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Discrete Cubic $\mathfrak{X}$-Splines*

By

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Abstract

In the present paper, we introduce the discrete cubic $\mathfrak{X}$-splines and study the existence, uniqueness and convergence properties of such $\mathfrak{X}$-splines which interpolate given functional values at the mesh points. An optimal interpolating discrete cubic $\mathfrak{X}$-spline and splines of special interest have also been discussed.

§ 1. Introduction

Clenshaw and Negus [2] have introduced cubic $\mathfrak{X}$-splines as cubic pp (piecewise polynomial) functions of class C^1 which satisfy certain conditions involving second and third derivatives of the polynomial pieces at the internal mesh points. Cubic $\mathfrak{X}$-splines in particular, reduce to the usual cubic splines and present some practical advantages in comparison with the conventional cubic splines (see [1], [2], [6]).

In the present paper, we define the discrete cubic $\mathfrak{X}$-splines and study the existence, uniqueness and convergence properties of such splines which interpolate given functional values at the mesh points.

We write for convenience, $f(x_i)=f_i$ for all i, and introduce the central difference operator D_h $(h>0)$ defined by

$$D_h^{\{0\}}f(x)=f(x);\ D_h^{\{1\}}f(x)=(f(x+h)-f(x-h))/2h \text{ and}$$
$$D_h^{\{k+1\}}f(x)=D_h^{\{1\}}(D_h^{\{k\}}f(x));\ k=0, 1, 2.$$

Further, the jump of the function f at, or across, the point α is denoted by

$$jump_\alpha f:=f(\alpha^+)-f(\alpha^-).$$

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Let a mesh on $[a, b]$ be defined by $\mathfrak{P}: a=x_0<x_1<\ldots<x_n=b$ with $x_i-x_{i-1}=p_i$ for $i=1, 2, \ldots, n$, $p=\max_i p_i$ and $p'=\min_i p_i$. Suppose that $\mathfrak{X}=\langle\alpha_i\rangle_{i=1}^n$ is a given set of real parameters. For a given $h>0$, a continuous function $s(x, h)$ such that its restriction s_i on $[x_{i-1}, x_i]$ is a polynomial of degree 3 or less for $i=1, 2, \ldots, n$, defines a discrete cubic $\mathfrak{X}$-spline, if,

$$(1.1)\qquad jump_{x_i} D_h^{[k]} s(x, h) = \alpha_i \delta_{k,2}\, jump_{x_i} D_h^{[3]} s(x, h),\ k=0, 1, 2 \quad i=1, 2, \ldots, n-1,$$

where $\delta_{k,2}$ is the usual Kronecker delta.

We shall denote by $\mathfrak{D}(3, \mathfrak{P}, \mathfrak{X}, h)$ the class of all discrete cubic $\mathfrak{X}$-splines over the mesh $\mathfrak{P}$. A discrete cubic $\mathfrak{X}$-spline $s(x, h)$ which is b-a periodic, is said to be a periodic discrete cubic $\mathfrak{X}$-spline. The class of all such splines is denoted by $\mathfrak{D}_1(3, \mathfrak{P}, \mathfrak{X}, h)$.

Considering the interpolatory condition

$$(1.2)\qquad s(x_i, h) = f_i,\ i=0, 1, \ldots, n,$$

where $\langle f_i\rangle$ are given periodic functional values, we propose to study the following:

Problem A. *Given $h>0$, for what restrictions on $\mathfrak{P}$ and $\mathfrak{X}$ does there exist a unique spline in $\mathfrak{D}_1(3, \mathfrak{P}, \mathfrak{X}, h)$ satisfying the interpolatory condition* (1.2)?

