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Cohomologies of Lie Algebras of Formal Vector Fields Preserving a Foliation

By

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§ 0. Introduction

Let $\mathfrak{a}_n$ be the topological Lie algebra of all formal vector fields on $\boldsymbol{R}^n$ with the Krull topology; i. e.

$$\mathfrak{a}_n=\{\sum_{i=1}^n f_i(x_1,\cdots,x_n)\partial/\partial x_i \mid f_i\in\boldsymbol{R}[[x_1,\cdots,x_n]]\}$$

where $\boldsymbol{R}[[x_1,\cdots,x_n]]$ is the ring of formal power series in n variables. In [3] I. M. Gel'fand and D. B. Fuks have calculated the entire cohomology of $\mathfrak{a}_n$. In this paper we shall study the following subalgebra of $\mathfrak{a}_n$:

$$\begin{aligned}\mathfrak{a}_{r,n-r}=\{&\sum_{i=1}^r f_i(x_1,\cdots,x_r)\partial/\partial x_i\\&+\sum_{j=1}^{n-r} g_j(x_1,\cdots,x_r,y_1,\cdots,y_{n-r})\partial/\partial y_j \mid\\&f_i\in\boldsymbol{R}[[x_1,\cdots,x_r]], g_j\in\boldsymbol{R}[[x_1,\cdots,x_r,y_1,\cdots,y_{n-r}]]\}.\end{aligned}$$

The cohomology of this subalgebra was first studied by B. L. Feigin in [2] in order to construct the characteristic classes of flags of foliations. In the same paper the entire cohomology of $\mathfrak{a}_{1,n-1}$ was calculated by using a result about a cohomology with nontrivial coefficients (cf. [4]). Concerning a more general case A. Haefliger questioned whether

$$H^i(\mathfrak{a}_{n,r},\boldsymbol{R})\cong H^i(\mathfrak{a}_r,\boldsymbol{R}) \text{ for } i\leqq 2n \text{ (canonically).}$$

In [12] K. Sithanantham proved this isomorphism for $i\leqq n-r$ by adopting the method of [13]. In this paper we prove this isomorphism for $i\leqq n+r$ using the tool which they employed and the result obtained in [7]. The Main theorem of this paper is the following:

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Theorem. *Let* $\iota : \mathfrak{a}_r \longrightarrow \mathfrak{a}_{r,n-r}$ *be the natural inclusion. Then* ι *induces an isomorphism of cohomology*

$$\iota^* : H^i(\mathfrak{a}_{r,n-r}; \boldsymbol{R}) \longrightarrow H^i(\mathfrak{a}_r; \boldsymbol{R}) \quad \text{for } i \leqq n+r.$$

It is known that $H^*(\mathfrak{a}_r; \boldsymbol{R})$ is 2r-connected ([3]).

Corollary.

$$H^i(\mathfrak{a}_{r,n-r}; \boldsymbol{R}) = 0 \qquad \text{for } i \leqq 2r,$$

and

$$H^{2r+1}(\mathfrak{a}_{r,n-r}; \boldsymbol{R}) \neq 0.$$

This paper consists of 5 sections. In § 1, we prove a key proposition which is a useful tool to calculate the cohomology of the classical infinite dimensional Lie algebras. In § 2, we recall the definition of the Weil algebra and a spectral sequence converging to it. In § 3, we make the theorem obtained in [7] appropriate to the general infinite dimensional case. In § 4, we shall prove the main theorem. In § 5, we give a result and a conjecture concerning the Weil algebra of an infinite dimensional Lie algebra.

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§ 1. Proof of a Key Proposition

In this section we prove a key proposition which plays an important role in calculating the stable cohomology of transitive infinite Lie subalgebras of $\mathfrak{a}_n$.

First we recall a definition and notations. Let k be a commutative field of characteristic zero. In this section cochain complexes and algebras are defined over k.

Definition 1.1. An *operation* of a Lie algebra $\mathfrak{g}$ in a cochain complex $\{C^q, d\}_{q=0,1,2,}$ is a pair (ι, θ) where:

(i) θ is a representation of $\mathfrak{g}$ in the graded module C^*, homogeneous degree zero.

(ii) ι is a linear map of $\mathfrak{g}$ to the space of endomorphisms of

C^*, such that each $\iota(X)$ $(X \in \mathfrak{g})$ is homogeneous of degree -1.

(iii) The following relations hold:

$$\theta(X) = \iota(X)d + d\iota(X) \qquad X \in \mathfrak{g} \qquad (1.2)$$
$$\iota([X, Y]) = \theta(X)\iota(X) - \iota(X)\theta(X) \qquad X, Y \in \mathfrak{g}.$$

When there is given an operation of a Lie algebra $\mathfrak{g}$ in a cochain complex C^*, we say that C^* is a $\mathfrak{g}$-cochain complex. The subcomplex of C^* consisting of $\mathfrak{g}$-invariant elements annihilated by $\iota(X)$ for all $X \in \mathfrak{g}$ is called the basic subcomplex of C^*, denoted by $C^*_{\mathfrak{g}-basic}$ or C^*_{basic}.

