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On the Mod p J-Homology of Complex Projective Space

Dedicated to Professor Hirosi Toda on his 60th birthday

By

Takashi WATANABE*

§ 1. Introduction

For each odd prime p there is a connective spectrum J which is called the image of J spectrum (see [3] or [7, 1.5]). Let $\tilde{J}_*(CP^\infty)$ be the reduced J-homology of an infinite-dimensional complex projective space CP^∞ and let $\tilde{J}_*(CP^\infty; Z/(p))$ be that with $Z/(p)$ coefficients. They are rings, since J is a ring spectrum and CP^∞ is an H-space. Knapp [6] showed that for every $n \geq 1$, $\tilde{J}_{2n}(CP^\infty) = Z_{(p)}$, the ring of integers localized at p, and $\tilde{J}_{2n-1}(CP^\infty)$ is a finite direct sum of cyclic p-groups. He also described its order and the number of direct summands in it. Therefore the additive structure of $\tilde{J}_*(CP^\infty; Z/(p))$ is known. After that, using complex K-theory, Schwartz [9] determined the multiplicative structure of the even dimensional part of $\tilde{J}_*(CP^\infty; Z/(p))$.

The purpose of this paper is to give an explicit $Z/(p)$-basis for the odd dimensional part of $\tilde{J}_*(CP^\infty; Z/(p))$. The result is stated in Theorem 9. We also give an alternative description of Schwartz' generators in Proposition 12. More precisely, we will compute the kernel and the cokernel of a certain operation in the reduced, connective, first Morava K-homology groups $\tilde{k}(1)_*(CP^\infty)$ (see [7]). In fact, if $SZ/(p)$ denotes the Moore spectrum of type $Z/(p)$, then $JZ/(p) = J \wedge SZ/(p)$ is the fiber of a map $\theta: k(1) \to \Sigma^{2(p-1)} k(1)$ which induces the above operation and comes from a suitable Adams operation in $k(1)$-theory (see §2).

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* Department of Mathematics, Osaka Women's University, Osaka 590, Japan.

The paper is organized as follows. In §2 we describe the spectrum J briefly and collect basic results about $\tilde{k}(1)_*(CP^\infty)$. From them we derive some data on the behavior of the operation θ in $\tilde{k}(1)_*(CP^\infty)$. In §3 we introduce several functions from the ring of integers $\boldsymbol{Z}$ into itself for use of the next section. §4 is devoted to prove Theorem 9. In §5 we review the result of Schwartz. Proposition 12 is proved there.

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§ 2. The $k(1)$-Homology of CP^∞

For the most part we follow the notation and terminology of [7].

Let $BP\langle 1\rangle$ be the first Johnson-Wilson spectrum at an odd prime p, which is a summand of the localization at p of the spectrum representing connective complex K-theory (see [5]). Then $k(1)=BP\langle 1\rangle\wedge S\boldsymbol{Z}/(p)$ represents connective first Morava K-theory. Both $BP\langle 1\rangle$ and $k(1)$ are complex oriented ring spectra, and their coefficient rings are

$$\pi_*(BP\langle 1\rangle)=\boldsymbol{Z}_{(p)}[v]$$

and

$$\pi_*(k(1))=\boldsymbol{Z}/(p)[v]$$

respectively, where the degree of v is $2(p-1)$. The following lemma is well known (e.g., see [7, 4.1]).

Lemma 1. *Let E be a complex oriented ring spectrum. Write $E_*=E_*(pt.)$ and $E^*=E^*(pt.)$. Then*

(a) *There is an element $x\in E^2(CP^\infty)$ such that $E^*(CP^\infty)=E^*[[x]]$ and $E^*(CP^\infty\times CP^\infty)=E^*[[x\otimes 1, 1\otimes x]]$.*

(b) *There is a formal group law F_E over E^* which is defined by $\mu^*(x)=F_E(x\otimes 1, 1\otimes x)$, where $\mu: CP^\infty\times CP^\infty\to CP^\infty$ is the H-space structure map.*

(c) *$E_*(CP^\infty)$ is a free E_*-module on $\beta_n\in E_{2n}(CP^\infty)$, $n\geq 0$, dual to x^n; that is, $\langle x^m, \beta_n\rangle=\delta_{mn}$ where $\langle\ ,\ \rangle$ denotes the Kronecker product.*

We recall the definition of Adams operations in $BP\langle 1\rangle$-theory.

According to Araki [2], the logarithm for $F_{BP\langle 1\rangle}$ is

$$\log(x)=\sum_{i\geq 0} p^{-i}v^{1+p+\cdots+p^{i-1}}x^{p^i}$$

(in $BP\langle 1\rangle^*(\boldsymbol{CP}^\infty;\boldsymbol{Q})$). For $q\in\boldsymbol{Z}$ prime to p, there is a map ψ^q: $BP\langle 1\rangle\to BP\langle 1\rangle$ satisfying the following properties:

$$\text{(2.1)}\qquad \begin{array}{ll}\text{(i)} & \log(q\cdot\psi^q(x))=q\cdot\log(x);\\ \text{(ii)} & \psi^q \textit{ is multiplicative; and}\\ \text{(iii)} & \psi^q(v)=q^{p-1}v.\end{array}$$

By Lemma 1(a), in $B\tilde{P}\langle 1\rangle^*(\boldsymbol{CP}^\infty)$ we may write

$$\text{(2.2)}\qquad \psi^q(x^m)=\sum_{i\geq 0} b_{m,i}v^i x^{m+i(p-1)}$$

for some $b_{m,i}\in\boldsymbol{Z}_{(p)}$. In the case $m=1$ we have

$$\text{(2.3)}\qquad \begin{array}{l} b_{1,0}=1,\\ b_{1,1}=-(q^{p-1}-1)/p \textit{ and}\\ b_{1,2}=q^{p-1}(q^{p-1}-1)/p\end{array}$$

