

# The Orders of Invariant Eigendistributions

By

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## Introduction

In a series of papers, Harish-Chandra studied the invariant eigendistributions and in particular established a fundamental theorem which states that any invariant eigendistribution on a connected semisimple Lie group is locally  $L^1$  (cf. [H]). It may be available to reconsider this result from the view point of microlocal analysis.

On the other hand, recently Hotta and Kashiwara [HK] have shown that the system of differential equations which governs an invariant eigendistribution on a semisimple Lie algebra is regular holonomic (=a holonomic system with regular singularities in the sense of [KK]). Among other things they showed, by using this result, that the holonomic system in question corresponds to the intersection cohomology complex defining Springer's representations of the Weyl group through the Riemann-Hilbert correspondence.

In this paper we examine a microlocal property of the invariant eigendistributions. The results of this paper is quite unsatisfactory in comparison with those mentioned above. But the author hopes that our attempts will be developed in future.

We now explain the contents shortly. In the first half we consider the holonomic system  $\mathcal{M}_\chi$  which governs an invariant eigendistribution on a connected linear semisimple Lie group. An invariant of a holonomic system is the set  $\text{ord}_A(u)$  of the orders along each irreducible component  $A$  of the characteristic variety of the system in question. Here  $u$  is a section of the system on the generic points of  $A$ . We attempt to determine  $\text{ord}_A(u)$  for the system  $\mathcal{M}_\chi$ . Unfortunately, we cannot do it for every irreducible component  $A$  of the characteristic  $\text{Ch}(\mathcal{M}_\chi)$  but if an irreducible component  $A$  of  $\text{Ch}(\mathcal{M}_\chi)$  satisfies the condition (A) in (3.1), we can calculate the orders along  $A$ . In this case, though  $\mathcal{M}_\chi$  is not a simple holonomic system in the sense of Sato-Kashiwara,  $\text{ord}_A(u)$  along such an irreducible component  $A$  consists of only one element 0. This is the main result of the first half (Theorem (3.4)). It rarely occurs that

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the set of orders are determined exactly. The author believes that this distinguished result connects with Harish-Chandra's theorem on the local integrability mentioned above. In the second half, we restrict our attention to the system  $\mathcal{N}_\lambda$  of the differential equations on  $\mathfrak{sl}(n, \mathbf{R})$  to which an invariant eigendistribution is a solution. In this case we have a sufficient information on the nilpotent orbits of  $\mathfrak{sl}(n, \mathbf{C})$  (cf. [He, KP]). Accordingly, we can examine a microlocal property of the system  $\mathcal{N}_\lambda$  in some detail by using the structure of nilpotent orbits (Theorem (4.5)). Furthermore we show that 0 is always contained in  $\text{ord}_A(u)$  for any irreducible component  $A$  of the characteristic variety of the system  $\mathcal{N}_\lambda$ .

The author wishes his hearty thanks to Professor M. Kashiwara for showing his unpublished result on holonomic systems (Theorem (2.8)) which plays a fundamental role in the proof of Theorem (3.4).

### §1. The Characteristic Variety of an Invariant Eigendistribution

(1.1) Let  $G$  be a connected linear semisimple Lie group and let  $\mathfrak{g}$  be its Lie algebra. For any element  $A$  of  $\mathfrak{g}$ , we define vector fields  $R_A$  and  $L_A$  on  $G$  in the following manner. If  $f(g)$  is a  $C^\infty$ -function on  $G$ , then

$$(R_A f)(g) = \frac{d}{dt} f(g e^{tA})|_{t=0},$$

$$(L_A f)(g) = \frac{d}{dt} f(e^{-tA} g)|_{t=0}.$$

Here  $A \rightarrow e^A$  denotes the exponential mapping of  $\mathfrak{g}$  to  $G$ . We frequently use the notation  $(R_A f)(g) = \langle A, D_g \rangle f$ .

Identifying  $\mathfrak{g}$  with the totality of left invariant vector fields on  $G$ , we have an isomorphism of the tangent bundle  $TG$  over  $G$  to  $G \times \mathfrak{g}$ . Then the cotangent bundle  $T^*G$  over  $G$  is identified with  $G \times \mathfrak{g}^*$ , where  $\mathfrak{g}^*$  is the dual of  $\mathfrak{g}$ .

(1.2) Let  $f(g)$  be a  $C^\infty$ -function on  $G$ . Then for any  $g \in G$ , we define the element  $d_g f$  of  $\mathfrak{g}^*$  by the formula

$$(1.2.1) \quad (d_g f)(X) = (R_X f)(g) \quad (\forall X \in \mathfrak{g}).$$

Similarly we define for any  $C^\infty$ -function  $\phi(\lambda)$  on  $\mathfrak{g}^*$ , the element  $d_\lambda \phi$  of  $\mathfrak{g}$  by the formula

$$(1.2.2) \quad (d_\lambda \phi)(\mu) = \frac{d}{dt} \phi(\lambda + t\mu)|_{t=0} \quad (\forall \mu \in \mathfrak{g}^*).$$

We frequently use the notation  $(d_\lambda \phi)(\mu) = \langle \mu, D_\lambda \rangle \phi$ .

(1.3) Let  $\mathfrak{g}_\mathbf{C}$  be a complexification of  $\mathfrak{g}$  and let  $S = S(\mathfrak{g}_\mathbf{C})$  be the symmetric algebra over  $\mathfrak{g}_\mathbf{C}$ . Let  $I$  be the subalgebra of all invariants of  $S$  by the action

of  $G$ . Then it follows from Chevalley's theorem that there exist homogeneous elements  $\phi_1, \dots, \phi_r$  of  $I$  such that  $I = \mathbb{C}[\phi_1, \dots, \phi_r]$ , where  $r$  is the rank of  $\mathfrak{g}$ . For later use, we denote by  $I_+$  the ideal of  $I$  generated by  $\phi_1, \dots, \phi_r$ .

(1.4) Let  $\mathfrak{g}_{\mathbb{C}}^*$  be the complexification of  $\mathfrak{g}^*$  and let  $N_{\mathbb{C}}^*$  be the totality of the nilpotent elements of  $\mathfrak{g}_{\mathbb{C}}^*$ . Then it follows from [Ko] that  $N_{\mathbb{C}}^* = \{\lambda \in \mathfrak{g}_{\mathbb{C}}^*; \phi(\lambda) = 0 \text{ for any } \phi \in I_+\}$ .

Let  $U$  be the universal enveloping algebra over  $\mathfrak{g}_{\mathbb{C}}$ . Then there is a (linear) bijection  $s$  of  $S$  onto  $U$ . For the sake of convenience, we set  $P_{\phi} = s(\phi)$  for any  $\phi \in S$ . Then it also follows from Chevalley's theorem that  $s|I$  is a bijection of  $I$  onto the center  $Z$  of  $U$ .

(1.5) Let  $P(g, D_g)$  be a differential operator on  $G$  with analytic coefficients. Then the principal symbol  $\sigma(P)$  of  $P$  is a function on  $T^*G \cong G \times \mathfrak{g}^*$  (see [B]). As usual, we identify  $U$  with the totality of left invariant differential operators on  $G$ . Then we have the following lemmas.

**Lemma (1.6).** (1) For any  $A \in \mathfrak{g}$ , we have

$$\sigma(R_A)(g, \lambda) = \langle \lambda, A \rangle, \quad \sigma(L_A)(g, \lambda) = -\langle \lambda, g^{-1}A \rangle.$$

(2) Let  $\phi$  be a homogeneous element of  $S$ . Then

$$\sigma(P_{\phi})(g, \lambda) = \phi(\lambda).$$

*Proof.* Obvious.