Next we consider a special case; i.e. an operation of a finite dimensional abelian Lie algebra T. It is well-known that any representation of Lie algebra T can be extended to an action of the universal enveloping algebra of T, which is denoted by $U(T)$. Now we state the key proposition and prove it (cf. [11]).

Proposition 1.3. *Let T be a finite dimensional abelian Lie algebra and $\{C^q, d\}_{q=0,1,2,\dots}$ be a T-cochain complex. Then if each C^q is a projective $U(T)$-module, we have*

$$H^i(C) = 0 \quad \text{for } i < \dim T.$$

Proof. Since T is abelian, $U(T)$ is isomorphic to a polynomial algebra. Hence we can consider the Koszul resolution (cf. [9, p 204]):

$$0 \longrightarrow \wedge^n T \otimes U(T) \xrightarrow{\delta} \wedge^{n-1} T \otimes U(T) \xrightarrow{\delta} \cdots \xrightarrow{\delta} U(T) \xrightarrow{\varepsilon} k \longrightarrow 0$$

where $n = \dim T$ and $\wedge^p$ is the p-th exterior product of T. Since both operators δ and d are commutative with action of $U(T)$, we can define the following double complex:

$$A = \{(\wedge T \otimes U(T)) \otimes_{U(T)} C, d', d''\},$$
$$A^{p,q} = (\wedge^{-p} T \otimes U(T)) \otimes_{U(T)} C^q, \qquad d' = \delta \otimes_{U(T)} 1,\ d'' = 1 \otimes_{U(T)} d.$$

Then we have the following two spectral sequences (cf. MacLane [9, XI. 6] and Cartan-Eilenberg [1, XIII. 2]):

$$'E_2^{p,q} = \mathrm{Tor}_{U(T)}^{-p}(k, H^q(C^*));$$
$$''E_2^{p,q} = H^q(\mathrm{Tor}_{U(T)}^{-p}(k, C^*)).$$

Consider the first spectral sequence. By the Cartan formula (1.2) the operation of $U(T)$ on $H^*(C)$ is trivial because for any cocycle

c and $X\in T$, $\theta(X)c=\iota(X)dc+d\iota(X)c=d(\iota(X)c)$ is a coboundary. Hence we have

$$'E_2^{p,q}\cong Tor_{U(T)}^{-p}(k, k)\otimes H^q(C)\cong H_{-p}(T)\otimes H^q(C).$$

Note that $H_\ell(T)=0$ for $\ell<0$ or $\ell>n=\dim T$. Consider the second spectral sequence. Since each C^q is a projective $U(T)$-module, this spectral sequence collapses and we obtain

$$''E_2\cong H^*(k\otimes_{U(T)}C).$$

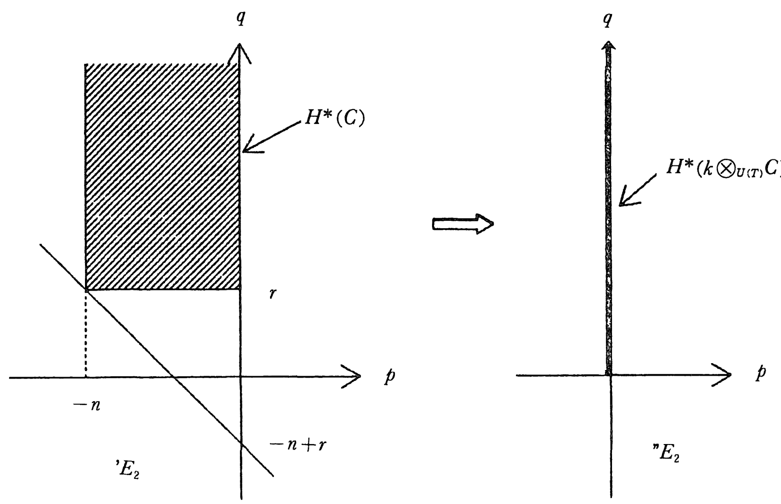

Figure 1

Therefore for the first spectral sequence,

$$'E_2^{p,q}\Rightarrow H^{p+q}(k\otimes_{U(T)}C).$$

Let r be the first integer such that $H^r(C)\neq 0$. Then

$$'E_2^{-n,r}='E_\infty^{-n,r}=H^r(C)\neq 0,$$
$$'E_2^{-n-i,r+i}='E_2^{-n+i,r-i}=0.$$

Consider E_∞-term. Then we have $'E_\infty^{-n,r}\cong H^{-n+r}(k\otimes_{U(T)}C)$. Since $H^i(k\otimes_{U(T)}C)=0$ for $i<0$, we obtain $-n+r\geqq 0$, i.e., $r\geqq n$. Hence $H^i(C)=0$ for $i<n=\dim T$. This completes the proof.

