,\mu^{1/2}\sum_{j,\bar{S}} E_{\bar{S}}(j,\nu).$$

Here we estimate $|u\,;\beta_j Q|_{S,\bar{e}+\langle\bar{e}_j\rangle}$ from below. Recall that $\beta_j = D(A, \varphi_j) \geqq c$ with a positive constant c (note that we have fixed λ in $b(x',\lambda)$). Then Melin's inequality gives that (cf. (5.1))

$$\mathrm{Re}(\beta_j Q v, v) + \|\langle\mu D'\rangle^{1/2} v\|^2 + c(\mu)\|v\|^2 \geqq c\,\mathrm{Re}(Gv, v).$$

Taking $v = J_{s(j)} I_S(\bar{e}) w_S$ and noting (5.1) and (5.4) it follows that

$$|u\,;\beta_j Q|_{S,\bar{e}+\langle\bar{e}_j\rangle} \geqq c\,|u\,;Q|_{S,\bar{e}+\langle\bar{e}_j\rangle} - (c+c(n)\,\mu) E_S^\sharp(\nu)$$

for any $\nu \in I$. Applying Remark 5.2 with $u_1 = J_{s(j)} I_S(\bar{e}) w_S$, $u_2 = J_{\bar{s}(j)} I_{\bar{S}}(\bar{e}) w_{\bar{S}}$ to (5.15) it follows that for any $\nu \in I$

$$(IV) \geqq cn\,|u\,;Q|_{S,\bar{e}+(1/2)} - c(n)\,\mu^{1/2}\sum_S E_S^\sharp(\nu)$$

for $0 < \mu \leqq \hat{\mu}(n)$. Collecting the estimates of (I), (II), (III) and

(IV) and summing up over S, we have

Proposition 5.1. *For any $\nu\in I$ we have*

$$2\mathrm{Im}\sum_S (I_S(\tilde{e})Qw_S, I_S(\tilde{e})Aw_S) \geqq \partial_0[u\,;Q]_{\tilde{e}}+\theta[u\,;Q]_{\tilde{e}}+c_1 n[u\,;Q]_{\tilde{e}+(1/2)}$$
$$-c_2\{[u]^2_{A,\tilde{e}+(1/2),0}+[u]^2_{\tilde{e}-e_\nu,1}+[u]^2_{\tilde{e}+(3/2),0}+[u]^2_{\tilde{e}-e_\nu,1}\}$$
$$-c(n,\mu)\{[u]^2_{\tilde{e}+(1/2),0}+[u]^2_{\tilde{e}+(1),0}\}$$

for $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\theta(n,\mu)\leqq\hat{\theta}$, $|x_0|\leqq\mu^{1/2}$, $u\in C^\infty(I,H^\infty)$ where c_i are positive constants independent of n, μ and θ.

§6. Estimate of Commutators

Let L, A, Q be as in §§4 and 5. We start with

Lemma 6.1. *Let $S\in\{-1,1\}^l$ and $M\subset I$. Assume that $T^{(\alpha)}\in S(\tilde{m},g_\mu)$ and $T_{(\alpha)}\in S(\tilde{m}\langle\xi'\rangle,g_\mu)$ for $|\alpha|=1$. Then*

$$[T,\alpha_{S\circ M}]=\sum_K T_{MK}\alpha_{S\circ K}$$

where the sum is taken over $K\subset M$ with $|K|\leqq|M|-1$ and

$$T_{MK}\in S(\tilde{m}\langle\xi'\rangle^{-(|M|-1-|K|)}\langle\mu\xi'\rangle^{(|M|-1-|K|)+|\varepsilon(K,M)\circ Q|/2+1/2}\times m(\varphi)^{\varepsilon(K,M)\circ Q},g_\mu)$$

for any $Q\in\boldsymbol{R}^l$. Here when $|K|=|M|-1$, $K=M\setminus\{i\}$ we have

$$T_{KM}=\pm[T,\alpha_{s(i)}].$$

Proof. Taking into account (3.14) the lemma will be proved by induction on $|M|$.

Corollary 6.1. *Let $S\in\{-1,1\}^l$. Then*

$$[L,\alpha_S]L=\sum_{|K|\leqq l-1}T_K L\alpha_{S\circ K}+\sum_{|K|\leqq l-2}\tilde{T}_K\alpha_{S\circ K}$$

where

$$T_K\in S(\langle\xi'\rangle^{-(l-1-|K|)}\langle\mu\xi'\rangle^{(l-1-|K|)+|\varepsilon(K)\circ Q|/2+1/2}m(\varphi)^{\varepsilon(K)\circ Q},g_\mu)$$
$$\tilde{T}_K\in S(\langle\xi'\rangle^{-(l-2-|K|)}\langle\mu\xi'\rangle^{(l-2-|K|)+|\varepsilon(K)\circ Q|/2+1}m(\varphi)^{\varepsilon(K)\circ Q},g_\mu)$$

for any $Q\in\boldsymbol{R}^l$ Further

$$T_K=\pm[L,\alpha_{s(i)}]\quad \text{if } |K|=l-1,\ K=I\setminus\{i\},$$

$$\tilde{T}_K = \pm [L, \alpha_{s(i)}][L, \alpha_{s(j)}] \quad \textit{if} \ |K| = l-2, K = I \setminus \{i, j\}.$$

Corollary 6.2. *Let* $S \in \{-1, 1\}^l$. *Then*

$$[L, [L, \alpha_S]] = \sum_{|K| \leqq l-1} \hat{T}_K \alpha_{S \circ K} + \sum_{|K| \leqq l-2} \tilde{T}_K \alpha_{S \circ K}$$

where $\hat{T}_K = \pm [L, [L, \alpha_{s(i)}]]$ *if* $|K| = l-1, K = I \setminus \{i\}$,

$$\hat{T}_K \in S(\langle \xi' \rangle^{-(l-1-|K|)} \langle \mu \xi' \rangle^{(l-1-|K|)+|\varepsilon(K) \circ Q|/2+1} m(\varphi)^{\varepsilon(K) \circ Q}, g_\mu)$$

and $\tilde{T}_K$ *satisfies the same properties as in Corollary* 6.1.