**Lemma (1.7).** Let  $\phi(g, \lambda)$  be a function on  $T^*G \cong G \times \mathfrak{g}$ . Then the Hamilton vector field  $H_{\phi}$  is expressed by

$$(1.7.1) \quad H_{\phi} = \langle d_{\lambda}\phi, D_g \rangle - \langle d_g\phi, d_{\lambda}\phi \rangle \lambda, D_{\lambda} \rangle.$$

*Proof.* Let  $\phi, \psi$  be functions on  $T^*G$ . We now calculate the Poisson bracket  $\{\phi, \psi\}$ . The result is

$$(1.7.2) \quad \{\phi, \psi\}(g, \lambda) = \langle d_{\lambda}\phi, d_g\psi \rangle - \langle d_g\phi, d_{\lambda}\psi \rangle + \langle \lambda, [d_{\lambda}\phi, d_{\lambda}\psi] \rangle.$$

This is shown as follows. The formula (1.7.2) is obvious in the case where  $\phi(g, \lambda)$  and  $\psi(g, \lambda)$  are independent of  $\lambda$  and in the case where  $\phi(g, \lambda)$  is independent of  $g$  and  $\psi(g, \lambda)$  is independent of  $\lambda$ . Accordingly it suffices to show (1.7.2) in the case where  $\phi = \sigma(R_A)$  and  $\psi = \sigma(R_{A'})$  for  $A, A' \in \mathfrak{g}$ . In this case, we have

$$\{\phi, \psi\} = \{\sigma(R_A), \sigma(R_{A'})\} = \sigma([R_A, R_{A'}]) = \sigma(R_{[A, A']}).$$

This implies (1.7.2). Since  $H_{\phi}(\psi) = \{\phi, \psi\}$ , the lemma follows from (1.7.2).

q. e. d.

**Lemma (1.8).** (1) For any  $A \in \mathfrak{g}$ , we have

$$H_{\sigma(R_A)} = R_A - \langle A\lambda, D_\lambda \rangle, \quad H_{\sigma(L_A)} = L_A.$$

(2) If  $\phi$  is a homogeneous element of  $I$ , then  $H_{\sigma(P_\phi)} = \langle d_\lambda \phi, D_g \rangle$ .

*Proof.* (1) is a direct consequence of Lemma (1.6) (1) and Lemma (1.7).

We now prove (2). Let  $\phi$  be a homogeneous element of  $I$ . Then it follows from the definition and (1.2.2) that

$$(d_\lambda \phi)(A\lambda) = \frac{d}{dt} \phi(\lambda + tA\lambda)|_{t=0} = 0 \quad (\forall A \in \mathfrak{g}).$$

This implies that  $\langle (d_\lambda \phi)\lambda, A \rangle = \langle A\lambda, d_\lambda \phi \rangle = 0$  for any  $A \in \mathfrak{g}$ . Therefore we have  $(d_\lambda \phi)\lambda = 0$ . On the other hand, since  $\phi$  is independent of  $g$ , we have  $d_g \phi = 0$  for any  $g \in G$ . Hence, in virtue of Lemma (1.7), we conclude that (2) holds.

q. e. d.

(1.9) Take an algebra homomorphism  $\chi$  of  $Z$  into  $\mathbb{C}$  and define the system of differential equations on  $G$ :

$$(1.9.1) \quad \begin{cases} (P - \chi(P))u = 0 & (\forall P \in Z), \\ (R_A + L_A)u = 0 & (\forall A \in \mathfrak{g}). \end{cases}$$

It should be noted here that if  $T$  is an invariant eigendistribution on  $G$ , then  $T$  is a solution of the system (1.9.1) for an appropriate infinitesimal character  $\chi$ .

Let  $G_C$  be a connected complex semisimple Lie group whose Lie algebra is  $\mathfrak{g}_C$  and contains  $G$ . Let  $\mathcal{D}$  be the sheaf of holomorphic differential operators on  $G_C$ . Corresponding to the system (1.9.1), we define a coherent left Ideal  $\mathcal{I}_\chi$  of  $\mathcal{D}$  by

$$\mathcal{I}_\chi = \sum_{P \in Z} \mathcal{D}(P - \chi(P)) + \sum_{A \in \mathfrak{g}} \mathcal{D}(R_A + L_A).$$

Then  $\mathcal{M}_\chi = \mathcal{D} / \mathcal{I}_\chi$  is a coherent left  $\mathcal{D}$ -Module on  $G_C$ . Let, further,  $\mathcal{E}$  be the sheaf of microdifferential operators on  $T^*G_C$  and define  $\tilde{\mathcal{M}}_\chi = \mathcal{E} / \mathcal{E} \otimes_{\mathcal{D}} \mathcal{I}_\chi$ .

**Proposition (1.10).** *The characteristic variety of  $\tilde{\mathcal{M}}_\chi$  is contained in the analytic subset*

$$A = \{(g, \lambda) \in G_C \times \mathfrak{g}_C^* : g\lambda = \lambda \text{ and } \lambda \in N_C^*\} \text{ of } G_C \times \mathfrak{g}_C^*.$$

*Proof.* Since

$$\sigma(R_A + L_A)(g, \lambda) = \langle \lambda, A - g^{-1}A \rangle \quad (\forall A \in \mathfrak{g}),$$

$$\sigma(P_\phi)(g, \lambda) = \phi(\lambda) \quad (\forall \phi \in I, \text{ homogeneous}),$$

it follows that if  $(g, \lambda)$  is contained in the characteristic variety of  $\tilde{\mathcal{M}}_\chi$ , then

$$\langle \lambda, A - g^{-1}A \rangle = 0 \quad (\forall A \in \mathfrak{g}), \quad \phi(\lambda) = \phi(0) \quad (\forall \phi \in I).$$

From these equations, we find that  $g\lambda=\lambda$  and  $\lambda\in N\mathfrak{g}^*$ . q. e. d.

(1.11) If  $(g, \lambda)$  is contained in  $A$  and  $h\in G_C$ , then  $(hgh^{-1}, h\lambda)$  is also contained in  $A$ . In this way,  $G_C$  acts on  $A$ . Let  $\{\lambda_1, \dots, \lambda_N\}$  be a complete set of representatives of nilpotent orbits in  $\mathfrak{g}^*$  and set  $C_i=\{g\lambda_i: g\in G_C\}$ , the  $G_C$ -orbit of  $\lambda_i$  and  $A'_i=\{(g, \lambda)\in G_C\times\mathfrak{g}^*; \lambda\in C_i, g\lambda=\lambda\}$ .

It follows from the Jacobson-Morozov lemma that for each  $i$ , there exist elements  $\mu_i$  and  $\lambda_i^-$  of  $\mathfrak{g}^*$  such that  $\langle\lambda_j, \mu_i, \lambda_i^-\rangle$  is an S-triple, that is,  $[\mu_i, \lambda_i]=2\lambda_i$ ,  $[\mu_i, \lambda_i^-]=-2\lambda_i^-$ ,  $[\lambda_i, \lambda_i^-]=\mu_i$ . Now fix  $i$  and define  $\mathfrak{a}_+=Z_{\mathfrak{g}_C}(\lambda_i)=\{A\in\mathfrak{g}_C; A\lambda_i=\lambda_i\}$  and  $\mathfrak{a}_-=Z_{\mathfrak{g}_C}(\lambda_i^-)$ . Let  $\mathfrak{m}$  be the orthogonal complement of  $\mathfrak{a}_+\oplus\mathfrak{a}_-$  with respect to the Killing form on  $\mathfrak{g}_C$ . Take any element  $g_0$  of  $G_C$  such that  $(g_0, \lambda_i)\in A'_i$  and define the mapping

$$F: (\mathfrak{a}_-\oplus\mathfrak{m})\times\mathfrak{a}_+ \longrightarrow A'_i$$

by  $F(A, B)=(\exp A(g_0 \exp B) \exp (-A), \exp A\cdot\lambda_i)$ . Then  $F(0, 0)=(g_0, \lambda_i)$  and  $dF(0, 0)$  is non-singular, so we get a coordinate system near  $(g_0, \lambda_i)$  by choosing a basis in  $\mathfrak{a}_-\oplus\mathfrak{m}, \mathfrak{a}_+$ . Accordingly,  $A'_i$  is a complex manifold and  $\dim A'_i=\dim G_C$ . Since  $Z_{G_C}(\lambda_i)=\{g\in G_C; g\lambda_i=\lambda_i\}$  is not connected in general (cf. [Ko, p. 363]), so is  $A'_i$ . Let  $A'_{i,1}, A'_{i,2}, \dots, A'_{i,k_i}$  be all the connected components of  $A'_i$  and set  $A_i=\bar{A}'_i$  and  $A_{i,j}=\bar{A}'_{i,j}$  ( $j=1, \dots, k_i$ ). Then each  $A_{i,j}$  is an irreducible analytic subset of  $G_C\times\mathfrak{g}^*$  and  $A=\bigcup_{i=1}^N \bigcup_{j=1}^{k_i} A_{i,j}$  is the decomposition of  $A$  into irreducible components.