Remark. If $C^0=0$, then $H^i(k\otimes_{U(T)}C)=0$ for $i\leqq 0$. Hence $H^i(C)=0$ for $i\leqq n$.

§ 2. The Weil Algebra

In this section we consider a spectral sequence associated to a filtration of the Weil algebra of $\mathfrak{A}_n$.

Let k be a field of characteristic zero. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ a dual space of $\mathfrak{g}$ with respect to a canonical topology.

Definition 2.1. The *Weil algebra* of a Lie algebra $\mathfrak{g}$, denoted by $W(\mathfrak{g})$, is $\wedge\mathfrak{g}^*\otimes S\mathfrak{g}^*$ as algebra, where the exterior algebra $\wedge\mathfrak{g}^*$ is generated by 1-forms $\alpha\in\mathfrak{g}^*$, and the symmetric algebra $S\mathfrak{g}^*$ by 2-forms Ω_α for $\alpha\in\mathfrak{g}^*$.

Its differential is defined by $d\alpha=d_1\alpha+\Omega_\alpha$, where $d_1\alpha\in\wedge^2\mathfrak{g}^*$ is the differential of α in the cochain complex of the Lie algebra $\mathfrak{g}$ with coefficients in k.

Its $\mathfrak{g}$-operation is defined by making $\iota(X)$ (for $X\in\mathfrak{g}$) operate as the obvious anti-derivation on $\wedge\mathfrak{g}^*$ and trivially on $S\mathfrak{g}^*$.

Consider the bidegree

$$W^{2p,q}(\mathfrak{g})=\wedge^q\mathfrak{g}^*\otimes S^p\mathfrak{g}^* \quad \text{for } q, p>0.$$

Then we have a natural filtration $W=F^0\supset F^1\supset F^2\supset\cdots$, where $F^s=\sum_{2p\geqq s}W^{2p,*}$, which is compatible with the differential d.

The associated graded module is

$$E_0^s(\mathfrak{g})=\begin{cases}\wedge^*\mathfrak{g}^*\otimes S^p\mathfrak{g}^* & \text{for } s=2p, p=0,1,2,\cdots\\ 0 & \text{otherwise.}\end{cases}$$

By the calculation we have

$$E_1^s(\mathfrak{g})=\begin{cases}H^*(\mathfrak{g}:S^p(\mathfrak{g}^*)) & \text{for } s=2p, p=0,1,2,\cdots\\ 0 & \text{otherwise.}\end{cases}$$

Now we consider the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\boldsymbol{R}^n$ and its maximal abelian subalgebra

$$T=\{\lambda_1D_1+\cdots+\lambda_nD_n\,;\,\lambda_i\in k\} \quad \text{where } D_i=\partial/\partial x_i.$$

Lemma 2.2. *The continuous dual space $\mathfrak{a}_n^*$ and the m-th tensor product $\otimes^m\mathfrak{a}_n^*$ $(m>0)$ of $\mathfrak{a}_n^*$ are free $U(T)$-modules.*

The proof of this lemma can be found in [12].

Since $\wedge^q\mathfrak{a}_n^*\otimes S^p\mathfrak{a}_n^*$ $(q+p>0)$ is a direct summand of $\otimes^{q+p}\mathfrak{a}_n^*$, we get the following corollary:

Corollary 2.3. *$E_0^{2p,q}(\mathfrak{a}_n)$ is a projective $U(T)$-module where $p+q>0$.*

It is easy to check that the $\mathfrak{a}_n$ cochain algebra structure of $W(\mathfrak{a}_n)$ induces a T-cochain complex structure on $E_0^{2p}(\mathfrak{a}_n)$. Hence from Proposition 1.3 and corollary 2.3 we have the following proposition:

Proposition 2.4.

$$E_1^{2p,q}(\mathfrak{a}_n)=H^q(\mathfrak{a}_n;S^p(\mathfrak{a}_n^*))=0 \quad for\ p>0, q<n.$$

Remark. Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra. The following fact is well-known:

$$E_1^{2*,q}(\mathfrak{g})=H^q(\mathfrak{g})\otimes I^*(\mathfrak{g})$$

where $I^*(\mathfrak{g})$ is the algebra of polynomials on $\mathfrak{g}$ invariant under coadjoint operation. In the case of $\mathfrak{a}_n$, it seems that the following holds:

$$E_2^{2p,*}(\mathfrak{a}_n)=H^{*-2n}(\mathfrak{gl}(n,\boldsymbol{R}))\otimes I^{p+n}\mathfrak{g}(\mathfrak{l}(n,\boldsymbol{R})).$$

When $p=1$, this is true (see [4]).