Now we observe $[L^2, \alpha_S] = 2[L, \alpha_S]\, L + [L, [L, \alpha_S]]$. Applying Corollaries 6.1 and 6.2 one obtains

$$[L^2, \alpha_S] = \sum_{|K| \leqq l-1} T_K L \alpha_{S \circ K} + \sum_{|K| \leqq l-1} \hat{T}_K \alpha_{S \circ K} + \sum_{|K| \leqq l-2} \tilde{T}_K \alpha_{S \circ K}.$$

Noting that $I_S^*(\tilde{e}) I_S(\tilde{e}) T_K$ is in

$$S(\mu^{(l-1-|K|)} \langle \mu \xi' \rangle^{2n\overset{*}{S}+|\varepsilon(K) \circ Q|/2+1/2} m(\varphi)^{-2nS-2\tilde{e}+\varepsilon(K) \circ Q}, g_\mu).$$

Corollary 3.1 gives that

(6.1) $\quad I_S^*(\tilde{e}) I_S(\tilde{e}) T_K \equiv I_{\tilde{S}}^*(R_1) B_{K\tilde{S}} I_{\tilde{S}}(R_2), \qquad B_{K\tilde{S}} \in S(\mu^{(l-1-|K|)}, g)$

where $R_1 + R_2 = e - \varepsilon(K) \circ U$, $|\varepsilon(K) \circ U| = -1$. In particular when $|K| = l-1$, hence $T_K = \pm [L, \alpha_{s(i)}]$ with some i, one has

$$B_{K\tilde{S}} = B_{K\tilde{S}}^1 + C_{K\tilde{S}}, \qquad B_{K\tilde{S}}^1 \in S(1, g_\mu), \qquad C_{K\tilde{S}} \in S(\mu, g)$$

where $|\sigma(B_{K\tilde{S}}^1)| \leqq c n^{1/2}$ with a positive constant independent of n, μ when $n \geqq 16$ which follows from the proof of Proposition 6.1 in [9]. As for $I_S^*(\tilde{e}) I_S(\tilde{e}) \hat{T}_K$ the same argument shows that

(6.2) $\quad I_S^*(\tilde{e}) I_S(\tilde{e}) \hat{T}_K \equiv I_S^*(R_1) B_{K\tilde{S}} I_{\tilde{S}}(R_2), \qquad B_{K\tilde{S}} \in S(\mu^{l-1-|K|}, g)$

where $R_1 + R_2 = e - \varepsilon(K) \circ U$, $|\varepsilon(K) \circ U| = -2$. When $|K| = l-1$ one has

(6.3) $\quad B_{K\tilde{S}} = B_{K\tilde{S}}^1 + C_{K\tilde{S}}, \ B_{K\tilde{S}}^1 \in S(1, g_\mu), \ C_{K\tilde{S}} \in S(\mu, g)$

and $|\sigma(B_{K\tilde{S}}^1)| \leqq cn$. Repeating similar arguments we can write

(6.4) $\quad I_S^*(\tilde{e}) I_S(\tilde{e}) \tilde{T}_K \equiv I_S^*(R_1) B_{K\tilde{S}} I_{\tilde{S}}(R_2), \qquad B_{K\tilde{S}} \in S(\mu^{l-2-|K|}, g)$

with $R_1 + R_2 = e - \varepsilon(K) \circ U$, $|\varepsilon(K) \circ U| = -2$. If $|K| = l-2$, $B_{K\tilde{S}}$ verifies (6.3). Since $|K| \leqq l-1$ and hence $\varepsilon(K) \neq 0$ we can choose U so that $R_1 = \tilde{e} + \tilde{e}_j$, $R_2 = \tilde{e} + \tilde{e}_j$ in (6.1) and $R_1 = \tilde{e} + \tilde{e}_j$, $R_2 = \tilde{e} + 3\tilde{e}_j$ in (6.2) and (6.4) with some j which depends of course on K. Hence $I_S^*(\tilde{e}) I_S(\tilde{e}) [L^2, \alpha_S]$ turns out to the sum

$$\sum_{j,\tilde{S}} I_S^*(\tilde{e}+\tilde{e}_j)B_{j\tilde{S}}I_{\tilde{S}}(\tilde{e}+\tilde{e}_j)L\alpha_{\tilde{S}}+\sum_{j,\tilde{S}} I_S^*(\tilde{e}+\tilde{e}_j)B_{j\tilde{S}}I_{\tilde{S}}(\tilde{e}+3\tilde{e}_j)\alpha_{\tilde{S}}.$$

From above discussions it follows that

Lemma 6.2.

$$|(I_S(\tilde{e})[L^2, \alpha_S]u, I_S(\tilde{e})A\alpha_S u)|\leqq cn^{1/2}|u|^2_{A,S,\tilde{e}+(1/2),0} + cn^{1/2}[u]^2_{L,\tilde{e}+(1/2),0}+cn^{3/2}[u]^2_{\tilde{e}+(3/2),0}$$

for $0<\mu\leqq\hat{\mu}(n)$ *where* c *is a positive constant independent of* n, μ, θ.

