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On Bounded Part of an Algebra of Unbounded Operators

By

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§1. Introduction

By an Op^*-algebra A is meant a collection of linear operators, not necessarily bounded, all defined in a dense subspace D of a Hilbert Space H, satisfying $TD \subset D$ for all $T \in A$, which forms *a* * algebra with vectorwise operations $(T+S)x = Tx + Sx$, $(\lambda T)x = \lambda Tx$ (λ *a* scalar), $(TS)x = T(Sx)$ and the involution $T \to T^* = T^{\#}|_D$, $T^{\#}$ denoting the operator adjoint of T. It is also assumed that the identity operator $1 \in A$. The * subalgebra $A_b = \{T \in A | T \text{ is bounded}\}$ is the *bounded part of* A. Throughout, $\|\cdot\|$ denote operator norm on A_b.

Op^*-algebras have been investigated in the contect of quantum theory and representation theory of abstract (Non-Banach)* algebras, in particular, enveloping algebras of Lie algebras. Among selfadjoint Op^*-algebra [13], there are two classes that are better behaved viz. symmetric algebras ([5], [8], [9]) and countably dominated algebras ([2], [10], [11]). The objective of this paper is to examine role of A_b in the structure of these two classes of algebras.

An Op^*-algebra A is symmetric if for each $T \in A$, $(1+T^*T)^{-1}$ exists and $(1+T^*T)^{-1} \in A_b$. We prove the following that shows that in a symmetric algebra A, A_b is very closely tied up with A, algebraically as well as topologically; and this infact characterizes symmetry.

Theorem 1. *Let* A *be an* Op^*-*algebra.*

(a) *If* A *is symmetric, then* A_b *is sequentially dense in* A *in any * algebra topology* τ *on* A *such that* $B_0 = \{T \in A_b | \|T\| \leq 1\}$ *is* τ-*bounded.*

(b) *Let* τ *be any * algebra topology on* A *such that the multiplication in* A *is* τ-*hypocontinuous, and* B_0 *is* τ-*bounded and* τ-*sequentially complete. If* $(A_b, \|\cdot\|)$ *is sequentially* τ-*dense in* A, *then* A *is symmetric.*

An Op^*-algebra A is *countably dominated* if the positive cone A^+ of A contains a cofinal sequence (A_n) in its natural ordering. We assume $A_n \geq 1$ and

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$A_1=1$. Additionally, $A_n^{-1}\in A$ for all n, then A is said to satisfy *condition* (I) [2]. In non commutative integration with countably dominated algebras, significant role is being played by the σ-weak topology and ρ-topology (introduced in [3]) defined below.

For each n define normed linear space (n_{A_n}, ρ_{A_n}) by

$$n_{A_n}=\left\{T\in A\,|\,\rho_{A_n}(T)=\sup_{x\in D}\frac{|\langle Tx, x\rangle|}{\langle A_n x, x\rangle}<\infty\right\}.$$

Then $A=\cup n_{A_n}$, and *ρ-topology* on A is the inductive topology [14] defined by the embeddings $id_n:(n_{A_n}, \rho_{A_n})\to A$. Note that $A_b=n_1$. Now consider $\beta(D, D)$, the space of bounded sesquilinear forms on $D\times D$ with *bibounded topology* τ_{bb} defined by the seminorms $\beta\to\sup\{|\beta(x, y)|\mid x\in K_1, y\in K_2\}$, K_1 and K_2 varying over bounded subsets of (D, t_A), where t_A is the *induced topology* on D defined by the seminorms $x\to\|Tx\|$ $(T\in A)$. (Note that t_A is metrizable due to countable domination and (D, t_A) can be assumed Frechet without loss of generality [11]). Then $(\beta(D, D), \tau_{bb})$ is the strong dual of the Frechet Space $D\hat{\otimes}D$ (projective tensor product). The σ-weak topology on A [2] is the relative topology induced on A by the weak topology $\sigma(\beta(D, D), D\hat{\otimes}D)$, A being naturally embedded in $\beta(D, D)$ by $T\to\beta^T:(x, y)\to\langle Tx, y\rangle$. We also consider the following two other topologies, the first one defined completely in terms of the bounded part A_b of A; and the other in terms of order structure on A.

(a) *Dixon topology*: Let Δ be the collection of all strictly positive functions on $A\times A$. For each $\delta\in\Delta$, let $N(\delta)=|co|\cup\{\delta(S, T)SB_0T\,|\,S, T \text{ in } A\}$, $|co|$ denoting the absolutely convex hull. Let $\theta=\{N(\delta)\mid\delta\in\Delta\}$. Since $1\in B_0$, each $N(\delta)$ is absorbing. Thus θ forms a 0-neighbourhood base for a locally convex linear topology $\mathcal{T}$ on A. It was considered first in [4] for a class of abstract topological * algebras called generalized B^*-algebras which are realizable as EC^*-algebras [8] viz. symmetric Op^*-algebras A with bounded part A_b a C^*-algebra.