(cf. [4, p. 379]). This follows by comparing both sides of the equation (2.1)(i). Since $\psi^q(x^m)=\psi^q(x)^m$ by (2.1)(ii), it follows that for $m\geq 2$

$$\text{(2.4)}\qquad \begin{array}{l} b_{m,0}=1,\\ b_{m,1}=m\cdot b_{1,1} \textit{ and}\\ b_{m,2}=m\cdot b_{1,2}+2^{-1}m(m-1)\cdot b_{1,1}^2.\end{array}$$

From now on, for each odd prime p, we choose q so that its image under the reduction $\boldsymbol{Z}\to\boldsymbol{Z}/(p^2)$ generates the multiplicative group of units in $\boldsymbol{Z}/(p^2)$, and use such a q. It is known that if $q\not\equiv 0 \mod(p)$ then $q^{p-1}\equiv 1 \mod(p)$, so

$$q^{p-1}-1=cp$$

for some $c\in\boldsymbol{Z}$. Our choice of q is equivalent to the assertion that $c\not\equiv 0 \mod(p)$.

By [5] there is a fibration

$$\Sigma^{2(p-1)}BP\langle 1\rangle\xrightarrow{\ v\ }BP\langle 1\rangle\xrightarrow{\ \rho\ }H\boldsymbol{Z}_{(p)}$$

where $H\boldsymbol{Z}_{(p)}$ denotes the integral Eilenberg-MacLane spectrum localized at p. Then the map $\psi^q-1\colon BP\langle 1\rangle\to BP\langle 1\rangle$ can be lifted to a unique map $\theta\colon BP\langle 1\rangle\to\Sigma^{2(p-1)}BP\langle 1\rangle$. Let J be the fiber of θ. Thus one has a fibration

$$J \xrightarrow{\eta_\theta} BP\langle 1\rangle \xrightarrow{\theta} \Sigma^{2(p-1)} BP\langle 1\rangle$$

where η_θ is known to be multiplicative (see [10]). Smashing $S\boldsymbol{Z}/(p)$ with this fibration gives rise to a fiber sequence

$$(2.5)\qquad \Sigma^{2p-3}k(1) \xrightarrow{\delta_\theta} J\boldsymbol{Z}/(p) \xrightarrow{\eta_\theta} k(1) \xrightarrow{\theta} \Sigma^{2(p-1)}k(1).$$

By (ii) and (iii) of (2.1), $(\psi^q-1)(v^i)=(q^{i(p-1)}-1)v^i$ in $\pi_*(BP\langle 1\rangle)$. Therefore by the definition of θ

$$(2.6)\qquad \theta(v^i)=0 \ \ in\ \ \pi_*(k(1)).$$

In general, if X is an H-space, then $k(1)_*(X)$ is a ring and by (2.1)(ii) the operation $\psi^q\colon k(1)_i(X)\to k(1)_i(X)$ is a ring homomorphism. Therefore the operation $\theta\colon k(1)_i(X)\to k(1)_{i-2(p-1)}(X)$ satisfies a formula

$$(2.7)\qquad \theta(\alpha\cdot\beta)=\theta(\alpha)\cdot\beta+\alpha\cdot\theta(\beta)+v\cdot\theta(\alpha)\cdot\theta(\beta)$$

for all α, $\beta\in k(1)_i(X)$, where $\cdot$ denotes the Pontrjagin product (cf. [10]). Take $X=\boldsymbol{C}P^\infty$. It follows from Lemma 1(c) and degree considerations that

$$(2.8)\qquad \tilde{k}(1)_i(\boldsymbol{C}P^\infty)=\begin{cases}\boldsymbol{Z}/(p)\ \{v^j\beta_{n-j(p-1)}\mid 0\le j\le n'\} & \text{if } i=2n \text{ and } n\ge 1\\ 0 & otherwise\end{cases}$$

where $n'=[(n-1)/(p-1)]$, the largest integer not exceeding $(n-1)/(p-1)$. Combining this with (2.5) yields exact sequences

$$(2.9)\qquad \begin{aligned}&0\longrightarrow \tilde{J}_{2n}(\boldsymbol{C}P^\infty;\boldsymbol{Z}/(p)) \xrightarrow{\eta_\theta} \tilde{k}(1)_{2n}(\boldsymbol{C}P^\infty)\\ &\xrightarrow{\theta} \tilde{k}(1)_{2n-2(p-1)}(\boldsymbol{C}P^\infty) \xrightarrow{\delta_\theta} \tilde{J}_{2n-1}(\boldsymbol{C}P^\infty;\boldsymbol{Z}/(p)) \longrightarrow 0\end{aligned}$$

for all $n\in\boldsymbol{Z}$. It follows from (2.6) and (2.7) that

$$(2.10)\qquad \theta(v^i\beta_j)=v^i\cdot\theta(\beta_j)$$

for all $i, j\ge 0$.

Let n be a positive integer. Multiplication by n in $\boldsymbol{Z}$ induces on $\boldsymbol{C}P^\infty=K(\boldsymbol{Z},2)$ the map $n\colon \boldsymbol{C}P^\infty\to\boldsymbol{C}P^\infty$ which is factored as a composition

$$\boldsymbol{C}P^\infty \xrightarrow{\Delta} \underbrace{\boldsymbol{C}P^\infty\times\ldots\times\boldsymbol{C}P^\infty}_{n} \xrightarrow{\mu'} \boldsymbol{C}P^\infty$$

where Δ is the diagonal and μ' the iterated multiplication. On the other hand, with the notation of Lemma 1(b), let $[1]_F(x)=x$ and inductively let $[n]_F(x)=F([n-1]_F(x),x)$ where $F=F_E$. Then the induced homomorphism $n^*: E^2(CP^\infty)\to E^2(CP^\infty)$ sends x to $[n]_F(x)$. Specializing to the case $E=k(1)$ and $n=p$, we have the following important result from [2, Corollary 11. 4] (or [8, Theorem 5. 5 with $n=1$]).