**Theorem (1.12).** *The system  $\mathcal{M}_\lambda$  is holonomic for any  $\lambda$ .*

*Proof.* Due to Proposition (1.10) and the definition of a holonomic system, it suffices to show that for each irreducible component  $A_{i,j}$  of  $A$ ,  $\dim A_{i,j}=\dim G_C$ . But this is already shown in (1.11).

*Remark (1.13).* It is known that  $\mathcal{M}_\lambda$  is regular holonomic (cf. [HK, p. 28]).

### § 2. A Theorem on a Holonomic System

(2.1) Let  $X$  be a complex manifold and let  $T^*X$  be the cotangent bundle over  $X$ . Let  $\omega$  be the fundamental 1-form on  $T^*X$ . Then its differential  $d\omega$  is the symplectic form on  $T^*X$  and  $d\omega$  gives a 1-1 correspondence between tangent and cotangent vectors on  $T^*X$  and this extends to a 1-1 correspondence between holomorphic vector fields and holomorphic differential 1-forms. Thus we obtain an identification  $H: T^*(T^*X)\rightarrow T(T^*X)$ . We set  $\mathcal{X}=-H(\omega)$ .

As usual  $\mathcal{E}_X$  denotes the sheaf of microdifferential operators of finite order on  $T^*X$  (cf. [B]). For any  $P(x, D_x)\in\mathcal{E}_X$  with  $\text{ord } P=m$ , we set

$$P(x, D_x)=P_m(x, D_x)+P_{m-1}(x, D_x)+\dots$$

where  $P_k(x, \xi)$  is homogeneous with respect to the cotangent variable  $\xi$  of degree  $k$ .

(2.2) Let  $V$  be an involutory submanifold of  $T^*X$  and consider a coherent  $\mathcal{E}_X$ -Module  $\mathcal{M} = \mathcal{E}_X/\mathcal{I}$  such that  $\text{Supp } \mathcal{M} \subset V$ , where  $\mathcal{I}$  is a left Ideal of  $\mathcal{E}_X$ .

**Definition (2.3)** (cf. [KO]). *Let  $u$  be a section of  $\mathcal{M}$ . Then a principal symbol of  $u$  is a solution  $\phi$  of the following system of differential equations on  $\mathcal{L}_V$ :*

$$(2.3.1) \quad L_P^{(m)}\phi = 0 \quad \text{for every } P \in \mathcal{E}_V^{(m)} \text{ which annihilates } u.$$

(For the definitions of  $\mathcal{L}_V$ ,  $L_P^{(m)}$  and  $\mathcal{E}_V^{(m)}$ , see [KO].)

(2.4) From now on we always assume that  $\mathcal{M}$  is holonomic and  $V$  is Lagrangian.

**Definition (2.5)** (cf. [KK]). *Let  $u$  be a section of  $\mathcal{M}$ . Then an order of  $u$  along  $V$  is a complex number  $\alpha$  such that  $(\mathcal{X} - \alpha)\phi = 0$  for a principal symbol  $\phi$  of  $u$ . We denote by  $\text{ord}_V(u)$  the set of orders of  $u$  along  $V$ .*

(2.6) Some properties of  $\text{ord}_V(u)$  are examined in [KK, Chap. I]. The purpose of this section is to prove the following theorem due to M. Kashiwara. We reproduce its proof with his permission.

**Theorem (2.7)** (M. Kashiwara). *Let  $\mathcal{M}$  be a holonomic system as in (2.2). Let  $u$  be a section of  $\mathcal{M}$  and let  $p = (x_0, \xi_0)$  be a point of  $V$ . Assume that there exists a microdifferential operator  $P(x, D_x) = P_m(x, D_x) + P_{m-1}(x, D_x) + \dots$  defined in a neighbourhood of  $p$  in  $V$  such that  $Pu = 0$  and that  $H_{\sigma(P)} = -\mathcal{X}$  at  $p$ . Then*

$$\text{ord}_V(u) = \left\{ P_{m-1}(p) - \frac{1}{2} \sum_{j=1}^n \frac{\partial^2 P_m}{\partial x_j \partial \xi_j}(p) + \frac{1}{2} \text{Tr}(H_{\sigma(P)} + \mathcal{X}; T_p V) \right\}.$$

Here for a vector field  $v$  leaving  $p$  fixed,  $\text{Tr}(v; T_p V)$  is the trace of the linear endomorphism  $T_p V \ni w \rightarrow [w, v] \in T_p V$ .

*Proof.* First we shall prove the following lemma.

**Lemma.** *If  $v$  is a vector field on  $V$  leaving  $p$  fixed, then we have  $v(f) = (1/2) \text{Tr}(v; T_p V)f$  at  $p$  for any  $f \in \mathcal{L}_V$ .*

In fact, choosing a coordinate system  $(t_1, \dots, t_n)$  of  $V$  near  $p$  such that  $p = 0$ , we can write  $v = \sum_{j=1}^n t_j v_j$  for some vector fields  $v_1, \dots, v_n$ . Then it follows from the definition that  $\text{Tr}(v; T_p V) = \sum_{j=1}^n v_j(t_j)(0)$ . Set  $dt = dt_1 \wedge \dots \wedge dt_n$  and let  $a(t)$  be a function on  $V$  such that  $f = a(t)\sqrt{dt}$ . Then we have

$$\begin{aligned} v(a(t)\sqrt{dt})|_{t=0} &= \{v(a(t))\sqrt{dt} + a(t)v(\sqrt{dt})\}|_{t=0} \\ &= a(0)v(\sqrt{dt})|_{t=0} \end{aligned}$$

and

$$\begin{aligned} v(\sqrt{dt})|_{t=0} &= \frac{1}{2\sqrt{dt}}v(dt)|_{t=0} \\ &= \frac{1}{2\sqrt{dt}}\left(\sum_{j=1}^n v_j(t_j)dt\right)|_{t=0} \\ &= \frac{1}{2}\left(\sum_{j=1}^n v_j(t_j)(0)\right)\sqrt{dt} \\ &= \frac{1}{2}\text{Tr}(v; T_p V)\sqrt{dt}. \end{aligned}$$

Thus we obtain the lemma.

Now we return to the proof of Theorem (2.8). Let  $\phi$  be a solution of the system (2.3.1). Since  $H_{\sigma(P)} + \mathcal{X}$  leaves  $p$  fixed, the lemma above induces that

$$(H_{\sigma(P)} + \mathcal{X})(\phi) = \frac{1}{2}\text{Tr}(H_{\sigma(P)} + \mathcal{X}; T_p(V))\phi \quad \text{at } p.$$

Noting that

$$H_{\sigma(P)}(\phi) = -\left(P_{m-1}(x_0, \xi_0) - \frac{1}{2}\sum_{j=1}^n \frac{\partial^2 P_m}{\partial x_j \partial \xi_j}(x_0, \xi_0)\right)\phi,$$

we conclude that

$$\begin{aligned} \mathcal{X}(\phi) &= -H_{\sigma(P)}(\phi) + (H_{\sigma(P)} + \mathcal{X})(\phi) \\ &= \left(P_{m-1}(x_0, \xi_0) - \frac{1}{2}\sum_{j=1}^n \frac{\partial^2 P_m}{\partial x_j \partial \xi_j}(x_0, \xi_0) + \frac{1}{2}\text{Tr}(H_{\sigma(P)} + \mathcal{X}; T_p(V))\right)\phi. \end{aligned}$$

Thus the theorem follows from (2.3). q. e. d.

### §3. Main Theorem

(3.1) Let us return to the situation in §1. Fix an irreducible component  $A_{i,j}$  of  $A$  and consider the following condition for  $A_{i,j}$ .

**Condition (A).** Fix  $(g, \lambda) \in A'_{i,j}$  and we take  $X \in \mathfrak{g}_C$  such that  $\langle \lambda, Z \rangle = B(X, Z)$  ( $Z \in \mathfrak{g}_C$ ). Here  $B$  denotes the Killing form on  $\mathfrak{g}_C$ . Then there exist  $H, Y$  in  $\mathfrak{g}_C$  and  $\phi \in I_+$  such that

$$(3.1.1) \quad [H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H,$$

$$(3.1.2) \quad H - g^{-1}H + d_\lambda \phi = 0.$$

We note that this condition depends only on  $A_{i,j}$  but does not on the choice

of  $(g, \lambda)$ .

