Duality for Maximal Op*-Algebras on Fréchet Domains

By

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Abstract

The completion with respect to the uniform topology of the maximal Op*-algebra $L^+(D)$ on a Fréchet domain D is denoted by $\mathscr{L}$. It is isomorphic to the second strong dual of the complete injective tensor product $D' \tilde{\otimes}_\varepsilon \bar{D}'$ of the strong duals of D and $\bar{D}$, where D is endowed with the topology generated by the graph norms of operators belonging to $L^+(D)$ and $\bar{D}$ denotes the complex conjugate space of D. The predual of $\mathscr{L}$, i. e., the dual of $D' \tilde{\otimes}_\varepsilon \bar{D}'$ is isomorphic to the space $\mathscr{N}(\bar{D}', D)$ of nuclear operators mapping $\bar{D}'$ into D. These facts, together with the fact that the positive cone of $\mathscr{L}$ is normal with respect to the order topology, are applied to the study of bounded, positive, and continuous linear functionals on $\mathscr{L}$. It is also shown that $D' \tilde{\otimes}_\varepsilon \bar{D}'$ is a barrelled DF-space, that $L^+(D)$ is a DF-space, and that the subspace $\mathscr{F} \subset L^+(D)$ of finite rank operators is a bornological DF-space. There are given several characterizations of the Montel property of the Fréchet domain D. One of them is the reflexivity of $\mathscr{L}$.

§ 1. Introduction

The present paper is concerned with the study of the completion with respect to the uniform topology of the maximal Op*-algebra $L^+(D)$ on a Fréchet domain D. (For precise definitions, see Section 2.) It is a continuation of [13]. There it was shown that this completion is the space $\mathscr{L}(D, D^+)$ of continuous linear operators from D into the space D^+ of continuous anti-linear functionals on D.

It is the aim of the present paper to investigate the following dualities: The space $\mathscr{L}(D, D^+)$ is the second strong dual of its subspace of completely continuous operators. It is the strong dual of the space $\mathscr{N}(D^+, D)$ of nuclear operators from D^+ into D.

Some applications concern properties of several locally convex subspaces of $\mathscr{L}(D, D^+)$. E. g., $L^+(D)$ is a DF-space. Its subspace $\mathscr{F}$

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of finite rank operators is even a bornological DF-space.

These results, together with the fact that the positive cone of $\mathscr{L}(D, D^+)$ is normal with respect to the order topology τ_0, are applied to the investigation of bounded, positive, and continuous linear functionals on $\mathscr{L}(D, D^+)$ and on $L^+(D)$.

We assume throughout that D is a Fréchet domain, i. e., that D is a Fréchet space in the topology defined by the graph norms of operators belonging to $L^+(D)$. By [15], some of the results can be generalized to more general domains. However, in this more general situation, the results and the proofs become more complicated. Therefore they will be published separately.

The pattern of the paper is as follows. In Section 2, we introduce definitions and notations and recall some known or easy results. In Section 3 and Section 4, we investigate in more detail the space of completely continuous operators from D into D^+ (denoted by $D'\varepsilon D^+$) and the space $\mathscr{N}(D^+, D)$ of nuclear operators from D^+ into D, respectively. Section 5 deals with the dualities $(D'\varepsilon D^+)' = \mathscr{N}(D^+, D)$ and $(\mathscr{N}(D^+, D))' = \mathscr{L}(D, D^+)$. One of the applications is the orthogonal decomposition of uniformly continuous linear functionals on $L^+(D)$ into their ultraweakly continuous and singular parts (Corollary 5.3). It will also be shown in Section 5 that the locally convex subspaces $L^+(D)$ and $\mathscr{F}$ of $\mathscr{L}(D, D^+)$ are DF-spaces and that $D'\varepsilon D^+$ is even a barrelled DF-space. In Section 6, we prove that the positive cone of $\mathscr{L}(D, D^+)$ is normal with respect to the order topology τ_0. Consequently, bounded linear functionals on $\mathscr{L}(D, D^+)$ are linear combinations of positive linear functionals. Moreover, they admit also an orthogonal decomposition into an ultraweakly continuous and a singular part. Section 7 contains some equivalent characterizations of the property that D is a Montel space with respect to the topology defined by the graph norms of operators belonging to $L^+(D)$. Such characterizations are the reflexivity of $\mathscr{L}(D, D^+)$, the ultraweak continuity of all continuous, positive, or bounded linear functionals on $\mathscr{L}(D, D^+)$ and the condition that D is of type I in the sense of G. Lassner and W. Timmermann [20].

Note that the problem of investigation of the completion of $L^+(D)$ arose in [18] in connection with the study of the time devel-

opment of thermodynamical systems in quantum statistics.

Under some additional condition, W. Timmermann [28] described the dual of $\mathscr{F}$. This description is similar to the algebraical part of Theorem 5.4.a). Some aspects of the duality described in Theorem 5.4.b) were previously investigated by J.-P. Jurzak [8]. The order topology was investigated by K. Schmüdgen [25] and H. Araki and J.-P. Jurzak [2]. For Fréchet Montel domains K. Schmüdgen [23, 24] proved the ultraweak continuity of uniformly continuous and of positive linear functionals on $L^+(D)$ by other methods. This was a generalization of earlier results of S. L. Woronowicz [29], T. Sherman [27], and G. Lassner and W. Timmermann [19]. Concrete Fréchet domains have been investigated, e. g., by G. Lassner [16], G. Lassner and W. Timmermann [20], and K. Schmüdgen [23, 25].

Acknowledgements

§ 2. Notations and Preliminary Results

In this section, we fix some definitions and notations. Moreover, we collect some well-known or simple facts for later use.

Suppose that D is a dense linear subspace of a complex Hilbert space H. We denote the norm, the unit ball, and the scalar product of H by $\| \cdot \|$, U_H, and $\langle .,.\rangle$, respectively. We assume the scalar product to be linear in the second argument. For a closable linear operator A on H, let $D(A)$, A^*, $\overline{A}=A^{**}$, and $\|A\|$ denote the domain, the adjoint, the closure, and the norm of A, respectively. If $\overline{A}$ is a trace class operator, $\nu(A) =\text{trace}\,((A^*\overline{A})^{1/2})$ denotes the nuclear norm of $\overline{A}$. If A is unbounded, we set $\|A\|=\infty$. Similarly, if $\overline{A}$ is not a trace class operator, we set $\nu(A) =\infty$.

The following definition was introduced by G. Lassner [16]:

$$L^+(D) = \{A\in\text{End}(D) : D\subset D(A^*) \text{ and } A^*(D)\subset D\}.$$

Note that $L^+(D)$ is a *-algebra of closable operators with involution

$A \rightarrow A^* \restriction D$ (the restriction to D of A^*).

We provide D with the weakest locally convex topology such that the seminorms

$$D \ni \varphi \rightarrow ||A\varphi||$$

are continuous for all $A \in L^+(D)$. This topology is denoted by t. It is called the graph topology.

The space D is said to be a Fréchet domain if it is a Fréchet space with respect to the graph topology. In this case, there exists a sequence (A_n) in $L^+(D)$ such that the following conditions are satisfied:

a) *The topology of D is generated by the sequence of seminorms*
   $(D \ni \varphi \rightarrow ||A_n\varphi||)$. *Moreover*, $D = \bigcap\limits_{n=1}^{\infty} D(\overline{A_n})$.

b) *For each $A \in L^+(D)$, there exists $n \in N$ such that*

$$|\langle A\varphi, \varphi \rangle| \leq \langle A_n\varphi, \varphi \rangle \quad (\varphi \in D).$$

c) $A_1\varphi = \varphi$, $||A_n\varphi||^2 \leq \langle A_{n+1}\varphi, \varphi \rangle$, *and* $||A_n\varphi|| \leq ||A_{n+1}\varphi||$ *for all* $n \in N$ *and* $\varphi \in D$.

We assume throughout that D is a Fréchet domain. Moreover, we fix a sequence (A_n) satisfying the conditions a), b), and c). Note that every Fréchet domain is reflexive ([4, 22]).

Let $\bar{D}$ denote the complex conjugate locally convex space of D, i. e., the space which arises when the multiplication with complex numbers is replaced by the multiplication with the complex conjugate numbers.

Let D^+ be the space of all continuous antilinear functionals on D, endowed with the topology of uniform convergence on bounded subsets of D. D^+ is exactly the strong dual of $\bar{D}$. The elements of D or $\bar{D}$ are denoted by Greek letters φ, ψ, η, ... Elements of D^+ (and of H) are denoted by Latin letters f, g, h, ... The value of $f \in D^+$ at the point $\varphi \in D$ is denoted by $\langle \varphi, f \rangle$, its complex conjugate number by $\langle f, \varphi \rangle$. Note that the correspondence between $f \in D^+$ and the linear functional

$$D \ni \varphi \rightarrow \langle f, \varphi \rangle$$

sets up an antilinear topological isomorphism between D^+ and the

strong dual D' of D. Therefore, the definition of D^+ here is equivalent to the definition of D^+ in [13].

We always identify $f \in H$ with the functional

$$D \ni \varphi \to \langle \varphi, f \rangle$$

which belongs to D^+. Thus, H and D become dense linear subspaces of D^+ and the imbeddings $D \subset H$ and $H \subset D^+$ are continuous.

If E and F are locally convex spaces, $\mathscr{L}(E, F)$ denotes the linear space of all continuous linear operators mapping E into F. We use the abbreviation $\mathscr{L}$ for the space $\mathscr{L}(D, D^+)$. Note that the mapping which assigns to $T \in \mathscr{L}$ the sesquilinear form

$$D \times D \ni (\varphi, \psi) \to \langle \varphi, T\psi \rangle$$

is an isomorphism of $\mathscr{L}$ onto the space of all continuous sesquilinear forms defined on $D \times D$ (see, e. g., [12] §40). Therefore, we can define an involution $T \to T^+$ of $\mathscr{L}$ by the equation

$$\langle \varphi, T^+\psi \rangle = \langle T\varphi, \psi \rangle \qquad (\varphi, \psi \in D).$$

An operator $T \in \mathscr{L}$ is said to be hermitian if $T = T^+$. We define a partial order relation on the real linear space $\mathscr{L}_h$ of all hermitian elements of $\mathscr{L}$ as follows:

$$T_1 \leq T_2 \text{ if and only if } \langle \varphi, T_1\varphi \rangle \leq \langle \varphi, T_2\varphi \rangle$$

for all $\varphi \in D$.

For $E, F \in \{D, H, D^+\}$, we denote

$$\mathscr{C}(E, F) = \{T \in \mathscr{L} : \text{There exists } R \in \mathscr{L}(E, F) \text{ such that} \\ T\varphi = R\varphi \text{ for all } \varphi \in D\}.$$

We regard the elements of $L^+(D)$ as operators from D into D^+. Then $L^+(D)$ becomes a subspace of $\mathscr{L}$. More precisely,

$$L^+(D) = \mathscr{C}(D, D) \cap \mathscr{C}(D^+, D^+)$$

(see [17]). Consequently, there are extensions of the operators A_n which belong to $\mathscr{L}(D^+, D^+)$. We denote these extensions by $\tilde{A}_n$.

