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C^∞ Ill-Posedness of the Mixed Problem for Hyperbolic Equations of Second Order

By

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§1. Introduction

We study the C^∞ ill-posedness of the mixed problem for hyperbolic equations of second order

$$
(1.1)\quad \begin{cases}
L[u]=D_t^2u-2\sum_{j=1}^{n}h_j(t,x)D_tD_{x_j}u-\sum_{i,j=1}^{n}a_{ij}(t,x)D_{x_i}D_{x_j}u+a_0(t,x)D_tu \\
\qquad +\sum_{j=1}^{n}a_j(t,x)D_{x_j}u+d(t,x)u=f(t,x) \quad \text{in } \Omega \\
B[u]|_{x_1=0}=a_{11}(t,0,x')^{-1/2} \\
\qquad \cdot\left\{a_{11}(t,0,x')D_{x_1}u+\sum_{j=2}^{n}a_{1j}(t,0,x')D_{x_j}u+h_1(t,0,x')D_tu\right\} \\
\qquad +\sum_{j=2}^{n}b_j(t,x')D_{x_j}u-c(t,x')\left(1+\frac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{1/2} \\
\qquad \cdot\left\{D_tu-\left(1+\frac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{-1}\right. \\
\qquad \left.\cdot\sum_{j=2}^{n}\left(h_j(t,0,x')-\frac{h_1(t,0,x')}{a_{11}(t,0,x')}a_{1j}(t,0,x')\right)D_{x_j}u\right\} \\
\qquad +\gamma(t,x')u|_{x_1=0}=g(t,x') \quad \text{in } \hat{\Omega} \\
\operatorname{supp}[u]\subset\Omega_{T_0}^+
\end{cases}
$$

where $x=(x_1, x_2, \cdots, x_n)$, $x'=(x_2, \cdots, x_n)$ $(n\geqq 2)$, $D_t=\partial/\partial t$, $D_{x_j}=\partial/\partial x_j$, Ω be an open neighborhood of $(t, x_1, x')=(0, 0, 0, \cdots, 0)$ in $\boldsymbol{R}^1\times\overline{\boldsymbol{R}_+^1}\times\boldsymbol{R}^{n-1}$, $\hat{\Omega}=\{(t, x') \mid (t, 0, x')\in\Omega\}$, T_0 is a fixed positive number, $\Omega_{T_0}^{\pm}=\{(t, x_1, x')\in\Omega \mid \pm(t-T_0)\geqq 0\}$, the coefficients of operators L and B are smooth in Ω and $\hat{\Omega}$ respectively.

We assume the following conditions:

(A. I) The operator L is regularly hyperbolic in $\Omega_{T_0}^+$, $a_{11}(t, 0, x')>0$ in $\hat{\Omega}_{T_0}^+$ and $\hat{\Omega}_{T_0}^+$ is connected where $\hat{\Omega}_{T_0}^+=\{(t, x') \mid (t, 0, x')\in\Omega_{T_0}^+\}$.

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(A. II) $c(t, x') \neq -1$ in $\hat{\Omega}^{+}_{T_0}$.

(A. III) The quadratic equation

$$(c(T, x_0')+1)z^2+2b(T, x_0', \eta_0')z+c(T, x_0')-1=0 \tag{1.2}$$

has two roots z_1 and z_2 such that $|z_2|>1$ and $I_m z_2 \neq 0$, where $x_0'=(x_{02}, \cdots, x_{0n})$, $(T, x_0') \in \hat{\Omega}^{+}_{T_0}$, $\eta_0'=(\eta_{02}, \cdots, \eta_{0n}) \in R^{n-1}-\{0\}$, and

$$\begin{cases} b(T, x_0', \eta_0')=\sum_{j=2}^{n} b_j(T, x_0')\eta_{0j}/\tilde{d}(T, x_0', \eta_0') \\ \tilde{d}(T, x_0', \eta_0')=\Big[\sum_{i,j=2}^{n} a_{ij}(T, 0, x_0')\eta_{0i}\eta_{0j}-\dfrac{1}{a_{11}(T, 0, x_0')} \\ \qquad \cdot\Big(\sum_{j=2}^{n} a_{1j}(T, 0, x_0')\eta_{0j}\Big)^2+\Big(1+\dfrac{h_1(T, 0, x')^2}{a_{11}(T, 0, x')}\Big)^{-1} \\ \qquad \cdot\Big(\sum_{j=2}^{n} h_j(T, 0, x')\eta_{0j}-\dfrac{h_1(T, 0, x')}{a_{11}(T, 0, x')}\sum_{j=2}^{n} a_{1j}(T, 0, x')\eta_{0j}\Big)^2\Big]^{1/2}. \end{cases} \tag{1.3}$$