Theorem 2. *$p^*(x)=vx^p$ in $k(1)^2(CP^\infty)$.*

Corollary 3. *For $n\geq 1$ $p_*: \tilde{k}(1)_{2n}(CP^\infty)\to\tilde{k}(1)_{2n}(CP^\infty)$ is given by*

$$p_*(\beta_n)=\begin{cases}v^i\beta_i & \text{if } n=ip \text{ for some } i\geq 1\\ 0 & \text{otherwise.}\end{cases}$$

Proof. For $i\geq 1$ we have

$$\begin{aligned}\langle x^i, p_*(\beta_n)\rangle &=\langle p^*(x^i), \beta_n\rangle\\ &=\langle p^*(x)^i, \beta_n\rangle\\ &=\langle (vx^p)^i, \beta_n\rangle \qquad \text{by Theorem 2}\\ &=\langle v^i x^{ip}, \beta_n\rangle\\ &=\begin{cases}v^i & \text{if } n=ip\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

By Lemma 1 (c), in $B\tilde{P}\langle 1\rangle_*(CP^\infty)$ we may write

$$\psi^q(\beta_n)=\sum_{j=0}^{n'} c_{n,j}v^j\beta_{n-j(p-1)} \tag{2.11}$$

for some $c_{n,j}\in Z_{(p)}$. Then in $\tilde{k}(1)_*(CP^\infty)$ we have

$$\theta(\beta_n)=\sum_{j=1}^{n'} c_{n,j}v^{j-1}\beta_{n-j(p-1)} \tag{2.12}$$

where $c_{n,j}\in Z/(p)$.

Lemma 4. *In $Z/(p)$*

(i) $c_{n,1}=c(n-(p-1))$ *where* $c=(q^{p-1}-1)/p$.

(ii) $c_{n,j}=c_{n/p,j/p}$ *if* $n\equiv 0 \bmod (p)$ *and* $j\equiv 0 \bmod (p)$.

(iii) $c_{n,j}=0$ *if* $n\not\equiv 0 \bmod (p)$ *and* $j\equiv -n \bmod (p)$.

(iv) $c_{n,2}\neq 0$ *if* $n\equiv -1 \bmod (p)$.

Proof. In general it follows from (2. 1) (ii) and [1, Lecture 3] that

$$\psi^q(\langle a, \alpha\rangle) = \langle \psi^q(a), \psi^q(\alpha)\rangle$$

for all $a \in BP\langle 1\rangle^*(X)$ and $\alpha \in BP\langle 1\rangle_*(X)$. Apply this formula to the case $X = \boldsymbol{C}P^\infty$ by setting $a = x^m$ and $\alpha = \beta_n$. Then the left-hand side is

$$\psi^q(\langle x^m, \beta_n\rangle) = \begin{cases} \psi^q(1) & \text{if } m=n \\ \psi^q(0) & \text{otherwise} \end{cases} \qquad \text{by Lemma 1(c)}$$

$$= \begin{cases} 1 & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$$

and the right-hand side is

$$\begin{aligned} &\langle \psi^q(x^m), \psi^q(\beta_n)\rangle \\ &= \langle \sum_i b_{m,i} v^i x^{m+i(p-1)}, \sum_j c_{n,j} v^j \beta_{n-j(p-1)}\rangle \qquad \text{by (2.2) and (2.11)} \\ &= \sum_{i,j} b_{m,i} c_{n,j} v^{i+j} \langle x^{m+i(p-1)}, \beta_{n-j(p-1)}\rangle \\ &= \begin{cases} \sum_{j=0}^k b_{n-k(p-1),k-j} c_{n,j} & \text{if } m = n-k(p-1) \text{ and } k \geq 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $b_{n,0} c_{n,0} = 1$ and for $k \geq 1$

$$\sum_{j=0}^k b_{n-k(p-1),k-j} c_{n,j} = 0.$$

In particular for $k=1$

$$b_{n-p+1,1} c_{n,0} + b_{n-p+1,0} c_{n,1} = 0$$

and for $k=2$

$$b_{n-2p+2,2} c_{n,0} + b_{n-2p+2,1} c_{n,1} + b_{n-2p+2,0} c_{n,2} = 0.$$

By these equalities, (2.3) and (2.4), we have

$$\begin{aligned} c_{n,0} &= b_{n,0} = 1, \\ c_{n,1} &= -b_{n-p+1,1} = -(n-p+1) b_{1,1} = c(n-p+1) \end{aligned}$$

and

$$\begin{aligned} c_{n,2} &= -b_{n-2p+2,2} - b_{n-2p+2,1} c_{n,1} \\ &= -(n-2p+2) b_{1,2} - 2^{-1}(n-2p+2)(n-2p+1) b_{1,1}^2 \\ &\quad - (n-2p+2) b_{1,1} c(n-p+1) \\ &= (n-2p+2)\{-c(cp+1) - 2^{-1}c^2(n-2p+1) + c^2(n-p+1)\} \\ &= c(n-2p+2)\{-(cp+1) + 2^{-1}c(n+1)\} \end{aligned}$$

in $\boldsymbol{Z}_{(p)}$. Hence part (i) follows. For part (iv), suppose $n \equiv -1 \bmod (p)$.

Then in $Z/(p)$

$$c_{n,2}=c(-1+2)\{-1+2^{-1}c(-1+1)\}=-c.$$

Thus part (iv) follows from the choice of q.