Let $\mathscr{F} \subset \mathscr{L}$ be the space of all finite rank operators belonging to $L^+(D)$, i. e., of operators of the form

$$D \ni \varphi \to \sum_{n=1}^{k} \langle \varphi_n, \varphi \rangle \psi_n \qquad (k \in N, \varphi_n, \psi_n \in D).$$

Define also

$$\mathfrak{B}_n = \{T \in \mathscr{L} : |\langle \varphi, T\psi \rangle| \le \|A_n\varphi\| \|A_n\psi\|$$
$$\text{for all } \varphi, \psi \in D\},$$

$$\mathscr{B} = \{B \in \mathscr{C}(D^+, D) : B \ge 0\},$$

$$\mathscr{P} = \{P \in \mathscr{B} : P^2 = P\}.$$

Now, we state an easy characterization of the elements of $\mathscr{P}$.

Proposition 2.1. *$\mathscr{P}$ is the set of the restrictions to D of orthogonal projections onto complete (with respect to the norm topology induced by H) linear subspaces of D.*

Proof: Each $P \in \mathscr{P}$ has a continuous extension $\tilde{P} \in \mathscr{L}(D^+, D)$. Clearly, the range of $\tilde{P} \upharpoonright H = \bar{P}$ is a norm-complete subspace of D. Conversely, let Q be an orthogonal projection onto a norm-complete linear subspace of D. According to the closed graph theorem, $A\,Q$ $(= (Q\,A^+)^*)$ belongs to $\mathscr{L}(H, H)$ for all $A \in L^+(D)$. Since $\|A(Qf)\|$ $\le \|AQ\| \|f\|$, Q belongs to $\mathscr{L}(H, D)$. In particular, $Q(U_H)$ is a bounded (with respect to the graph topology t) subset of D. Therefore, the seminorm

$$q : D^+ \ni f \to \sup\{|\langle Qg, f \rangle| : g \in U_H\}$$

is continuous. Since

$$\|Qf\| \le q(f)$$

for all $f \in H$, Q has a continuous extension $\tilde{Q} \in \mathscr{L}(D^+, H)$. Finally, the assertion follows from the fact that $Q\tilde{Q} \in \mathscr{L}(D^+, D)$ is an extension of $Q \upharpoonright D$.

The following definition of a partial multiplication on $\mathscr{L}$ is taken from [13].

Definition 2.2. *We say that the product $T_n \circ \ldots \circ T_1$ of elements of $\mathscr{L}$ is defined if there are spaces $E_0, \ldots, E_n$ belonging to $\{D, H, D^+\}$ such that $T_j \in \mathscr{C}(E_{j-1}, E_j)$. Let $R_j \in \mathscr{L}(E_{j-1}, E_j)$ denote the continuous extension of T_j. Then the product $T_n \circ \ldots \circ T_1$ is defined by*

$$T_n \circ \ldots \circ T_1 \varphi = R_n(\ldots (R_1\varphi)\ldots) \qquad\qquad (\varphi \in D).$$

This partial multiplication has the following property (cf. [13]).

Proposition 2.3. *If $n > k \geq 1$ and if the product $T_n \circ \ldots \circ T_1$ is defined, then*

$$\langle \varphi, T_n \circ \ldots \circ T_1 \psi \rangle = \langle (T_{k+1})^+ \circ \ldots \circ (T_n)^+ \varphi, T_k \circ \ldots \circ T_1 \psi \rangle$$

for all φ, $\psi \in D$.

Usually, we endow the space $\mathscr{L} = \mathscr{L}(D, D^+)$ and its subsets with the topology of uniform convergence on bounded subsets of D. This topology is called the uniform topology. It is denoted by τ_D. Next, we collect some topological properties of D, D^+, and $\mathscr{L}[\tau_D]$.

Proposition 2.4.

a) *The system*

$$\{\tilde{B}(U_H) : B \in \mathscr{B}\}$$

is a fundamental system of bounded subsets of D.

b) *For $B \in \mathscr{B}$, let $\tilde{B} \in \mathscr{L}(D^+, D)$ denote the continuous extension of B. Then the system of seminorms*

$$D^+ \ni f \rightarrow \|\tilde{B}f\| \qquad\qquad (B \in \mathscr{B})$$

defines the topology of D^+.

c) *The sequence of sets*

$$(\tilde{A}_n(U_H))_{n \in N}$$

is a fundamental sequence of bounded subsets of D^+.

d) *The system of seminorms*

$$\mathscr{L} \ni T \rightarrow \|B \circ T \circ B\| \qquad\qquad (B \in \mathscr{B})$$

defines the topology τ_D.

e) *The sequence*

$$(\mathfrak{B}_n)_{n \in N}$$

is a fundamental sequence of bounded subsets of $\mathscr{L}[\tau_D]$.

The assertions a), b), and d) were obtained in [13]. It was also mentioned in [13] that e) follows from the theory of locally convex spaces.

Proof of 2.4.c) : Since D is reflexive, it suffices to show that the polar of $\{\varphi \in D : \|A_n\varphi\| \leq 1\}$ is contained in $\tilde{A}_n(U_H)$. For, let f be an

element of this polar. This means that $|\langle f, \varphi \rangle| \le ||A_n\varphi||$ for all $\varphi \in D$. Consequently, there is a linear functional h on $A_n(D)$ such that

$$h(A_n\varphi) = \langle f, \varphi \rangle \qquad\qquad (\varphi \in D).$$

Since the norm of h is not larger than 1, there exists $g \in U_H$ such that

$$\langle g, A_n\varphi \rangle = h(A_n\varphi) = \langle f, \varphi \rangle$$

for all $\varphi \in D$. Since D is dense in D^+, the equation

$$\langle A_n\psi, \varphi \rangle = \langle \psi, A_n\varphi \rangle \qquad\qquad (\varphi, \psi \in D)$$

implies

$$\langle \tilde{A}_n g, \varphi \rangle = \langle g, A_n\varphi \rangle = \langle f, \varphi \rangle \qquad\qquad (\varphi \in D).$$

This completes the proof.

We refer to [1] for the theory of operators in a Hilbert space and to [11, 12, 21] for the theory of locally convex spaces.

§ 3. Completely Continuous Operators

In this section, we investigate a space of completely continuous operators.

We make use of the ε-product $D'\varepsilon D^+$ defined by L. Schwartz [26]. Let E'_c denote the dual of the locally convex space E, endowed with the topology of uniform convergence on compact convex circled sets. By definition, the ε-product $D'\varepsilon D^+$ is the space $\mathscr{L}_e((D')'_c, D^+)$ of all continuous linear operators from $(D')'_c$ into D^+, endowed with the topology of uniform convergence on equicontinuous sets. Since D is reflexive, the equicontinuous subsets of $(D')'$ are exactly the bounded subsets of D. Therefore, $D'\varepsilon D^+$ is a topological linear subspace of $\mathscr{L}[\tau_D]$. Note that D' and D^+ are complete locally convex spaces. They have the approximation property because the semi-norms described in Proposition 2.4.b) can be given by semi-definite scalar products. This implies that $D'\varepsilon D^+$ is isomorphic to the complete injective tensor product $D'\tilde{\otimes}_\varepsilon D^+$ (see [12], §43.3.(7)).

We use the following generalization of the familiar concept of completely continuous operators between Banach spaces. A continuous linear operator between locally convex spaces is said to be completely continuous if it transforms weakly convergent sequences into con-

vergent sequences. Our next proposition shows, in particular, that $D'\varepsilon D^+$ is the space of all completely continuous operators from D into D^+.

Proposition 3.1. *For $T \in \mathscr{L}$, the following assertions are equivalent :*

a) $T \in D'\varepsilon D^+$.

b) *T belongs to the closure of $\mathscr{F}$ in $\mathscr{L}[\tau_D]$.*

c) *For each bounded subset $M \subset D$, $T(M)$ is relatively compact in D^+.*

d) *For each weakly convergent sequence (φ_n) in D, $(T\varphi_n)$ converges in D^+.*

Proof : Since D' and D^+ fulfil the approximation property, $D'\varepsilon D^+$ is the closure of the set of all finite rank operators in $\mathscr{L}[\tau_D]$ ([12] §43. 3. (6)). Since D is dense in D^+, $\mathscr{F}$ is dense in the set of all finite rank operators. Hence, a) and b) are equivalent.

The equivalence of a) and c) follows from the fact that a weakly continuous operator belongs to $D'\varepsilon D^+$ if and only if it maps the equicontinuous subsets of $(D')'$ into relatively compact subsets of D^+ ([12] §43. 3. (2)).

The implication c) $\Rightarrow$ d) follows from the fact that a weakly convergent sequence converges if it is contained in a compact set.

Remember that each bounded subset of D is contained in a weakly compact and weakly sequentially compact set of the form $\bar{B}(U_H)$ $(B \in \mathscr{B})$. Therefore, the implication d) $\Rightarrow$ c) follows from the fact that weakly compact and sequentially compact sets are compact (see, e. g., [10] 7. 7., 17. 8.). This completes the proof.

Remark : If T belongs to $(D'\varepsilon D^+) \cap L^+(D)$, then it transforms weakly convergent sequences of D into convergent sequences of D. Furthermore, it maps bounded subsets of D into relatively compact subsets of D (see [14]).

§4. Nuclear Operators

In this section, we investigate properties of the space $\mathscr{N}(D^+, D)$ of nuclear operators from D^+ into D.

Since D is assumed to be a Fréchet space, each element of the

complete projective tensor product $\bar{D} \tilde{\otimes}_\pi D$ is the sum of an abso-lutely convergent series $\sum \varphi_n \otimes \psi_n$ $(\varphi_n \in \bar{D}, \psi_n \in D)$. The correspondence between $\sum \varphi_n \otimes \psi_n$ and the operator

$$D^+ \ni f \to \sum_{n=1}^{\infty} \langle \varphi_n, f \rangle \psi_n \in D$$

sets up a linear isomorphism J between the tensor product $\bar{D} \tilde{\otimes}_\pi D$ and the space $\mathcal{N}(D^+, D)$ of nuclear operators from D^+ into D (see, e. g., [3] Theorem 2.1 or Lemma 4.1 below).

Let τ_π be the weakest locally convex topology on $\mathcal{N}(D^+, D)$ such that the seminorms

$$\nu_n : \mathcal{N}(D^+, D) \ni S \to \nu(A_n S \tilde{A}_n \upharpoonright H) \qquad (n \in N)$$

are continuous. We want to show that τ_π is essentially the topology of the projective tensor product.

Lemma 4.1. *Let p_n denote the Minkowski functional of the convex hull in $\bar{D} \otimes D$ of*

$$\{\varphi \in \bar{D} : \|A_n \varphi\| \leq 1\} \otimes \{\psi \in D : \|A_n \psi\| \leq 1\}.$$

Then the seminorm

$$\bar{D} \tilde{\otimes}_\pi D \ni x \to \nu_n(Jx)$$

is the continuous extension of p_n.

