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Minimizing Indices of Conditional Expectations onto a Subfactor

By

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Abstract

For a pair of factors $M \supseteq N$, let $\mathscr{E}(M, N)$ be the set of all conditional expectations from M onto N. We characterize $E_0 \in \mathscr{E}(M, N)$ whose index is the minimum of $\{\text{Index } E : E \in \mathscr{E}(M, N)\}$. When $M \supseteq N$ are II_1 factors, we establish the relation between Index E_0 and $[M : N]$.

Introduction

Jones [5] developed the index theory for type II_1 factors using the coupling constant and Umegaki's conditional expectation [10]. Kosaki [6] extended it to arbitrary factors. Let M be a factor and N a subfactor of M. We denote by $\mathscr{E}(M, N)$ the set of all faithful normal conditional expectations from M onto N. The index Index E of $E \in \mathscr{E}(M, N)$ was introduced in [6] based on Connes' spatial theory [3] and Haagerup's theory on operator valued weights [4] as follows: Index $E = E^{-1}(1)$ where E^{-1} is the operator valued weight from N' to M' characterized by the equation $d(\varphi \circ E)/d\psi = d\varphi/d(\psi \circ E^{-1})$ of spatial derivatives. Here φ and ψ are faithful normal semifinite weights on N and M', respectively. See also [9, 12. 11].

As shown in [2, Théorème 1. 5. 5], $\mathscr{E}(M, N)$ contains at most one element if the relative commutant $N' \cap M$ is $\boldsymbol{C}1$. But $\mathscr{E}(M, N)$ has many elements in general. Indeed, when $\mathscr{E}(M, N) \neq \emptyset$, the map $E \mapsto E|N' \cap M$ is a bijection from $\mathscr{E}(M, N)$ onto the set of all faithful normal states on $N' \cap M$ (see [1, Théorème 5. 3]). The aim of this paper is to discuss the problem when Index E takes

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the minimum value for a fixed pair $M \supseteq N$.

§ 1. Main Result

From now on, let M be a σ-finite factor and N a subfactor of M with $\mathscr{E}(M, N) \neq \emptyset$. We first note the following two facts:

1° If Index $E<4$ for some $E \in \mathscr{E}(M, N)$, then $N' \cap M = \boldsymbol{C}1$ (see [6, Theorem 4.4]). In this case, $\mathscr{E}(M, N)$ consists of one element.

2° If Index $E<\infty$ for some $E \in \mathscr{E}(M, N)$, then $N' \cap M$ is finite dimensional (see [6, Proposition 4.3]). In this case, since $(\text{Index } E)^{-1} E^{-1} \in \mathscr{E}(N', M')$, it follows from [4, Theorem 6.6] that each operator valued weight from N' to M' is bounded. Hence Index $E'<\infty$ for every $E' \in \mathscr{E}(M, N)$.

The fact 2° shows that either Index $E<\infty$ for all $E \in \mathscr{E}(M, N)$ or Index $E=\infty$ for all $E \in \mathscr{E}(M, N)$.

Theorem 1. *Assume that* Index $E<\infty$ *for some* (*hence all*) $E \in \mathscr{E}(M, N)$.

(1) *There exists a unique* $E_0 \in \mathscr{E}(M, N)$ *such that*

$$\text{Index } E_0 = \min\{\text{Index } E : E \in \mathscr{E}(M, N)\}.$$

(2) *If* $E \in \mathscr{E}(M, N)$, *then the following conditions are equivalent*:

(i) $E = E_0$;

(ii) $E|N' \cap M$ *and* $E^{-1}|N' \cap M$ *are traces and*

$$E^{-1}|N' \cap M = (\text{Index } E) E|N' \cap M;$$

(iii) $E^{-1}|N' \cap M = cE|N' \cap M$ *for some constant c.*

(3) *If* $N' \cap M \neq \boldsymbol{C}1$, *then*

$$\{\text{Index } E : E \in \mathscr{E}(M, N)\} = [\text{Index } E_0, \infty).$$

Proof. We first show that there exists an $E_0 \in \mathscr{E}(M, N)$ satisfying condition (ii) of (2). Let φ and ψ be faithful normal semifinite weights on N and M', respectively. By [1, Théorème 5.3], we can choose an $E \in \mathscr{E}(M, N)$ such that $E|N' \cap M$ is a trace. For every unitary u in $N' \cap M$, we have by [3, Proposition 8 and Theorem 9]

$$\begin{aligned} \frac{d(\varphi \circ uEu^*)}{d\psi} &= u \frac{d(\varphi \circ E)}{d\psi} u^* \\ &= \left(u \frac{d(\psi \circ E^{-1})}{d\varphi} u^*\right)^{-1} \end{aligned}$$