To prove parts (ii) and (iii) we use a relation

$$p_*\theta(\beta_n)=\theta p_*(\beta_n)$$

in $\tilde{k}(1)_{2n}(CP^\infty)$, $n\geq 1$. First suppose $n\equiv 0 \mod (p)$, i.e., $n=ip$ for some $i\geq 1$. Then the left-hand side is

$$\begin{aligned} p_*\theta(\beta_{ip}) &= p_*(\sum_{k=1}^{(ip)'} c_{ip,k}v^{k-1}\beta_{ip-k(p-1)}) && \text{by (2.12)} \\ &= \sum_{k=1}^{(ip)'} c_{ip,k}v^{k-1}p_*(\beta_{ip-k(p-1)}) \\ &= \sum_{1\leq jp\leq (ip)'} c_{ip,jp}v^{jp-1}v^{i-j(p-1)}\beta_{i-j(p-1)} && \text{by Corollary 3} \\ &= \sum_{j=1}^{i'} c_{ip,jp}v^{i+j-1}\beta_{i-j(p-1)} \end{aligned}$$

and the right-hand side is

$$\begin{aligned} \theta p_*(\beta_{ip}) &= \theta(v^i\beta_i) && \text{by Corollary 3} \\ &= v^i\cdot\theta(\beta_i) && \text{by (2.10)} \\ &= v^i\sum_{j=1}^{i'} c_{i,j}v^{j-1}\beta_{i-j(p-1)} && \text{by (2.12)} \\ &= \sum_{j=1}^{i'} c_{i,j}v^{i+j-1}\beta_{i-j(p-1)}. \end{aligned}$$

By equating the coefficients of $v^{i+j-1}\beta_{i-j(p-1)}$ in both sides, part (ii) follows. Next suppose $n\not\equiv 0 \mod(p)$. Then the left-hand side is

$$\begin{aligned} p_*\theta(\beta_n) &= p_*(\sum_{j=1}^{n'} c_{n,j}v^{j-1}\beta_{n-j(p-1)}) && \text{by (2.12)} \\ &= \sum_{j=1}^{n'} c_{n,j}v^{j-1}p_*(\beta_{n-j(p-1)}) \\ &= \sum_{\substack{1\leq j\leq n' \\ n-j(p-1)=kp}} c_{n,j}v^{j+k-1}\beta_k && \text{by Corollary 3} \end{aligned}$$

and the right-hand side is equal to zero, by Corollary 3. Since $n-j(p-1)=kp$ for some k if and only if $j\equiv -n \mod(p)$, part (iii) follows.

Part (i) of this lemma determines a constant c in [10, Lemma 1.1].

§ 3. Several Functions

In this section we make some notation and conventions.

Let N be the set of positive integers and let M be the set of nonnegative integers. Throughout the rest of this paper we will use the letters

$$h, i, j, k, l, m, n, r, s, t$$

to denote integers.

For each l with $1 \leq l \leq p-1$, we define a function $f_l: \boldsymbol{Z} \to \boldsymbol{Z}$ by

$$f_l(m) = l + m(p-1)$$

for $m \in \boldsymbol{Z}$. Then every $n \in N$ can be uniquely expressed as $f_l(m)$ where $m \in M$. For each k with $1 \leq k \leq p$ and for each l as above, we define a function $g_{k,l}: \boldsymbol{Z} \to \boldsymbol{Z}$ by

$$g_{k,l}(i) = ip + k + l$$

for $i \in \boldsymbol{Z}$. For each $r \geq 0$ and for each l as above, we define a subset $M_{r,l}$ of M by

$$M_{r,l} = \begin{cases} \{m \mid 0 \leq m < l+1\} & \text{if } r=0 \\ \left\{ m \,\middle|\, \begin{array}{l} (l+1)(p^{r-1}+p^{r-2}+\dots+1) \leq m < \\ \qquad (l+1)(p^r+p^{r-1}+\dots+1) \end{array} \right\} & \text{if } r>0. \end{cases}$$

Obviously, for any l as above, the $M_{r,l}$ with $r \geq 0$ constitute a partition of M.

Proposition 5. *Let $1 \leq l \leq p-1$ and $r \geq 1$. Then every $m \in M_{r,l}$ can be uniquely expressed as $g_{k,l}(i)$ where $i \in M_{r-1,l}$ and $1 \leq k \leq p$.*

Proof. If $m \in M_{r,l}$ for $r \geq 1$, then $m \geq l+1$, i.e., $m-l \geq 1$, so there is a unique expression

$$m - l = ip + k$$

where $i \geq 0$ and $1 \leq k \leq p$. That is, $m = q_{k,l}(i)$. Thus the proposition follows from the observation that, when k runs over $\{1, 2, \dots, p\}$, $g_{k,l}(i)$ belongs to $M_{r,l}$ if and only if i belongs to $M_{r-1,l}$.

Corollary 6. *Every $n \in N$ can be uniquely expressed as*

$$f_l g_{k_1,l} g_{k_2,l} \dots g_{k_r,l}(i)$$

where $l\equiv n \mod(p-1)$ *with* $1\leq l\leq p-1$; $1\leq k_s\leq p$ *for* $1\leq s\leq r$; $r\geq 0$; *and* $i\in M_{0,l}$ (*where if* $r=0$ *then* $n=f_l(i)$).

Proof. Use Proposition 5 repeatedly.

Hereafter we will fix an integer l with $1\leq l\leq p-1$ and suppress it in related notation. So we simply write f, g_k and M_r for $f_l, g_{k,l}$ and $M_{r,l}$ respectively.

For each $h\geq 0$ we define three functions

$$F_h, G_h, H_h: \boldsymbol{Z}\longrightarrow\boldsymbol{Z}$$

by

$$F_h=\begin{cases} f & \text{if } h=0\\ f\underbrace{g_1\dots g_1}_{h} & \text{if } h>0,\end{cases}$$

$$G_h(j)=\begin{cases} j-1 & \text{if } h=0\\ jp & \text{if } h=1\\ (jp^{h-1}+p^{h-2}+p^{h-3}+\dots+1)p & \text{if } h>1,\end{cases}$$

$$H_h(j)=f(j)p^h$$

for $j\in\boldsymbol{Z}$, respectively. Then

$$F_0(m)=f(m),\ G_0(j)=j-1,\ H_0(j)=f(j) \tag{3.1}$$

and if $h\geq 0$

$$\begin{aligned} &F_{h+1}(m)=F_h g_1(m),\\ &G_{h+1}(j)=G_h(jp)+G_h(2)-G_h(1),\\ &H_{h+1}(j)=H_h g_p(j-1)\end{aligned} \tag{3.2}$$

for $m, j\in\boldsymbol{Z}$. Furthermore for $1\leq j\leq m$ we have

$$(p-1)G_h(j)+H_h(m-j)=f(\underbrace{g_1\dots g_1}_{h}(m)-1). \tag{3.3}$$

The proofs are immediate.