Corollary 6.3.

$$|(I_S(\tilde{e})[A^2, \alpha_S]u, I_S(\tilde{e})A\alpha_S u)|\leqq cn^{1/2}[u]^2_{A,\tilde{e}+(1/2),0}+cn^{3/2}[u]^2_{\tilde{e}+(3/2),0}$$

for $0<\mu\leqq\hat{\mu}(n)$.

Lemma 6.3. *Let* $S\in\{-1, 1\}^l$ *and* $M\subset I$. *Then*

$$[Q, \alpha_{S\circ M}]=\sum T^j_{MK}\langle D'\rangle^{-1}Q_{(j)}\alpha_{S\circ K}+\sum T_{jMK}Q^{(j)}\alpha_{S\circ K}+\sum\tilde{T}_{MK}\alpha_{S\circ K}$$

where the sum is taken over $K\subset M$ *with* $|K|\leq|M|-1$ *and*

$$T^j_{MK}, T_{jMK}\in S(\mu^{|M|-|K|-1}\langle\mu\xi'\rangle^{|\varepsilon(K,M)\circ Q|/2+1/2}m(\varphi)^{\varepsilon(K,M)\circ Q}, g_\mu),$$
$$\tilde{T}_{MK}\in S(\mu\langle\mu\xi'\rangle^{|\varepsilon(K,M)\circ Q|/2+1}m(\varphi)^{\varepsilon(K,M)\circ Q}, g_\mu)$$

for any $Q\in\boldsymbol{R}^l$. *Further*

$$T^j_{MK}=\pm i\alpha^{(j)}_{s(i)}\langle\xi'\rangle,\ T_{jMK}=\pm i\alpha_{s(i)(j)}\ \textit{if}\ |K|=|M|-1,\ K=M\setminus\{i\}.$$

Proof. Noting (3.11), (3.14) and (5.1) the lemma will be proved by induction on $|M|$.

We shall estimate $I_S^*(\tilde{e})I_S(\tilde{e})[Q, \alpha_S]$. By Lemma 6.3 we have

$$[Q, \alpha_S]\equiv\sum T^j_K\langle D'\rangle^{-1}Q_{(j)}\alpha_{S\circ K}+\sum T_{jK}Q^{(j)}\alpha_{S\circ K}+\sum\tilde{T}_K\alpha_{S\circ K}.$$

Corollary 3.1 gives an expression of $I_S^*(\tilde{e})I_S(\tilde{e})T^j_K$ as

$$I_S^*(R_1)B_{jK\tilde{S}}I_{\tilde{S}}(R_2), \qquad B_{jK\tilde{S}}\in S(1, g)$$

with $R_1+R_2=e-\varepsilon(K)\circ U$, $|\varepsilon(K)\circ U|=-1$. Repeating similar argument for $I_S^*(\tilde{e})I_S(\tilde{e})\tilde{T}_K$ we conclude that $I_S^*(\tilde{e})I_S(\tilde{e})[Q, \alpha_S]$ is expressed as

$$\sum_{j,k,\tilde{S}} I_S^*(\tilde{e}+\tilde{e}_k)B_{jk\tilde{S}}I_{\tilde{S}}(\tilde{e}+\tilde{e}_k)\langle D'\rangle^{-1}Q_{(j)}\alpha_{\tilde{S}}+\sum_{j,k,\tilde{S}} I_S^*(\tilde{e}+\tilde{e}_k)B_{jk\tilde{S}}$$
$$\times I_{\tilde{S}}(\tilde{e}+\tilde{e}_k)Q^{(j)}\alpha_{\tilde{S}}+\sum_{k,\tilde{S}} I_S^*(\tilde{e}+\tilde{e}_k)B_{k\tilde{S}}I_{\tilde{S}}(\tilde{e}+3\tilde{e}_k)\alpha_{\tilde{S}}.$$

Here we note that $\mu^{-1/2}\|I_{\tilde{S}}(\tilde{e}+\tilde{e}_k)T\alpha_{\tilde{S}}u\|^2$ where $T=\langle D'\rangle^{-1}Q_{(j)}$ or

$T=Q^{(j)}$ are estimated by $c(n)\mu^{1/2}\sum_{j,S} E_S(j,\nu)$ for any $\nu\in I$. In fact writing $I_{\bar{S}}(\tilde{e}+\tilde{e}_k)T\equiv(1+r)J_{\bar{S}(k)}(1/2)I_{\bar{S}}(\tilde{e})T$ with $r\in S(\mu,g)$ it suffices to estimate $[J_{\bar{S}(k)}(1/2)I_{\bar{S}}(\tilde{e}),T]$. This is estimated by $c(n)\mu|u|^2_{\bar{S},\tilde{e}+(3/2),0}$. Then applying Lemma 5.1 we get the desired estimate. Now we have

Lemma 6.4. *For any* $\nu\in I$ *we have*

$$|(I_S(\tilde{e})[Q,\alpha_S]u, I_S(\tilde{e})A\alpha_S u)|\leqq c(n)\mu^{1/2}|u|^2_{A,S,\tilde{e}+(1/2),0}+c(n)\mu^{1/2}\sum_{j,S}E_S(j,\nu).$$