$$=\frac{d\varphi}{d(\psi\circ uE^{-1}u^*)},$$

where $uEu^*=E(u^*\cdot u)$. Hence $(uEu^*)^{-1}=uE^{-1}u^*$. Since $uEu^*|N'\cap M=E|N'\cap M$, we get $uEu^*=E$ by [1, Théorème 5.3] again, so that $uE^{-1}u^*=E^{-1}$. This shows that $E^{-1}|N'\cap M$ is a trace. Choosing minimal projections $f_1, \cdots, f_n$ in $N'\cap M$ with $\sum_i f_i=1$, we define a positive invertible element h in the center of $N'\cap M$ by $h=\sum_{i=1}^n \alpha_i f_i$ where

$$\alpha_i=(\sum_{i=1}^n E(f_i)^{1/2}E^{-1}(f_i)^{1/2})^{-1}\frac{E^{-1}(f_i)^{1/2}}{E(f_i)^{1/2}}, \quad 1\leqq i\leqq n.$$

Now let $E_0=h^{1/2}Eh^{1/2}$. Then $E_0\in \mathscr{E}(M, N)$ follows from

$$E_0(y)=E(hy)=E(h)y=y, \quad y\in N.$$

Since

$$\begin{aligned}\frac{d(\varphi\circ E_0)}{d\psi}&=h^{1/2}\frac{d(\varphi\circ E)}{d\psi}h^{1/2}\\&=\left(h^{-1/2}\frac{d(\psi\circ E^{-1})}{d\varphi}h^{-1/2}\right)^{-1}\\&=\frac{d\varphi}{d(\psi\circ h^{-1/2}E^{-1}h^{-1/2})},\end{aligned}$$

we get $E_0^{-1}=h^{-1/2}E^{-1}h^{-1/2}$ and hence

$$\frac{E_0^{-1}(f_i)}{E_0(f_i)}=\frac{E^{-1}(h^{-1}f_i)}{E(hf_i)}=\alpha_i^{-2}\frac{E^{-1}(f_i)}{E(f_i)}=c, \quad 1\leqq i\leqq n,$$

where $c=(\sum_{i=1}^n E(f_i)^{1/2}E^{-1}(f_i)^{1/2})^2$. Therefore $E_0^{-1}|N'\cap M=cE_0|N'\cap M$, so that $c=\text{Index } E_0$.

(1) For each $E\in \mathscr{E}(M, N)$, let h be the Radon-Nikodym derivative of $E|N'\cap M$ with respect to the trace $E_0|N'\cap M$. Since $E=h^{1/2}E_0h^{1/2}$ follows from $E|N'\cap M=h^{1/2}E_0h^{1/2}|N'\cap M$, we obtain $E^{-1}=h^{-1/2}E_0^{-1}h^{-1/2}$ as above. Hence

$$\begin{aligned}\text{Index } E&=E_0^{-1}(h^{-1})\\&=(\text{Index } E_0)E_0(h^{-1})\\&\geqq \text{Index } E_0,\end{aligned}$$

because

$$1=E_0(1)\leqq E_0(h)^{1/2}E_0(h^{-1})^{1/2}=E_0(h^{-1})^{1/2}.$$

Moreover it is readily checked that $E_0(h^{-1})=1$ holds if and only if $h=1$, i.e. $E=E_0$. Therefore (1) is proved.

(2) (i)$\Rightarrow$(ii) is seen from the construction of E_0, and (ii)$\Rightarrow$(iii) is trivial. To show (iii)$\Rightarrow$(i), assume that $E\in\mathscr{E}(M, N)$ satisfies (iii). Then $c=$Index E. Let h be as in the proof of (1). Because $E_0^{-1}=h^{1/2}E^{-1}h^{1/2}$ and

$$\begin{aligned}\text{Index } E_0 &= E^{-1}(h)\\ &= (\text{Index } E)E(h)\\ &= (\text{Index } E)E_0(h^2)\\ &\geqq (\text{Index } E_0)E_0(h^2),\end{aligned}$$

we get

$$E_0((h-1)^2)=E_0(h^2)-1\leqq 0,$$

implying $h=1$ and thus $E=E_0$.