Definition. We say that the mixed problem (1.1) is C^∞ well-posed in $\Omega^{+}_{T_0}$ if the following conditions hold:

(i) For every $f \in C^\infty(\Omega)$ and $g \in C^\infty(\hat{\Omega})$ with $\mathrm{supp}[f] \subset \Omega^{+}_{T_0}$ and $\mathrm{supp}[g] \subset \hat{\Omega}^{+}_{T_0}$, there exists u in $C^\infty(\Omega)$ which satisfies (1.1).

(ii) Provided that $u \in C^\infty(\Omega)$, $\mathrm{supp}[u] \subset \Omega^{+}_{T_0}$, $L[u]=0$ in $\Omega^{-}_{t_0}$ and $B[u]|_{x_1=0}=0$ in $\hat{\Omega}^{-}_{t_0}$ for every $t_0>T_0$, $u=0$ in $\Omega^{-}_{t_0}$ where $\Omega^{-}_{t_0}=\{(t, x) \in \Omega \mid t \leqq t_0\}$ and $\hat{\Omega}^{-}_{t_0}=\{(t, x') \mid (t, 0, x') \in \Omega^{-}_{t_0}\}$.

Now, we state our result:

Theorem. *Assume the conditions* (A. I), (A. II) *and* (A. III). *Then, the mixed problem* (1.1) *is not C^∞ well-posed in* $\Omega^{+}_{T_0}$.

By Wakabayashi's result [13: Theorem 1.4], we know that the Lopatinski determinant of the system $\{L^0, B^0\}$ is not zero if the problem (1.1) is C^∞ well-posed, where L^0 and B^0 are the principal part of L and B in (1.1) respectively. In [2], [8], [9], [10], [11], and [12], we discovered and developed the method to reduce the mixed problem for hyperbolic equations of second order to the one for symmetric hyperbolic pseudo differential systems of first order with non-negative boundary condition or non-negative type boundary condition. In [3], Kreiss studied the Lopatinski determinant for the mixed problem for a simple hyperbolic system of first order with two space variables. Here, we can easily prove that the Lopatinski determinant $R(t, x', \tau, \eta')$ for the system $\{L^0, B^0\}$ is zero at $(T, x_0', \tau_0, \eta_0')$ or $(T, x_0', \tau_0, -\eta_0')$ $(\mathrm{Re}\,\tau_0>0)$ by the above systematization of the problem (1.1) and the similar method as the one in [3]. Therefore, we have the above theorem.

Remark 1. Instead of Ω, we can obtain the same result as our theorem for a domain Ω_* such that Ω_* is an open neighborhood of $(t, x)=(0, 0)$ in $\boldsymbol{R}^1\times\bar{\omega}$ and ω is a domain in $\boldsymbol{R}^n$ with smooth boundary.

Remark 2. We consider the mixed problem

$$\begin{cases} L[u]=f & \text{in } \Omega \\ B[u]|_{x_1=0}=g & \text{in } \hat{\Omega} \end{cases} \tag{1.4}$$

where $(t, x)=(t, x_1, x')\in\Omega=\boldsymbol{R}^1\times\overline{\boldsymbol{R}^1_+}\times\boldsymbol{R}^{n-1}$, $(t, x')\in\hat{\Omega}=\boldsymbol{R}^1\times\boldsymbol{R}^{n-1}$, L and B are the same operators in (1.1), the coefficients h_j, a_{ij}, a_0, a_j and d belong to $\mathscr{B}(\Omega)$ and are constant outside a compact set in Ω, and the coefficients b_j, c and γ belong to $\mathscr{B}(\hat{\Omega})$ and are constant outside a compact set in $\hat{\Omega}$.

We assume the following condition for the problem (1.4):

(A. I′) The operator L is regularly hyperbolic in Ω and $a_{11}(t, 0, x')>0$ in $\hat{\Omega}$.

Then, the mixed problem (1.4) is L^2 well-posed in Ω if and only if for every $(t, x')\in\hat{\Omega}$, the quadratic equation

$$(c(t, x')+1)z^2+2b(t, x', \eta')z+c(t, x')-1=0 \tag{1.5}$$

has roots in $\bar{D}=\{z\in\boldsymbol{C} \mid |z|\leqq 1\}$ if they are different and in $D=\{z\in\boldsymbol{C} \mid |z|<1\}$ if they are equal.