Recall that $\tilde{P}_j=-L_{j0}^2+\sum_{k=1}^{p-1}L_{jk}^2+Q_j$. Set

$$\tilde{P}_{j\theta}(x,D,\mu)=\tilde{P}_j(x,D_0-i\theta,D',\mu).$$

Combining Lemmas 6.2, 6.4 and Corollary 6.3 one obtains

Proposition 6.1. *For any* $\nu\in I$ *we have*

$$\begin{aligned}&|(I_S(\tilde{e})[\tilde{P}_{j\theta},\alpha_S]u, I_S(\tilde{e})A\alpha_S u)|\leqq cn^{1/2}\{|u|^2_{A,S,\tilde{e}+(1/2),0}\\&+\sum_k|u|^2_{L_k,S,\tilde{e}+(1/2),0}+n[u]^2_{\tilde{e}+(3/2),0}\}+c(n)\mu^{1/2}([u]^2_{\tilde{e}-\tilde{e}_\nu,1}+[u;Q]_{\tilde{e}+\langle\tilde{e}_j\rangle})\\&+c(n,\mu)[u]^2_{\tilde{e}+(1/2),0}\end{aligned}$$

for $0<\mu\leqq\hat{\mu}(n)$.

Noting that $S(1,g)\subset S(\langle\mu\xi'\rangle^{|R|/2}m(\varphi)^R,g)$ for any $R\in(\boldsymbol{R}^+)^l$, Lemmas 6.1 and Lemma 3.3 prove the following lemma.

Lemma 6.5. *Let* $T^{(\alpha)}\in S(\tilde{m},g_\mu)$, $T_{(\alpha)}\in S(\tilde{m}\langle\xi'\rangle,g_\mu)$ *for* $|\alpha|=1$. *Then we have*

$$I_S(R)\alpha_S T\equiv TI_S(R)\alpha_S+\sum_{j,\bar{S}}B_{j\bar{S}}\langle\mu D'\rangle^{|R(j)|/2-k}I_{\bar{S}}(R+e_j-R(j))\alpha_{\bar{S}}\langle\mu D'\rangle^k$$

for any $R(j)\in(\boldsymbol{R}^+)^l$, $k\in\boldsymbol{R}^l$ *where* $B_{j\bar{S}}\in S(\tilde{m},g_\mu)$.

§7. Energy Estimate for $-A^2+\sum L_k^2+Q$

In this section we derive energy estimate which absorbs $[u]^2_{\tilde{e}-\tilde{e}_\nu,1}$ combining estimates in §§4 and 5. Let A, L_k, Q be as in §§4 and 5. Put

$$T_j(x,\xi',\mu)=\varphi_j(x,\xi',\mu)\langle\mu\xi'\rangle.$$

Since $T_j^*-T_j\in S(\mu^{1/2},g_\mu)$ by (3.4) one has

$$T_j^* T_j = \mathrm{Op}(\varphi_j^2 \langle \mu\xi' \rangle^2) + r, \qquad r \in S(\mu^{1/2} \langle \mu\xi' \rangle, g_\mu).$$

As in § 5 we write J_ε instead of $J_\varepsilon(1/2)$. Using Melin's inequality the same argument as in the proof of Lemma 5.1 shows that

$$(7.1) \qquad ||T_j J_{s(j)} I_S(\tilde{e}) \alpha_S u||^2 \leqq c\,|u\,;Q|_{S,\tilde{e}+\langle \tilde{e}_j \rangle} + c(n)\,\mu^{1/2} E_S^{\sharp}(\nu)$$

for any $\nu \in I$. We shall estimate $(E) = ||T_j J_{s(j)} I_S(\tilde{e}) \alpha_S u||^2 + ||I_S(\tilde{e}+3\tilde{e}_j) \alpha_S u||^2$ from below. From Corollary 3.1 one can write

$$I_S^*(\tilde{e}) J_{s(j)}^* T_j^* T_j J_{s(j)} I_S(\tilde{e}) + I_S^*(\tilde{e}+3\tilde{e}_j) I_S^*(\tilde{e}+3\tilde{e}_j)$$
$$\equiv I_S^*(\tilde{e}-\tilde{e}_j) \langle \mu D' \rangle (B_{jS} + C_{jS}) \langle \mu D' \rangle I_S(\tilde{e}-\tilde{e}_j).$$

Noticing that the left-hand side belongs to $S(\langle \mu\xi' \rangle^{2+2nS^*} \times m(\varphi)^{-2nS+e_j-e}, g_\mu)$ we see that $B_{jS} \in S(1, g_\mu)$, $C_{jS} \in S(\mu, g)$. We observe $\sigma(B_{jS})$. In view of Corollary 3.1 it follows that

$$\sigma(B_{jS}) = \langle \mu\xi' \rangle^{-2} T_j^2 J_{s(j)}(2) + \langle \mu\xi' \rangle^{-2} J_{s(j)}(4)$$
$$= J_{s(j)}(4) \{\varphi_j^2 J_{s(j)}^2(\varphi_j) + \langle \mu\xi' \rangle^{-2}\} \geqq c J_{s(j)}^{-4}(\varphi_j) m(\varphi_j)^4 \geqq \hat{c} > 0.$$

The last inequality follows from (3.13). Using this inequality, Lemma 4.6 in [12] gives that

$$(7.2) \qquad \mathrm{Re}(B_{jS} v, v) \geqq (\hat{c} - c(B_{jS})\mu) ||v||^2.$$

Taking $v = J_{s(j)} I_S(\tilde{e}) \alpha_S u$ in (7.2) we get an estimate of (E) from below by

$$(\hat{c} - c(B_{jS})\mu)\, |u|_{S,\tilde{e}-\tilde{e}_j,1}^2.$$

Combining this estimate with (6.1) $(\nu = j)$ we obtain

Lemma 7.1.

$$c[u\,;Q]_{\tilde{e}+\langle \tilde{e}_j \rangle} + c[u]_{\tilde{e}+(3/2),0}^2 + c(n,\mu)[u]_{\tilde{e}+(1/2),0}^2$$
$$\geqq (\hat{c} - c(n)\mu^{1/2})[u]_{\tilde{e}-\tilde{e}_j,1}^2$$

for $0 < \mu \leqq \hat{\mu}(n)$ *where* Q *is defined with* G_j.