**Lemma (3.2).** *Assume that  $\lambda_1$  is a regular nilpotent element of  $\mathfrak{g}_\mathbb{C}^*$  and therefore  $C_1$  is the regular nilpotent orbit of  $N_\mathbb{C}^*$ . Then for any  $A_{1,j}$  ( $j=1, \dots, k_1$ ), the condition (A) holds.*

*Proof.* It follows from the Jacobson-Morozov lemma that the condition (3.1.1) always holds. Accordingly, it suffices to show the existence of an element  $\phi$  of  $I_+$  satisfying (3.1.2). It follows from [Ko, Th. 9] that in this case, the set  $M = \{d_\lambda \phi; \phi \in I_+\}$  is an  $r$ -dimensional vector space ( $r = \text{rank } \mathfrak{g}_\mathbb{C}$ ). Since  $X$  is regular,  $\dim Z_{\mathfrak{g}_\mathbb{C}}(X) = r$ . Therefore we find that  $M = Z_{\mathfrak{g}_\mathbb{C}}(X)$ . But  $H - g^{-1}H$  is contained in  $Z_{\mathfrak{g}_\mathbb{C}}(X)$  and (3.1.2) is shown. q. e. d.

*Remark (3.3).* (1) The system  $\mathcal{M}_\lambda|_{A_{1,j}}$  is simple in the sense of Sato-Kashiwara.

(2) The condition (A) does not hold for every irreducible component  $A_{i,j}$  (cf. § 4).

**Theorem (3.4).** *Let  $u$  be the generator of the system  $\mathcal{M}_\lambda$  such that  $u \equiv 1 \pmod{\mathcal{G}_\lambda}$ . Take an irreducible component  $A_{i,j}$  of  $A$  such that the condition (A) holds for  $A_{i,j}$ . Then we have  $\text{ord}_{A_{i,j}}(u) = \{0\}$ .*

*Proof.* Let  $p = (g_0, \lambda_i)$  be a point of  $A'_{i,j}$ . If  $\lambda_i = 0$ , it is easy to show that  $\text{ord}_{A_{i,j}}(u) = \{0\}$  because, in this case  $A_{i,j}$  is contained in the base space. Thus we may assume that  $\lambda_i \neq 0$  without loss of generality. Then there is an element  $A$  of  $\mathfrak{g}_\mathbb{C}$  such that  $\langle \lambda_i, A \rangle = 1$ . We take  $X \in \mathfrak{g}_\mathbb{C}$  such that  $B(X, Z) = \langle \lambda_i, Z \rangle$  for any  $Z \in \mathfrak{g}_\mathbb{C}$ . Then it follows from the condition (A) that there exist  $H, Y \in \mathfrak{g}_\mathbb{C}$  and  $\phi \in I_+$  such that  $X, H, Y$  satisfy (3.1.1) and (3.1.2) for  $(g, \lambda) = (g_0, \lambda_i)$ . Now we write  $\phi = \sum_{k=2}^m \phi_k$ , where  $\phi_k \in I$  is homogeneous of degree  $k$ . It should be noted that there exists no non-zero homogeneous element of  $I$  with degree 1. Set  $Q_k = s(\phi_k)$  and

$$P = \frac{1}{2}(L_H + R_H) + \sum_{k=2}^m (R_A)^{-(k-1)}(Q_k - \chi(Q_k)).$$

Then  $P$  is contained in  $\mathcal{E}_{\mathfrak{G}_\mathbb{C}}$  and  $Pu = 0$ . Since

$$\sigma(P)(g, \lambda) = \frac{1}{2} \langle \lambda, H - g^{-1}H \rangle + \sum_{k=2}^m \langle \lambda, A \rangle^{-(k-1)} \phi_k(\lambda).$$

it follows from Lemma (1.8) that

$$\begin{aligned} H_{\sigma(P)} &= \frac{1}{2} (\langle H - g^{-1}H, D_g \rangle - \langle H\lambda, D_\lambda \rangle) \\ &\quad + \sum_{k=2}^m \{ \langle \lambda, A \rangle^{-(k-1)} \langle d_\lambda \phi_k, D_g \rangle \\ &\quad - (k-1) \langle \lambda, A \rangle^{-k} \phi_k(\lambda) (\langle A, D_g \rangle - \langle A\lambda, D_\lambda \rangle) \}. \end{aligned}$$



Hence we have

$$\begin{aligned}
 H_{\sigma(P)} &= \frac{1}{2} (\langle H - g_0^{-1}H, D_g \rangle - \langle H\lambda_i, D_\lambda \rangle) + \sum_{k=2}^m \langle d_\lambda \phi_k, D_g \rangle \\
 &= \left\langle \frac{1}{2} (H - g_0^{-1}H) + d_\lambda \phi, D_g \right\rangle - \frac{1}{2} \langle H\lambda_i, D_\lambda \rangle
 \end{aligned}$$

at  $p$ . From the definition it follows that  $H\lambda_i = 2\lambda_i$ . Therefore we find that  $H_{\sigma(P)} \equiv -\langle \lambda_i, D_\lambda \rangle$  at  $p$ . Since  $\mathcal{X} = \langle \lambda, D_\lambda \rangle$ , this implies that  $H_{\sigma(P)} = -\mathcal{X}$  at  $p$ . Then applying Theorem (2.8) to the operator  $P$ , we find that

$$\text{ord}_{\lambda_i, j}(u) = \left\{ \frac{1}{2} \{ \sigma_0(P + P^*) - \text{Tr}(H_{\sigma(P)} + \mathcal{X}; T_p A_{i, j}) \} \right\},$$

where  $P^*$  is the adjoint of  $P$ . Accordingly, to prove the theorem, it suffices to show

$$(3.4.1) \quad \sigma_0(P + P^*) = 0.$$

$$(3.4.2) \quad \text{Tr}(H_{\sigma(P)} + \mathcal{X}; T_p A_{i, j}) = 0.$$

We now prove (3.4.1). For this purpose, set

$$\begin{aligned}
 P_1 &= \frac{1}{2} (L_H + R_H) + \sum_{k=2}^m (R_A)^{-(k-1)} Q_k, \\
 P_{-1} &= - \sum_{k=2}^m \chi(Q_k) (R_A)^{-(k-1)}.
 \end{aligned}$$

Then  $P = P_1 + P_{-1}$  and  $\text{ord } P_{-1} \leq -1$ . Since  $P_1^* = -P_1$ , we have  $\sigma_0(P + P^*) = \sigma_0(P_1 + P_1^*) = 0$ . This shows (3.4.1).

The equality (3.4.2) will be shown in the following lemma. Hence the theorem is proved.

**Lemma (3.5).** *Under the assumption in Theorem (3.4), we have*

$$\text{Tr}(H_{\sigma(P)} + \mathcal{X}; T_p A_{i, j}) = 0.$$

*Proof.* We first note that

$$T_p A_{i, j} = (0, \mathfrak{g}_C \lambda_i) + ((\mathfrak{g}_C \lambda_i)^\perp, 0) \cong (\mathfrak{g}_C \lambda_i) \oplus (\mathfrak{g}_C \lambda_i)^\perp.$$

Under this identification, any tangent vector  $v \in T_p A_{i, j}$  is expressed by  $v = \langle E, D_g \rangle + \langle F\lambda_i, D_\lambda \rangle$  with  $E \in (\mathfrak{g}_C \lambda_i)^\perp$  and  $F\lambda_i \in \mathfrak{g}_C \lambda_i$ . Then

$$\begin{aligned}
 & [v, H_{\sigma(P)} + \mathcal{X}] \text{ (at } p) \\
 &= ([\langle E, D_g \rangle, H_{\sigma(P)} + \mathcal{X}] + [\langle F\lambda_i, D_\lambda \rangle, H_{\sigma(P)} + \mathcal{X}]) \text{ (at } p) \\
 &= \frac{1}{2} \langle [E, g_0^{-1}H], D_g \rangle + \left\langle F\lambda_i - \frac{1}{2} HF\lambda_i, D_\lambda \right\rangle - \sum_{k=2}^m (k-1) \langle F\lambda_i, A \rangle \langle d_\lambda \phi_k, D_g \rangle \\
 &= \left\langle \frac{1}{2} [E, g_0^{-1}H] - \sum_{k=2}^m (k-1) \langle F\lambda_i, A \rangle d_\lambda \phi_k, D_g \right\rangle + \frac{1}{2} \langle [F, H]\lambda_i, D_\lambda \rangle.
 \end{aligned}$$