§4. The Odd Dimensional Part of $\tilde{J}_*(CP^\infty;Z/(p))$

Our objective of this section is to compute $\tilde{J}_{2n-1}(CP^\infty;\boldsymbol{Z}/(p))$ for $n\geq 1$.

By (2.9) it is isomorphic to the cokernel of $\theta:\tilde{k}(1)_{2n}(CP^\infty)\rightarrow \tilde{k}(1)_{2(n-(p-1))}(CP^\infty)$. Let us compute this cokernel. In doing so, we deal with $n\in N$ such that $n\equiv l \bmod (p-1)$, where l is a fixed integer with $1\leq l\leq p-1$. (This reflects the existence of a p-equivalence

$$\Sigma CP^{\infty}\simeq\bigvee_{l=1}^{p-1}X_{l}.)$$

In that case, $n=f(m)$ for some $m\in M$, and then $n-j(p-1)=f(m-j)$ for all $j\in \boldsymbol{Z}$. Therefore (the upper part of) (2. 8) and (2. 12) are rewritten as

$$\tilde{k}(1)_{2f(m)}(\boldsymbol{C}P^{\infty})=\boldsymbol{Z}/(p)\{v^{j}\beta_{f(m-j)}|0\leq j\leq m\}$$

and

$$\theta(\beta_{f(m)})=\sum_{j=1}^{m}c_{f(m),j}v^{j-1}\beta_{f(m-j)}$$

respectively, and by (2. 10)

$$\theta(v^{j}\beta_{f(m-j)})=v^{j}\cdot\theta(\beta_{f(m-j)})$$

for $1\leq j\leq m$. These facts lead to the following situation for $m\in M$.

$(0;m)'$: *Let* $n=f(m)$. *Then* $\tilde{J}_{2n-1}(\boldsymbol{C}P^{\infty};\boldsymbol{Z}/(p))$ *has elements*

$$\delta_{\theta}(v^{j-1}\beta_{f(m-j)}),\qquad j=1,\ldots,m$$

(where if $m=0$ *it has no elements) among which the only relation that contains the first term is*

$$\sum_{j=1}^{m}c_{f(m),j}\delta_{\theta}(v^{j-1}\beta_{f(m-j)})=0.$$

Especially (2. 10) enables us to use an inductive method for computing the cokernel of $\theta:\tilde{k}(1)_{2f(m)}(\boldsymbol{C}P^{\infty})\rightarrow\tilde{k}(1)_{2f(m-1)}(\boldsymbol{C}P^{\infty})$ for $m\in M$. For example, it follows from (2. 9) and (2. 10) that if $\beta\in\tilde{k}(1)_{2f(m-1)}(\boldsymbol{C}P^{\infty})$ satisfies $\delta_{\theta}(\beta)=0$, then $\delta_{\theta}(v^{i}\beta)=0$ for all $i\geq 1$. We will use this fact frequently but implicitly.

Theorem 7. *For every* $h\geq 0$ *we have*

(a) *For* $m\in M$ *there exists the following situation.*

$(h;m)$: *Let* $n=F_{h}(m)$. *Then* $\tilde{J}_{2n-1}(\boldsymbol{C}P^{\infty};\boldsymbol{Z}/(p))$ *has elements*

$$\delta_{\theta}(v^{G_{h}(j)}\beta_{H_{h}(m-j)}),\qquad j=1,\ldots,m$$

(see (3. 3)) among which the only relation that contains the first term is

$$\sum_{j=1}^{m}c_{f(m),j}\delta_{\theta}(v^{G_{h}(j)}\beta_{H_{h}(m-j)})=0.$$

(b) *In the situation* $(h; m)$ *for* $m \in M_0$,

$$\delta_\theta(v^{G_h(j)}\beta_{H_h(m-j)}) = 0$$

for all $j = 1, \ldots, m$.

We first show part (b).

Proof of Theorem 7(b). We argue by induction on m. (For $m=0$ there is nothing to prove.) Assume that

$$\delta_\theta(v^{G_h(1)}\beta_{H_h(m-1)}) = 0 \qquad \text{for } 1 \leq m \leq i-1$$

where $1 \leq i \leq l$. Consider the situation $(h; i)$. Then the relevant elements are

$$\delta_\theta(v^{G_h(j)}\beta_{H_h(i-j)}) \qquad \text{for } 1 \leq j \leq i$$

and the relevant relation is

$$c_{f(i),1}\delta_\theta(v^{G_h(1)}\beta_{H_h(i-1)}) + \sum_{j=2}^{i} c_{f(i),j}\delta_\theta(v^{G_h(j)}\beta_{H_h(i-j)}) = 0.$$

It follows from the inductive hypothesis that

$$\delta_\theta(v^{G_h(j)}\beta_{H_h(i-j)}) = 0 \qquad \text{for } 2 \leq j \leq i.$$

By Lemma 4(i),

$$c_{f(i),1} = c \cdot f(i-1) = c(l-i+1)$$

which is nonzero, since $1 \leq l-i+1 \leq l$. Hence

$$\delta_\theta(v^{G_h(1)}\beta_{H_h(i-1)}) = 0.$$

This proves the case $m=i$ and the result follows.

Proof of Theorem 7(a). We argue by induction on h. By (3.1) the situation $(0; m)'$ coincides with the situation $(0; m)$, which begins our induction. Assume that the situations $(h; m)$ with $m \in M$ are given. Since for $m \in M_0$ there is a trivial result as in Theorem 7(b), we may suppose that $m \in M - M_0$. Then by Proposition 5

$$m = g_k(i) = ip + k + l$$

where $i \geq 0$ and $1 \leq k \leq p$.

Lemma 8. *In the situation* $(h; ip+k+l)$, *the following results hold* *for all* $i\geq 0$ *and* $k=1, 2, \ldots, p$.