Proof: Let $q_n : \bar{D} \tilde{\otimes}_\pi D \to R$ denote the continuous extension of p_n. Fix $n \in N$, $\varepsilon \in (0, 1)$ and $x \in \bar{D} \tilde{\otimes}_\pi D$. There exists a sequence (x_k) in $\bar{D} \otimes D$ such that

$$q_{n+k}(x - x_k) < \varepsilon^k \qquad (k \in N).$$

It follows that

$$p_n(x_1) \leq q_n(x) + q_n(x - x_1) < q_n(x) + \varepsilon.$$
$$p_{n+k}(x_{k+1} - x_k) \leq q_{n+k}(x - x_k) + q_{n+k+1}(x - x_{k+1})$$
$$< \varepsilon^k + \varepsilon^{k+1} \qquad (k \in N).$$

Consequently, there are representations

$$x_1 = \sum_{l=1}^{m_1} \varphi_l \otimes \psi_l,$$

$$x_{k+1} - x_k = \sum_{l=m_k+1}^{m_{k+1}} \varphi_l \otimes \psi_l \qquad (k \in N),$$

where $\varphi_l \in \bar{D}$, $\psi_l \in D$, and

$$\sum_{l=1}^{m_1} \|A_n \varphi_l\| \; \|A_n \psi_l\| < q_n(x) + \varepsilon,$$

$$\sum_{l=m_k+1}^{m_{k+1}} \|A_{n+k} \varphi_l\| \; \|A_{n+k} \psi_l\| < \varepsilon^k + \varepsilon^{k+1}.$$

Therefore, the series $\sum \varphi_l \otimes \psi_l$ converges in $\bar{D} \tilde{\otimes}_\pi D$ to x and we have

$$\nu_n(Jx) = \nu(\sum_{l=1}^{\infty} \langle A_n \varphi_l, \cdot \rangle A_n \psi_l)$$

$$\leq \sum_{l=1}^{\infty} \|A_n \varphi_l\| \; \|A_n \psi_l\| \leq q_n(x) + 2 \sum_{k=1}^{\infty} \varepsilon^k.$$

Letting $\varepsilon \to 0$, we get

$$\nu_n(Jx) \leq q_n(x).$$

In particular, the seminorm $\bar{D} \tilde{\otimes}_\pi D \ni x \to \nu_n(Jx)$ is continuous. Therefore, it suffices to establish the converse inequality for elements of $\bar{D} \otimes D$.

Let

$$y = \sum_{l=1}^{m} \varphi_l \otimes \psi_l \in \bar{D} \otimes D$$

be fixed. The operator

$$A_n(Jy) \tilde{A}_n \upharpoonright H : H \ni f \to \sum_{l=1}^{m} \langle A_n \varphi_l, f \rangle A_n \psi_l$$

has a representation

$$f \to \sum_{l=1}^{k} \langle \eta_l, f \rangle \zeta_l$$

such that

$$\sum_{l=1}^{k} \|\eta_l\| \; \|\zeta_l\| = \nu_n(Jy).$$

Moreover, we can assume that the vectors η_l and ζ_l belong to the linear span of $\{A_n \varphi_l, A_n \psi_l\}_{ll=1}^{m}$. Define

$$z = \sum_{l=1}^{k} (A_n)^{-1} \eta_l \otimes (A_n)^{-1} \zeta_l.$$

Clearly, $A_n(Jy) \tilde{A}_n = A_n(Jz) \tilde{A}_n$. Note that

$$\langle \tilde{A}_n f, \varphi \rangle = \langle f, A_n \varphi \rangle$$

for all $f \in D^+$ and $\varphi \in D$. Note also that A_n is injective. Therefore, the operator $\tilde{A}_n$ has a dense range. Now, the equality $A_n(J(y-z))\tilde{A}_n = 0$ implies $J(y-z) = 0$, i. e., $y = z$. Finally, we get

$$q_n(y) = q_n(z) \leq \sum_{l=1}^{k} ||\eta_l|| \; ||\zeta_l|| \leq \nu_n(Jy).$$

This completes the proof.

Remark: G. Lassner and W. Timmermann [19] defined the space
$$\mathfrak{S}_1(D) = \{S \in \mathscr{L}(H, H) : A\,S \text{ and } A\,S^* \text{ are trace class operators for} \\ \text{all } A \in L^+(D)\}.$$
Clearly, $S \restriction H$ belongs to $\mathfrak{S}_1(D)$ if S is an element of $\mathscr{N}(D^+, D)$. Conversely, each element of $\mathfrak{S}_1(D)$ is the restriction to H of an operator belonging to $\mathscr{N}(D^+, D)$. Indeed, it has been shown by K. Schmüdgen ([23] proof of Lemma 1 (5)) that each element of $\mathfrak{S}_1(D)$ is a linear combination of operators of the form

$$H \ni f \to \sum_{n=1}^{\infty} \langle \varphi_n, f \rangle \varphi_n,$$

where (φ_n) is an orthogonal sequence in D such that

$$\sum_{n=1}^{\infty} ||A_l \varphi_n||^2 < \infty$$

for all $l \in N$, i. e., where $\sum \varphi_n \otimes \varphi_n$ is an absolutely convergent series in $\bar{D} \tilde{\otimes}_\pi D$.

For $S \in \mathscr{N}(D^+, D)$, $S \restriction D$ belongs to $\mathscr{C}(D^+, D)$. Our next theorem characterizes such elements of $\mathscr{C}(D^+, D)$, which can be extended to nuclear operators from D^+ into D. It also characterizes bounded subsets of $\mathscr{N}(D^+, D)[\tau_\pi]$.

Theorem 4.2. *Let* $\mathfrak{M} \subset \mathscr{C}(D^+, D)$ *be a set of operators such that*
$$\sup \{\nu(A_n \circ S \circ A_n) : S \in \mathfrak{M}\} < \infty$$
for all $n \in N$. *Then there exist an operator* $B \in \mathscr{C}(H, D)$ *such that* $B \geq 0$ *and*
$$\mathfrak{M} \subset \{B^2 \circ R \circ B^2 : R \in \mathscr{C}(H, H) \text{ and } \nu(R) \leq 1\}.$$

Remark: If $B \in \mathscr{C}(H, D)$ and $B \geq 0$ then $B = B^+$ belongs also to $\mathscr{C}(D^+, H)$ ([13] Proposition 3.5). In this case, $B \circ B = B^2$ belongs to $\mathscr{C}(D^+, D)$. Consequently, it belongs to $\mathscr{B}$.

Proof of Theorem 4.2: Since $\nu(A_n \circ S \circ A_n) = \nu(A_n \circ S^+ \circ A_n)$, we can assume that $S^+ \in \mathfrak{M}$ for all $S \in \mathfrak{M}$.

For arbitrary B_1, B_2 in $L^+(D)$ there are operators B_3, B_4 in $\mathscr{C}(H, H)$ and a natural number n such that $B_1 = B_3 \circ A_n$ and $B_2 = A_n \circ B_4$ ([13] Proposition 5.5 and Corollary 5.6). Consequently,

$$\sup \{\nu(B_1 \circ S \circ B_2) : S \in \mathfrak{M}\}$$
$$= \sup \{\nu(B_3 \circ A_n \circ S \circ A_n \circ B_4) : S \in \mathfrak{M}\}$$
$$\leq \|B_3\| \ \|B_4\| \ \sup \{\nu(A_n \circ S \circ A_n) : S \in \mathfrak{M}\} < \infty.$$

In particular, the values

$$c_n = 1 + \sup \{\nu((A_k)^2 \circ S \circ (A_l)^2) : S \in \mathfrak{M}, \ k \leq n, \ l \leq n\}$$

are finite. We choose a sequence (ε_n) of positive real numbers such that

$$\sum_{n=1}^{\infty} \varepsilon_n c_n < 1.$$

Since

$$\sup \{\|A_l \bar{S} \varphi\| : S \in \mathfrak{M}, \varphi \in U_H\}$$
$$\leq \sup \{\|A_l \circ S\| : S \in \mathfrak{M}\} \leq \sup \{\nu(A_l \circ S) : S \in \mathfrak{M}\} < \infty,$$

the set

$$M = \cup \{\bar{S}(U_H) : S \in \mathfrak{M}\}$$

is bounded in D. By a similar argument,

$$\sup \{\|(A_n)^2 \varphi\| : \varphi \in M\} \leq c_n.$$

On the domain

$$D_t = \{\varphi \in D : \sum_{n=1}^{\infty} \varepsilon_n \|A_n \varphi\|^2 < \infty\},$$

we define the hermitian form

$$t : (\varphi, \psi) \to t(\varphi, \psi) = \sum_{n=1}^{\infty} \varepsilon_n \langle A_n \varphi, A_n \psi \rangle.$$

We prove that t is closed. For, let (φ_k) be a sequence in D_t such

that

$$\lim_{k,l\to\infty} t(\varphi_k-\varphi_l,\ \varphi_k-\varphi_l)=0.$$

Since

$$||A_n(\varphi_k-\varphi_l)||^2\leq(\varepsilon_n)^{-1}t(\varphi_k-\varphi_l,\ \varphi_k-\varphi_l)$$

for all k, l, $n\in N$, the sequence (φ_k) is a Cauchy sequence in D. Hence, there exists $\varphi_0\in D$ such that the sequence (φ_k) converges to φ_0 with respect to the graph topology.

We show that $\varphi_0\in D_t$ and

$$\lim_{k\to\infty} t(\varphi_k-\varphi_0,\ \varphi_k-\varphi_0)=0. \tag{1}$$

Given $\varepsilon>0$, there exists k_0 in N such that

$$t(\varphi_k-\varphi_l,\ \varphi_k-\varphi_l)=\sum_{n=1}^{\infty}\varepsilon_n||A_n(\varphi_k-\varphi_l)||^2<\varepsilon$$

if $k>k_0$ and $l>k_0$. Keeping $k>k_0$ fixed and letting $l\to\infty$, we get

$$t(\varphi_k-\varphi_0,\ \varphi_k-\varphi_0)=\sum_{n=1}^{\infty}\varepsilon_n||A_n(\varphi_k-\varphi_0)||^2\leq\varepsilon.$$

In particular, this implies $\varphi_k-\varphi_0\in D_t$. Since D_t is a linear space, $\varphi_0\in D_t$. Moreover, (1) is satisfied.

Hence, D_t is complete with respect to the norm $\varphi\to(t(\varphi,\varphi))^{1/2}$. But this means that t is closed.

Let H_1 denote the closure of D_t in H. By the representation theorem for closed positive sesquilinear forms (see, e. g., [9] Chap. VI Theorem 2. 23), there exists a positive self-adjoint operator T acting on H_1, with domain D_t, such that

$$\langle T\varphi,\ T\psi\rangle=t(\varphi,\psi)$$

for all $\varphi,\psi\in D_t$.

For $\varphi\in M$ and $\psi\in D(T)=D_t$, we have

$$\sum_{n=1}^{\infty}\varepsilon_n||A_n\varphi||^2\leq\sum_{n=1}^{\infty}\varepsilon_n||(A_n)^2\varphi||\ ||\varphi||$$

$$\leq\sum_{n=1}^{\infty}\varepsilon_nc_n||\varphi||<\infty,$$

$$\varphi\in D_t,$$

$$|\langle T\varphi,\ T\psi\rangle|=|\sum_{n=1}^{\infty}\varepsilon_n\langle A_n\varphi,\ A_n\psi\rangle|$$

$$= |\langle \sum_{n=1}^{\infty} \varepsilon_n (A_n)^2 \varphi, \psi \rangle| \leq ||\psi||,$$

$$T\varphi \in D(T^*) = D(T),$$

$$\langle T^2 \varphi, \psi \rangle = \langle \sum_{n=1}^{\infty} \varepsilon_n (A_n)^2 \varphi, \psi \rangle. \tag{2}$$

Clearly, (2) holds also for $\psi \in D(T)$ and φ in the linear hull of M.