§ 3. The Cohomology of Formal G-invariant Vector Fields

In this section we recall the result obtained in [7] where a similar type of cohomology was studied.

First we recall a few facts about topological vector spaces over discrete fields, which are useful in studying infinite dimensional Lie algebras.

Let Δ be a topological field with the discrete topology. We say that a topological vector space E over Δ is *linearly compact* when E is a projective limit of finite dimensional discrete vector spaces.

Let E be a topological vector spaces over Δ, and E^* the topological dual of E. We topologize E^* by prescribing, for a system of neighborhoods of the origin, the collection of all sets of the form $F^{\perp}$, where F is a linearly compact subspace of E and $F^{\perp}$ is its annihilator in E^*.

Let E and F be topological vector spaces. Consider the ordinary tensor product $E^*\otimes F^*$ and give it the discrete topology. We define the topological tensor product of E and F to be the space $(E^*\otimes F^*)^*$, which will be denoted by $E\hat{\otimes}F$. We note that when E and F are linearly compact, so is $E\hat{\otimes}F$(see [5]).

Now we recall a formal G-invariant vector fields. Let $\mathfrak{g}$ be a linearly compact Lie algebra; that is, a topological Lie algebra and linearly compact as a topological vector space. Consider the direct sum $\mathfrak{a}_n\oplus \boldsymbol{R}[[x]]\hat{\otimes}\mathfrak{g}$, denoted by $\mathfrak{a}_{n,\mathfrak{g}}$, where $\boldsymbol{R}[[x]]$ is the ring of all formal power series in n-variables over $\boldsymbol{R}$ and linearly compact with respect to the Krull topology. From definition, the canonical action of $\mathfrak{a}_n$ on $\boldsymbol{R}[[x]]$ induces the action of $\mathfrak{a}_n$ on $\boldsymbol{R}[[x]]\hat{\otimes}\mathfrak{g}$. We define the bracket operation as follows:

$$[(X_1, H_1), (X_2, H_2)]=([X_1, X_2], X_1H_2-X_2H_1-[H_1, H_2])$$

where $X_i\in\mathfrak{a}_n$, $H_i\in\boldsymbol{R}[[x]]\hat{\otimes}\mathfrak{g}$ and the bracket $[H_1, H_2]$ is induced by the bracket operation $\mathfrak{g}\hat{\otimes}\mathfrak{g}\to\mathfrak{g}$. Hence we may give $\mathfrak{a}_{n,\mathfrak{g}}$ the structure of linearly compact Lie algebra.

Before we state the fact concerning the cohomology of $\mathfrak{a}_{n,\mathfrak{g}}$ we review a notation. Define:

$$W_n(\mathfrak{g})\cong W(\mathfrak{g})/W\cdot S^{n+1}(\mathfrak{g}^*)$$

where $W\cdot S^{n+1}(\mathfrak{g}^*)$ is the ideal of Weil algebra $W(\mathfrak{g})$ generated by the $(n+1)$-*th* symmetric product space $S^{n+1}(\mathfrak{g}^*)$. Let $\mathfrak{gl}_n$ be the Lie algebra of all $n\times n$ real matrices.

Let $\pi:\mathfrak{a}_{n,\mathfrak{g}}\longrightarrow\mathfrak{gl}_n\oplus\mathfrak{g}$ be the projection defined by

$$\mathfrak{a}_n\ni\textstyle\sum_{i=1}^n(a^i+a^i_jx^j+(\text{higher order}))\ \partial/\partial x^i\longrightarrow(-a^i_j)_{i,j}\in\mathfrak{gl}_n,$$

and

$$\varepsilon\hat{\otimes}id:\boldsymbol{R}[[x_1,\cdots,x_n]]\hat{\otimes}\mathfrak{g}\longrightarrow\boldsymbol{R}\otimes\mathfrak{g}=\mathfrak{g}$$

where ε is a canonical projection. Then π induces a cochain map

$$\Phi(\pi):W_n(\mathfrak{gl}_n\oplus\mathfrak{g})\longrightarrow C^*(\mathfrak{a}_{n,\mathfrak{g}})$$

where $C^*(\mathfrak{a}_{n,\mathfrak{g}})$ is a cochain complex of $\mathfrak{a}_{n,\mathfrak{g}}$ with values in $\boldsymbol{R}$. (see Hamasaki [7, p. 408]).