Therefore  $H_{\sigma(P)} + \mathcal{X}$  transforms the vector  $\langle E, D_g \rangle + \langle F\lambda_i, D_\lambda \rangle$  to

$$\left\langle \frac{1}{2}[E, g_0^{-1}H] - \sum_{k=2}^m (k-1)\langle F\lambda_i, A \rangle d_\lambda \phi_k, D_g \right\rangle + \frac{1}{2}\langle [F, H]\lambda_i, D_\lambda \rangle.$$

Hence  $H_{\sigma(P)} + \mathcal{X}$  induces the mapping of  $(\mathfrak{g}_C\lambda_i) \oplus (\mathfrak{g}_C\lambda_i)^\perp$  to itself defined by

$$(F\lambda_i, E) \mapsto \left( \frac{1}{2}[F, H]\lambda_i, \frac{1}{2}[E, g_0^{-1}H] - \sum_{k=2}^m (k-1)\langle F\lambda_i, A \rangle d_\lambda \phi_k \right).$$

Accordingly we have

$$\begin{aligned} & \text{Tr}(H_{\sigma(P)} + \mathcal{X}; T_p A_{i,j}) \\ &= \text{Tr}_{\mathfrak{g}_C\lambda_i}(F\lambda_i \rightarrow \frac{1}{2}[F, H]\lambda_i) + \text{Tr}_{(\mathfrak{g}_C\lambda_i)^\perp}(E \rightarrow \frac{1}{2}[E, g_0^{-1}H]) \\ &= \text{Tr}_{[\mathfrak{g}_C, X]}([F, X] \rightarrow \frac{1}{2}[[F, H], X]) + \text{Tr}_{Z_{\mathfrak{g}_C}(X)}(E \rightarrow \frac{1}{2}[E, g_0^{-1}H]). \end{aligned}$$

We now calculate

$$\begin{aligned} & \text{Tr}_{[\mathfrak{g}_C, X]}([F, X] \rightarrow \frac{1}{2}[[F, H], X]) \\ &= \text{Tr}_{[\mathfrak{g}_C, X]}([F, X] \rightarrow \frac{1}{2}[[F, X], H] + [F, X]) \\ &= \dim [\mathfrak{g}_C, X] + \text{Tr}_{[\mathfrak{g}_C, X]}([F, X] \rightarrow \frac{1}{2}[[F, X], H]) \\ &= \dim [\mathfrak{g}_C, X] - \text{Tr}_{Z_{\mathfrak{g}_C}(X)}(F \rightarrow \frac{1}{2}[F, H]). \end{aligned}$$

Here we used that  $\text{Tr}_{\mathfrak{g}_C} \text{ad } H = 0$ . We recall that  $\text{ad } H|_{Z_{\mathfrak{g}_C}(Y)}$  is an endomorphism of  $Z_{\mathfrak{g}_C}(Y)$  and if we take a basis  $u_1, \dots, u_p$  of  $Z_{\mathfrak{g}_C}(Y)$  such that  $\text{ad } (H)u_i$

$= -n_i u_i$  with a non-negative integer  $n_i$  ( $1 \leq i \leq p$ ), then  $\dim \mathfrak{g}_C = \sum_{i=1}^p (n_i + 1)$ . Hence we have

$$\begin{aligned} \text{Tr}_{Z_{\mathfrak{g}_C}(Y)}(F \rightarrow \frac{1}{2}[F, H]) &= \frac{1}{2} \sum_{i=1}^p n_i \\ &= \frac{1}{2} \dim [\mathfrak{g}_C, Y] = \frac{1}{2} \dim [\mathfrak{g}_C, X]. \end{aligned}$$

On the other hand, since  $g_0 X = X$  and therefore  $g_0$  is an automorphism of  $Z_{\mathfrak{g}_C}(X)$ , we have

$$\text{Tr}_{Z_{\mathfrak{g}_C}(X)}(E \rightarrow \frac{1}{2}[E, g_0^{-1}H]) = \text{Tr}_{Z_{\mathfrak{g}_C}(X)}(E \rightarrow \frac{1}{2}[E, H]).$$

Then by an argument similar to the above, we find that

$$\text{Tr}_{Z_{\mathfrak{g}_C}(X)}(E \rightarrow \frac{1}{2}[E, g_0^{-1}H]) = -\frac{1}{2} \dim [\mathfrak{g}_C, X].$$

From these equations, we finally obtain

$$\begin{aligned} & \text{Tr}(H_{\sigma(C)} + \mathcal{X}; T_p A_{i,j}) \\ &= \text{Tr}_{[\mathfrak{g}_C, X]} \left( [F, X] \rightarrow \frac{1}{2} [[F, H], X] \right) + \text{Tr}_{Z_{\mathfrak{g}_C}(X)} \left( E \rightarrow \frac{1}{2} [E, g_{\bar{v}}^{-1} H] \right) \\ &= \dim [\mathfrak{g}_C, X] - \frac{1}{2} \dim [\mathfrak{g}_C, X] - \frac{1}{2} \dim [\mathfrak{g}_C, X] = 0. \end{aligned} \quad \text{q. e. d.}$$

§ 4. An Example

(4.1) In the previous sections, we treat a holonomic system  $\mathcal{M}_\chi$  on a connected linear semisimple Lie group. Needless to say, for the holonomic system which governs an invariant eigen-distribution on a semisimple Lie algebra, a claim similar to Theorem (3.4) holds. We note here that Hotta and Kashiwara [HK] studied the system in quite a detail.

In this section, we mainly concentrate on the holonomic system on  $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$  which governs an invariant eigendistribution on  $\mathfrak{g}$  a little more.

(4.2) We first introduce some notation. Set  $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$ ,  $\mathfrak{g}_C = \mathfrak{sl}(n, \mathbf{C})$ ,  $G = SL(n, \mathbf{R})$  and  $G_C = SL(n, \mathbf{C})$ . Let  $S$  be the symmetric algebra over  $\mathfrak{g}_C$  and let  $P[\mathfrak{g}_C]$  be the algebra of polynomials on  $\mathfrak{g}_C$ . Let, further,  $I = S^G$  and  $I^* = P[\mathfrak{g}_C]^G$  denote the subalgebra of  $S$  and  $P[\mathfrak{g}_C]$  consisting of  $G$ -invariant ones, respectively.

Let  $E_{i,j}$  be an  $n \times n$  matrix whose  $(i, j)$ -entry is 1 and others are 0 and set  $E_i = E_{ii} - \frac{1}{n} \sum_{j=1}^n E_{jj}$ . We define polynomials  $P_2(X), \dots, P_n(X)$  on  $\mathfrak{g}_C$  by the following formula:

$$(4.2.1) \quad \det(\lambda I_n - X) = \lambda^n + P_2(X)\lambda^{n-2} - P_3(X)\lambda^{n-3} + \dots + (-1)^n P_n(X).$$

It should be noted that  $P_k$  is a homogeneous polynomial of degree  $k$ . We denote by  $P_k^*$  the element of  $S$  obtained by substituting the  $(i, j)$ -entry of  $X$  for  $E_{i,j}$  ( $i \neq j$ ) and the  $(i, i)$ -entry of  $X$  for  $E_i$  in the polynomial  $P_k$ . Then the following is well-known:

$$(4.2.2) \quad \begin{aligned} P[\mathfrak{g}_C]^G &= \mathbf{C}[P_2, \dots, P_n], \\ S^G &= \mathbf{C}[P_2^*, \dots, P_n^*]. \end{aligned}$$

Let  $\mathfrak{h}$  be the Cartan subalgebra of  $\mathfrak{g}_C$  consisting of diagonal matrices. As usual, we identify  $\mathfrak{h}$  with  $\mathbf{C}^{n-1}$  by the correspondence  $\text{diag}(t_1, \dots, t_n) \rightarrow (t_1, \dots, t_n)$  ( $t_1 + \dots + t_n = 0$ ). Let  $\mathfrak{h}^*$  be the dual of  $\mathfrak{h}$  which is also identified with  $\mathbf{C}^{n-1}$ . For any  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathfrak{h}^*$  ( $\lambda_1 + \dots + \lambda_n = 0$ ), we define an algebra homomorphism  $\chi_\lambda$  of  $S^G$  to  $\mathbf{C}$  in the following way. Let  $p_i(\lambda)$  be the  $i$ -th fundamental symmetric polynomial of  $\lambda_1, \dots, \lambda_n$ . Then  $\chi_\lambda(\phi(P_2, \dots, P_n)) = \phi(p_2(\lambda), \dots, p_n(\lambda))$  for any  $\phi(P_2, \dots, P_n) \in S^G$  (cf. (4.2.2)).