Similar arguments give that

Lemma 7.2.

$$c[u\,;Q]_{\tilde{e}} + c[u]_{\tilde{e}+(1),0}^2 + c(n,\mu)[u]_{\tilde{e},0}^2 \geqq (\hat{c} - c(n)\mu^{1/2})[u]_{\tilde{e}-e_j,1}^2$$

for $0 < \mu \leqq \hat{\mu}(n)$ *where* Q *is defined with* G_j.

In Propositions 4.1 and 5.1 taking $\nu=j$ and adding these we obtain an estimate of $2\mathrm{Im}\sum_S (I_S(\tilde{e})\tilde{P}_{j\theta}\alpha_S u,\ I_S(\tilde{e})\alpha_S u)$ from below. To simplify notation we set

$$\begin{aligned} e(u;j) &= [u]^2_{A,\tilde{e},0}+\sum_k [u]^2_{L_k,\tilde{e},0}+[u;Q]_{\tilde{e}}+c_1 n\theta[u]^2_{\tilde{e}+(1/2),0} \\ &\quad +c_2 n^2[u]^2_{\tilde{e}+(1),0}, \\ E_s^{(1)}(u;j) &= [Au]^2_{\tilde{e},s}+\sum_k [L_k u]^2_{\tilde{e},s}+\mu^{-1}[\nabla Qu]^2_{\tilde{e},s}+[u]^2_{\tilde{e}-e_j,s+1} \\ &\quad +n\theta[u]^2_{\tilde{e}+(1/2),s}+n^2[u]^2_{\tilde{e}+(1),s}, \\ E_s^{(2)}(u;j) &= [Au]^2_{\tilde{e}+(1/2),s}+\sum_k [L_k u]^2_{\tilde{e}+(1/2),s}+\mu^{-1}[\nabla Qu]_{\tilde{e}+(1/2),s} \\ &\quad +[u]^2_{\tilde{e}-\tilde{e}_j,s+1}+n\theta[u]^2_{\tilde{e}+(1),s}+n^2[u]^2_{\tilde{e}+(3/2),s}. \end{aligned}$$

Note that for a fixed ν, $0<\nu<1$ we have

$$(7.3)\qquad \begin{aligned} & c\theta E_0^{(1)}(\langle\mu D'\rangle^k u;j)\geqq \theta^\nu E_k^{(1)}(u;j) \\ & c\{\theta^\nu E_0^{(1)}(\langle\mu D'\rangle^k u;j)+nE_0^{(2)}(\langle\mu D'\rangle^k u;j)\}\geqq nE_k^{(2)}(u;j) \end{aligned}$$

for $0<\mu\leqq\hat{\mu}(n)$, $\hat{\theta}(n,\mu,k)\leqq\theta$. In view of Lemmas 5.1, 6.1 and 6.5 $E_0^{(1)}(u;j)$ is estimated by constant times of $e(u;j)$ for $0<\mu\leqq\hat{\mu}(n)$, $\hat{\theta}(n,\mu)\leqq\theta$. $E_0^{(2)}(u;j)$ is estimated similarly. Then taking Proposition 6.1 into account we get

Proposition 7.1.

$$2\mathrm{Im}\sum_S (I_S(\tilde{e})\alpha_S\tilde{P}_{j\theta}u,\ I_S(\tilde{e})A\alpha_S u)\geqq \partial_0 e(u;j)+c_3\theta E_0^{(1)}(u;j)+c_3 nE_0^{(2)}(u;j)$$

for $(16\leqq)\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\hat{\theta}(n,\mu)\leqq\theta$, $|x_0|\leqq\mu^{1/2}$, $u\in C^\infty(I,H^\infty)$.

Next we estimate lower order terms. It will suffice to handle

$$(I_S(\tilde{e})\alpha_S B_i u^i,\ I_S(\tilde{e})A_j\alpha_S u^j),\qquad (I_S(\tilde{e})\alpha_S A_i u^i,\ I_S(\tilde{e})A_j\alpha_S u^j)$$

where $A_k=L_{k0}(x,D_0-i\theta,D',\mu)$, $B_i\in S(\langle\mu\xi'\rangle, dx_0^2+\tilde{G}_\mu)$. In view of Lemma 6.1 the second term is estimated by

$$c(n)\sum_{k=i,j}\{|A_k u^k|^2_{S,\tilde{e},0}+[u^k]^2_{\tilde{e}+(1),0}\}$$

when $0<\mu\leqq\hat{\mu}(n)$. Again by Lemma 6.1 the first term is estimated by

$$(7.4)\qquad \begin{aligned} & c\{|u^i|^2_{S,\tilde{e}-\tilde{e}_i,0}+|A_j u^j|^2_{S,\tilde{e}+(1/2),0}+n[u^j]^2_{\tilde{e}+(3/2),0}\} \\ & \qquad +c(n,\mu)\sum_{k=i,j}[u^k]^2_{\tilde{e}+(1),0} \end{aligned}$$

when $0<\mu\leqq\hat{\mu}(n)$.