Proposition 3.1. *If* $\mathfrak{g}$ *is a linearly compact Lie algebra, then* π *induces an isomorphism*

$$\Phi(\pi)^* : H^*(W_n(\mathfrak{gl}_n \oplus \mathfrak{g})) \cong H^*(\mathfrak{a}_{n,\mathfrak{g}}).$$

Remark. In [7] this proposition was proved on condition that $\mathfrak{g}$ is a finite dimensional Lie algebra. But the finite dimensionality is not essential. We need the following two conditions:

i) $(\boldsymbol{R}[[x]]\hat{\otimes}\mathfrak{g})^* \cong \boldsymbol{R}[[x]]^* \otimes \mathfrak{g}^*$,

ii) with respect to a basis $\{\omega_i\}_{i\in I}$ of $\mathfrak{g}^*$, there is a family $\{\xi_i \in \mathfrak{g}\,; i\in I\}$ such that $\omega_i(\xi_j)=1$ $(i=j)$, 0 (otherwise).

It is known that, when E is linearly compact or discrete, $E\cong E^{**}$ (see [5]). Since $\boldsymbol{R}[[x]]^*\otimes\mathfrak{g}^*$ is discrete,

$$(\boldsymbol{R}[[x]]\hat{\otimes}\mathfrak{g})^* \cong ((\boldsymbol{R}[[x]]^*\otimes\mathfrak{g}^*)^*)^* \cong \boldsymbol{R}[[x]]^*\otimes\mathfrak{g}^*.$$

Since $\mathfrak{g}^*$ is discrete, we can find $\{\xi_i\in\mathfrak{g}^{**}\,; i\in I\}$ such that $\omega_i(\xi_j) = 1$ $(i=j)$, 0 (otherwise). Hence above two conditions are satisfied.

§ 4. The Main Theorem

In this section we will state the main theorem and prove it.

Let $\iota : \mathfrak{a}_r = \boldsymbol{R}[[x]]\otimes\boldsymbol{R}^r \longrightarrow \mathfrak{a}_{r,n-r} = \boldsymbol{R}[[x]]\otimes\boldsymbol{R}^r \oplus \boldsymbol{R}[[x,y]]\otimes\boldsymbol{R}^{n-r}$ be the inclusion map to the first factor of the direct sum.

Theorem 4.1. *The inclusion map ι induces an isomorphism of cohomology*

$$\iota^* : H^i(\mathfrak{a}_{r,n-r}) \cong H^i(\mathfrak{a}_r) \quad \textit{for } i\leqq n+r.$$

First consider the Lie algebra $\mathfrak{a}_{r,\mathfrak{a}_{n-r}} = \mathfrak{a}_r \oplus \boldsymbol{R}[[x]]\hat{\otimes}\mathfrak{a}_{n-r}$ $(x=(x^1,\dots,x^r))$ defined in § 3. Since $\boldsymbol{R}[[x]]\hat{\otimes}\boldsymbol{R}[[y]] = \boldsymbol{R}[[x,y]]$ $(y=(y^1,\dots,y^{n-r}))$, $\mathfrak{a}_{r,\mathfrak{a}_{n-r}}$ is isomorphic to $\mathfrak{a}_{r,n-r}$. Using this fact we consider the following commutative diagram concerning the canonical projections:

$$\begin{array}{ccc} \mathfrak{a}_{r,n-r} & \xrightarrow{\kappa} & \mathfrak{a}_r \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathfrak{gl}_r\oplus\mathfrak{a}_{n-r} & \longrightarrow & \mathfrak{gl}_r \end{array}$$

where κ is a canonical projection which is a left inverse of ι and π, π' are projection introduced in § 3. By the naturality of construction we obtain the following commutative diagram:

$$\begin{array}{ccc} W_r(\mathfrak{gl}_r) & \longrightarrow & W_r(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r}) \\ \downarrow & & \downarrow \\ C^*(\mathfrak{a}_r) & \xrightarrow{\kappa^*} & C^*(\mathfrak{a}_{r,n-r}) \end{array}$$

where $C^*(\mathfrak{a})$ is the continuous cohomology of $\mathfrak{a}$. On the other hand by the Proposition 3. 1, we have:

Lemma 4. 2.

$$H^*(\mathfrak{a}_{r,n-r}) \cong H^*(W_r(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r})).$$

Taking cohomology of the above diagram we have the following commutative diagram:

$$\begin{array}{ccc} H^*(W_r(\mathfrak{gl}_r)) & \longrightarrow & H^*(W_r(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r})) \\ \downarrow \cong & & \downarrow \cong \\ H^*(\mathfrak{a}_r) & \longrightarrow & H^*(\mathfrak{a}_{r,n-r}). \end{array} \tag{4. 3}$$

Hence in order to prove the theorem we consider the relation between the truncated Weil algebras $W_r(\mathfrak{gl}_r)$ and $W_r(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r})$.