(3) Assuming $N'\cap M\neq \boldsymbol{C}1$, we choose nonzero projections p_1 and p_2 in $N'\cap M$ with $p_1+p_2=1$. For each $h=\alpha_1p_1+\alpha_2p_2$ with $\alpha_1, \alpha_2>0$ and $\alpha_1E_0(p_1)+\alpha_2E_0(p_2)=1$, letting $E=h^{1/2}E_0h^{1/2}$ we obtain $E\in\mathscr{E}(M, N)$ and

$$\begin{aligned}\text{Index } E &= E_0^{-1}(h^{-1})\\ &= (\text{Index } E_0)(\alpha_1^{-1}E_0(p_1)+\alpha_2^{-1}E_0(p_2)).\end{aligned}$$

Therefore Index E can take any real numbers in [Index E_0, ∞). □

§2. Case of II_1 Factors

Now let M be a type II_1 factor with the normalized trace τ. For a subfactor N of M, let $E_N\in\mathscr{E}(M, N)$ be Umegaki's conditional expectation [10] with respect to τ. Then Index E_N coincides with Jones' index $[M:N]$ (see [6]). By definition of Jones' index [5], $[M:N]<\infty$ if and only if N' on $L^2(M, \tau)$ is finite. In this case, let τ' be the normalized trace on N'.

Theorem 2. *Let $M\supseteq N$ be factors of type* II_1 *with* $[M:N]<\infty$, *and* $f_1, \cdots, f_n$ *be minimal projections in* $N'\cap M$ *with* $\sum_i f_i=1$.

(1) *If* $E_0\in\mathscr{E}(M, N)$ *is as in Theorem* 1, *then*

$$\text{Index } E_0=[M:N]\Big(\sum_{i=1}^{n}\tau(f_i)^{1/2}\tau'(f_i)^{1/2}\Big)^2.$$

(2) *The following conditions are equivalent*:

(i) $[M:N]=\min\{\text{Index } E : E\in\mathscr{E}(M, N)\}$;

(ii) $\tau' | N' \cap M = \tau | N' \cap M$.

(3) *If* $[M : N] \geqq 4$, *then*

$$[M : N] \geqq 4 (\sum_{i=1}^{n} \tau(f_i)^{1/2} \tau'(f_i)^{1/2})^{-2}.$$

Proof. (1) We can take $E = E_N$ in the first part of the proof of Theorem 1. Since $E_N | N' \cap M = \tau | N' \cap M$ and $E_N^{-1} | N' \cap M = [M : N] \tau' | N' \cap M$ (see [6]), we have $E_0 = h^{1/2} E_N h^{1/2}$ where $h = \sum_{i=1}^{n} \alpha_i f_i$ and

$$\alpha_i = (\sum_{i=1}^{n} \tau(f_i)^{1/2} \tau'(f_i)^{1/2})^{-1} \frac{\tau'(f_i)^{1/2}}{\tau(f_i)^{1/2}}, \qquad 1 \leqq i \leqq n.$$

Therefore

$$\begin{aligned} \text{Index } E_0 &= E_N^{-1}(h^{-1}) \\ &= [M : N] \sum_{i=1}^{n} \alpha_i^{-1} \tau'(f_i) \\ &= [M : N] (\sum_{i=1}^{n} \tau(f_i)^{1/2} \tau'(f_i)^{1/2})^2. \end{aligned}$$

(2) Because condition (i) means $E_N = E_0$, it follows from Theorem 1(2) that (i) is equivalent to $E_N^{-1} | N' \cap M = [M : N] E_N | N' \cap M$, that is, $\tau' | N' \cap M = \tau | N' \cap M$.

(3) Since $[M : N] \geqq 4$, we get Index $E_0 \geqq 4$ (see 1° before Theorem 1). Then the desired inequality follows from (1). □

Remarks. Let $M \supseteq N$ be type II_1 factors with $[M, N] < \infty$.

(1) Let $\mathrm{H}(M | N)$ be the entropy considered in [7]. It was shown in [7, Corollary 4.5] that condition (ii) of Theorem 2 is equivalent to the equality $\mathrm{H}(M | N) = \log[M : N]$. In particular if $[M : N] = 4$, then $\sum_i \tau(f_i)^{1/2} \tau'(f_i)^{1/2} = 1$ by Theorem 2(3), so that (ii) holds. In this connection, see [7, Corollary 4.8].

(2) Let $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \cdots$ be the sequence of type II_1 factors obtained by iterating the basic construction [5]. The following result is in [8]: If $\mathrm{H}(M | N) = \log[M : N]$ (equivalently $[M : N] = \min\{\text{Index } E : E \in \mathscr{E}(M, N)\}$), then $\mathrm{H}(M_n | N) = \log[M_n : N]$ (equivalently $[M_n : N] = \min\{\text{Index } E : E \in \mathscr{E}(M_n, N)\}$) for every $n \geqq 1$.

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