Publ. RIMS, Kyoto Univ.
24 (1988), 739-757

On a Banach Space of Functions Associated with a Homogeneous Additive Process

By

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Introduction

Let $(X(t))_{t\in T}$ be a time homogeneous additive process on a probability space $(\Omega, \mathcal{F}, P)$, where $T=\boldsymbol{N}$ or $\boldsymbol{R}_+$ (the nonnegative real numbers). We consider stochastic integrals of Wiener type:

$$\int x(t)\, dX(t)\,.$$

The object of study of the present paper is the Banach space consisting of integrands of such a integral. Our result gives a correspondence between the spectrum (after [S1]) of this Banach space and the equivalency of $L_p(\Omega, P)$-norms of these stochastic integrals. Spectrum is one of those characteristics which describe geometrical properties of a Banach space. There exists a celebrated general theoryin cluding this concept, which is often called local theory of Banach spaces (we refer to e.g. [S1], [P1]). We investigate the case of discrete time (i.e. $T=\boldsymbol{N}$) in §2, and, using the results, deal with the case of continuous time (i.e. $T=\boldsymbol{R}_+$) in §3.

We note that the space of differentiable shifts for a measure on an infinite dimensional space is closely related to the above Banach space associated with $X(t)$ in the case of discrete time. More precisely, as we show in §4, these Banach spaces are realized as spaces of the differentiable shifts for stationary product measures on $\boldsymbol{R}^{\boldsymbol{N}}$. As for differentiation of measures, we refer to [B1], [Y-H1] and [Sh1].

Throughout the present note, we try to write as explicitly as possible the constants which appear in inequalities though they may be far from the best possible ones.

§1. Preliminaries

In the case of discrete time, our study is nothing but that of the L_1-convergence of sums of independent random variables. We propose in this sec-

Communicated by S. Matsuura, January 14, 1988. Revised March 29, 1988.
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tion an inequality which will play an important role later. This inequality is essentially due to H. Shimomura ([Sh1], § 2).

Let X be a real-valued random variable such that $0<E|X|<\infty$ and $EX=0$, and $X_0, X_1, X_2, \cdots$ independent copies of X. The set of all real sequences with finitely many nonzero terms is denoted by $\boldsymbol{R}_0^{\boldsymbol{N}}$. For $x=(x(k))_{k\in\boldsymbol{N}}\in\boldsymbol{R}_0^{\boldsymbol{N}}$, put

$$\|x\|_{D_X}=\|\sum_k x(k)X_k\|_{L_1(\Omega,P)}\left(=E|\sum_k x(k)X_k|\right). \tag{1.1}$$

Then $\|\cdot\|_{D_X}$ is a norm on $\boldsymbol{R}_0^{\boldsymbol{N}}$ stronger than the weak topology. The completion of $\boldsymbol{R}_0^{\boldsymbol{N}}$ with respect to $\|\cdot\|_{D_X}$ is considered as a subspace of $\boldsymbol{R}^{\boldsymbol{N}}$ and denoted by D_X. For $x=(x(k))_{k\in\boldsymbol{N}}\in\boldsymbol{R}^{\boldsymbol{N}}$, we see that $x\in D_X\Leftrightarrow\sup_{n\in\boldsymbol{N}} E|\sum_{k=0}^{n} x(k)X_k|<\infty$ from the martingale convergence theorem.

H. Shimomura showed in [Sh1] that D_X is an Orlicz sequence space in the following sense. Put

$$M_X(t)=E\left(\int_0^1\left(1-\exp\left(-\frac{1}{2}t^2u^2X^2\right)\right)u^{-2}du\right), \quad t\geqq 0. \tag{1.2}$$

$M_X(t)$ is an Orlicz function satisfying the Δ_2-condition at $t=0$. For $x=(x(k))_{k\in\boldsymbol{N}}\in\boldsymbol{R}_0^{\boldsymbol{N}}$, put $\|x\|_{M_X}=\inf\left\{\rho>0\,;\ \sum_k M_X(|x(k)|/\rho)\leqq 1\right\}$. The completion of $\boldsymbol{R}_0^{\boldsymbol{N}}$ with respect to the norm $\|\cdot\|_{M_X}$ is the Orlicz sequence space associated with the Orlicz function M_X.