We have the above fact by the Hermite theorem in [4] and the conformal mapping from the upper half plane to the unit disk (see [2]).

In [7: Remark 1.6], Taira treated the mixed problem for wave equation when the (1.5) has a double root in $D_1=\{z\in\boldsymbol{C} \mid |z|=1, I_m z\neq 0\}$, and obtained a well-posedness of the problem.

Remark 3. In [6: Theorem 2], Soga considered the mixed problem for wave equation in a half space ($n\geqq 3$) where the problem is C^∞ well-posed in our sense and has the finite propagation property, and obtained the condition which give the ill-posedness of the problem. By Wakabayashi's result [13: Theorem 1.6] and the method of our proof of the theorem, we can easily obtain the same result as the one in [6].

Remark 4. When the coefficients of L and B are all constant and $\Omega=\boldsymbol{R}^1\times\overline{\boldsymbol{R}^1_+}\times\boldsymbol{R}^{n-1}$, we know a necessary and sufficient condition for C^∞ well-posedness by Sakamoto [5]. In [1], Agemi and Shirota studied the mixed problem for wave equation in a half space precisely when $n=2$, b_2 and c are real constants, and $\gamma=0$.

An outline of this paper is as follows. In § 2, we treat the systematization of the wave equation. In § 3, we give the proof of the theorem.

§ 2. Some Examples for Reduction to Symmetric Hyperbolic System

In this section, we prepare the systematization of the wave equation $u_{tt}-u_{xx}-u_{yy}=0$, which have some parameters and enable us to study the Cauchy problem and the mixed problem for hyperbolic equations of second order by choosing parameters as appropriate functions and operators (see [2], [8], [9], [10], [11], [12]).

We set

$$U=\begin{pmatrix}U_1\\U_2\end{pmatrix}=\begin{pmatrix}u_t-u_x+z_1u_y\\z_1(u_t+u_x)+u_y\end{pmatrix} \tag{2.1}$$

$$V=\begin{pmatrix}V_1\\V_2\\V_3\end{pmatrix}=\begin{pmatrix}u_t-u_x+z_2(p_2-iq_2)u_y\\z_2(u_t+u_x)+(p_2+iq_2)u_y\\\sqrt{1+|z_2|^2}\,r_2u_y\end{pmatrix} \tag{2.2}$$

and

$$W=\begin{pmatrix}W_1\\W_2\\W_3\end{pmatrix}=\begin{pmatrix}u_t-u_x+z_3(p_3-iq_3)u_y\\z_3(u_t+u_x)+(p_3+iq_3)u_y\\u_y\end{pmatrix} \tag{2.3}$$

where $u_{tt}-u_{xx}-u_{yy}=0$, z_1, z_2 and z_3 are complex constants, p_2, q_2, p_3 and q_3 are real constants with $p_2^2+q_2^2<1$ and $p_3^2+q_3^2\leqq 1$, and $r_2=(1-p_2^2-q_2^2)^{1/2}$.

Proposition 2.1. *U, V and W satisfy the following equations respectively:*

$$U_t=\begin{pmatrix}-1&0\\0&1\end{pmatrix}U_x+\begin{pmatrix}0&1\\1&0\end{pmatrix}U_y=A_1U_x+B_1U_y \tag{2.4}$$

$$\begin{cases} V_t=A_2V_x+B_2V_y \\ A_2=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&\dfrac{1-|z_2|^2}{1+|z_2|^2}\end{pmatrix} \\ B_2=\begin{pmatrix}0&p_2-iq_2&\dfrac{r_2}{\sqrt{1+|z_2|^2}}\\p_2+iq_2&0&\dfrac{z_2r_2}{\sqrt{1+|z_2|^2}}\\\dfrac{r_2}{\sqrt{1+|z_2|^2}}&\dfrac{\bar{z}_2r_2}{\sqrt{1+|z_2|^2}}&\dfrac{-2(p_2\operatorname{Re}z_2+q_2\operatorname{Im}z_2)}{1+|z_2|^2}\end{pmatrix}\end{cases} \tag{2.5}$$

and

$$\begin{cases} M_3W_t=A_3W_x+B_3W_y \\ M_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&(1+|z_3|^2)r_3\end{pmatrix},\quad A_3=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&(1-|z_3|^2)r_3\end{pmatrix}\end{cases} \tag{2.6}$$

$$B_3=\begin{pmatrix} 0 & p_3-iq_3 & r_3 \\ p_3+iq_3 & 0 & z_3r_3 \\ r_3 & \bar{z}_3r_3 & -2r_3(p_3 \operatorname{Re} z_3+q_3 \operatorname{Im} z_3) \end{pmatrix}$$

where $r_3=1-p_3^2-q_3^2$.