Now we rewrite our inequality thus obtained in terms of matrix representation. Set

$$E_s^{(i)}(u)=\sum_j E_s^{(i)}(u^j;j),\quad e(u)=\sum_j e(u^j;j),\quad u=(u^1,\cdots,u^l),$$

$$\tilde{L}_{0\theta}=\mathrm{diag}(L_{10}(x,D_0-i\theta,D',\mu),\cdots,L_{l0}(x,D_0-i\theta,D',\mu)).$$

Since (7.4) implies that $(I_S(\tilde{e})\alpha_S(\hat{P}_\theta-\tilde{P}_\theta)u,\ I_S(\tilde{e})\tilde{L}_{0\theta}\alpha_S u)$ is estimated by $cE_0^{(2)}(u)+c(n,\mu)E_0^{(1)}(u)$ we have from Proposition 7.1 that

Theorem 7.1.

$$2\mathrm{Im}\sum_S(I_S(\tilde{e})\alpha_S\hat{P}_\theta u,\ I_S(\tilde{e})\tilde{L}_{0\theta}\alpha_S u)\geqq\partial_0 e(u)+c_3\theta E_0^{(1)}(u)+c_3 nE_0^{(2)}(u)$$

for $(16\leqq)\hat{n}\leqq n,\ 0<\mu\leqq\hat{\mu}(n),\ \hat{\theta}(n,\mu)\leqq\theta,\ |x_0|\leqq\mu^{1/2},\ u\in(C^\infty(I,H^\infty))^l$ *where* c_3 *is a positive constant independent of* $n,\ \mu,\ \theta$.

We shall examine that Theorem 7.1 holds with $E_k^{(i)}(u)$ for any $k\in\boldsymbol{R}$ after obvious modification. In virtue of Lemmas 6.1 and 6.5 it is easy to see that $(I_S(\tilde{e})\alpha_S[\langle\mu D'\rangle^k,\ L^2]u,\ I_S(\tilde{e})A\alpha_S\langle\mu D'\rangle^k u)$ and $(I_S(\tilde{e})\alpha_S[\langle\mu D'\rangle^k,\ Q]u,\ I_S(\tilde{e})A\alpha_S\langle\mu D'\rangle^k u)$ are estimated by

$$c(n,\mu)E_k^{(1)}(u;j)$$

for $0<\mu\leqq\hat{\mu}(n)$. Fix ν, $0<\nu<1$ then taking (7.3) into account it follows that

$$2\mathrm{Im}\sum_S(I_S(\tilde{e})\alpha_S\langle\mu D'\rangle^k\hat{P}_\theta u,\ I_S(\tilde{e})\tilde{L}_{0\theta}\alpha_S\langle\mu D'\rangle^k u)\geqq\partial_0 e(\langle\mu D'\rangle^k u)$$
$$+c_3\theta^\nu E_k^{(1)}(u)+c_3 nE_k^{(2)}(u)$$

for $\hat{n}\leqq n,\ 0<\mu\leqq\hat{\mu}(n),\ \hat{\theta}(n,\mu,k)\leqq\theta,\ k\in\boldsymbol{R}$. Applying Lemmas 6.1 and 6.5 the left-hand side of the above is estimated by

$$2\mathrm{Im}\sum_S(\langle\mu D'\rangle^k I_S(\tilde{e})\alpha_S\hat{P}_\theta u,\ \langle\mu D'\rangle^k I_S(\tilde{e})\tilde{L}_{0\theta}\alpha_S u)$$
$$+c(n,\mu,k)[\hat{P}_\theta u]^2_{\tilde{e}-(1/2),k-1/2}+cE_k^{(2)}(u)+c(n,\mu,k)E_k^{(1)}(u).$$

We summarize with notation $(u,v)_{(k)}=(\langle\mu D'\rangle^k u,\ \langle\mu D'\rangle^k v)$;

Proposition 7.2. *Fix* $0<\nu<1$. *Then we have*

$$c(n,\mu,k)[\hat{P}_\theta u]^2_{\tilde{e}-(1/2),k-1/2}+2\mathrm{Im}\sum_S(I_S(\tilde{e})\alpha_S\hat{P}_\theta u,\ I_S(\tilde{e})\tilde{L}_{0\theta}\alpha_S u)_{(k)}$$
$$\geqq\partial_0 e(\langle\mu D'\rangle^k u)+c_3\theta^\nu E_k^{(1)}(u)+c_3 nE_k^{(2)}(u)$$

for $\hat{n}\leqq n,\ 0<\mu\leqq\hat{\mu}(n),\ \hat{\theta}(n,\mu,k)\leqq\theta,\ k\in\boldsymbol{R},\ |x_0|\leqq\mu^{1/2},\ u\in(C^\infty(I,H^\infty))^l$.

§ 8. Estimate of Wave Front Sets

In this section we shall give an estimate of wave front sets. Since argument in this section is parallel to scalar operators we only skech the argument. Recall

$$\begin{aligned}\tilde{P}_{j\theta}&=-A_j^2(x, D, \mu)+\sum_{k=1}^{p-1} L_{jk}^2(x, D', \mu)+Q_j(x, D', \mu)\\&=-A_j^2(x, D, \mu)+\hat{Q}_j(x, D', \mu).\end{aligned}$$

Denote by $\hat{q}_j(x, \xi', \mu)$ the principal symbol of $\hat{Q}_j(x, D', \mu)$. Since $\hat{Q}_j(x, D', \mu)-\hat{q}_j(x, D', \mu)\in S(\langle\mu\xi'\rangle, dx_0^2+\tilde{G}_\mu)$ we canr eplace $\hat{Q}_j$ by $\hat{q}_j$ in the estimate of Proposition 7. 1. Then in the following we assume that $\hat{Q}_j(x, D', \mu)=\hat{q}_j(x, D', \mu)$ without loss of generality. Fix j and we write P, A, $\hat{Q}$ instead of $\tilde{P}_{j\theta}$, A_j, $\hat{Q}_j$.