Corresponding to the orthogonal direct sum $H = H_1 \oplus (H \ominus H_1)$, we define the positive self-adjoint operator $T^{-1} \oplus 0$. This operator is $||\cdot||-$bounded because of $||T\varphi||^2 \geq \varepsilon_1 ||\varphi||^2$. Moreover, it maps H into $D_t \subset D$. By the closed graph theorem, it belongs to $\mathcal{L}(H, D)$. We define $B = (T^{-1} \oplus 0) \restriction D$.

Now, for a fixed $S \in \mathfrak{M}$, we consider the operator

$$R : D \ni \varphi \to \sum_{k, l=1}^{\infty} \varepsilon_k \varepsilon_l (A_k)^2 \circ S \circ (A_l)^2 \varphi.$$

The estimate

$$\sum_{k, l=1}^{\infty} \varepsilon_k \varepsilon_l \nu ((A_k)^2 \circ S \circ (A_l)^2) \leq \sum_{k, l=1}^{\infty} \varepsilon_k \varepsilon_l c_k c_l < 1$$

implies $R \in \mathcal{C}(H, H)$ and $\nu(R) < 1$.

Finally, we show that $S = B^2 \circ R \circ B^2$. For, let $\xi, \eta \in D$. Using several times (2) for φ in the range of S or S^+ and ψ in the range of B, we get

$$\langle B^2 \circ R \circ B^2 \xi, \eta \rangle = \langle R B^2 \xi, B^2 \eta \rangle$$

$$= \sum_{k, l=1}^{\infty} \varepsilon_k \varepsilon_l \langle (A_k)^2 \circ S \circ (A_l)^2 B^2 \xi, B^2 \eta \rangle$$

$$= \sum_{l=1}^{\infty} \varepsilon_l \langle \sum_{k=1}^{\infty} \varepsilon_k (A_k)^2 (S(A_l)^2 B^2 \xi), B^2 \eta \rangle$$

$$= \sum_{l=1}^{\infty} \varepsilon_l \langle T^2 (S(A_l)^2 B^2 \xi), B^2 \eta \rangle$$

$$= \sum_{l=1}^{\infty} \varepsilon_l \langle S(A_l)^2 B^2 \xi, \eta \rangle = \langle B^2 \xi, \sum_{l=1}^{\infty} \varepsilon_l (A_l)^2 S^+ \eta \rangle$$

$$= \langle B^2 \xi, T^2 (S^+ \eta) \rangle = \langle \xi, S^+ \eta \rangle = \langle S\xi, \eta \rangle.$$

Consequently, $S = B^2 \circ R \circ B^2$. This completes the proof.

Corollary 4.3. *For $B \in \mathcal{B}$, let $\tilde{B} \in \mathcal{L}(D^+, D)$ denote the continuous extension of B. Then the sets*

$$\mathfrak{N}_B = \{\bar{B} R \tilde{B} : R \in \mathcal{C}(H, H) \text{ and } \nu(R) \leq 1\} \qquad (B \in \mathcal{B})$$

form a fundamental system of bounded subsets of $\mathcal{N}(D^+, D)[\tau_\pi]$.

Proof : If $\mathfrak{N}$ is a bounded subset of $\mathcal{N}(D^+, D)[\tau_\pi]$, the set of operators

$$\mathfrak{M} = \{S \upharpoonright D : S \in \mathfrak{N}\}$$

satisfies the assumptions of Theorem 4.2. Consequently, there exists B in $\mathscr{B}$ such that

$$\mathfrak{M} \subset \{B \circ R \circ B : R \in \mathscr{C}(H, H) \text{ and } \nu(R) \leq 1\}.$$

This implies

$$\mathfrak{N} \subset \{\bar{B} R \tilde{B} : R \in \mathscr{C}(H, H) \text{ and } \nu(R) \leq 1\} = \mathfrak{N}_B$$

because $\bar{B} R \tilde{B}$ is an extension of $B \circ R \circ B$ and belongs to $\mathscr{L}(D^+, D)$.
Conversely,

$$\nu_n(\bar{B} R \tilde{B}) = \nu(A_n \bar{B} R \tilde{B} \tilde{A}_n \upharpoonright H) = \nu(A_n \circ B \circ R \circ B \circ A_n)$$
$$\leq \|A_n B\| \; \|B A_n\| \nu(R) = \|A_n B\| \; \|(A_n B)^+\| \nu(R)$$

and

$$\|(A_n B)^+\| = \|A_n B\| < \infty$$

for all $B \in \mathscr{B}$ and $n \in N$. Consequently, for each $B \in \mathscr{B}$, $\mathfrak{N}_B$ is a bounded subset of $\mathcal{N}(D^+, D)[\tau_\pi]$. This completes the proof.

Remark : In particular, it follows from Corollary 4.3 that each element of $\mathcal{N}(D^+, D)$ has a representation

$$\bar{B} R \tilde{B}$$

with $B \in \mathscr{B}$, $R \in \mathscr{C}(H, H)$, and $\nu(R) \leq 1$.

We conclude this section by showing that nuclear operators on D or on D^+ have uniquely defined traces.

Lemma 4.4. *Let $(a_n) \in l_1$ and let (f_n) and (φ_n) be bounded sequences in D^+ and D, respectively, such that*

$$\sum_{n=1}^{\infty} a_n \langle f_n, \varphi \rangle \langle \psi, \varphi_n \rangle = 0$$

for all $\varphi, \psi \in D$. Then

$$\sum_{n=1}^{\infty} a_n \langle f_n, \varphi_n \rangle = 0.$$

Proof: By Proposition 2.4.a), there exist $B \in \mathcal{B}$ and a bounded sequence (g_n) in H such that $\tilde{B}(g_n) = \varphi_n$. We can also assume that g_n belongs to the norm-closure in H of the range of $\bar{B}$. Let $\tilde{B} \in \mathcal{L}(D^+, D)$ denote the continuous extension of B. Taking $\varphi = B\eta$, we get

$$\sum_{n=1}^{\infty} a_n \langle \tilde{B} f_n, \eta \rangle \langle B\psi, g_n \rangle = \sum_{n=1}^{\infty} a_n \langle f_n, B\eta \rangle \langle \psi, \bar{B} g_n \rangle = 0.$$

Therefore, the trace class operator

$$H \ni f \to \sum_{n=1}^{\infty} a_n \langle \tilde{B} f_n, f \rangle g_n \in H$$

is the zero operator. This implies

$$\sum_{n=1}^{\infty} a_n \langle f_n, \varphi_n \rangle = \sum_{n=1}^{\infty} a_n \langle f_n, \bar{B} g_n \rangle = \sum_{n=1}^{\infty} a_n \langle \tilde{B} f_n, g_n \rangle = 0.$$

Our next proposition is an immediate consequence of the previous lemma and of elementary properties of nuclear operators (see, e.g., [21] III 7.1).

Proposition 4.5. *Let* $S \in \mathcal{N}(D, D)$ *and* $T \in \mathcal{N}(D^+, D^+)$ *be nuclear operators. Then there are nullsequences* (φ_n), (ψ_n) *in* D, *nullsequences* (f_n), (g_n) *in* D^+, *and sequences* (a_n), $(b_n) \in l_1$ *such that*

$$S\varphi = \sum_{n=1}^{\infty} a_n \langle f_n, \varphi \rangle \varphi_n$$

$$Tf = \sum_{n=1}^{\infty} b_n \langle \psi_n, f \rangle g_n$$

for all $\varphi \in D$ *and* $f \in D^+$. *The traces of* S *and* T *are uniquely defined by the formulas*

$$\operatorname{tr}(S) = \sum_{n=1}^{\infty} a_n \langle f_n, \varphi_n \rangle \tag{3}$$

$$\operatorname{tr}(T) = \sum_{n=1}^{\infty} b_n \langle \psi_n, g_n \rangle. \tag{4}$$

Remark: It follows from (3) and (4) that $\operatorname{tr}(S\,T) = \operatorname{tr}(T\,S)$ if $S \in \mathcal{N}(D^+, D)$ and $T \in \mathcal{L}$. Furthermore, $\operatorname{tr}(A\,S) = \operatorname{tr}(S\,A)$ if $S \in \mathcal{N}(D, D)$ and $A \in \mathcal{L}(D, D)$, and $\operatorname{tr}(R\,T) = \operatorname{tr}(T\,R)$ if $T \in \mathcal{N}(D^+, D^+)$ and $R \in \mathcal{L}(D^+, D^+)$.

§5. Duality

The pairs $(\mathscr{L}, \mathscr{N}(D^+, D))$ and $(D'\varepsilon D^+, \mathscr{N}(D^+, D))$ are dual pairs with respect to the bilinear mapping

$$(T, S) \to \mathrm{tr}\,(T\,S). \tag{5}$$

In this section, we consider some properties of these dual pairs.

Lemma 5. 1. *For $S\in\mathscr{N}(D^+, D)$ and $k\in\mathbf{N}$,*

$$\nu_k(S) = \sup\{\,|\mathrm{tr}\,(T\,S)\,| : T\in\mathfrak{B}_k\cap\mathscr{F}\}$$
$$= \sup\{\,|\mathrm{tr}\,(T\,S)\,| : T\in\mathfrak{B}_k\}.$$

Proof: Let $\sum \varphi_n\otimes\psi_n$ be an absolutely convergent series in $\bar{D}\tilde{\otimes}_\pi D$ such that

$$Sf = \sum_{n=1}^{\infty} \langle\varphi_n, f\rangle\psi_n \qquad\qquad (f\in D^+).$$

For $T\in\mathscr{C}(H, H)$,

$$\mathrm{tr}\,((A_k\circ T\circ A_k)S) = \sum_{n=1}^{\infty} \langle\varphi_n, A_k\circ T\circ A_k\psi_n\rangle$$
$$= \sum_{n=1}^{\infty} \langle A_k\varphi_n, T\,A_k\psi_n\rangle = \mathrm{tr}\,(\bar{T}(A_k\circ(S\upharpoonright D)\circ A_k)^{**}).$$

Therefore,

$$\nu_k(S) = \nu(A_k\circ(S\upharpoonright D)\circ A_k)$$
$$= \sup\{\,|\mathrm{tr}\,(\bar{T}(A_k\circ(S\upharpoonright D)\circ A_k)^{**})\,| : T\in\mathscr{F} \text{ and } \|T\|\leq 1\}$$
$$= \sup\{\,|\mathrm{tr}\,((A_k\circ T\circ A_k)S)\,| : T\in\mathscr{F} \text{ and } \|T\|\leq 1\}$$
$$\leq \sup\{\,|\mathrm{tr}\,((A_k\circ T\circ A_k)S)\,| : T\in\mathscr{C}(H, H) \text{ and } \|T\|\leq 1\}$$
$$= \sup\{\,|\mathrm{tr}\,(\bar{T}(A_k\circ(S\upharpoonright D)\circ A_k)^{**})\,| : T\in\mathscr{C}(H, H), \|T\|\leq 1\}$$
$$= \nu(A_k\circ(S\upharpoonright D)\circ A_k) = \nu_k(S).$$

Finally, the assertion follows from the fact that

$$\mathfrak{B}_k = \{A_k\circ T\circ A_k : T\in\mathscr{C}(H, H) \text{ and } \|T\|\leq 1\}$$

(see [13] Proposition 5. 1).