Proof. By simple calculations, we obtain Proposition 2.1. Q. E. D.

Remark 5. Let M be

$$M=D_t^2-2\Big(\alpha_1 D_{x_1}+\sum_{j=2}^n \alpha_j D_{x_j}\Big)D_t - \Big(\beta_{11}D_{x_1}^2+2\sum_{j=2}^n \beta_{1j}D_{x_1}D_{x_j}+\sum_{i,j=2}^n \beta_{ij}D_{x_i}D_{x_j}\Big). \tag{2.7}$$

If M is regularly hyperbolic with respect to t and $\beta_{11}>0$, we obtain

$$\begin{cases} \sigma(M)=\tilde{\tilde{\xi}}^2+\tilde{\tilde{d}}(\eta')^2-\tilde{\tilde{\tau}}^2 \\ \tilde{\tilde{d}}(\eta')>0 \quad (\eta'\neq 0) \end{cases} \tag{2.8}$$

which correspond to the symbol of the wave equation $\xi^2+|\eta'|^2-\tau^2$ $(\eta'=(\eta_2, \cdots, \eta_n))$ where

$$\begin{cases} \tilde{\tilde{\xi}}=\dfrac{1}{\sqrt{\beta_{11}}}\Big(\beta_{11}\xi+\sum_{j=2}^n \beta_{1j}\eta_j+\alpha_1\tau\Big) \\ \tilde{\tilde{\tau}}=\Big(1+\dfrac{\alpha_1^2}{\beta_{11}}\Big)^{1/2}\Big\{\tau-\Big(1+\dfrac{\alpha_1^2}{\beta_{11}}\Big)^{-1}\Big(\sum_{j=2}^n \alpha_j\eta_j-\dfrac{\alpha_1}{\beta_{11}}\sum_{j=2}^n \beta_{1j}\eta_j\Big)\Big\} \\ \tilde{\tilde{d}}(\eta')=\Big[\sum_{i,j=2}^n \beta_{ij}\eta_i\eta_j-\dfrac{1}{\beta_{11}}\Big(\sum_{j=2}^n \beta_{1j}\eta_j\Big)^2 \\ \qquad +\Big(1+\dfrac{\alpha_1^2}{\beta_{11}}\Big)^{-1}\Big(\sum_{j=2}^n \alpha_j\eta_j-\dfrac{\alpha_1}{\beta_{11}}\sum_{j=2}^n \beta_{1j}\eta_j\Big)^2\Big]^{1/2} \end{cases} \tag{2.9}$$

for $(\xi, \eta')\in \boldsymbol{R}^n$.

§3. Proof of the Theorem

In this section, we shall prove the theorem.

Let z_1 and z_2 be two roots of the quadratic equation (1.2) such that $|z_2|>1$ and $\operatorname{Im} z_2\neq 0$. We set

$$z_2=re^{i\theta} \quad (r>1,\ 0<\theta<\pi \text{ or } \pi<\theta<2\pi) \tag{3.1}$$

and

$$
(3.2)\quad \begin{cases} \tilde{\tau}_{\pm}=\left(1+\dfrac{h_1(T,\,0,\,x_0')^2}{a_{11}(T,\,0,\,x_0')}\right)^{1/2}\Bigl\{\tau\mp i\left(1+\dfrac{h_1(T,\,0,\,x_0')^2}{a_{11}(T,\,0,\,x_0')}\right)^{-1} \\ \qquad\qquad \cdot\left(\sum\limits_{j=2}^{n} h_j(T,\,0,\,x_0')\eta_{0j}-\dfrac{h_1(T,\,0,\,x_0')}{a_{11}(T,\,0,\,x_0')}\sum\limits_{j=2}^{n} a_{1j}(T,\,0,\,x_0')\eta_{0j}\right)\Bigr\} \\ \tilde{d}(\eta_0')=\tilde{d}(T,\,x_0',\,\eta_0')\quad (=\tilde{d}(-\eta_0')) \\ \tau=\mu+i\sigma\quad (\mu>0,\ \sigma\in \boldsymbol{R}^1) \end{cases}
$$