Let $f(x, \xi', \mu)\in S(1, dx_0^2+\tilde{G}_\mu)$. We set following [7]

$$\Psi(x, \xi', \mu)=\begin{cases}\exp(1/f(x, \xi', \mu)) & \text{if } f(x, \xi', \mu)<0\\ 0 & \text{if } f(x, \xi', \mu)\geqq 0.\end{cases}$$

Define $\Psi^{(i)}(x, \xi', \mu)$ by

$$\Psi^{(1)}(x, \xi', \mu)=f^{-1}(x, \xi', \mu)\{A, f\}^{1/2}(x, \xi', \mu)\Psi(x, \xi', \mu)$$
$$\Psi^{(2)}(x, \xi', \mu)=f^{-1}(x, \xi', \mu)\{A, f\}^{-1/2}(x, \xi', \mu)\Psi(x, \xi', \mu).$$

Our basic hypothese on $f(x, \xi', \mu)$ is; there is a positive constant δ such that (see [7])

$$4(1-\delta)\hat{Q}\{A, f\}^2\geqq\{\hat{Q}, f\}^2. \tag{8. 1}$$

Let $f(x, \xi', \mu)$ satisfy (8. 1). Then we shall estimate $E_k^{(2)}(\Psi u; j)$ by $[\Psi Pu]_{\tilde{e}-(1/2), k}^2$ and $E_{k-1/4}^{(2)}(u; j)$ applying Proposition 7. 1. We use the notation $\sim$ and $\lesssim$ to indicate equality and inequality which hold modulo a term that is estimated by

$$c(n, \mu, k)E_{k-1/4}^{(2)}(u; j).$$

By repeated use of Lemmas 6. 1 and 6. 5 we shall first obtain the following estimate

$$\begin{aligned}2\mathrm{Im}(I_S(\tilde{e})\alpha_S[\hat{Q}, \Psi]u,\ I_S(\tilde{e})A\alpha_S\Psi u)_{(k)}&\sim 2|\Psi^{(1)}u|_{A,S,\tilde{e},k}^2\\&+2^{-1}|\Psi^{(2)}u|_{[,\hat{Q}f],S,\tilde{e},k}^2\\&-2\|\langle\mu D'\rangle^k I_S(\tilde{e})\{A\alpha_S\Psi^{(1)}-2^{-1}i[\hat{Q}, f]\alpha_S\Psi^{(2)}\}u\|^2.\end{aligned} \tag{8. 2}$$

Similar arguments give that

(8. 3) $$-2\mathrm{Im}(I_S(\tilde{e})\alpha_S[A, \Psi]Au, I_S(\tilde{e})A\alpha_S\Psi u)_{(k)}\sim -2|\Psi^{(1)}u|^2_{A,S,\tilde{e},k}.$$

Next again using Lemmas 6. 1 and 6. 5 we obtain

$$\begin{aligned}&-2\mathrm{Im}(I_S(\tilde{e})\alpha_S A[A, \Psi]u, I_S(\tilde{e})A\alpha_S\Psi u)_{(k)}\\&\sim 2\partial_0\mathrm{Re}(I_S(\tilde{e})\alpha_S[A, \Psi]u, I_S(\tilde{e})\alpha_S\Psi Au)_{(k)}\\&-2\mathrm{Im}(I_S(\tilde{e})\alpha_S[A, \Psi]u, I_S(\tilde{e})\alpha_S\Psi A^2u)_{(k)}.\end{aligned}$$

Since $A^2=\hat{Q}-P$, replacing A^2 by $\hat{Q}-P$ it turns out to

(8. 4) $$\begin{aligned}&-2\mathrm{Im}(I_S(\tilde{e})\alpha_S A[A, \Psi]u, I_S(\tilde{e})A\alpha_S\Psi u)_{(k)}\\&-2\mathrm{Im}(I_S(\tilde{e})\alpha_S[A, \Psi]u, I_S(\tilde{e})\alpha_S\Psi Pu)_{(k)}\\&\sim 2\partial_0\mathrm{Re}(I_S(\tilde{e})\alpha_S[A, \Psi]u, I_S(\tilde{e})\alpha_S\Psi Au)_{(k)}\\&-2\mathrm{Im}(I_S(\tilde{e})\alpha_S[A, \Psi]u, I_S(\tilde{e})\alpha_S\Psi\hat{Q}u)_{(k)}.\end{aligned}$$

We estimate the second term in the right-hand side of (8. 4).

Lemma 8. 1. *For any* $\varepsilon>0$ *we have*

$$\begin{aligned}&-2\mathrm{Im}\sum_S(I_S(\tilde{e})\alpha_S[A, \Psi]u, I_S(\tilde{e})\alpha_S\Psi\hat{Q}u)_{(k)}\\&\leqq -2(1-\varepsilon)\mathrm{Re}\sum_S(\hat{Q}\langle\mu D'\rangle^k I_S(\tilde{e})\alpha_S\Psi^{(1)}u, \langle\mu D'\rangle^k I_S(\tilde{e})\alpha_S\Psi^{(1)}u)\end{aligned}$$

modulo $c(n, \mu, k, \varepsilon)E^{(2)}_{k-1/4}(u; j)$.