(b) *Order topology*: ([12], [14]): Let τ_{oh} be the order topology on $A^b=\{T\in A\mid T=T^*\}$ viz. the largest locally convex linear topology making each order interval bounded. Let ϕ be a τ-neighbourhood base for τ_{oh}. For $U\in\phi$, let $\tilde{U}$ be the complex absolutely convex hull of U in A. Then $\tilde{\phi}=\{\tilde{U}\,|\,U\in\phi\}$ is a o-neighbourhood base for a locally convex linear topology τ_o (complexification of τ_{oh}).

We also prove the following.

Theorem 2. *Let A be a countably dominated Op^*-algebra satisfying condition* (I). *Then on A,*

$$\rho=\mathcal{T}=\tau_{bb}=\tau_o.$$

Proofs of both the theorems are presented in Part 3. In Part 2, we give a couple of lemmas that are needed and that appear to be of some independent interest. Finally the results are applied to countably dominated algebras.

§ 2. Preliminary Lemmas

Recall that in a topological * algebra (A, t) (viz. a topological vector space with separately continuous multiplication and continuous involution), multiplication is called *hypocontinuous* if given a bounded set B in A, and a 0-neighbourhood U, there exists 0-neighbourhoods V_1 and V_2 such that $BV_1 \subset U$, $V_2 B \subset U$. The following is wellknown.

Lemma 2.1. *In a topological algebra, joint continuity of multiplication implies hypocontinuity, and hypocontinuity of multiplication implies sequential joint continuity.*

Lemma 2.2. *Let A be a countably dominated Op*-algebra on a dense subspace D of a Hilbert Space H. On $A_b = n_1$, the norm $\rho_1(\cdot)$ is equivalent to the operator norm $\|\cdot\|$.*

Proof. For a $T \in A_b$,

$$\begin{aligned}\rho_1(T) &= \sup\{|\langle Tx, x\rangle| : x \in D,\ \|x\|=1\}\\ &= \sup\{|\langle Tx, x\rangle| : x \in H,\ \|x\|=1\}\\ &= w(T)\end{aligned}$$

where $w(T)$ is the numerical radius of T. By Halmos [7, p. 173], numerical radius defines a norm on A_b equivalent to the operator norm; in fact, $1/2\|T\| \leqq w(T) \leqq \|T\|$. Note that the validity of the lemma can also be alternatively seen by noting that for $T = T^*$ in A_b, $\rho_1(T) = \|T\|$, and so for any T in A_b, $\rho_1(T) \leqq \|T\| \leqq 2\rho_1(T)$.

*Op**-algebra A is *ρ-closed* [3] if each (n_{A_n}, ρ_{A_n}) is a Banach space. Also as discussed in [2, Proposition 5.1], in the presence of condition (I), there exists an onto isometric isomorphism $T^{(k)}: (n_{A_k}, \rho_{A_k}) \to (n_1, \rho_1)$. Hence the following is immediate.

Corollary 2.3. *Let A satisfies condition* (I). *Then A is ρ-closed iff $(A_b, \|\cdot\|)$ is a C*-algebra.*

The following lemma sheds some light on the role of A_b in the structure of A.

Lemma 2.4. *Let A be an Op*-algebra.*

(a) *If B is a ρ-bounded * idempotent in A, then $B \subset B_0$. In particular, B_0*

is closed in ρ.

(b) *If a *subalgebra* B *of* A *is a Banach *algebra under any norm* $|.|$ *such that* $(B, |.|)\to(A, \rho)$ *is a continuous embeding, then* B *consists of bounded operators.*

(c) *Let* A *be* ρ*-closed. Let* $T\in A$, $T\geqq 0$. *If* n_T *is an algebra, then* n_T *consists of bounded operators; in particular,* T *is bounded.*

Proof. By Proposition 1.2 in [3], boundedness of B, together with the fact that B is an idempotent, implies that there exists $S>0$ in A such that $|\langle T^n x\rangle| \leqq (Sx, x)$ for all $x\in D$, $T\in B$, for all $n=1, 2, 3, \cdots\cdots$. Given $T\in B$, $Q=T^*T\in B$; and so for all n,

$$\|Q^n x\|^2=\langle Q^{2n}x, x\rangle\leqq\|x\|\|Q^{2n}x\| .$$

Hence iterating, for all $x\in D$, we obtain

$$\begin{aligned}\|Qx\|&\leqq\|x\|^{(1-1/2^n)}\|Q^{2^n}x\|^{1/2^n}\\&\leqq\|x\|^{(1-1/2^n)}\langle Sx, x\rangle^{1/2^n}\end{aligned}$$

for all n. This gives $\|Qx\|\leqq\|x\|$ for all x, and so $\|Tx\|\leqq\|x\|$ for all x, showing T to be bounded and $B\subset B_0$. (b) is immediate from (a); and (c) follows from (b).