where T, x_0, η_0' and $\tilde{d}(T,\,x_0',\,\eta_0')$ are the same as in (1.3). Then, we have the Lopatinski determinant $R(T,\,x_0',\,\tau,\,\pm\eta_0')$ of the $\{L^0,\,B^0\}$ at $(T,\,x_0',\,\tau,\,\pm\eta_0')$,

$$
(3.3)\quad R(T,\,x_0',\,\tau,\,\pm\eta_0')=-\sqrt{\tilde{\tau}_{\pm}^2+\tilde{d}(\eta_0')^2}\pm i\sum_{j=2}^{n} b_j(T,\,x_0')\eta_{0j}-c(T,\,x_0')\tilde{\tau}
$$
$$
=-\frac{c(T,\,x_0')+1}{2}(1+z_2)\begin{pmatrix}\tilde{\tau}_{\pm}\sqrt{\tilde{\tau}_{\pm}^2+\tilde{d}(\eta_0')^2}\pm z_1 i\tilde{d}(\eta_0') \\ z_1\left(\tilde{\tau}_{\pm}-\sqrt{\tilde{\tau}_{\pm}^2+\tilde{d}(\eta_0')^2}\right)\pm i\tilde{d}(\eta_0')\end{pmatrix}
$$

where L^0 and B^0 are the principal part of L and B in (1.1) respectively, and $\mathrm{Re}\{\sqrt{\tilde{\tau}_{\pm}^2+\tilde{d}(\eta_0')^2}\}>0$. The systematization (2.1) is used in (3.3) (see [2]). From now on, we shall obtain a similar result as in [3: Lemma 2.2] and prove $R(T,\,x_0',\,\tau_0,\,\eta_0')=0$ or $R(T,\,x_0',\,\tau_0,\,-\eta_0')=0$ for a $\tau=\tau_0$ ($\mathrm{Re}\,\tau_0>0$).

(i) *The case* $0<\theta<\pi$.

By $\tilde{d}(\eta_0')>0$, we can set

$$
(3.4)\quad \tilde{\tau}_{+}+\sqrt{\tilde{\tau}_{+}^2+\tilde{d}(\eta_0')^2}+z_1 i\tilde{d}(\eta_0')=i(p+iq)\tilde{d}(\eta_0')
$$

where p and q are real numbers. Then, by (3.4), we get

$$
(3.5)\quad z_1\left(\tilde{\tau}_{+}-\sqrt{\tilde{\tau}_{+}^2+\tilde{d}(\eta_0')^2}\right)+i\tilde{d}(\eta_0')
$$
$$
=\frac{1}{\tilde{d}(\eta_0')}[z_1\tilde{d}(\eta_0')(\tilde{\tau}_{+}-\sqrt{\tilde{\tau}_{+}^2+\tilde{d}(\eta_0')^2})]+i\tilde{d}(\eta_0')
$$
$$
=\frac{1}{\tilde{d}(\eta_0')}[\{(p+iq)\tilde{d}(\eta_0')+i(\tilde{\tau}_{+}+\sqrt{\tau_{+}^2+\tilde{d}(\eta_0')^2})\}(\tilde{\tau}_{+}-\sqrt{\tilde{\tau}_{+}^2+\tilde{d}(\eta_0')^2})]+i\tilde{d}(\eta_0')
$$
$$
=(p+iq)(\tilde{\tau}_{+}-\sqrt{\tilde{\tau}_{+}^2+\tilde{d}(\eta_0')^2})+\frac{i}{\tilde{d}(\eta_0')}[\tilde{\tau}_{+}^2-\tilde{\tau}_{+}^2-\tilde{d}(\eta_0')^2]+i\tilde{d}(\eta_0')
$$
$$
=(p+iq)(\tilde{\tau}_{+}-\sqrt{\tilde{\tau}_{+}^2+\tilde{d}(\eta_0')^2}).
$$

By (3.3), (3.4) and (3.5), we have

$$
(3.6)\quad R(T,\,x_0',\,\tau,\,\eta_0')=-\frac{c(T,\,x_0')+1}{2}(p+iq)(1\ \ z_2)\begin{pmatrix} i\tilde{d}(\eta_0') \\ \tilde{\tau}_{+}-\sqrt{\tilde{\tau}_{+}^2+\tilde{d}(\eta_0')^2}\end{pmatrix}.
$$