(4.3) For any  $\phi \in S$ , let  $\partial(\phi)$  denote the constant coefficient differential operator on  $\mathfrak{g}_C$  which corresponds to  $\phi$  (cf. [HC]). On the other hand, for any  $A \in \mathfrak{g}_C$ , we define a vector field  $\tau(A)$  by the condition that  $\tau(A)$  assigns to any  $X \in \mathfrak{g}_C$  the tangent vector  $[A, X]$ .

Using these notation, we introduce a system of differential equations. For an element  $\lambda$  of  $\mathfrak{h}^*$ , let  $\chi_\lambda$  be the algebra homomorphism of  $S^G$  to  $C$ . Define a system on  $\mathfrak{g}$ :

$$(4.3.1) \quad \begin{cases} (\partial(\phi) - \chi_\lambda(\phi))u = 0 & (\forall \phi \in S^G), \\ \tau(A)u = 0 & (\forall A \in \mathfrak{g}). \end{cases}$$

Needless to say, an invariant eigendistribution  $T$  on  $\mathfrak{g}$  corresponding to the infinitesimal character  $\chi_\lambda$  is a solution of the differential equations in (4.3.1).

Let  $\mathcal{D}$  denote the sheaf of differential operators on  $\mathfrak{g}_C$ . Corresponding to the system (4.3.1), we define the coherent Ideal  $\mathcal{I}_\lambda$  of  $\mathcal{D}$  by

$$\mathcal{I}_\lambda = \sum_{\phi \in S^G} \mathcal{D}(\partial(\phi) - \chi_\lambda(\phi)) + \sum_{A \in \mathfrak{g}} \mathcal{D}\tau(A).$$

Then  $\mathcal{N}_\lambda = \mathcal{D}/\mathcal{I}_\lambda$  is the coherent  $\mathcal{D}$ -Module on  $\mathfrak{g}_C$  corresponding to the system on  $\mathfrak{g}_C$  defined in (4.3.1). Let  $\mathcal{E}$  be the sheaf of microdifferential operators on  $T^*\mathfrak{g}_C$  and define  $\tilde{\mathcal{N}}_\lambda = \mathcal{E}/\mathcal{E} \otimes_{\mathcal{D}} \mathcal{I}_\lambda$ . We note that  $\tilde{\mathcal{N}}_\lambda$  is an  $\mathcal{E}$ -Module.

By the correspondence  $\partial(E_{ij}) \rightarrow E_{ji}$  ( $i \neq j$ ),  $\partial(E_i) \rightarrow E_i$ , we identify  $T^*\mathfrak{g}_C$  with  $\mathfrak{g}_C \times \mathfrak{g}_C$ . Then, by an argument similar to that (1.10), we see that the characteristic variety of  $\tilde{\mathcal{N}}_\lambda$  is contained in the analytic set

$$A = \{(X, Y) \in \mathfrak{g}_C \times \mathfrak{g}_C; Y \in N, [X, Y] = 0\}.$$

Here  $N$  denotes the set of nilpotent elements of  $\mathfrak{g}_C$ . Later we shall show that  $\dim A = \dim \mathfrak{g}_C$  and therefore the  $\mathcal{D}$ -Module  $\mathcal{N}_\lambda$  is holonomic for any  $\lambda \in \mathfrak{h}^*$  (cf. Theorem (4.5)).

(4.4) To examine the structure of  $A$  in detail, we review on nilpotent matrices and their commuting matrices (cf. [G, He]).

Let  $\eta = (p_1, \dots, p_n)$  be a partition of  $n$ , that is,  $p_1 \geq \dots \geq p_n \geq 0$  and  $p_1 + \dots + p_n = n$ . Associated with  $\eta$ , we define a matrix

$$J_\eta = \begin{pmatrix} J_{p_1} & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & J_{p_k} \end{pmatrix}, \quad \text{where } J_p = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \text{ a } p \times p \text{ matrix.}$$

Let  $P_n$  be the totality of partitions of  $n$ . Set  $\mathcal{C}_\eta = \{gJ_\eta g^{-1}; g \in G_C\}$ , the conjugate class of  $J_\eta$  and

$$A'(\eta) = \{(X, Y) \in A; Y \in \mathcal{C}_\eta, p(X) = \eta\}, \quad A(\eta) = \overline{A'(\eta)}.$$

Here  $p(X)$  is defined as follows (cf. [G]). Let  $m_1, \dots, m_k$  be the multiplicities of different eigenvalues of  $X$  in the decreasing order. Then  $p(X) = (m_1, \dots, m_k, \underbrace{0, \dots, 0}_{n-k})$  and is regarded as a partition of  $n$ .

We introduce a partial ordering on  $\mathbf{P}_n$ . For any two partitions  $\eta = (p_1, \dots, p_n)$  and  $\nu = (q_1, \dots, q_n)$  of  $n$ , define  $\eta \geq \nu$  if and only if  $\sum_{i=1}^j p_i \geq \sum_{i=1}^j q_i$  for any  $j$ . Then the following lemma is known.

**Lemma (4.4.1)** ([He, KP]). *Given  $\eta = (p_1, \dots, p_n), \nu = (q_1, \dots, q_n) \in \mathbf{P}_n$ . If  $\eta > \nu$  and no partition is between them (i.e.  $\eta$  and  $\nu$  are adjacent in the ordering), then one of the following two possibilities holds for  $\eta$  and  $\nu$ :*

- (I) *There is an  $i \in \mathbf{N}^+$  such that  $p_k = q_k$  for  $k \neq i, i+1$  and  $q_i = p_i - 1 \geq q_{i+1} = p_{i+1} + 1$ .*
- (II) *There are  $i, j \in \mathbf{N}^+, i < j$ , such that  $p_k = q_k$  for  $k \neq i, j$  and  $q_i = p_i - 1 = q_j = p_j + 1$ .*

Furthermore, in the case (I), we have  $\text{codim}_{\overline{\mathcal{C}}_\eta} \mathcal{C}_\nu = 2$  and in the case (II), we have  $\text{codim}_{\overline{\mathcal{C}}_\eta} \mathcal{C}_\nu = 2(j-i)$ .

**Theorem (4.5).**

- (1) *For any  $\eta \in \mathbf{P}_n$ ,  $A(\eta)$  is an irreducible component of  $A$  and  $\dim A(\eta) = n^2 - 1$  ( $= \dim \mathfrak{g}_\mathcal{C}$ ).*
- (2)  $A'(\eta) = A(\eta) - \bigcup_{\eta \neq \nu} A(\nu)$ .
- (3)  $A = \bigcup_{\eta \in \mathbf{P}_n} A(\eta)$  is the irreducible decomposition of  $A$ .
- (4) *Given  $\eta, \nu \in \mathbf{P}_n$ . If  $\eta > \nu$  and  $\eta$  and  $\nu$  are adjacent in the ordering, then  $\dim(A(\eta) \cap A(\nu)) = n^2 - 2$ .*

*Remark.* (1) The irreducible decomposition (3) of  $A$  is simpler than that of the corresponding analytic set in the Lie group case (cf. (1.11)).

(2) We conjecture that the converse of the statement (4) is valid. Namely, we conjecture that if  $\dim(A(\eta) \cap A(\nu)) = n^2 - 2$ , then  $\eta$  and  $\nu$  are adjacent in the ordering.