Now for a locally convex linear topology t on A, consider the following statements

(i) Involution in A is t-continuous.
(ii) Multiplication in A is separately t-continuous.
(iii) B_0 is t-closed.
(iv) B_0 is t-bounded.
(v) B_0 is the greatest member, under inclusion, of $\mathscr{B}^*(t)$ where $\mathscr{B}^*(t)$ is the collection of all absolutely convex, t-closed, t-bounded *idempotents in A.

The following describes basic properties of $\mathscr{T}$. This can be proved as in Dixon [4]. Part (d) is a consequence of Lemma 2.4(a).

Lemma 2.5. *Let* A *be an Op*-algebra*

(a) $\mathscr{T}$ *satisfies* (i)—(v).
(b) $\mathscr{T}$ *is finer than any locally convex topology* t *satisfying* (i)—(v).
(c) *If* A_b *is a C*-algebra, then* $\mathscr{T}$ *is barrelled.*
(d) $\mathscr{T}$ *is finer than* ρ.

§3. Proofs of the Theorems

Proof of theorem 1(a). Let A be symmetric. Then for all $T\in A$, $\bar{T}^*=\bar{T}^\#$. This is a standard argument as in Lemma 2.1 in [8(I)] or Lemma 7.9 in [4].

Now given T in A, take $T_n=T(1+(1/n)T^*T)^{-1}$ for $n=1, 2, 3, \cdots\cdots$ Note that

$$T_1=T(1+T^*T)^{-1}=T(1+\bar{T}^*\bar{T})^{-1}=T(1+\bar{T}^\#\bar{T})^{-1}$$

is in A and is a bounded operator with $\|T_1\|\leqq 1$. Hence

$$T_n=\sqrt{n}\left(\frac{T}{\sqrt{n}}\right)\left(1+\left(\frac{T}{\sqrt{n}}\right)^*\left(\frac{T}{\sqrt{n}}\right)\right)^{-1}$$

is in A_b with $\|T_n\|\leqq\sqrt{n}$. Further

$$T-T_n=\frac{1}{n}TT^*T\left(1+\frac{1}{n}T^*T\right)^{-1}$$

$$=\frac{1}{\sqrt{n}}(TT^*)\left(\frac{T}{\sqrt{n}}\right)\left(1+\left(\frac{T}{\sqrt{n}}\right)^*\left(\frac{T}{\sqrt{n}}\right)\right)^{-1}$$

which is in $1/\sqrt{n}\,TT^*B_0$. Let τ be any topology on A such that (A, τ) is a topological *algebra. Let B_0 be τ-bounded. Then given a 0-neighbourhood V in A, there exists a 0-neighbourhood U such that $TT^*U\subset V$. Further, for sufficiently large n, $1/\sqrt{n}\,B_0\subset U$. Hence for such n, $T-T_n\in TT^*U\subset V$ showing that $T_n\to T$.

(b) Given T in A, choose a sequence T_n in A_b such that $T_n\to T$ in τ. Then $T_n^*\to T^*$, and so by Lemma 2.1, $T_n^*T_n\to T^*T$. Let $S_n=(1+T_n^*T_n)^{-1}$ which are in A_b with $\|S_n\|\leqq 1$. Now

$$S_n-S_m=(1+T_n^*T_n)^{-1}(-T_n^*T_n+T_m^*T_m)(1+T_m^*T_m)^{-1}.$$

By hypocontinuity, given a 0-neighbourhood U, choose 0-neighbourhoods V_1 and V such that $B_0V_1\subset U$; $VB_0\subset V_1$; with the result, $B_0VB_0\subset U$. Since $T_n^*T_n$ is Cauchy, $T_m^*T_m-T_n^*T_n\in V$ eventually. Hence $S_n-S_m\in B_0VB_0\subset U$ eventually. Thus the sequence S_n in B_0 is Cauchy; hence $S_n\to S\in B_0$ by assumption. Then by sequential joint continuity of multiplication,

$$(1+T^*T)S=\lim(1+T_n^*T_n)\quad(1+T_n^*T_n)^{-1}=1=S(1+T^*T).$$

Thus $S=(1+T^*T)^{-1}\in B_0\subset A_0$ showing that A is symmetric.