(i) *In the situation* $(h; ip+k+l)$ *for* $1\leq k\leq p$,

$$\delta_\theta(v^{G_h(j)}\beta_{H_h(ip+k+l-j)})=0$$

if $2\leq j\leq ip+k+l$ *and* $j\not\equiv k \mod(p)$.

(ii) *In the situation* $(h; ip+1+l)$,

$$\delta_\theta(v^{G_h(1)}\beta_{H_h((i-1)p+p+l)})\neq 0.$$

(iii) *In the situation* $(h; ip+k+l)$ *for* $1<k<p$,

$$\delta_\theta(v^{G_h(1)}\beta_{H_h(ip+k-1+l)})=0$$

and

$$\delta_\theta(v^{G_h(k)}\beta_{H_h((i-1)p+p+l)})\neq 0.$$

(iv) *In the situation* $(h; ip+p+l)$, *there is a relation*

$$\delta_\theta(v^{G_h(1)}\beta_{H_h(ip+p-1+l)}) = -c^{-1}\sum_{j=1}^{i+1} c_{f(i+1),j}\delta_\theta(v^{G_h(jp)}\beta_{H_h((i-j)p+p+l)})$$

and

$$\delta_\theta(v^{G_h(p)}\beta_{H_h((i-1)p+p+l)})\neq 0.$$

Proof. We prove this by induction on i. Assume that the lemma is true for $i<t$.

Consider first the situation $(h; tp+1+l)$. Then it follows from part (i) for the case $i=t-1$ and $k=p$ that the relevant relation becomes

$$\begin{aligned} &c_{f(tp+1+l),1}\delta_\theta(v^{G_h(1)}\beta_{H_h((t-1)p+p+l)}) \\ &\quad +c_{f(tp+1+l),2}\delta_\theta(v^{G_h(2)}\beta_{H_h((t-1)p+p-1+l)}) \\ &\quad +\sum_{s=1}^{t} c_{f(tp+1+l),sp+1}\delta_\theta(v^{G_h(sp+1)}\beta_{H_h((t-s-1)p+p+l)})=0. \end{aligned}$$

By Lemma 4(i),

$$c_{f(tp+1+l),1}=c\cdot f(tp+l)=c\cdot f(l)=0.$$

Since $f(tp+1+l)\equiv -1 \mod(p)$, by Lemma 4(iv)

$$c_{f(tp+1+l),2}\neq 0$$

and by Lemma 4(iii)

$$c_{f(tp+1+l),sp+1}=0.$$

These imply that

$$\delta_\theta(v^{G_h(2)}\beta_{H_h((t-1)p+p-1+l)})=0$$

and

$$\delta_\theta(v^{G_h(1)}\beta_{H_h((t-1)p+p+l)})\neq 0.$$

This proves parts (i) and (ii) for the case $i=t$ and $k=1$.

Assume inductively that the lemma is true for $i=t$ and $k<j$ where $2\leq j\leq p-1$. Consider the situation $(h;tp+j+l)$. Then part (i) for the case $i=t$ and $k=j$ follows from parts (i) and (iii) for the case $i=t$ and $k=j-1$. Therefore the relevant relation becomes

$$c_{f(tp+j+l),1}\delta_\theta(v^{G_h(1)}\beta_{H_h(tp+j-1+l)}) + \sum_{s=0}^{t} c_{f(tp+j+l),sp+j}\delta_\theta(v^{G_h(sp+j)}\beta_{H_h((t-s-1)p+p+l)})=0.$$

By Lemma 4(i),

$$c_{f(tp+j+l),1}=c\cdot f(tp+j-1+l)=c\cdot f(j-1+l)=c(-j+1)$$

which is nonzero, since $1\leq j-1\leq p-2$. Since $f(tp+j+l)\equiv -j \bmod (p)$, by Lemma 4(iii)

$$c_{f(tp+j+l),sp+j}=0.$$

These, together with the inductive hypothesis, imply that

$$\delta_\theta(v^{G_h(1)}\beta_{H_h(tp+j-1+l)})=0$$

and

$$\delta_\theta(v^{G_h(j)}\beta_{H_h((t-1)p+p+l)})\neq 0.$$

This proves part (iii) for the case $i=t$ and $k=j$, so part (iii) for $i=t$ follows.

Consider finally the situation $(h;tp+p+l)$. Then part (i) for the case $i=t$ and $k=p$ follows from part (i) for the case $i=t$ and $k=p-1$. Therefore the relevant relation becomes

$$c_{f(tp+p+l),1}\delta_\theta(v^{G_h(1)}\beta_{H_h(tp+p-1+l)}) + \sum_{s=0}^{t} c_{f(tp+p+l),sp+p}\delta_\theta(v^{G_h(sp+p)}\beta_{H_h((t-s-1)p+p+l)})=0.$$

By Lemma 4(i),

$$c_{f(tp+p+l),1}=c\cdot f(tp+p-1+l)=c\cdot f(l-1)=c.$$

By Lemma 4(ii),

$$c_{f(tp+p+l),sp+p}=c_{f(t+1)p,(s+1)p}=c_{f(t+1),s+1}.$$

These, together with part (iii), imply that

$$c\cdot\delta_\theta(v^{G_h(1)}\beta_{H_h(tp+p-1+l)}) + \sum_{s=0}^{t} c_{f(t+1),s+1}\delta_\theta(v^{G_h((s+1)p)}\beta_{H_h((t-(s+1))p+p+l)})=0$$

and

$$\delta_\theta(v^{G_h(p)}\beta_{H_h((t-1)p+p+l)})\neq 0.$$

This proves part (iv) for the case $i=t$.

Thus we have shown the lemma for $i=t$, which completes the inductive step and Lemma 8 follows.