§ 2. Existence and Uniqueness

It is clear that $s(x, h)$ is piecewise cubic hence in the interval $[x_{i-1}, x_i]$, we have

$$(2.1)\qquad s(x, h) = a_i + b_i(x_i - x) + c_i(x_i - x)^2(x - x_{i-1}) + d_i(x_i - x)(x - x_{i-1})^2$$

where a_i, b_i, c_i, d_i are appropriate constants. Now using the interpolatory condition (1.2) we have

$$(2.2)\qquad a_i = f_i;\ b_i = (f_{i-1} - f_i)/p_i$$

Setting $m_i = m_i(h) = D_h^{[1]} s(x_i, h)$ for all i and taking the first difference of (2.1) we determine the constants c_i and d_i. Thus, (2.1) is rewritten as

$$(2.3)\quad p_i(p_i^2+2h^2)s(x,h)=(x_i-x)(x-x_{i-1})[(h^2+p_i(x_i-x))m_{i-1} - (h^2+p_i(x-x_{i-1}))m_i+(2x-x_i-x_{i-1})\Delta f_{i-1}] + (p_i^2+2h^2)((x_i-x)f_{i-1}+(x-x_{i-1})f_i)$$

where Δ is the usual forward difference operator.

We are now set to answer the problem A in the following.

Theorem 1. *Suppose that $p'\geq h>0$ and $\mathfrak{X}=\langle\alpha_i\rangle_{i=1}^n$ is such that $|\alpha_i|\leq p'/3$ for all i. Then there exists a unique discrete cubic $\mathfrak{X}$-spline $s(x,h)$ in the class $\mathfrak{D}_1(3,\mathfrak{P},\mathfrak{X},h)$ satisfying* (1.2).

Remark 2.1. It may be observed that as $h\to 0$ a discrete cubic $\mathfrak{X}$-spline reduces to a continuous cubic $\mathfrak{X}$-spline. If $\alpha_i=0$ for all i, a discrete cubic $\mathfrak{X}$-spline reduces to a discrete cubic spline. If both the foregoing conditions are assumed simultaneously then a discrete cubic $\mathfrak{X}$-spline reduces to a cubic spline.

Proof of Theorem 1. In view of the defining condition (1.1) with $k=2$ for the discrete cubic $\mathfrak{X}$-spline we have the following system of equations:

$$(2.4)\quad r_i[p_{i+1}^2+3p_{i+1}\alpha_i-h^2]m_{i+1}+[r_i(2p_{i+1}^2+3p_{i+1}\alpha_i+h^2) +r_{i+1}(2p_i^2-3p_i\alpha_i+h^2)]m_i+r_{i+1}[p_i^2-3p_i\alpha_i-h^2]m_{i-1}=3F_i(h)$$

where $r_i=p_i(p_i^2+2h^2)$ and $F_i(h)=r_i\Delta f_i(p_{i+1}+2\alpha_i)+r_{i+1}\Delta f_{i-1}(p_i-2\alpha_i)$.

In order to prove Theorem 1, it is sufficient to show that the system of equations (2.4) for $i=1,2,\ldots,n$ has a unique solution. For this, we shall consider the cases: (i) $\alpha_i\geq 0$ and (ii) $\alpha_i\leq 0$ separately. Assuming (i) we see from the condition $p'\geq h$ that the coefficient of m_{i+1} is nonnegative. Further, in view of the condition $\alpha_i\leq p'/3$, we observe that the coefficient of m_i is also nonnegative and the absolute value of the coefficient of m_{i-1} is

$$|r_{i+1}(p_i^2-3p_i\alpha_i-h^2)|<r_{i+1}(p_i^2-3p_i\alpha_i+h^2).$$

Thus, the excess of the positive value of the coefficient of m_i over the sum of the positive values of the coefficients of m_{i-1} and m_{i+1} in (2.4) is not less than $t_i(h)=r_i(p_{i+1}^2+2h^2)+r_{i+1}p_i^2$ which is clearly positive.