Note that $W(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r}) \cong \wedge^*(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r})^* \otimes S^*(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r})^* \cong \wedge^* \mathfrak{gl}_r^* \otimes S^* \mathfrak{gl}_r^* \otimes \wedge^* \mathfrak{a}_{n-r}^* \otimes S^* \mathfrak{a}_{n-r}^* \cong W(\mathfrak{gl}_r) \otimes W(\mathfrak{a}_{n-r})$. Since the ideal in $W(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r})$ generated by $S^{r+1}(\mathfrak{gl}_r^*)$ is contained in the ideal generated by $S^{r+1}(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r})^*$, we have a canonical homomorphism and a commutative diagram

$$\begin{array}{ccc} \Phi: W_r(\mathfrak{gl}_r) \otimes W(\mathfrak{a}_{n-r}) & \longrightarrow & W_r(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r}). \\ \uparrow & \nearrow & \\ W_r(\mathfrak{gl}_r) & & \end{array}$$

Lemma 4. 4.

$$\Phi^*: H^i(W_r(\mathfrak{gl}_r) \otimes W(\mathfrak{a}_{n-r})) \longrightarrow H^i(W_r(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r}))$$

is an isomorphism for $i \leqq n+r$.

Proof. The truncated Weil algebra $W_r(\mathfrak{gl}_r) \otimes W(\mathfrak{a}_{n-r})$ has a natural filtration induced by that of the Weil algebra $W(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r})$. By the calculation we have

$$E_2^{2p+1,q} = 0,$$
$$E_2^{2p,q} = \sum_{j \in Z, j \leqq r} H^i(\mathfrak{gl}_r; S^j(\mathfrak{gl}_r)) \otimes H^{q-i}(\mathfrak{a}_{n-r}; S^{p-j}(\mathfrak{a}_{n-r})).$$

Consider the case that $p>r$ and $q<n-r$. Then $p-j\geqq p-r>0$ and $q-i<n-r$ or $i<0$. From Proposition 2.3, in this case

$$E_2^{2p,q}=0 \text{ for } p>r \text{ and } q<n-r.$$

On the other hand the truncated Weil algebra $W_r(\mathfrak{gl}_r\oplus\mathfrak{a}_{n-r})$ also has a natural filtration. The E_2-term of the corresponding spectral sequence is

$$\begin{cases} {}'E_2^{2p,q}=\sum_{i,j\in z}H^i(\mathfrak{gl}_r\ ;S^j(\mathfrak{gl}_r))\otimes H^{q-i}(\mathfrak{a}_{n-r}\ ;S^{p-j}(\mathfrak{a}_{n-r})) & \text{for } p\leqq r, \\ {}'E_2^{p,q}=0 & \text{when } p \text{ is odd or } p>2r. \end{cases}$$

By the construction, Φ preserves the filtrations. Hence we have a homomorphism of spectral sequences where

$$\Phi:(E,d)\longrightarrow({}'E,{}'d),$$
$$E\Rightarrow H(W_r(\mathfrak{gl}_r)\otimes W(\mathfrak{a}_{n-r}))$$
$${}'E\Rightarrow H(W_r(\mathfrak{gl}_r\oplus\mathfrak{a}_{n-r})).$$

Note that

(4.5) $\qquad E_{2k-1}\cong E_{2k}$ and ${}'E_{2k-1}\cong{}'E_{2k}$ for $k\geqq 1$.

In order to prove the lemma, we shall show the following by induction:

$$(*_k)\ \Phi_{2k}^{p,q}: E_{2k}^{p,q}\longrightarrow{}'E_{2k}^{p,q}$$

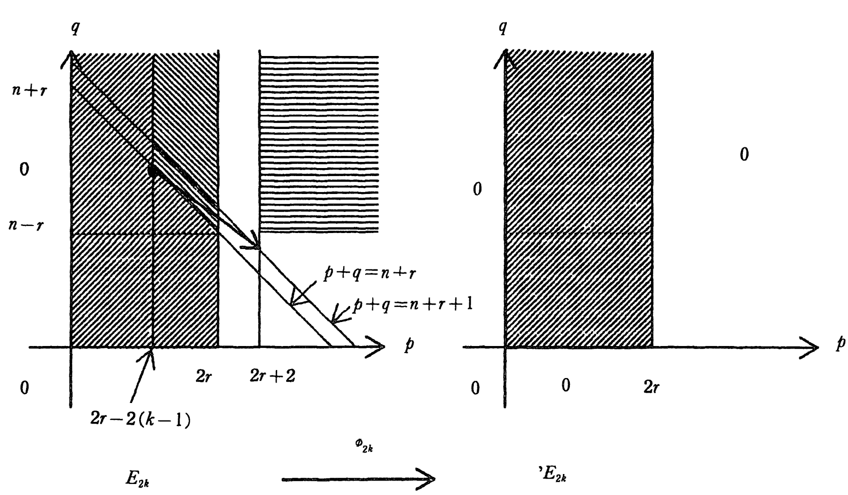

Figure 2

is a monomorphism for $p \leqq 2r$,
and an isomorphism for $p \leqq 2r-2(k-1)$ or $p+q \leqq n+r$.