We return to the proof of Theorem 7(a). In the situation $(h;(i-1)p+p+l)$, by Lemma 8(iv)

$$\delta_\theta(v^{G_h(1)}\beta_{H_h((i-1)p+p-1+l)}) = -c^{-1}\sum_{j=1}^{i} c_{f(i),j}\delta_\theta(v^{G_h(jp)}\beta_{H_h((i-1-j)p+p+l)}).$$

In the situation $(h;ip+1+l)$, by Lemma 8(i)

$$\delta_\theta(v^{G_h(2)}\beta_{H_h((i-1)p+p-1+l)})=0.$$

From these equalities and the proof of Lemma 8 it follows that in the situation $(h;ip+1+l)$,

$$\sum_{j=1}^{i} c_{f(i),j}\delta_\theta(v^{G_h(jp)+G_h(2)-G_h(1)}\beta_{H_h((i-j-1)p+p+l)})=0$$

and there are no other relations which contain the term

$$\delta_\theta(v^{G_h(p)+G_h(2)-G_h(1)}\beta_{H_h((i-2)p+p+l)}).$$

In this way for $i\in M$ we have

$(h+1;i)'$: *Let* $n=F_hg_1(i)$. *Then* $\tilde{J}_{2n-1}(CP^\infty;Z/(p))$ *has elements*

$$\delta_\theta(v^{G_h(jp)+G_h(2)-G_h(1)}\beta_{H_hg_p(i-j-1)}),\qquad j=1,\ldots,i$$

among which the only relation that contains the first term is

$$\sum_{j=1}^{i} c_{f(i),j}\delta_\theta(v^{G_h(jp)+G_h(2)-G_h(1)}\beta_{H_hg_p(i-j-1)})=0.$$

Rewriting this by using (3.2), we can find the situation $(h+1;i)$. This completes the inductive step and the result follows.

Consequences of the above argument in the situation $(h; g_k(i))$ can be stated as follows.

(4. 1) *Let* $n=F_h g_k(i)$. *Then in* $\tilde{J}_{2n-1}(CP^\infty; \boldsymbol{Z}/(p))$

(i) *among the elements*

$$\delta_\theta(v^{G_h(m)}\beta_{H_h(g_k(i)-m)}) \qquad \text{with } 1\leq m\leq g_k(i),$$

the only possible nonzero elements are

$$\delta_\theta(v^{G_h(jp+k)}\beta_{H_h g_p(i-j-1)})$$
$$=\delta_\theta(v^{G_{h+1}(j)+(k-1)p^h}\beta_{H_{h+1}(i-j)}), \qquad j=0, 1, \ldots, i$$

(and

$$\delta_\theta(v^{G_h(1)}\beta_{H_h g_{p-1}(i)}) \qquad \text{if } k=p);$$

(ii) *the element*

$$\delta_\theta(v^{G_h(k)}\beta_{H_h g_p(i-1)})=\delta_\theta(v^{G_h(k)}\beta_{H_{h+1}(i)})$$
$$=\delta_\theta(v^{G_h(k)}\beta_{f(i)p^{h+1}})$$

is nonzero; and

(iii) *the relation in* $(h+1; i)'$, *which occurs in the situation* $(h; g_1(i))$, *yields*

$$\sum_{j=1}^{i} c_{f(i),j}\delta_\theta(v^{G_{h+1}(j)+(k-1)p^h}\beta_{H_{h+1}(i-j)})=0.$$

Now we come to the main result.

Theorem 9. *Let* $n\geq 1$ *and write it in the form*

$$fg_{k_1}\ldots g_{k_r}(i)$$

(for details see Corollary 6). Then a $\boldsymbol{Z}/(p)$*-basis for* $\tilde{J}_{2n-1}(CP^\infty; \boldsymbol{Z}/(p))$ *is given by the elements*

$$\delta_\theta(v^{-1+\sum_{t=1}^{s}k_t p^{t-1}}\beta_{fg_{k_{s+1}}\ldots g_{k_r}(i)p^s})$$

where $1\leq s\leq r$.

Proof. Since $n=F_0(m)$ *where*

$$m=g_{k_1}\ldots g_{k_r}(i),$$

we start with the situation $(0; m)$. Since

$$m=g_{k_1}(m_1) \text{ where } m_1=g_{k_2}\dots g_{k_r}(i),$$

by (4.1)(ii)

$$\delta_\theta(v^{k_1-1}\beta_{f(m_1)p})$$

must be a basis element in $\tilde{J}_{2n-1}(CP^\infty; Z/(p))$, and in view of (4.1) (iii) the problem turns on the situation $(1; m_1)$. Since

$$m_1=g_{k_2}(m_2) \text{ where } m_2=g_{k_3}\dots g_{k_r}(i),$$

by (4.1)(ii)

$$\delta_\theta(v^{k_2p}\beta_{f(m_2)p^2})$$

must be a basis element in $\tilde{J}_{2n_1-1}(CP^\infty; Z/(p))$ where $n_1=F_1(m_1)$, from which we see that

$$\delta_\theta(v^{k_2p+k_1-1}\beta_{f(m_2)p^2})$$

is a basis element in $\tilde{J}_{2n-1}(CP^\infty; Z/(p))$, and in view of (4.1)(iii) the problem turns on the situation $(2; m_2)$. Continue this procedure, whose end is given by Theorem 7(b). Thus the result is obtained.

This theorem provides information about the CW-filtration degree of generators of $\tilde{J}_{2n-1}(CP^\infty)$.

§5. The Even Dimensional Part of $\tilde{J}_*(CP^\infty; Z/(p))$

Our objective of this section is to compute $\tilde{J}_{2n}(CP^\infty; Z/(p))$ for $n\geq 1$.

By (2.9) it is isomorphic to the kernel of $\theta: \tilde{k}(1)_{2n}(CP^\infty)\to \tilde{k}(1)_{2(n-(p-1))}(CP^\infty)$. Let us compute this kernel. To do so we will need the following, which is the connective version of [8, Theorem 5.6 with $n=1$].

Theorem 10. *As a $Z/(p)[v]$-algebra $k(1)_*(CP^\infty)$ is generated by the elements β_{p^i} for $i\geq 0$ with relations*

$$\beta_{p^i}^p=v^{p^i}\beta_{p^i}.$$

Recall that

$$\pi_{2*}(JZ/(p))=Z/(p)[\alpha]$$

where $\alpha\in\pi_{2(p-1)}(JZ/(p))$ is a unique element such that $\eta_\theta(\alpha)=v$. (This follows from (2.5) and (2.6).) The following result was proved in Schwartz [9] as Théorème 7(ii).