For the other case in which $\alpha_i\leq 0$, we observe that the coefficient of m_{i-1} is nonnegative for $p'\geq h$ while the coefficient of m_i is nonnegative for $\alpha_i \leq p'/3$. Further, the absolute value of the coefficient of m_{i+1} is

$$|r_i(p_{i+1}^2+3p_{i+1}\alpha_i-h^2)|<r_i(p_{i+1}^2+3p_{i+1}\alpha_i+h^2).$$

Writing $t_i^*(h)=r_ip_{i+1}^2+r_{i+1}(p_i^2+2h^2)$, we see that the excess of the positive value of the coefficient of m_i over the sum of the positive values of the coefficients of m_{i-1} and m_{i+1} in (2. 4) is not less than $t_i^*(h)$ which is clearly positive. We, thus conclude that the coefficient matrix of the system of equations (2. 4) is invertible. This completes the proof of Theorem 1.

§ 3. Discrete Error Bounds

For a given $h>0$, we introduce the set

$$\mathfrak{R}_{ha}=\{a+jh: j \text{ is an integer}\},$$

and define a discrete interval as follows,

$$[a, b]_h=[a, b]\cap\mathfrak{R}_{ha}.$$

For a function f and two distinct points x_1, x_2 in its domain, the first divided difference is defined by

$$[x_1, x_2]f=\{f(x_1)-f(x_2)\}/(x_1-x_2).$$

For convenience, we write $f^{(r)}$ for $D_h^{(r)} f$, $r=1, 2$, and $w(f, p)$ for the modulus of continuity of f. The discrete norm of a function f over the interval $[a, b]_h$ is defined by

$$||f||=\max_{x\in[a,b]_h}|f(x)|.$$

Without assuming any smoothness condition on the function f, we shall obtain in this section the bounds for the error function $e(x)=s(x, h)-f(x)$ over the discrete interval $[a, b]_h$ where $s(x, h)$ is the discrete periodic cubic $\mathfrak{X}$-spline interpolant of f under the conditions of Theorem 1.

It may be observed that the system of equations (2. 4) may be written as

(3. 1) $$A(h)M(h)=3F(h)$$

where $A(h)$ is the coefficient matrix, $M(h)=(m_i(h))$ and $F(h)$ denotes the single column matrix $(F_i(h))$. In view of the diagonally dominant property of $A(h)$, which has already been established in Section 2 of the paper, it follows that

$$(3.2) \qquad ||A^{-1}(h)|| \leq t(h)$$

where $t(h) = \max_i \{t_i^{-1}(h),\ t_i^{*-1}(h)\}$.

(3.1) may be rewritten as,

$$(3.3) \qquad A(h)(e_i^{(1)}) = 3(F_i(h)) - A(h)(f_i^{(1)}).$$

In order to estimate the row-max norm of the right hand side matrix in (3.3), we shall need particular case of the following results due to Lyche ([5] Corollary 5.2 and [4] Lemma 2.1 (discrete Taylor formula) respectively) (see also Dikshit and Rana [3]).

Lemma 3.1. *Let a_j, $j=1,2,3,4$ and b_j, $j=1,2,3$ be given sequences of nonnegative real numbers such that $\sum a_j = \sum b_j$. Then for a real valued function f defined on a given $[\alpha, \beta]_h$, we have*

$$(3.4) \qquad \left|\sum_{j=1}^{4} a_j[x_{j0}, x_{j1}]f - \sum_{j=1}^{3} b_j[y_{j0}, y_{j1}]f\right| \leq \sum a_j w(f^{(1)}, |\beta-\alpha-h|)$$

where $x_{j0}, x_{j1}, y_{j0}, y_{j1} \in [\alpha, \beta]_h$ for relevant values of j.

Lemma 3.2. *Let f be a function defined over $[a, b]_h$ for some $h>0$ and a be a given real number. Then for any $x \in \mathfrak{R}_{ha}$, we have*

$$(3.5) \qquad f(x) = f(a) + \theta(x-a) f^{(1)}(y);\ 0 \leq \theta \leq 1$$

where $y \in [a, x-h]_h$.