When $k=1$, $\Phi_2^{p,q}$ is an identity map for $p \leqq 2r$ or $p+q \leqq n+r$. Assume that $(*_k)$ holds. Consider the first case i. e. $p \leqq 2r$. Then $\Phi_{2k}^{p,q}$ is a monomorphism, and $\Phi_{2k}^{p-2k,q+2k-1}$ is an isomorphism because $p-2k \leqq 2r-2(k-1)$.

$$\begin{array}{ccccc} E_{2k}^{p-2k,q+2k-1} & \xrightarrow{d_{2k}} & E_{2k}^{p,q} & \xrightarrow{d_{2k}} & E_{2k}^{p+2k,q-2k+1} \\ \downarrow \Phi_{2k}^{p-2k,q+2k-1} & & \downarrow \Phi_{2k}^{p,q} & & \downarrow \Phi_{2k}^{p+2k,q-2k+1} \\ {}'E_{2k}^{p-2k,q+2k-1} & \xrightarrow{'d_{2k}} & {}'E_{2k}^{p,q} & \xrightarrow{'d_{2k}} & {}'E_{2k}^{p+2k,q-2k+1} \end{array}$$

By diagram chasing and considering (4. 5) we can see that

$$\Phi_{2(k+1)}^{p,q} : E_{2(k+1)}^{p,q} \longrightarrow E_{2(k+1)}^{p,q}$$

is a monomorphism for $p \leqq 2r$. Next consider the second case $p \leqq 2r-2\{(k+1)-1\}$. Then $\Phi_{2k}^{p,q}$ and $\Phi_{2k}^{p-2k,q+2k-1}$ are isomorphisms, and $\Phi_{2k}^{p+2k,q-2k+1}$ is monomorphism because $p+2k \leqq 2r$. By diagram chasing and considering (4. 5) we can see that $\Phi_{2(k+1)}^{p,q}$ is an isomorphism for $p \leqq 2r-2\{(k+1)-1\}$. Next consider the last case, i. e. $p+q \leqq n+r$. Then $\Phi_{2k}^{p,q}$ and $\Phi_{2k}^{p-2k,q+2k-1}$ are isomorphisms. If $p+2k > 2r$ and $q-2k+1 < n-r$, then $E_{2k}^{p+2k,q-2k+1} \cong {}'E_{2k}^{p+2k,q-2k+1} \cong 0$. Hence $\Phi_{2k}^{p+2k,q-2k+1}$ is monomorphism. When $q-2k+1 \geqq n-r$, considering $p+q \leqq n+r$, we have

$$p+2k+n-r \leqq p+2k+q-2k+1 \leqq n+r+1.$$

Hence $p+2k \leqq 2r+1$. Since $E_{2k}^{2r+1,*} \cong {}'E_{2k}^{2r+1,*} \cong 0$ and $(*_k)$, $\Phi_{2k}^{p+2k,q-2k+1}$ is monomorphism. In the same way as the second case we see that $\Phi_{2(k+1)}^{p,q}$ is an isomorphism for $p+q \leqq n+r$. This proves $(*_{k+1})$. Hence $(*_k)$ holds for any positive integer k. Using this, now we prove the Lemma. Consider the E_∞-term. Then we see that

$$\Phi_\infty^{p,q} : E_\infty^{p,q} \longrightarrow {}'E_\infty^{p,q}$$

is an isomorphism for $p+q \leqq n+r$. Since each spectral sequence converges, we see that

$$\Phi^* : H^i(W_r(\mathfrak{gl}_r) \otimes W(\mathfrak{a}_{n-r})) \longrightarrow H^i(W_r(\mathfrak{gl}_r \oplus \mathfrak{a}_{n-r}))$$

is an isomorphism for $i \leqq n+r$. This completes the proof of Lemma 4. 4.

It is well-known that the Weil algebra is acyclic (cf. Natsume

[10]). By the Kunneth formula, we have

$$H^i(W_r(\mathfrak{gl}_r)\otimes W(\mathfrak{a}_{n-r}))\cong H^i(W_r(\mathfrak{gl}_r)).$$

By the diagram (4. 3) we obtain

$$H^i(\mathfrak{a}_r)\cong H^i(\mathfrak{a}_{r,n-r}) \quad \text{for } i\leqq n+r.$$

This completes the proof of the main Theorem 4. 1.

§ 5. Some Remarks

In this section we shall give a result and a conjecture concerning the Weil algebra of an infinite dimensional Lie algebra.