Lemma 5. 2. *Suppose that $B\in\mathscr{B}$. Let $\tilde{B}\in\mathscr{L}(D^+, D)$ denote the continuous extension of B.*

a) *We have*

$$\mathrm{tr}\,(T(\bar{B}R\tilde{B})) = \mathrm{tr}\,((B\circ T\circ B)^{**}\bar{R})$$

for $T\in\mathscr{L}$ and $R\in\mathscr{C}(H, H)$ with $\nu(R)<\infty$.

b) *We have*

$$\|B\circ T\circ B\| = \sup\{|\mathrm{tr}\,(T(\bar{B}R\tilde{B}))| : R\in\mathscr{C}(H,\ H)\ \mathrm{and}\ \nu(R)\le1\}$$

for $T\in\mathscr{L}$.

Proof: a) R has a representation

$$R : D\ni\varphi\to\sum_{n=1}^{\infty}a_n\langle f_n, \varphi\rangle g_n,$$

where f_n, $g_n\in U_H$ and $\sum|a_n|<\infty$. Then $\bar{B}R\tilde{B}$ is the operator

$$D^+\ni f\to\sum_{n=1}^{\infty}a_n\langle\bar{B}f_n, f\rangle\bar{B}g_n\in D.$$

Consequently,

$$\mathrm{tr}\,(T(\bar{B}R\tilde{B})) = \sum_{n=1}^{\infty}a_n\langle\bar{B}f_n, T\bar{B}g_n\rangle.$$

Note that

$$\langle B\varphi, T\,B\psi\rangle = \langle\varphi, B\circ T\circ B\psi\rangle$$

for all φ, $\psi\in D$. This implies

$$\langle\bar{B}f, T\bar{B}g\rangle = \langle f, (B\circ T\circ B)^{**}g\rangle$$

for all $f, g\in H$. Further we obtain

$$\mathrm{tr}\,(T(\bar{B}R\tilde{B})) = \sum_{n=1}^{\infty}a_n\langle f_n, (B\circ T\circ B)^{**}g_n\rangle = \mathrm{tr}\,((B\circ T\circ B)^{**}\bar{R}).$$

b) Assertion b) is a consequence of assertion a) and of the fact that

$$\|V\| = \sup\{|\mathrm{tr}\,(V\,R)| : R\in\mathscr{L}(H, H)\ \mathrm{and}\ \nu(R)\le1\}$$

for all $V\in\mathscr{L}(H, H)$. The proof is complete.

Lemma 5.3. *Suppose that $B\in\mathscr{B}$. Let $\tilde{B}\in\mathscr{L}(D^+, D)$ be the continuous extension of B. Suppose further that θ is a linear functional on $D'\varepsilon D^+$ such that*

$$|\theta(T)|\le\|B\circ T\circ B\| \tag{6}$$

for all $T\in D'\varepsilon D^+$. Let P denote the orthogonal projection onto the norm-

closure of the range of B. Then there exists an operator R in $\mathscr{C}(H, H)$ with $\nu(R) \leq 1$ and $P\bar{R}P = \bar{R}$ such that

$$\theta(T) = \mathrm{tr}(T(\bar{B}R\tilde{B}))$$

for all $T \in D'\varepsilon D^+$. This operator is uniquely defined. If the functional θ is positive, R is also positive.

Proof: Because of (6), the mapping

$$\omega: (B \circ T \circ B)^{**} \to \theta(T)$$

is a well-defined linear functional on the linear subspace

$$\mathscr{X} = \{(B \circ T \circ B)^{**} : T \in D'\varepsilon D^+\}$$

of $\mathscr{L}(H, H)$. Moreover,

$$\|\omega\| \leq 1.$$

It follows from Proposition 3.1 that the elements of $\mathscr{X}$ are compact operators on H. Therefore, there exists an operator R_1 on H with $\nu(R_1) \leq 1$ such that

$$\theta(T) = \omega((B \circ T \circ B)^{**}) = \mathrm{tr}((B \circ T \circ B)^{**}R_1)$$

for all $T \in D'\varepsilon D^+$. Defining the operator

$$R: D \ni \varphi \to PR_1P\varphi,$$

we have

$$\nu(R) \leq 1,$$
$$P\bar{R}P = \bar{R},$$
$$\theta(T) = \mathrm{tr}(P(B \circ T \circ B)^{**}PR_1) = \mathrm{tr}((B \circ T \circ B)^{**}\bar{R})$$
$$= \mathrm{tr}(T(\bar{B}R\tilde{B})).$$

For $f, g \in H$, the operator

$$T_{f,g}: D \ni \varphi \to \langle f, \varphi \rangle g$$

belongs to $D'\varepsilon D^+$. Since

$$\theta(T_{f,g}) = \mathrm{tr}((B \circ T \circ B)^{**}\bar{R}) = \langle \bar{B}f, R\bar{B}g \rangle,$$

the sesquilinear form

$$(\varphi, \psi) \to \langle \varphi, R\psi \rangle \tag{7}$$

is uniquely defined on the range of B. This implies that R is uniquely defined. Similarly, if θ is positive, the corresponding sesqui-

linear form (7) is also positive. This implies that R is positive. This completes the proof.

Theorem 5.4. a) *The correspondence between* $s \in \mathcal{N}(D^+, D)$ *and the linear functional*

$$\text{tr}(.\ S): D'\varepsilon D^+ \ni R \to \text{tr}(R\ S)$$

defines a topological linear isomorphism between $\mathcal{N}(D^+, D)[\tau_\pi]$ *and the strong dual of* $D'\varepsilon D^+$.

b) *The correspondence between* $T \in \mathcal{L}$ *and the linear functional*

$$\text{tr}(T.\ ): \mathcal{N}(D^+, D) \ni R \to \text{tr}(T\ R)$$

defines a topological linear isomorphism between $\mathcal{L}[\tau_D]$ *and the strong dual of* $\mathcal{N}(D^+, D)[\tau_\pi]$.

Proof: a) According to [3] Theorem 2.1, the map

$$S \to \text{tr}(.\ S)$$

is a linear isomorphism of $\mathcal{N}(D^+, D)$ $(\cong \bar{D} \tilde{\otimes}_\pi D)$ onto the strong dual of $D'\varepsilon D^+$. By Lemma 5.1, it maps the set

$$\{S \in \mathcal{N}(D^+, D): \nu_n(S) \le 1\}$$

onto the polar of $\mathfrak{B}_n \cap (D'\varepsilon D^+)$. Therefore, it is also a topological isomorphism.

b) Let $\boldsymbol{B}(\bar{D}, D)$ denote the space of continuous bilinear forms on $\bar{D} \times D$. Using the isomorphisms $\mathcal{N}(D^+, D)[\tau_\pi] \cong \bar{D} \tilde{\otimes}_\pi D$, $\mathcal{L} \cong \boldsymbol{B}(\bar{D}, D)$, and the well-known representation of the dual of $\bar{D} \tilde{\otimes}_\pi D$ (cf. Lemma 4.1 and [12] §§40, 41), we see that the mapping

$$\mathcal{L} \ni T \to \text{tr}(T.\ ) \tag{8}$$

is an isomorphism of the linear space $\mathcal{L}$ onto the dual of $\mathcal{N}(D^+, D)[\tau_\pi]$.

It follows from Lemma 5.2.b) that the polar of the bounded subset $\mathfrak{N}_B$ of $\mathcal{N}(D^+, D)[\tau_\pi]$, defined in Corollary 4.3, is the set of functionals

$$\{\text{tr}(T.\ ): T \in \mathcal{L} \text{ and } \|B \circ T \circ B\| \le 1\}.$$

This means that the mapping (8) is a topological isomorphism between $\mathcal{L}[\tau_D]$ and the strong dual of $\mathcal{N}(D^+, D)[\tau_\pi]$. The proof is com-

plete.

Remarks : 1. It is not difficult to show directly that the dual of $D'\varepsilon D^+$ is isomorphic to $\mathcal{N}(D^+, D)$ as a linear space. Indeed, each $S \in \mathcal{N}(D^+, D)$ has a representation

$$S = \bar{B} R \tilde{B}$$

with $B \in \mathscr{B}$, $R \in \mathscr{C}(H, H)$, and $\nu(R) < \infty$. It follows, therefore, from Lemma 5.2.b) that the functionals $\mathrm{tr}(.\ S)$ are continuous on $D'\varepsilon D^+$. By Lemma 5.3, the linear mapping

$$\mathcal{N}(D^+, D) \ni S \to \mathrm{tr}(.\ S) \in (D'\varepsilon D^+)'$$

is a surjective mapping. As a consequence of Lemma 5.1, it is also injective.

2. On condition that $L^+(D)$ is a self-adjoint Op^*-algebra (i. e. that $D = \cap \{D(A^*) : A \in L^+(D)\}$), W. Timmermann [28] proved that the dual of $\mathscr{F}[\tau_D]$ is algebraically isomorphic to $\mathfrak{S}_1(D)$. This implies also the isomorphism of the linear spaces $(D'\varepsilon D^+)'$ and $\mathcal{N}(D^+, D)$.

3. It is well-know that, for arbitrary Fréchet spaces E and F, the dual of $E \tilde{\otimes}_\pi F$ is algebraically isomorphic to the space $\boldsymbol{B}(E, F)$ of all continuous bilinear forms on $E \times F$. The problem whether this linear isomorphism is also a topological isomorphism with respect to the strong topology of the dual of $E \tilde{\otimes}_\pi F$ and the bibounded topology of $\boldsymbol{B}(E, F)$ is the "problem of topologies" of A. Grothendieck. In [6] p. 70 Remark 2, A. Grothendieck remarked without giving a proof that the "problem of topologies" has a positive solution for subspaces of the product of a sequence of Hilbert spaces. On the basis of this remark, J.-P. Jurzak [8] stated a result which is equivalent to Theorem 5.4.b). Actually Corollary 4.3 contains the positive solution of the "problem of topologies" for Fréchet domains of Op^*-algebras.

4. The functionals

$$\mathrm{tr}(.\ S) : \mathscr{L} \ni T \to \mathrm{tr}(T\ S) \qquad (S \in \mathcal{N}(D^+, D))$$

were called normal functionals by G. Lassner, W. Timmermann and K. Schmüdgen ([19, 23]). We will use the terminology of H. Araki and J.-P. Jurzak [2] : The weak topology $\sigma(\mathscr{L}, \mathcal{N}(D^+, D))$ is refered to as the ultraweak topology. A linear functional on $\mathscr{L}$ is said to be

ultraweakly continuous if and only if it is of the form $\text{tr}(.\ S)$ for some $S \in \mathcal{N}(D^+, D)$.