Proof of Theorem 2. Note that as discussed in [2], $\rho=\tau_{bb}$ follows from the ρ-normality of the positive cone in A, a consequence of condition (I). We show $\tau_0=\mathcal{T}$. Since τ_0 is the largest locally convex linear topology on A making each order interval in A^h bounded, $\tau_0\geqq\mathcal{T}$ follows if each order bounded set M (which is contained in $[A_n, A_n]$ for some n) is $\mathcal{T}$-bounded, where (A_n), $A_n\geqq 1$ and $A_n^{-1}\in A$, is the sequence in A^+ defined by condition (I). Thus for all $Z\in M$, $-A_n^2\leqq -A_n\leqq Z\leqq A_n\leqq A_n^2$; and so $-1\leqq A_n^{-1}ZA_n^{-1}\leqq 1$, $A_n^{-1}ZA_n^{-1}\in B_0$. Now let $N(\delta)$ be a and $\mathcal{T}$-neighbourhood of 0. Then

$$N(\delta)=|co|\cup\{\delta(X, Y)XB_0Y|X, Y \text{ in } A\}\supseteqq\delta(A_n, A_n)A_nB_0A_n\,.$$

But

$$\delta(A_n, A_n)Z=\delta(A_n, A_n)A_n(A_n^{-1}ZA_n^{-1})A_n\in\delta(A_n, A_n)A_nB_0A_n\subset N(\delta)\,.$$

Thus $M\subset(\delta(A_n, A_n))^{-1}N(\delta)$ showing that M is $\mathcal{T}$-bounded. Now to show $\tau_0\leqq\mathcal{T}$, we apply Lemma 2.5(b). The topology τ_0 is easily seen to satisfy (i) and (ii). Let $B\in\mathcal{B}^*(\tau_0)$. Then B $\mathcal{T}$-bounded as $\mathcal{T}\leqq\tau_0$; hence its $\mathcal{T}$-closure $\bar{B}$ is in $\mathcal{B}^*(\mathcal{T})$. Lemma 2.5(a) implies that $B\subset B_0$. The same argument applied to B_0 shows that τ_0 satisfies (iii) and (iv); and so also (v). Thus $\tau_0=\mathcal{T}$; which, in view of Lemma 2.5(d), gives $\tau_0\geqq\rho$. On the other hand, for each n, each bounded subset in the normed linear space (n_{A_n}, ρ_{A_n}) is τ_0-bounded. Hence the embedings $id_n(n_{A_n}, \rho_{A_n})\to(A, \tau_0)$ are continuous, (n_{A_n}, ρ_{A_n}) being bornological. Since ρ is the largest locally convex linear topology on A making each of these embedings continuous, $\tau_0\leqq\rho$, and the proof is complete.

Corollary. *Let A be an Op^*-algebra.*

(a) *If A is symmetric, then A_b is sequentially dense in (A, ρ)*

(b) *Assume the following*

(i) *A satisfies condition* (I)

(ii) *The domain D of A is quasi-normable in the induced topology t_A.*

(iii) *A_b is a C^*-algebra (in particular, A is ρ-closed or σ-weakly closed).*

If A_b is sequentially dense in ρ-topology, then A is symmetric.

Proof. Theorem 1(a) gives (a). Note that if A satisfies (I) and is σ-weakly closed, then A_b is von Neumann algebra by [2]. Now (i) and (iii) in (b) together with Theorem 2 and Lemma 2.5(c) imply that (A, ρ) is barelled, and in a barelled algebra, multiplication is known to be hypocontinuous. Further, due to quasi-normability, (A, ρ) satisfies strict condition of Mackey convergence [11]. From this, using Corollary 2.3 and Lemma 2.4(a), it is easily seen that B_0 is sequentially ρ-complete. Now the conclusion follows from Theorem 1(b).

Remarks. (a) Symmetry in an Op^*-algebra A is a very stringent requirement, since as noted in the proof of Theorem 1(a), $\bar{T}^*=\bar{T}^\#$ holds, for all $T\in A$ which implies that in such an A, every hermitian element is essentially self-adjoint.

(b) Sequential density of A_b in (A, ρ) is not sufficient to make A symmetric. As shown in [11, Part 3], in the maximal Op^*-algebra $L(D)$ on Schwartz domain D, every $T\in A$ can be approximated, even in some normed space (n_{A_k}, ρ_{A_k}), by a sequence of finite rank operators in $L(D)$.

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