Thus, we see that the $i-$th row of (3.3) may be written as

$$\sum_{j=1}^{4} a_j[x_{j0}, x_{j1}]f - \sum_{j=1}^{3} b_j[y_{j0}, y_{j1}]f$$

where

$a_1 = r_{i+1}h^2$, $a_2 = 3r_{i+1}p_i(p_i - 2\alpha_i)$, $a_3 = 3r_i p_{i+1}(p_{i+1} + 2\alpha_i)$, $a_4 = r_i h^2$, $b_1 = r_{i+1}p_i(p_i - 3\alpha_i)$, $b_2 = r_i(2p_{i+1}^2 + 3p_{i+1}\alpha_i + h^2) + r_{i+1}(2p_i^2 - 3p_i\alpha_i - h^2)$, $b_3 = r_i p_{i+1}(p_{i+1} + 3\alpha_i)$, $\langle y_{j0}\rangle = \langle x_{i+j-2} - h\rangle_{j=1}^3$, $\langle y_{j1}\rangle = \langle x_{i+j-2} + h\rangle_{j=1}^3$, $x_{10} = x_{i-1} - h$, $x_{11} = x_{i-1} + h$, $x_{20} = x_{i-1}$, $x_{21} = x_i = x_{30}$, $x_{31} = x_{i+1}$, $x_{40} = x_{i+1} - h$ and $x_{41} = x_{i+1} + h$.

Clearly $\langle a_j\rangle_{j=1}^4$ and $\langle b_j\rangle_{j=1}^3$ are sequences of nonnegative real numbers such that

$\sum a_j = \sum b_j = r_i(3p_{i+1}^2 + 6p_{i+1}\alpha_i + h^2) + r_{i+1}(3p_i^2 - 6p_i\alpha_i + h^2) = (R_i)$, say.

In order to highlight the advantages of $\mathfrak{X}$-splines, we shall now determine the lower bounds of (R_i) by choosing α_i appropriately. It is a general observation that error function, for a class of splines, can be minimized by increasing the number of mesh points or decreasing the length of each meshes of a given mesh. In fact, the expression for (R_i) is linear in α_i and nonnegative in the range from $-p'/3$ to $p'/3$. Thus, the choice $\alpha_i = -p'/3$ minimizes (R_i) when $p_i > p_{i+1}$. The other case $p_{i+1} > p_i$ is not sufficiently interesting in which the choice $\alpha_i = p'/3$ gives the minimum value of (R_i). Further, it may be mentioned here that the expression for (R_i) will be independent of α_i for the case of uniform mesh ($p_i = p_{i+1}$ for all i). Thus choosing α_i suitably and then applying Lemma 3.1, we have

$$\|(e_i)^{(1)}\| \leq \max_i (\min_i |R_i|) t(h) w(f^{(1)}, 2p) \tag{3.6}$$

where $\min_i |R_i| = r_i(3p_{i+1}^2 + h^2) + r_{i+1}(3p_i^2 + h^2) + 2p_i p_{i+1}\ p'(p_{i+1}^2 - p_i^2)$.

Now taking the first difference of $s(x, h)$ in Equation (2.3) we replace m_i by $e_i^{(1)}$ and $D_h^{(1)}s(x, h)$ by $e^{(1)}$ to get the following,

$$\begin{aligned} r_i(e)^{(1)} &= 6p_i(x - x_{i-1})(x_i - x)[x_{i-1}, x_i]f - r_i[x-h, x+h]f \\ &+ (x - x_{i-1})(2h^2 - 2p_i^2 + 3p_i(x - x_{i-1}))((e_i)^{(1)} + [x_i - h, x_i + h]f) \\ &+ (x_i - x)(2h^2 - 2p_i^2 + 3p_i(x_i - x))((e_{i-1})^{(1)} + [x_{i-1} - h, x_{i-1} + h]f). \end{aligned} \tag{3.7}$$