Corollary 5.5. *Each continuous linear functional θ on $\mathcal{L}[\tau_D]$ is the sum of an ultraweakly continuous linear functional θ_1 and a continuous linear functional θ_2 which vanishes on $D'\varepsilon D^+$.*

Proof: The functional $\theta \upharpoonright (D'\varepsilon D^+)$ has a representation

$$T \to \text{tr}(T\ S),$$

where $S \in \mathcal{N}(D^+, D)$. We can define

$$\theta_1 = \text{tr}(.\ S) \quad \text{and} \quad \theta_2 = \theta - \theta_1.$$

Remarks: 1. The decomposition $\theta = \theta_1 + \theta_2$ in the above corollary is unique.

2. The following argument of W. Timmermann shows that θ_1 and θ_2 are positive if θ is positive. Let θ be a positive continuous linear functional on $\mathcal{L}[\tau_D]$. Note that the subspace $\mathcal{C}(H, H) \subset \mathcal{L}$ is isomorphic to the W*-algebra $\mathcal{L}(H, H)$. The restrictions $\theta_1 \upharpoonright \mathcal{C}(H, H)$ and $\theta_2 \upharpoonright \mathcal{C}(H, H)$ are positive because they are the normal and the singular part, respectively, of the positive linear functional $\theta \upharpoonright \mathcal{C}(H, H)$. Since the positive cone of $\mathcal{C}(H, H)$ is τ_D-dense in the positive cone of $\mathcal{L}$ by [13] Corollary 6.2, θ_1 and θ_2 are positive.

Now, we are going to prove some properties of the locally convex spaces $D'\varepsilon D^+$, $\mathcal{F}$, $L^+(D)$, and $\mathcal{C}(D^+, D)$.

Lemma 5.6. *Each closed convex circled subset of $D'\varepsilon D^+$ or $\mathcal{F}[\tau_D]$, which absorbs the sets $\mathfrak{B}_n \cap \mathcal{F}\ (n \in N)$, is a neighbourhood of zero in $D'\varepsilon D^+$ or $\mathcal{F}[\tau_D]$, respectively. In particular, the spaces $D'\varepsilon D^+$ and $\mathcal{F}[\tau_D]$ are quasi-barrelled locally convex spaces.*

Proof: We take the polars with respect to the pairing (5). Let $\mathcal{U}$ be a closed convex circled set which absorbs the sets $\mathfrak{B}_n \cap \mathcal{F}$. By Lemma 5.1, the polar of $\mathcal{U}$ in $\mathcal{N}(D^+, D)$ is τ_π-bounded. Therefore, the bipolar of $\mathcal{U}$ in $\mathcal{L}$ is a neighbourhood of zero for the uniform topology τ_D. But the intersection of this bipolar and $D'\varepsilon D^+$ or $\mathcal{F}$, respectively, is $\mathcal{U}$. This proves the lemma.

Theorem 5.7. a) *$D'\varepsilon D^+$ is a complete barrelled DF-space.*

b) *$\mathscr{F}[\tau_D]$ is a bornological DF-space.*

c) *$L^+(D)[\tau_D]$ and $\mathscr{C}(D^+, D)[\tau_D]$ are DF-spaces.*

Proof: All subspaces of $\mathscr{L}[\tau_D]$ admit fundamental sequences of bounded sets.

a) is a consequence of the facts that a complete quasi-barrelled space is barrelled and that a quasi-barrelled space with a fundamental sequence of bounded sets is a DF-space.

In the same way, it follows that $\mathscr{F}[\tau_D]$ is a DF-space. We show that the topology of $\mathscr{F}[\tau_D]$ is the Mackey topology. For, let $\mathfrak{M} \subset \mathscr{N}(D^+, D)$ be a convex circled subset which is compact with respect to the weak topology $\sigma(\mathscr{N}(D^+, D), \mathscr{F})$. (The duality between $\mathscr{N}(D^+, D)$ and $\mathscr{F}$ is given by (5).) For each $n \in N$, the set of $\sigma(\mathscr{N}(D^+, D), \mathscr{F})$-continuous linear functionals

$$\{\operatorname{tr}(T \, . \,) : T \in \mathfrak{B}_n \cap \mathscr{F}\}$$

is pointwise bounded on $\mathfrak{M}$. According to the principle of uniform boundedness (see, e. g., [10] 12. 4), we get

$$\sup \; \{|\operatorname{tr}(T, S)| : T \in \mathfrak{B}_n \cap \mathscr{F}, S \in \mathfrak{M}\} < \infty.$$

By Lemma 5.1, $\mathfrak{M}$ is τ_π-bounded. By Corollary 4.3, it is contained in

$$\mathfrak{N}_B = \{\tilde{B}R\tilde{B} : R \in \mathscr{C}(H, H) \text{ and } \nu(R) \leq 1\}$$

for some $B \in \mathscr{B}$. As a consequence of Lemma 5.2. b), $\mathfrak{N}_B$ is equicontinuous. Since each weakly compact convex circled subset of the dual of $\mathscr{F}[\tau_D]$ is equicontinuous, $\mathscr{F}[\tau_D]$ is a Mackey space.

Next, we show that every bounded linear functional on $\mathscr{F}[\tau_D]$ is continuous. By a bounded functional, we mean a functional which is bounded on each bounded set. Since $\mathscr{F}[\tau_D]$ is a DF-space, it suffices to show that a bounded linear functional is continuous on each of the bounded sets $\mathfrak{B}_n \cap \mathscr{F}$. Let θ be a bounded linear functional on $\mathscr{F}[\tau_D]$. For a fixed natural number n, let B be the inverse operator of Friedrich's extension of A_n. Furthermore, let H_1 be the closure in the Hilbert space H of the range of A_n and let P be the orthogonal projection onto H_1. For $T \in \mathscr{F}$, we define the linear mapping

$$J(T): H_1 \ni f \rightarrow PB\overline{T}Bf \in H_1.$$

Since

$$\langle A_n\varphi, PB\overline{T}BA_n\psi \rangle = \langle BPA_n\varphi, T\psi \rangle = \langle \varphi, T\psi \rangle$$

for all $\varphi, \psi \in D$, T coincides with the mapping

$$D \ni \varphi \rightarrow (A_n)^* J(T) A_n\varphi.$$

This implies that

$$J : T \rightarrow J(T)$$

is a bijective mapping of $\mathscr{F}$ onto some linear space $\mathscr{X}$ of continuous finite rank operators on H_1. Moreover, $T \in \mathfrak{B}_n$ if $\|J(T)\| \leq 1$. Since θ is bounded, the linear functional

$$\omega : \mathscr{X} \ni R \rightarrow \theta(J^{-1}(R))$$

is continuous with respect to the operator norm topology of $\mathscr{X}$. Consequently, it has a representation

$$\omega : R \rightarrow \sum_{k=1}^{\infty} \langle f_k, R\, g_k \rangle,$$

where (f_k) and (g_k) are sequences in H_1 such that

$$\sum_{k=1}^{\infty} \|f_k\|\, \|g_k\| < \infty.$$

Since $A_n(D)$ is dense in H_1, we can choose the sequences (f_k) and (g_k) in $A_n(D)$ (see, e. g., [21] III 6.4). For an arbitrary $\varepsilon > 0$, we find $m \in N$ such that

$$\sum_{k=m+1}^{\infty} \|f_k\|\, \|g_k\| < \varepsilon.$$

Then we define the nuclear operator

$$S : D^+ \ni f \rightarrow \sum_{k=1}^{m} \langle (A_n)^{-1}(f_k), f \rangle (A_n)^{-1}(g_k) \in D.$$

For $T \in \mathfrak{B}_n \cap \mathscr{F}$, we get the estimate

$$|\theta(T) - \mathrm{tr}(T\,S)| = |\omega(J(T)) - \mathrm{tr}(T\,S)|$$

$$= |\sum_{k=1}^{\infty} \langle f_k, PB\overline{T}Bg_k \rangle - \sum_{k=1}^{m} \langle (A_n)^{-1}f_k, T(A_n)^{-1}g_k \rangle|$$

$$= |\sum_{k=m+1}^{\infty} \langle (A_n)^{-1}(f_k), \overline{T}(A_n)^{-1}(g_k) \rangle|$$

$$\leq \sum_{k=m+1}^{\infty} \|A_n(A_n)^{-1}(f_k)\|\, \|A_n(A_n)^{-1}(g_k)\| < \varepsilon.$$

This proves that θ can be approximated uniformly on $\mathfrak{B}_n \cap \mathscr{F}$ by a τ_D-continuous linear functional. Thus, θ is τ_D-continuous on $\mathfrak{B}_n \cap \mathscr{F}$.

Now, part b) of the theorem follows from the fact that a locally convex space is bornological if and only if its topology is the Mackey topology and each bounded linear functional is continuous.

In order to prove c), it suffices to show that the intersection $\mathscr{U}$ $= \cap \mathscr{U}_n$ of a sequence $(\mathscr{U}_n)$ of closed convex circled neighbourhoods of zero in $L^+(D) [\tau_D]$ or $\mathscr{C}(D^+, D) [\tau_D]$, respectively, is again a neighbourhood of zero if it absorbs the sets $\mathfrak{B}_n \cap \mathscr{C}(D^+, D)$. Let $\overline{\mathscr{U}}_n$ be the closure of $\mathscr{U}_n$ in $\mathscr{L}[\tau_D]$. The intersection $\cap \overline{\mathscr{U}}_n$ absorbs the sets $\mathfrak{B}_n$ since $\mathfrak{B}_n \cap \mathscr{C}(D^+, D)$ is dense in $\mathfrak{B}_n$ ([13] Corollary 6.3). Consequently, it is a neighbourhood of zero in the DF-space $\mathscr{L}[\tau_D]$. Finally, the set $\mathscr{U}$ is a neighbourhood of zero because it is the intersection of $\cap \overline{\mathscr{U}}_n$ and $L^+(D)$ or $\mathscr{C}(D^+, D)$, respectively. This completes the proof of the theorem.

J.-P. Jurzak [8] considered Fréchet domains D which are quasi-normable in the sense of A. Grothendieck [5]. For such domains, it follows from [8] Theorem 2 that $\mathscr{L}[\tau_D]$ is a bornological locally convex space. We show that the same is true for $D' \varepsilon D^+$, $\mathscr{C}(D^+, D)$, and $L^+(D)$.

Proposition 5.8. *If D is a quasi-normable Fréchet space (with respect to the graph topology), then the spaces $D' \varepsilon D^+$, $\mathscr{C}(D^+, D) [\tau_D]$, $L^+(D) [\tau_D]$ and $\mathscr{L}[\tau_D]$ are bornological locally convex spaces.*

Proof: The quasi-normability of D implies the quasi-normability of $\overline{D} \tilde{\otimes}_\pi D$ ([6] I §1.3. Proposition 7). Consequently, each subspace of the space $\mathscr{L}[\tau_D]$ (which is isomorphic to the strong dual of $\overline{D} \tilde{\otimes}_\pi D$) satisfies the strict Mackey convergence condition (see [5] Definition 4). This implies that the associated bornological topology of an arbitrary subspace of $\mathscr{L}[\tau_D]$ coincides with τ_D on bounded sets. Note that a convex circled subset of a DF-space is a neighbourhood of zero if and only if its intersection with any convex circled bounded set is a neighbourhood of zero in this bounded set. Thus, each subspace of $\mathscr{L}[\tau_D]$, which is a DF-space, is also a bornological locally convex space. Finally, the assertion follows from

Theorem 5.7.