*Proof.* (1) and (3) are proved by arguments similar to those in (1.11). (2) is a consequence of Propositions 2.1 and 2.2 in [G]. What we must note here is that in this case, the set  $A'(\eta)$  is connected.

To prove (4), we need two lemmas.

**Lemma (4.5.1).** *Set  $Z = \mathbf{C}^3$  and  $S_k = \{(x, y, z) \in Z; x^{k+1} - yz = 0\}$ . We consider the cotangent bundle  $T^*Z$  of  $Z$  and denote by  $(\xi, \eta, \zeta)$  the conormal variables. Let  $A_1$  and  $A_2$  be the closure of the conormal bundle of  $S_k - \{0\}$  and the origin,*

respectively. Then  $A_1 \cap A_2 = \{(0; \xi, \eta, \zeta) \in T^*Z; \xi=0\}$ . In particular,  $\text{codim}_{A_i} A_1 \cap A_2 = 1$  ( $i=1, 2$ ).

**Lemma (4.5.2).** Set  $Z = \mathfrak{sl}(k, \mathbb{C})$  and  $S_k^*$  is the closure of the set  $\{X \in Z; X$  is conjugate to  $X_0\}$ , where  $X_0 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 0 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$ .

Identify the cotangent bundle  $T^*Z$  with  $Z \times Z$  as in (4.3). Let  $A_1$  and  $A_2$  be the closures of the conormal bundles of  $S_k^* - \{0\}$  and the origin, respectively. Then  $A_1 \cap A_2 = \{(0, Y) \in Z \times Z; D(Y)=0\}$ , where  $D$  denotes the discriminant polynomial on  $Z = \mathfrak{sl}(k, \mathbb{C})$ . In particular,  $\text{codim}_{A_i} A_1 \cap A_2 = 1$  ( $i=1, 2$ ).

Lemma (4.5.2) is less obvious than Lemma (4.5.1). Accordingly, we only give here a proof of Lemma (4.5.2). That of Lemma (4.5.1) may be accomplished by an argument similar to the one given below.

We are going to prove Lemma (4.5.2). Set  $G_k = SL(k, \mathbb{C})$  and  $I = \{A \in \mathfrak{sl}(k, \mathbb{C}); [A, X_0]=0\}$ . For the sake of convenience, we consider the element  $g = \begin{bmatrix} 1 & & & \\ \mathbf{g} & I_{k-2} & & \\ x & \mathbf{h} & & 1 \end{bmatrix}$  ( $\mathbf{g}, \mathbf{h} \in \mathbb{C}^{k-2}, x \in \mathbb{C}$ ). Then  $gI g^{-1} = \{gAg^{-1}; A \in I\}$  is the

centralizer of the element  $g(tX_0)g^{-1}$  ( $\in S_k^* - \{0\}$ ) for any  $t \in \mathbb{C}^x$ . Now we let the parameter  $t$  tend to zero. Then we find that the set  $M = \{gAg^{-1}; A \in I, \mathbf{g}, \mathbf{h} \in \mathbb{C}^{k-2}, x \in \mathbb{C}\}$  is contained in the set  $\{A \in Z; (0, A) \in A_1 \cap A_2\}$ . It is easy to check  $\dim M = \dim Z - 1$ . Furthermore it follows from the definition that for any  $A \in M$ , at least one eigenvalue of  $A$  has multiplicity  $\geq 2$ . This means that  $D(A)=0$  for any  $A \in M$ . Comparing the dimensions of  $M$  and the set  $S = \{A \in Z; D(A)=0\}$ , we find  $\bar{M} = S$ . Therefore it follows that  $A_1 \cap A_2 = \{(0, A) \in Z \times Z; D(A)=0\}$ . Hence Lemma (4.5.2) is shown.

We return the proof of Theorem (4.5) (4). Let  $\eta$  and  $\nu$  be the partitions of  $n$  such that  $\eta > \nu$  and that  $\eta$  and  $\nu$  are adjacent in the ordering. Then it follows from Lemma (4.4.1) that there are two possibilities. We first consider the case (I) in Lemma (4.4.1). Let  $Y \in \mathcal{C}_\eta$ . Then it follows from Theorem 3.2 in [KP] that there are an open neighbourhood  $U$  of  $Y$  in  $\mathfrak{g}_\mathbb{C}$  and a local coordinate system  $(x, y, z, t_1, \dots, t_p, u_1, \dots, u_q)$  on  $U$  ( $p+q+3 = \dim \mathfrak{g}_\mathbb{C}$ ) such that  $Y$  corresponds to the origin of  $U$  and

$$U \cap \mathcal{C}_\eta = \{(x, y, z, t, u) \in U; u_1 = \dots = u_q = 0, x^{k+1} + yz = 0\},$$

$$U \cap \mathcal{C}_\nu = \{(x, y, z, t, u) \in U; u_1 = \dots = u_q = x = y = z = 0\}.$$

Here  $k$  is the number uniquely determined from  $\eta$  and  $\nu$  (cf. Lemma (4.4.1) and [KP, Th. 3.2]). Let  $\pi$  be the projection of  $T^*\mathfrak{g}_\mathbb{C}$  to the base space  $\mathfrak{g}_\mathbb{C}$  and

set  $\pi^{-1}(U)=V$ . Then under the identification in (4.3),  $V \cong U \times \mathfrak{g}_C$  and

$$A(\eta) \cap \pi^{-1}(U) \cong A_1 \times \mathbf{C}^{p+q},$$

$$A(\nu) \cap \pi^{-1}(U) \cong A_2 \times \mathbf{C}^{p+q}.$$

Here we used the notation in Lemma (4.5.1). Accordingly, Lemma (4.5.1) implies that

$$\text{codim}_{A(\eta)} A(\eta) \cap A(\nu) = \text{codim}_{A(\nu)} A(\eta) \cap A(\nu) = 1.$$

If the case (II) of Lemma (4.4.1) occurs between  $\eta$  and  $\nu$ , one can show the claim in (4) by an argument similar to the above, using Lemma (4.5.2) instead of Lemma (4.5.1). Hence the proof of Theorem (4.5) is completed.

(4.6) We shall next obtain a theorem similar to Theorem (3.4).

Fix a partition  $\eta \in \mathbf{P}_n$ . A condition similar to (A) in (3.1) is then :

**Condition (A').** For any  $(A, X) \in A'(\eta)$ , there exist  $H, Y \in \mathfrak{g}_C$  and  $\phi \in I^*$  such that

$$(4.6.1) \quad [H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

$$(4.6.2) \quad [H, A] + d_X \phi = 0.$$

Here  $d_X \phi$  means the element of  $\mathfrak{g}_C$  defined by

$$(4.6.3) \quad (d_X \phi)Z = \frac{d}{dt} \phi(X + tZ)|_{t=0} \quad \text{for any } Z \in \mathfrak{g}_C.$$

Though (4.6.1) follows from the Jacobson-Morozov lemma, Condition (A') does not hold for every  $\eta \in \mathbf{P}_n$ .

**Theorem (4.7).** Let  $\eta = (p_1, \dots, p_n)$  be a partition of  $n$  such that  $p_1 = k, p_2 = \dots = p_{n-k+1} = 1, p_{n-k+2} = \dots = p_n = 0$ . Then the condition (A') holds for  $A(\eta)$ . And in this case, for any  $\lambda \in \mathfrak{h}^*$ , if  $u$  is the generator of the Ideal  $\mathcal{G}_\lambda$ , that is,  $u \equiv 1 \pmod{\mathcal{G}_\lambda}$ , we have  $\text{ord}_{A(\eta)}(u) = \{0\}$ .