In order to estimate $e^{(1)}$, we adjust suitably the terms of (3.7) and see that

$$p'(p'^2 + 2h^2)\|e^{(1)}\| \leq p(p^2 + 2h^2)\|(e_i)^{(1)}\| + p(3p^2 + 2h^2) w(f^{(1)}, p) \tag{3.8}$$

by virtue of Lemma 3.1. Combining (3.6) with (3.8), we have

$$\|e^{(1)}\| \leq pK(p, p', h) w(f^{(1)}, p) \tag{3.9}$$

where $p'(p'^2 + 2h^2)K(p, p', h) = 4p(p^2 + 2h^2)\{(p^2 + 2h^2)(3p^2 + h^2) + pp'(p^2 - p'^2)\} t(h) + (3p^2 + 2h^2)$.

We have thus proved the following:

Theorem 2. *Suppose $s(x, h)$ is the optimal discrete cubic $\mathfrak{X}$-spline interpolant of f in the class $\mathfrak{D}_1(3, \mathfrak{P}, \mathfrak{X}, h)$ under the assumption $\alpha_i = -p'/3$. Then over the discrete interval $[a, b]_h$,*

$$\|e^{(r)}\| \leq p^{2-r} K(p, p', h) w(f^{(1)}, p), \quad r = 0, 1 \tag{3.10}$$

where $K(p, p', h)$ *is a positive function of* p, p' *and* h.

(3.9) proves Theorem 2 for $r=1$. Now taking $f(x)=e(x)$, $a=x_{i-1}$ in Lemma 3.2 we observe that $e_{i-1}=0$ and, therefore, (3.10) follows for $r=0$ when we appeal to (3.5).

§ 4. Discrete Cubic $\mathfrak{X}$-Splines of Special Interest

A discrete cubic $\mathfrak{X}$-spline seems to be interesting from the practical point of view if it can be easily computed. Discrete cubic $\mathfrak{X}$-spline and the usual discrete cubic spline may be computed by solving a diagonally dominant system of linear equations which generally require more computational labour and some times diagonally dominant property does not hold. In order to minimise the computational labour and to avoid the possibility of singular matrix involved in the system of linear equations, we reduce the three terms recurrence relation (2.4) to a two term recurrence relation by choosing suitably the parameter α_i. It has been shown by Clenshaw and Negus [2] that computational simplication may be achieved at the cost of a small loss in accuracy. Thus, for the choice $3\alpha_i=(h^2-p_{i+1}^2)/p_{i+1}$, we have the resulting equations:

$$m_i=F_i^*(h)-A_i(h)m_{i-1};\ i=1,2,\ \ldots,n \tag{4.1}$$

where $A_i(h)=(p_ip_{i+1}-h^2)/(p_i^2+p_ip_{i+1}+h^2)$ and

$F_i^*(h)=3p_{i+1}F_i(h)/r_{i+1}(p_i+p_{i+1})(p_i^2+p_ip_{i+1}+h^2)$, $F_i(h)$ and r_{i+1} are the same as defined in Section 2. A similar two term recurrence relation can also be obtained for the choice $3\alpha_i=(p_i^2-h^2)/p_i$. From the recurrence relation (4.1) we determine m_n in terms of m_0. Thus,

$$\begin{aligned} m_n=F_n^*(h)-A_n(h)F_{n-1}^*(h)+\ldots+(-1)^{n-1}A_n(h)A_{n-1}(h)\ldots \\ A_2(h)F_1^*(h)+(-1)^nA_n(h)A_{n-1}(h)\ldots A_1(h)m_0. \end{aligned} \tag{4.2}$$

Now using the periodicity condition i.e. $m_n=m_0$ in (4.2), we get unique m_0 which together with the recurrence relation (4.1) are sufficient to determine all $\{m_i\}$ uniquely.

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