Remark: D is quasi-normable if it is a Schwartz space ([5] Proposition 17) or if it is of the form

$$D = \bigcap_{n=1}^{\infty} D(T^n),$$

where T is a self-adjoint operator ([7] 2, 7.16). D is not quasi-normable if it is a Montel space but not a Schwartz space ([5] Proposition 17) or if $L^+(D)[\tau_D]$ is not bornological. Such Fréchet domains exist as shown by K. Schmüdgen in [23, 25].

§ 6. Normality of the Positive Cone

The positive cone of $\mathscr{L}$ is normal with respect to the topology τ_D because the neighbourhood of zero

$$\{T \in \mathscr{L} : |\langle \varphi, T\varphi \rangle| \leq 1 \text{ for all } \varphi \in M\}$$

is saturated with respect to the positive cone for each bounded subset M of D. Consequently, each continuous linear functional is a linear combination of positive continuous liner functionals ([21] V). It is also known that each ultraweakly continuous linear functional is a linear combination of positive ultraweakly continuous linear functionals ([23, 2]). In this section, we establish an analogous result for bounded linear functionals on $\mathscr{L}[\tau_D]$. More precisely, we show that the positive cone of $\mathscr{L}$ is normal with respect to the associated bornological topology of $\mathscr{L}[\tau_D]$. Among other things, this result is applied to the decomposition of a positive linear functional into an ultraweakly continuous positive linear functional and a singular positive linear functional.

Let τ_0 denote the associated bornological topology of $\mathscr{L}[\tau_D]$, i. e., the strongest locally convex topology on $\mathscr{L}$ such that the sets $\mathfrak{B}_n$ are bounded. This topology coincides on $\mathscr{L}_h$ with the order topology in the sense of H. Schaefer [21]. It was called the ρ-topology by H. Araki and J. -P. Jurzak [2, 8]. A priori, the order topology of $L^+(D) \cap \mathscr{L}_h$ may be different from the topology induced by τ_0. However, these topologies coincide if the condition $A_n(D) = D$ of H. Araki and J. -P. Jurzak [2] is satisfied. Therefore it would be of

interest to answer the following question.

Does there exist a cofinal sequence (A_n) in $L^+(D) \cap \mathscr{L}_h$ with $A_n(D) = D$ for any Fréchet domain D?

So far the author knows, the only examples of Fréchet domains constructed in the literature are of the form

$$D = \bigcap_{n=1}^{\infty} f_n(T),$$

where T is a self–adjoint operator on H and (f_n) is a sequence of real–valued functions on the spectrum $\mathfrak{S}(T)$ of T, which are measurable with respect to the spectral measure of T and satisfy $(f_n(t))^2 \leq f_{n+1}(t)$ for all $t \in \mathfrak{S}(T)$ and $n \in N$. Such domains have been studied systematically by K. Schmüdgen [25]. They fulfil clearly the condition of H. Araki and J.-P. Jurzak.

Under the condition $A_n(D) = D$, H. Araki and J.-P. Jurzak [2] proved that the positive cone of $\mathscr{L}$ (and of some subspaces) is normal with respect to the topology τ_0. Here we prove this fact without using the condition $A_n(D) = D$.

Theorem 6.1. *The sets*

$$\mathscr{U}((\varepsilon_n)) = \{T \in \mathscr{L} : \text{There exists } k \in N \text{ such that}$$
$$|\langle \varphi, T\varphi \rangle| \leq \sum_{n=1}^{k} \varepsilon_n \langle \varphi, A_n \varphi \rangle$$
$$\text{for all } \varphi \in D.\}$$

form a basis of neighbourhoods of zero in $\mathscr{L}[\tau_0]$ if (ε_n) runs through the sequences of positive real numbers.

Proof: The sets $\mathscr{U}((\varepsilon_n))$ are convex circled sets which absorb each of the sets $\mathfrak{B}_k$. Hence, they are neighbourhoods of zero in $\mathscr{L}[\tau_0]$.

Conversely, let $\mathscr{U}$ be a convex circled neighbourhood of zero in $\mathscr{L}[\tau_0]$. Then $\delta_k \mathfrak{B}_k \subset \mathscr{U}$ for some positive numbers δ_k. We have to show that $\mathscr{U} \supset \mathscr{U}((\varepsilon_n))$ for some sequence (ε_n). We define the sequence (ε_n) by induction such that

$$\varepsilon_k > 0, \quad \varepsilon_k \sum_{n=1}^{k} \varepsilon_n < 2^{-k-2} \varepsilon_1 \delta_k$$

for all $k \in N$. This yields that

$$2 \sum_{n,m=1}^{k} \varepsilon_n \varepsilon_m (\varepsilon_1 \delta_{\max(n,m)})^{-1} < 1. \tag{9}$$

We show that $\mathscr{U}((\varepsilon_n)) \subset \mathscr{U}$. For let $T \in \mathscr{L}$ and $k \in N$ be given such that

$$|\langle \varphi, T\psi \rangle| \leq \sum_{n=1}^{k} \varepsilon_n \langle \varphi, A_n \varphi \rangle$$

for all $\varphi \in D$. Setting

$$A = \sum_{n=1}^{k} \varepsilon_n A_n,$$

we get

$$|\langle \varphi, T\varphi \rangle| \leq \langle \varphi, A\varphi \rangle \leq ||\varphi||\ ||A\varphi|| \leq (\varepsilon_1)^{-1} ||A\varphi||^2.$$

For each positive real number c, the inequality

$$|\langle \varphi, T\psi \rangle| = |4^{-1} \sum_{l=0}^{3} i^{-l} \langle c\varphi + i^l c^{-1}\psi, T(c\varphi + i^l c^{-1}\psi) \rangle|$$

$$\leq (\varepsilon_1)^{-1} 4^{-1} \sum_{l=0}^{3} ||A(c\varphi + i^l c^{-1}\psi)||^2$$

$$= (\varepsilon_1)^{-1}(c^2 ||A\varphi||^2 + c^{-2} ||A\psi||^2)$$

is satisfied. Taking the infimum over all positive real numbers c, we get

$$|\langle \varphi, T\psi \rangle| \leq 2(\varepsilon_1)^{-1} ||A\varphi||\ ||A\psi||.$$

Therefore, there exists an operator $R \in \mathscr{C}(H, H)$ with $||R|| \leq 2(\varepsilon_1)^{-1}$ such that

$$T = A \circ R \circ A = \sum_{n,m=1}^{k} \varepsilon_n \varepsilon_m A_n \circ R \circ A_m$$

(see, e. g., [13] §5). Since

$$|\langle \varphi, A_n \circ R \circ A_m \psi \rangle| = |\langle A_n \varphi, RA_m \psi \rangle|$$
$$\leq ||R||\ ||A_n \varphi||\ ||A_m \psi||$$
$$\leq 2(\varepsilon_1)^{-1} \max\{||A_n \varphi||\ ||A_n \psi||,\ ||A_m \varphi||\ ||A_m \psi||\},$$

it follows that

$$A_n \circ R \circ A_m \in 2(\varepsilon_1)^{-1} \mathfrak{B}_{\max(n,m)}.$$

In view of (9),

$$T = \sum_{n,m=1}^{k} 2\varepsilon_n \varepsilon_m (\varepsilon_1 \delta_{\max(n,m)})^{-1} 2^{-1} \varepsilon_1 \delta_{\max(n,m)} A_n \circ R \circ A_m$$

belongs to the convex hull of $\cup \delta_n \mathscr{U}_n$. Consequently, it belongs to $\mathscr{U}$. This proves the theorem.

Since the sets $\mathscr{U}((\varepsilon_n))$ are saturated with respect to the positive cone of $\mathscr{L}_h$, we get the following two corollaries.

Corollary 6.2. *The positive cone of $\mathscr{L}_h$ is normal with respect to the topology τ_0.*

Corollary 6.3. *Each bounded linear functional on $\mathscr{L}[\tau_D]$ is a linear combination of positive bounded linear functionals.*

It is known that the associated bornological topology of the strong dual of a Fréchet space has a basis of strongly closed neighbourhoods of zero ([11] §29.1). This implies the following three assertions.

Corollary 6.4. *The τ_D-closures of the sets $\mathscr{U}((\varepsilon_n))$ (defined in Theorem 6.1) form a basis of neighbourhoods of zero in $\mathscr{L}[\tau_0]$.*

Corollary 6.5. *The topologies τ_0 and τ_D induce the same topology on $D'\varepsilon D^+$.*

Proof. If $\mathscr{U}$ is a τ_D-closed neighbourhood of zero in $\mathscr{L}[\tau_0]$, then $\mathscr{U} \cap (D'\varepsilon D^+)$ is a barrel in $D'\varepsilon D^+$. According to Theorem 5.7. a), it is a neighbourhood of zero in $D'\varepsilon D^+$.

Corollary 6.6. *Each bounded linear functional θ on $\mathscr{L}[\tau_D]$ is the sum of an ultraweakly continuous linear functional θ_1 and a bounded linear functional θ_2 which vanishes on $D'\varepsilon D^+$. The functionals θ_1 and θ_2 are positive if θ is positive.*

Proof: Since θ is τ_0-continuous, the restriction $\theta \upharpoonright (D'\varepsilon D^+)$ is τ_D-continuous. By Theorem 5.4. a), it is of the form

$$D'\varepsilon D^+ \ni T \to \mathrm{tr}(T\,S),$$

where $S \in \mathscr{N}(D^+, D)$. We can define $\theta_1 = \mathrm{tr}(.\;S)$ and $\theta_2 = \theta - \theta_1$.

Now, suppose that θ is positive. In this case, the functionals

$$\omega_n : \mathscr{L} \ni T \to \theta (A_n \circ T \circ A_n)$$

are positive, as well. Note that $\theta_2(A_n \circ T \circ A_n) = 0$ for all $T \in \mathscr{F}$. This implies that the functionals

$$\mathscr{C}(H, H) \ni T \to \theta_1(A_n \circ T \circ A_n),$$
$$\mathscr{C}(H, H) \ni T \to \theta_2(A_n \circ T \circ A_n)$$

are the normal and the singular part of $\omega_n \upharpoonright \mathscr{C}(H, H)$. Consequently, they are positive. Finally, the assertion follows from the fact that each positive $T \in \mathscr{L}_h$ is of the form $T = A_n \circ R \circ A_n$ for some $n \in N$ and some $R \in \mathscr{C}(H, H)$ with $R \geq 0$ (see [13] Proposition 5. 1).

Remark : In the above corollary, the representation $\theta = \theta_1 + \theta_2$ is unique.

§ 7. Fréchet Montel Domains

In this section, we apply the preceding results to the special case, where D is a Fréchet Montel space. In particular, we give a new proof of a result on trace representation of linear functionals, which was obtained in its general form by K. Schmüdgen [23]. We collect the results in the following theorem.