*Proof.* Let us take an element  $(A, X) \in A'(\eta)$ . We may assume that  $X = \begin{bmatrix} J_k & \\ & 0 \end{bmatrix}$  without loss of generality. As is easily seen, there is an element  $g$  of  $G_C$  such that  $gXg^{-1} = X$  and  $gAg^{-1} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$  ( $A_1$  is a  $k \times k$  matrix and  $A_2$  is an  $(n-k) \times (n-k)$  matrix). Accordingly, from the first time we may set  $A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$ . Since  $A$  commutes with  $X$ , it follows that  $A_1 = a_0 I_k + \sum_{i=1}^{k-1} a_i (J_k)^i$  with complex numbers  $a_0, a_1, \dots, a_{k-1}$ . We take

$$H = \begin{pmatrix} k-1 & & & & \\ & k-3 & & & \\ & & \ddots & & \\ & & & -(k-1) & \\ & & & & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} & & & & 0 \\ 2(k-1) & 0 & & & \\ & 4(k-2) & 0 & & \\ & & \ddots & \ddots & \\ & & & & 0 \\ & & & 2(k-1) & 0 \\ & & & & 0 \end{pmatrix}$$

Then we find that (4.6.1) holds for  $X, H, Y$ . Furthermore  $[H, X] = \begin{bmatrix} B \\ 0 \end{bmatrix}$ , where  $B = \sum_{i=1}^{k-1} 2a_i(J_k)^i$ , a  $k \times k$  matrix. On the other hand, it follows from the definition that  $d_X P_i = \begin{bmatrix} (J_k)^{i-1} \\ 0 \end{bmatrix}$  ( $i=2, \dots, k$ ). Noting this, we set  $\phi = -\sum_{i=1}^{k-1} 2ia_i P_{i+1}$ . Then we obtain that  $[H, X] + d_X \phi = 0$  and that  $H, Y$  and  $\phi$  are required ones. The rest of the proof is shown by an argument similar to (3.3).  
 q. e. d.

*Remark (4.8).* If  $\eta \in \mathbf{P}_n$  does not satisfy the assumption in Theorem (4.7), the condition (A') does not hold for  $\mathcal{A}(\eta)$ .

(4.9) We conjecture that  $\text{ord}_{\mathcal{A}(\eta)}(u) = \{0\}$  for any  $\lambda \in \mathfrak{h}^*$  and  $\eta \in \mathbf{P}_n$  (cf. (3.5)). In the rest of this section we give a weak version of this conjecture. For this purpose we introduce some notation.

Let  $\eta$  be a partition of  $n$  and let  $I_\eta = \langle f_1, \dots, f_p \rangle$  be the defining ideal of the irreducible analytic subset  $\overline{\mathcal{C}}_\eta$  of  $\mathfrak{g}_c$ . Since each  $P_i(X)$  is contained in  $I_\eta$ , we write  $P_i(X) = \sum_{j=1}^p u_{ij}(X) f_j(X)$  ( $u_{ij}(X) \in P[\mathfrak{g}_c]$ ). Identifying  $S$  with  $P[\mathfrak{g}_c]$ , for any  $f \in P[\mathfrak{g}_c]$ , we denote by  $f^*$  the elements of  $S$  corresponding to  $f$ . Then it follows from the definition in [KO, p. 152] that the differential operator  $L_{\partial(P_i) - \chi(P_i)}$  on  $\mathcal{L}_{\mathcal{A}(\eta)}$  is of the form

$$L_{\partial(P_i) - \chi(P_i)} = \sum_{j=1}^p L'_{\partial(u_{ij}^*)} L_{\partial(f_j^*)} \quad (i=2, \dots, n),$$

where  $L'_{\partial(u_{ij}^*)} = L_{\partial(u_{ij}^*)}$  if  $u_{ij} \in I_\eta$  and  $L'_{\partial(u_{ij}^*)} = \sigma(\partial(u_{ij}^*)) = u_{ij}^*$  if otherwise (cf. [KO, Lemma 2.1]). Accordingly, any principal symbol  $\phi$  of the system  $\mathcal{N}_\lambda$  along  $\mathcal{A}(\eta)$  is a solution of the system of differential equations on  $\mathcal{L}_{\mathcal{A}(\eta)}$ :

$$(4.9.1) \quad \begin{cases} \sum_{j=1}^p L'_{\partial(u_{ij}^*)} L_{\partial(f_j^*)} \phi = 0 & (j=2, \dots, n), \\ L_{\tau(A)} \phi = 0 & (\forall A \in \mathfrak{g}_c). \end{cases}$$

Noting this, we consider the system of differential equations on  $\mathcal{L}_{\mathcal{A}(\eta)}$ :



$$(4.9.2) \quad \begin{cases} L_{\partial(f^*)}\phi=0 & (\forall f \in I_\eta), \\ L_{\tau(A)}\phi=0 & (\forall A \in \mathfrak{g}_C). \end{cases}$$

We note that the system (4.9.2) is in involution and that any solution of (4.9.2) is also that of (4.9.1).

**Theorem (4.10).** *For any  $\eta \in \mathbf{P}_n$ ,  $\text{ord}_{A(\eta)}(u)$  contains 0.*

*Proof.* To prove this, we recall a lemma.

**Lemma** (Tanisaki [T]). *Set  $\eta=(p_1, \dots, p_n)$ . Assume that  $p_1 \geq \dots \geq p_k > p_{k+1} = \dots = p_n = 0$ . We define polynomials which vanish on  $\bar{\mathcal{C}}_\eta$ . For any  $X \in \mathfrak{g}_C$ , we consider all the  $k$ -minors of the matrix  $tI_n - X$  ( $t$  is an indeterminate) and regard them as polynomials of  $t$ . Let  $f_{i,1}^m(X), \dots, f_{i,r(i,m)}^m(X)$  be the coefficients of the term  $t^m$  of these polynomials, where  $r(i,m)$  denotes the number of the minors above. Using these, we define an ideal  $I'_\eta$  of  $P[\mathfrak{g}_C]$  generated by  $f_{i,j}^m(X)$ ,  $i=1, \dots, n$ ,  $m=0, 1, \dots, u(i)$ ,  $j=1, 2, \dots, r(i,m)$ . Here we set  $u(i)=p_{n-i+1} + p_{n-i+2} + \dots$  ( $i=1, 2, \dots, n$ ). Then*

$$\bar{\mathcal{C}}_\eta = \{X \in \mathfrak{g}_C; f(X)=0 \text{ for any } f \in I'_\eta\}.$$

*Remark.* DeConcini-Procesi [DP] introduced an ideal of  $P[\mathfrak{g}_C]$  which has the same property as  $I'_\eta$ . The following is a conjecture:  $I'_\eta = I_\eta$ .

We return to the proof of Theorem (4.10). Let us take  $(A_0, X_0) \in A'(\eta)$ . As in the case of the proof of Theorem (4.7), we may assume that  $X_0 = J_\eta$  (cf. (4.4)) and  $A_0 = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{bmatrix}$  where each  $A_i$  is a  $p_i \times p_i$  matrix ( $i=1, \dots, k$ ).

Since  $[X_0, A_0]=0$ , each  $A_i$  is of the form  $A_i = \sum_{j=1}^{p_i} a_{i,j}(J_{p_i})^j$ . If we can choose  $H_0, Y_0 \in \mathfrak{g}_C$  and  $\phi \in I'_\eta$ , homogeneous, such that

$$(4.10.1) \quad \begin{aligned} [H_0, X_0] &= 2X_0, & [H_0, X_0] &= -2Y_0, & [X_0, Y_0] &= H_0, \\ [H_0, A_0] + d_{Y_0}\phi &= 0, \end{aligned}$$

we can prove the theorem by an argument similar to the proof of Theorem (3.3). Hence it suffices to show the existence of  $H_0, Y_0, \phi$  with the condition (4.10.1).

For this purpose, we define polynomials in  $I'_\eta$ . For any  $X=(x_{i,j}) \in \mathfrak{g}_C$ , we set  $X^{(m)}=(x_{i,j})_{u(m) \leq i, j \leq n}$  and define

$$\begin{aligned} \det(tI_{n-u(m)+1} + X^{(m)}) &= f_{m,0}(X) + f_{m,1}(X)t + \dots \\ &\quad + f_{m,p_m}(X)t^{p_m} + \dots + t^{n-u(m)+1}. \end{aligned}$$

Then due to the lemma above, we find that the polynomials  $f_{m,i}$  ( $i=1, 2, \dots, k$ ,  $i=0, 1, \dots, p_m$ ) are contained in  $I'_\eta$  and that  $d_{X_0}f_{m,i} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & J_m^{i-1} & \\ & & & & \ddots \\ & & & & & J_k^{i-1} \end{bmatrix}$ . Here

$J_p$  is the Jordan matrix of size  $p$ . Noting this we can take  $H_0, Y_0 \in \mathfrak{g}_C$  and  $\phi \in I'_\eta$  with the conditions above by an argument similar to the proof of Theorem (4.7). Hence the theorem is proved.

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