The structure of the Weil algebra $W(\mathfrak{g})$ of a finite dimensional reductive Lie algebra $\mathfrak{g}$ is almost completely determined. By contraries there seems to be no study of the infinite dimensional case but [10] where the cohomology $H^*(W(\mathfrak{a}_n), \mathfrak{gl}_n)$ of $\mathfrak{gl}_n$-basic subcochain algebra of $W(\mathfrak{a}_n)$ was determined by calculating the spectral sequences. The result is

$$H^*(W(\mathfrak{a}_n), \mathfrak{gl}_n)\cong I^*(\mathfrak{gl}_n)$$

where $I^*(\mathfrak{gl}_n)$ is the algebra of polynomials on $\mathfrak{g}$ invariant under coadjoint operation by $\mathfrak{g}$. Note that we can deduce this result using the following Theorem (see Kamber and Tondeur [8, Theorem 5. 64]) :

Theorem 5. 1. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a reduced pair of Lie algebras and* θ: $\mathfrak{g}\to\mathfrak{h}$ *an equivariant splitting of the exact h-module sequence* $0\to\mathfrak{h}\to\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}\to 0$. *Let* $K(\theta): w(\mathfrak{h})\to w(\mathfrak{g})$ *be the Weil homomorphism. Then the induced map on* $\mathfrak{h}$*-basic elements*

$$K(\theta)_{\mathfrak{h}}: I(\mathfrak{h})\to W(\mathfrak{g})_{\mathfrak{h}-basic}$$

is a homotopy equivalence.

Since the pair $(\mathfrak{a}_n, \mathfrak{gl}_n)$ satisfies above conditions (see [7]), we have

$$K(\theta)^*_{\mathfrak{gl}_n}: I^*(\mathfrak{gl}_n)\cong H^*(W(\mathfrak{a}_n), \mathfrak{gl}_n).$$

We can also apply the above theorem to the pair $(H_{2n}, \mathfrak{sp}_{2n})$ where H_{2n} is the Lie algebra of Hamiltonian vector fields on $\mathbf{R}^{2n}$ and $\mathfrak{sp}_{2n}$

is the Lie algebra of the Symplectic group.

Even if the total cohomology is calculated, we are rather interested in the E_2-term of the spectral sequence converging to $H^*(W(\mathfrak{a}_n), \mathfrak{gl}_n)$. Consider the filtration of $W(\mathfrak{a}_n)_{\mathfrak{gl}_n\text{-}basic}$ induced by the one of $W(\mathfrak{a}_n)$ studied in § 2. By the similar calculation we have

$$E_2^{p,1} \cong H^*(\mathfrak{a}_{n,\mathfrak{gl}_n}; S^p(\mathfrak{a}_n)).$$

In [3] and [4], the following was calculated:

$$E_2^{0,*} \cong H^*(\mathfrak{a}_{n,\mathfrak{gl}_n}; \boldsymbol{R}) \cong I^*(\mathfrak{gl}_n)/deg>2n,$$

$$E_2^{2,q} \cong H^q(\mathfrak{a}_{n,\mathfrak{gl}_n}; \mathfrak{a}_n^*) \cong \begin{cases} I^{n+1}(\mathfrak{gl}_n) & q=2n, \\ 0 & \text{otherwise.} \end{cases}$$

Using Proposition 2. 3 we can easily see that

$$E_2^{2p,q} \cong H^q(\mathfrak{a}_n, \mathfrak{gl}_n; S^p(\mathfrak{a}_n)) \cong 0 \qquad \text{for } p>0, q<n.$$

The vast range remains unknown. But studying the Theorem 5. 1 it seems that $I^{n+i} = E_2^{2i,2n}$.

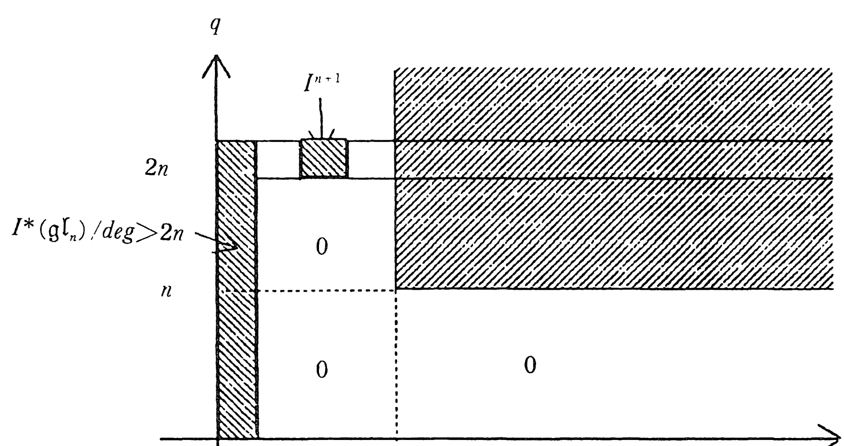

Figure 3

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