Theorem 7. 1. *For a Fréchet domain D, the following properties are equivalent* :
a) D *is a Montel space.*
b) $\mathscr{L}[\tau_D]$ *is a Montel space.*
c) $\mathscr{L}[\tau_D]$ *is reflexive.*
d) *Each bounded linear functional on $\mathscr{L}[\tau_D]$ is ultraweakly continuous.*
e) *Each positive linear functional on $L^+(D)$ has a representation*

$$\theta : L^+(D) \ni T \to \sum_{n=1}^{\infty} \langle \varphi_n, T\varphi_n \rangle,$$

where (φ_n) is a sequence in D.
f) *Each continuous linear functional on $L^+(D)[\tau_D]$ has a representation*

$$\theta : L^+(D) \ni T \to \mathrm{tr}(T\,S),$$

where $S \in \mathscr{N}(D^+, D)$.

g) $\mathscr{F}$ *is dense in $L^+(D)\ [\tau_D]$.*

h) *Each complete (with respect to the norm topology) linear subspace of D has finite dimension, i. e., D is of type I in the sense of G. Lassner and W. Timmermann* [20].

Proof: a)⇒b) : It follows from Theorem 4.2 that each bounded subset of $\mathcal{N}(D^+, D)[\tau_\pi]$ is contained in

$$\{\bar{B}^2 R \bar{B} \tilde{B} : R \in \mathcal{L}(H, H) \text{ and } \nu(R) \leq 1\}$$

for some positive $B \in \mathcal{C}(H, D)$ with the continuous extension $\tilde{B} \in \mathcal{L}(D^+, H)$. Let

$$\bar{B} = \int \lambda \, dE_\lambda$$

be the spectral representation of the positive self-adjoint operator $\bar{B}$. For each $n \in N$, the orthogonal projection

$$P_n = \int\limits_{(1/n, \infty)} dE_\lambda = \bar{B} \int\limits_{(1/n, \infty)} \lambda^{-1} dE_\lambda$$

has a continuous extension $\tilde{P}_n \in \mathcal{L}(D^+, H)$ because $P_n(H) \subset D$ (see Proposition 2.1). Now assume that D is a Montel space. Then the bounded set $\bar{B}(U_H)$ is relatively compact in D. Consequently, it is also compact with respect to the norm topology of H. This implies that P_n and $\tilde{P}_n$ are finite rank operators. The estimate

$$\nu_k(\bar{B}^2 R \bar{B} \tilde{B} - P_n \bar{B}^2 R \bar{B}^2 \tilde{P}_n) = \nu(A_k \circ B \circ (((\bar{B} - \bar{B} P_n) R \bar{B}$$
$$+ \bar{B} P_n R (\bar{B} - \bar{B} P_n)) \upharpoonright D) \circ B \circ A_k)$$
$$\leq ||A_k \circ B|| \; ||B \circ A_k|| \; ||\bar{B} - \bar{B} P_n|| (||\bar{B}|| + ||\bar{B} P_n||) \nu(R)$$
$$\leq 2n^{-1} ||B|| \; ||A_k \circ B|| \; ||B \circ A_k|| \nu(R)$$

shows that each bounded subset of $\mathcal{N}(D^+, D) [\tau_\pi]$ can be approximated uniformly by sets of finite dimension. Consequently, it is precompact. Since $\mathcal{N}(D^+, D)[\tau_\pi]$ is a Fréchet space, it is barrelled and each bounded subset is relatively compact. This means that $\mathcal{N}(D^+, D)[\tau_\pi]$ is a Montel space. Since $\mathcal{L}[\tau_D]$ is isomorphic to the strong dual of $\mathcal{N}(D^+, D) [\tau_\pi]$, it is also a Montel space.

The implication b)⇒c) is obvious.

c) ⇒d) : Assume that $\mathcal{L}[\tau_D]$ is reflexive. Note that barrelled subspaces of reflexive locally convex spaces are again reflexive. Therefore, $D' \varepsilon D^+$ is reflexive. Hence, $\mathcal{L}[\tau_D] = D' \varepsilon D^+$. Now, the assertion follows from Corollary 6.6.

d) $\Rightarrow$ e) : The positive linear functional θ on $L^+(D)$ has a positive linear extension ω to $\mathscr{L}$ because $L^+(D) \cap \mathscr{L}_h$ is cofinal in $\mathscr{L}_h$. The positive linear functional ω is necessarily bounded on $\mathscr{L}[\tau_D]$. As a consequence of Corollary 6.5, the restriction $\omega \upharpoonright (D'\varepsilon D^+)$ is continuous. By Lemma 5.3, there exist $B \in \mathscr{B}$ and $R \in \mathscr{C}(H, H)$ with $\nu(R) \leq 1$ and $R \geq 0$ such that

$$\omega(T) = \mathrm{tr}\,(T(\bar{B} R \tilde{B}))$$

for $T \in D'\varepsilon D^+$, where $\tilde{B} \in \mathscr{L}(D^+, D)$ denotes the continuous extension of B. It follows from Theorem 1.1 that $D'\varepsilon D^+$ is ultraweakly dense in $\mathscr{L}$. Since ω is ultraweakly continuous by assumption d),

$$w(T) = \mathrm{tr}\,(T(\bar{B} R \tilde{B}))$$

for all $T \in \mathscr{L}$. The operator R has a representation

$$R : D \ni \varphi \to \sum_{n=1}^{\infty} a_n \langle f_n, \varphi \rangle f_n ,$$

where $f_n \in U_H$, $a_n \geq 0$, and $\sum a_n < \infty$. Consequently, $\bar{B} R \tilde{B}$ has the representation

$$\bar{B} R \tilde{B} : D^+ \ni f \to \sum_{n=1}^{\infty} a_n \langle \bar{B} f_n, f \rangle \bar{B} f_n .$$

Setting $\varphi_n = (a_n)^{1/2} \bar{B} f_n$, we obtain

$$\theta(T) = \omega(T) = \mathrm{tr}\,(T(\bar{B} R \tilde{B}))$$

$$= \sum_{n=1}^{\infty} a_n \langle \bar{B} f_n, T \bar{B} f_n \rangle = \sum_{n=1}^{\infty} \langle \varphi_n, T \varphi_n \rangle$$

for all $T \in L^+(D)$.

e) $\Rightarrow$ f) : Since each continuous linear functional on $L^+(D)[\tau_D]$ is a linear combination of positive linear functionals, it suffices to show that each functional of the form

$$\omega : L^+(D) \ni T \to \sum_{n=1}^{\infty} \langle \varphi_n, T \varphi_n \rangle \qquad\qquad (\varphi_n \in D)$$

has also a representation

$$L^+(D) \ni T \to \mathrm{tr}\,(T\,S),$$

where $S \in \mathscr{N}(D^+, D)$. In order to do this, we note that the convergence of the series

$$\sum_{n=1}^{\infty} \langle \varphi_n, T \varphi_n \rangle$$

for all $T \in L^+(D)$ implies that

$$\sum_{n=1}^{\infty} ||A_k \varphi_n||^2 = \sum_{n=1}^{\infty} \langle \varphi_n, (A_k)^2 \varphi_n \rangle < \infty$$

for all $k \in N$. This means that the series $\sum \varphi_n \otimes \varphi_n$ converges absolutely in $\bar{D} \tilde{\otimes}_\pi D$. Thus, the nuclear operator S can be defined by

$$S : D^+ \ni f \to \sum_{n=1}^{\infty} \langle \varphi_n, f \rangle \varphi_n \in D.$$

f)$\Rightarrow$g) : If $\mathscr{F}$ is not dense in $L^+(D) [\tau_D]$, then there exists a continuous linear functional θ on $L^+(D) [\tau_D]$ such that $\theta \neq 0$ and $\theta(T) = 0$ for all $T \in \mathscr{F}$. It follows from Theorem 5.4.a) that θ cannot be of the form $T \to \text{tr}(TS)$ with $S \in \mathscr{N}(D^+, D)$ because $\mathscr{F}$ is dense in $D' \varepsilon D^+$.

g)$\Rightarrow$h) : Let H_1 be a complete (with respect to the norm topology) linear space, which is contained in D. According to Proposition 2.1, the orthogonal projection P of D onto H_1 belongs to $\mathscr{P}$. Therefore, the seminorm

$$L^+(D) \ni T \to ||P \circ T \circ P||$$

is τ_D-continuous. If $\mathscr{F}$ is dense in $L^+(D) [\tau_D]$, then

$$||P \circ (Id - T) \circ P|| < 1$$

for some $T \in \mathscr{F}$. This implies that the dimension of $H_1 = P(D)$ is finite.

h)$\Rightarrow$a) Assume that D contains a bounded, but not relatively compact subset M. Since D is a Fréchet space, there exist $\varepsilon > 0$. $k \in N$ and a sequence (φ_n) in M such that

$$||A_k(\varphi_n - \varphi_m)|| > \varepsilon \qquad (n \neq m). \tag{10}$$

The set $\{A_k \varphi_n\}_{n \in N}$ is bounded in D. By Proposition 2.4.a), it is contained in $\bar{B}(U_H)$ for some $B \in \mathscr{B}$. Let

$$\bar{B} = \int \lambda \, dE_\lambda$$

be the spectral representation of the positive self-adjoint operator $\bar{B}$. Because of (10), $\bar{B}(U_H)$ is not relatively compact in H. Therefore, the complete (with respect to the norm topology) linear subspace of D

$$H_1 = (\int_{(\delta, \infty)} dE_\lambda) (H) = \bar{B}(\int_{(\delta, \infty)} \lambda^{-1} dE_\lambda) (H)$$

is of infinite dimension for some $\delta > 0$. This completes the proof of the theorem.

Remarks : 1. The equivalence of a), e), f), and h) in the preceding theorem was essentially obtained by K. Schmüdgen ([23, 24]) who generalized earlier results of S. L. Woronowicz [29], T. Sherman [27], and G. Lassner and W. Timmermann [19].

2. Let $\mathscr{X}$ and $\mathscr{Y}$ be dense linear subspaces of $\mathscr{L}[\tau_D]$. Suppose that $\mathscr{X}$ is symmetric (i. e., $\mathscr{X} = \{T^+ : T \in \mathscr{X}\}$) and that $\mathscr{X} \cap \mathscr{X}_h$ is cofinal in $\mathscr{L}_h$.

It is a consequence of the proof of Theorem 7.1 that the following conditions are also equivalent to the Montel property of D :

– Each positive linear functional on $\mathscr{X}$ is of the form

$$\theta : \mathscr{X} \ni T \to \sum_{n=1}^{\infty} \langle \varphi_n, T\varphi_n \rangle,$$

where (φ_n) is a sequence in D.

– Each continuous linear functional on $\mathscr{Y}[\tau_D]$ has an ultraweakly continuous extension to $\mathscr{L}$.

– $\mathscr{N}(D^+, D) [\tau_\pi]$ is a Montel space.

– $\mathscr{N}(D^+, D) [\tau_\pi]$ is reflexive.

– $D' \varepsilon D^+$ is a Montel space.

– $D' \varepsilon D^+$ is reflexive.

Since the strong dual of a reflexive Frechet space is always a bornological locally convex space (see [11] §29. 4. (4)), we get the following result.

Corollary 7. 2. *If D is a Fréchet Montel space, then $\mathscr{L}[\tau_D]$ is a bornological locally convex space.*

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