By $i\tilde{d}(\eta_0')+z_2(\tilde{\tau}_{+}-\sqrt{\tilde{\tau}_{+}^2+\tilde{d}(\eta_0')^2})=0$, we obtain

$$\tilde{\tau}_+=\frac{z_2^2+1}{2iz_2}\tilde{d}(\eta_0')=-\frac{i}{2}\Big(z_2+\frac{1}{z_2}\Big)\tilde{d}(\eta_0') \tag{3.7}$$
$$=-\frac{i}{2}\Big[\Big(r+\frac{1}{r}\Big)\cos\theta+i\Big(r-\frac{1}{r}\Big)\sin\theta\Big]\tilde{d}(\eta_0').$$

Therefore, we have

$$\operatorname{Re}\tilde{\tau}_+=\frac{1}{2}\Big(r-\frac{1}{r}\Big)\sin\theta\cdot\tilde{d}(\eta_0')>0. \tag{3.8}$$

Also, we get

$$\begin{cases}\sqrt{\tilde{\tau}_+^2+\tilde{d}(\eta_0')^2}=-\dfrac{i}{2}\Big[\Big(r-\dfrac{1}{r}\Big)\cos\theta+i\Big(r+\dfrac{1}{r}\Big)\sin\theta\Big]\tilde{d}(\eta_0')\\ \operatorname{Re}\{\sqrt{\tilde{\tau}_+^2+\tilde{d}(\eta_0')^2}\}>0.\end{cases} \tag{3.9}$$

Then, we have $R(T, x_0', \tau_0, \eta_0')=0$ where τ_0 is a complex number with a positive real part which is determined by (3.7).

(ii) *The case* $\pi<\theta<2\pi$.

By $\tilde{d}(-\eta_0')=\tilde{d}(\eta_0')$, we have, at $(T, x_0', \tau, -\eta_0')$,

$$R(T, x_0', \tau, -\eta_0')=-\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2}-i\sum_{j=2}^{n}b_j(T, x_0')\eta_{0j}-c(t, x_0')\tilde{\tau}_- \tag{3.10}$$
$$=-\frac{c(T, x_0')+1}{2}(1\ \ z_2)\begin{pmatrix}\tilde{\tau}_-+\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2}-z_1i\tilde{d}(\eta_0')\\ z_1\Big(\tilde{\tau}_--\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2}\Big)-i\tilde{d}(\eta_0')\end{pmatrix}$$

where $\operatorname{Re}\{\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2}\}>0$.

By $\tilde{d}(\eta_0')>0$, we can set

$$\tilde{\tau}_-+\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2}-z_1i\tilde{d}(\eta_0')=-i(p+iq)\tilde{d}(\eta_0'). \tag{3.11}$$

Then, we have, by (3.11),

$$z_1\Big(\tilde{\tau}_--\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2}\Big)-i\tilde{d}(\eta_0')=\frac{1}{\tilde{d}(\eta_0')}[z_1\tilde{d}(\eta_0')(\tilde{\tau}_--\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2})]-i\tilde{d}(\eta_0') \tag{3.12}$$
$$=(p+iq)(\tilde{\tau}_--\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2}).$$

By (3.11) and (3.12), we obtain

$$R(T, x_0', \tau, -\eta_0')=-\frac{c(T, x_0')+1}{2}(p+iq)[-i\tilde{d}(\eta_0')+z_2(\tilde{\tau}_--\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2})]. \tag{3.13}$$

By $-i\tilde{d}(\eta_0')+z_2(\tilde{\tau}_--\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2})=0$, we get

$$\tilde{\tau}_-=\frac{z_2^2+1}{-2iz_2}\tilde{d}(\eta_0')=\frac{i}{2}\Big[\Big(r+\frac{1}{r}\Big)\cos\theta+i\Big(r-\frac{1}{r}\Big)\sin\theta\Big]\tilde{d}(\eta_0'). \tag{3.14}$$

Therefore, we have

(3.15) $$\operatorname{Re}\tilde{\tau}_{-}=-\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta\cdot\tilde{d}(\eta_0')>0\,.$$

Also, we obtain $\operatorname{Re}\{\sqrt{\tilde{\tau}_-^2+\tilde{d}(\eta_0')^2}\}>0$. Then, we have $R(T,\, x_0',\, \tau_0,\, -\eta_0')=0$ where τ_0 is a complex number with a positive real part which is determined by (3.14).

By the above results and Wakabayashi's result [13: Theorem 1.4], we get our theorem. Q.E.D.

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