In view of Lemma 8. 1 it follows from (8. 2) – (8. 4) that

$$\begin{aligned}&2\mathrm{Im}\sum_S(I_S(\tilde{e})\alpha_S[P, \Psi]u, I_S(\tilde{e})A\alpha_S\Psi u)_{(k)}\\&-2\mathrm{Im}\sum_S(I_S(\tilde{e})\alpha_S[A, \Psi]u, I_S(\tilde{e})\alpha_S\Psi Pu)_{(k)}\\&\lesssim 2\partial_0\mathrm{Re}\sum_S(I_S(\tilde{e})\alpha_S[A, \Psi]u, I_S(\tilde{e})\alpha_S\Psi Au)_{(k)}+2^{-1}[\Psi^{(2)}u]^2_{[\hat{Q},f],\tilde{e},k}\\&-2(1-\varepsilon)\sum_S\mathrm{Re}(\hat{Q}w_{Sk}, w_{Sk})\end{aligned}$$

where $w_{Sk}=\langle\mu D'\rangle^k I_S(\tilde{e})\alpha_S\Psi^{(1)}u$. We consider the last two terms in the right-hand side of the above. Since

$$|\Psi^{(2)}u|^2_{[\hat{Q},f],S,\tilde{e},k}\sim\mathrm{Re}(\mathrm{Op}(\{A, f\}^{-2}\{\hat{Q}, f\}^2)w_{Sk}, w_{Sk})$$

the sum of the last two terms is equal to

$$-2\mathrm{Re}((\mathrm{Op}(1-\varepsilon)\hat{Q}-4^{-1}\{A, f\}^{-2}\{\hat{Q}, f\}^2)w_{Sk}, w_{Sk})$$

modulo $c(n, \mu, k, \varepsilon)E^{(2)}_{k-1/4}(u; j)$. Hence taking ε so that $\varepsilon<\delta$ (δ is the constant in (8. 1)) this is non positive in view of (8. 1) modulo $c(n, \mu, k)E^{(2)}_{k-1/4}(u; j)$. We summarize;

Lemma 8.2.

$$\begin{aligned}&2\mathrm{Im}\sum_S (I_S(\tilde{e})\alpha_S[P,\Psi]u, I_S(\tilde{e})A\Psi\alpha_S u)_{(k)}\\&\leqq 2\partial_0 \mathrm{Re}\sum_S (I_S(\tilde{e})\alpha_S[A,\Psi]u, I_S(\tilde{e})\alpha_S\Psi Au)_{(k)}\\&+[\Psi Pu]^2_{\tilde{e}-(1/2),k-1/4}+c(n,\mu,k)E^{(2)}_{k-1/4}(u;j).\end{aligned}$$

Now from Lemma 8.2 and Proposition 7.2 it follows that

Proposition 8.1. *Assume* (8.1) *and fix* $0<\nu<1$. *Then*

$$\begin{aligned}&c(n,\mu,k)[\Psi\hat{P}_\theta u]^2_{\tilde{e}-(1/2),k}+c(n,\mu,k)E^{(2)}_{k-1/4}(u)\\&\geqq \partial_0\{e(\langle\mu D'\rangle^k\Psi u)-2\mathrm{Re}\sum_S (I_S(\tilde{e})\alpha_S[A,\Psi]u, I_S(\tilde{e})\alpha_S\Psi Au)_{(k)}\}\\&+c_3\theta^\nu E^{(1)}_k(\Psi u)+c_3 n E^{(2)}_k(\Psi u)\end{aligned}$$

for $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\hat{\theta}(n,\mu,k)\leqq\theta$, $k\in\boldsymbol{R}$, $|x_0|\leqq\mu^{1/2}$, $u\in(C^\infty(I,H^\infty))^l$.

Propositions 7.2 and 8.1 prove that there is a parametrix of $\hat{P}$ at $\rho'=(0,0,e_d')$ with finite propagation speed of wave front sets.

Proof of Theorem 1.2. Let $\langle H_\varphi(\rho)\rangle^\sigma=H_1$. We may assume that $\rho=(0,e_d)$. Since $H_\varphi(\rho)$ belongs to $\Gamma(p^i_\rho, H_{x_0})$ for every i and hence

$$p^i_\rho(H_\varphi(\rho))\neq 0. \tag{8.5}$$

In particular $H_\varphi(\rho)$ and the radial vector field at ρ are linearly independent. Put $X_0=\varphi(x,\xi)$ and extend it to a full homogeneous symplectic coordinates $\{X,\Xi\}$ such that $X(\rho)=0$, $\Xi(\rho)=e_d$. We write (x,ξ) instead of (X,Ξ). From (8.5) we have $H^2_{x_0}p^i(\rho)\neq 0$ and Malgrange's preparation theorem gives that

$$p^i(x,\xi)=e^i(x,\xi)\{\xi_0^2-2\tilde{a}^i(x,\xi')\xi_0+\tilde{b}^i(x,\xi')\}=e^i(x,\xi)\tilde{p}^i(x,\xi)$$

with $e^i(\rho)\neq 0$. Clearly the condition (1.4)′ is symplectically invariant and then $\{\tilde{p}^i(x,\xi)\}$ satisfy (1.4). A pseudodifferential operator analogue of Malgrange's division theorem shows that

$$P^i(x,D)\equiv E^i(x,D)\tilde{P}^i(x,D)$$

where $E^i(x,D)$ is non characteristic at ρ and $\tilde{p}^i(x,\xi)$ is the principal symbol of $\tilde{P}^i(x,D)$. Then we can apply Proposition 8.1 to $\{\tilde{P}^i(x,D)\}$ and we conclude Theorem 1.2.

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