Theorem 11. *$J_{2*}(CP^\infty;Z/(p))$ is a free $Z/(p)[\alpha]$-module on generators*

$$V_{i,l}\quad \text{for } i\geq 0 \text{ and } 1\leq l\leq p-1$$

where the degree of $V_{i,l}$ is $2((l+1)p^i-1)$. Its multiplicative structure is given by

$$V_{i,l}\cdot V_{j,m}=\begin{cases}\alpha^{(p^i-1)/(p-1)}V_{i,l+m} & \text{if } i=j \text{ and } l+m\leq p-1\\ \alpha^{p^i+(p^i-1)/(p-1)}V_{i,l+m-p+1} & \text{if } i=j \text{ and } l+m>p-1\\ 0 & \text{if } i\neq j\end{cases}$$

where $i,j\geq 0$ and $1\leq l,m\leq p-1$.

We can describe $V_{i,l}$ in terms of the β_{p^i}.

Proposition 12. *For $i\geq 0$ and $1\leq l\leq p-1$,*

$$\eta_\theta(V_{i,l})=(-1)^{i(l-1)}\beta_{p^i}^l\cdot\prod_{j=0}^{i-1}(\beta_{p^j}^{p-1}-v^{p^j}).$$

Denote by $V'_{i,l}$ the right-hand side of the above equality.

Lemma 13. *For $i\geq 0$ and $1\leq l\leq p-1$, $\theta(V'_{i,l})=0$.*

Proof. We first show that

$$\theta\Big(\prod_{j=0}^{i-1}(\beta_{p^j}^{p-1}-v^{p^j})\Big)=0 \tag{5.1}$$

by induction on i. Since $\theta(\beta_1)=0$ (by (2.8) and (2.9)) and $\theta(v)=0$, it follows from (2.7) that $\theta(\beta_1^{p-1}-v)=0$. Assume that (5.1) is valid for $i\leq k$. Then we have

$$\begin{aligned}&\theta\Big(\prod_{j=0}^{k}(\beta_{p^j}^{p-1}-v^{p^j})\Big)\\ &=\theta(\beta_{p^k}^{p-1}-v^{p^k})\cdot\prod_{j=0}^{k-1}(\beta_{p^j}^{p-1}-v^{p^j}) && \text{by (2.7) and the inductive hypothesis}\\ &=\theta(\beta_{p^k}^{p-1})\cdot\prod_{j=0}^{k-1}(\beta_{p^j}^{p-1}-v^{p^j}) && \text{by (2.6)}\end{aligned}$$

which is zero, because by Theorem 10 $\theta(\beta_{p^k}^{p-1})$ can be expressed as a linear combination of monomials including β_{p^i}, $0\leq i\leq k-1$, for degree reasons and if $0\leq i\leq k-1$

$$\beta_{p^i}\circ\prod_{j=0}^{k-1}(\beta_{p^j}^{p-1}-v^{p^j})=0.$$

Thus (5. 1) follows. A similar argument gives

$$\begin{aligned}\theta(\beta_{p^i}^l\cdot\prod_{j=0}^{i-1}(\beta_{p^j}^{p-1}-v^{p^j}))&=\theta(\beta_{p^i}^l)\cdot\prod_{j=0}^{i-1}(\beta_{p^j}^{p-1}-v^{p^j})\\&=0\end{aligned}$$

and the result follows.

Proof of Proposition 12. From (2. 6), (2. 7) and Lemma 13 it follows that $\theta(v^kV'_{i,l})=0$ for all $k\geq 0$. From this and Theorems 10, 11 it is clear that the $v^kV'_{i,l}$ form a $\boldsymbol{Z}/(p)$-basis for our kernel. In order to prove the proposition, it suffices to verify that the $V'_{i,l}$ satisfy the same relations as in Theorem 11. By Theorem 10 we have

$$\begin{aligned}&V'_{i,l}\cdot V'_{i,m}\\&=(-1)^{i(l-1)+i(m-1)}\beta_{p^i}^{l+m}\circ\prod_{j=0}^{i-1}(\beta_{p^j}^{p-1}-v^{p^j})^2\\&=(-1)^{i(l+m-2)}\beta_{p^i}^{l+m}\cdot\prod_{j=0}^{i-1}(\beta_{p^j}^{2(p-1)}-2v^{p^j}\beta_{p^j}^{p-1}+v^{2p^j})\\&=(-1)^{i(l+m-2)}\beta_{p^i}^{l+m}\cdot\prod_{j=0}^{i-1}(v^{p^j}\beta_{p^j}^{p-1}-2v^{p^j}\beta_{p^j}^{p-1}+v^{2p^j})\\&=(-1)^{i(l+m-2)}\beta_{p^i}^{l+m}\cdot\prod_{j=0}^{i-1}(-v^{p^j}(\beta_{p^j}^{p-1}-v^{p^j}))\\&=(-1)^{i(l+m-2)}\beta_{p^i}^{l+m}\circ(-1)^iv^{1+p+\dots+p^{i-1}}\cdot\prod_{j=0}^{i-1}(\beta_{p^j}^{p-1}-v^{p^j})\\&=(-1)^{i(l+m-1)}v^{(p^i-1)/(p-1)}\beta_{p^i}^{l+m}\circ\prod_{j=0}^{i-1}(\beta_{p^j}^{p-1}-v^{p^j}).\end{aligned}$$

Since

$$\beta_{p^i}^{l+m}=\begin{cases}\beta_{p^i}^{l+m} & \text{if } l+m\leq p-1\\ v^{p^i}\beta_{p^i}^{l+m-p+1} & \text{if } l+m>p-1,\end{cases}$$

we get the desired relations. The verification for the remaining case is similar to the proof of Lemma 13. So Proposition 12 follows.

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