From the elaborate estimation due to H. Shimomura, we can deduce the following

Theorem 1.1. *Under the above notations, an inequality*

$$K'\|x\|_{D_X}\leqq\|x\|_{M_X}\leqq K\|x\|_{D_X} \tag{1.3}$$

holds for all $x\in\boldsymbol{R}_0^{\boldsymbol{N}}$. *Here* K *and* K' *are positive absolute constants defined by*

$$K'=\frac{1}{2}\left(1+\frac{8}{b_1}\right)^{-1}, \qquad K=\left\{\left(1+\frac{32c_1}{b_2c_2(1+2a)}\right)^{1/2}-1\right\}^{-1} \tag{1.4}$$

where $a=\sup_{t>0}\left|\int_0^t\frac{\sin u}{u}du\right|$, $b_1=N_{0,1}((-1, 1)^C)$, $b_2=\int_{\boldsymbol{R}}(4u^2+|u|)N_{0,1}(du)$ ($N_{0,1}$ *is the standard normal distribution on* $\boldsymbol{R}$) *and* $c_1\leqq\frac{\log u}{u-1}\leqq c_2$ *on* $\left\{u\in\boldsymbol{R}\,;\ |u-1|\leqq\frac{1}{2}\right\}$ (*hence* $K'=0.019\cdots$, $K=2.836\cdots$).

The proof is omitted. See [Sh1], § 2, particularly the proof of Theorem 2.5. □

The following proposition shows the relation of comparison between Orlicz functions to that between Orlicz norms.

Proposition 1.2. *Let $p\geqq 1$ and M be an Orlicz function. If $\|x\|_M\leqq C\|x\|_{l_p}$ for all $x\in \boldsymbol{R}_0^N$ (C is a positive constant), then $M(t)\leqq 2C^p t^p$ for all $0\leqq t\leqq C^{-1}$.*

Proof. Let $\{e_0, e_1, \cdots\}$ be the standard basis of $\boldsymbol{R}_0^N$. From $\|e_0+\cdots+e_{n-1}\|_M \leqq C\|e_0+\cdots+e_{n-1}\|_{l_p}$, we get $nM(1/Cn^{1/p})\leqq 1$. This means that the desired result holds for $t=(Cn^{1/p})^{-1}$. Now we interpolate the other values of t. Let $0<t\leqq C^{-1}$. Take $n\in \boldsymbol{N}$ such that $(C(n+1)^{1/p})^{-1}\leqq t\leqq (Cn^{1/p})^{-1}$. Then $(C(n+1)^{1/p})^{-p} \leqq t^p$ and $M((Cn^{1/p})^{-1})\leqq 1/n=C^p(Cn^{1/p})^{-p}$. Since $\frac{(Cn^{1/p})^{-p}}{(C(n+1)^{1/p})^{-p}}=\frac{n+1}{n}\leqq 2$, we get $M(t)\leqq 2C^p t^p$. □

§ 2. The Case of Discrete Time

In this section we prove the correspondence between the spectrum of D_X and the equivalency of $L_p(\Omega, P)$-norms $(1\leqq p\leqq 2)$ of $\sum_k x(k)X_k$ (for $x\in D_X$). We have only to consider the case where the index p ranges between 1 and 2 because we will treat spaces of cotype 2 and their spectra. Later we will refer to this reason again after explaining spectra of Banach spaces (before Theorem 2.3).

First we show that we can estimate the moments of a random variable X by comparing the norm of D_X with the l_p-norm. The situations are somewhat different in the case $p=2$ and in the case $1\leqq p<2$.

(I) *The case* $p=2$. From [Y-H1] Theorem 9.1, we see that, if there exists a positive constant C such that $\|x\|_{D_X}\leqq C\|x\|_{l_2}$ for all $x\in \boldsymbol{R}_0^N$, then $E|X|^2\leqq 24C^2\varepsilon^{-2}\log 1/1-\varepsilon$ holds for $0<\varepsilon<1$. Computing the minimum in ε, we get

$$(E|X|^2)^{1/2}\leqq 3\sqrt{6\log 3}\,C. \tag{2.1}$$

We note that $\|x\|_{D_X}\geqq 1/2\sqrt{2}\, E|X|\,\|x\|_{l_2}$ always holds since L_1-space is of cotype 2.

(II) *The case* $1\leqq p<2$. We cannot claim the existence of p-th moment of X in this case.

Proposition 2.1. *Let $1\leqq p<2$. If there exists a positive constant C such that $\|x\|_{D_X}\leqq C\|x\|_{l_p}$ for all $x\in \boldsymbol{R}_0^N$, then*

$$(E|X|^q)^{1/q}\leqq KCA(p, q) \tag{2.2}$$

holds for all q satisfying $0<q<p$, where K is an absolute constant in (1.4) *and $A(p, q)$ is defined by*

$$A(p, q)=\left(1+\frac{2\sqrt{e}}{\sqrt{e}-1}\frac{q}{p-q}\right)^{1/q} \qquad (0<q<p). \tag{2.3}$$

Proof. We follow the method in [Sh1] Theorem 3.5. From the assumption, using Theorem 1.1 and Proposition 1.2, we get $M_X(t)\leqq 2K^pC^pt^p$ for $0\leqq t \leqq 1/KC$ (M_X is an Orlicz function defined by (1.2)). Then, from the estimation due to H. Shimomura, we obtain for $0<q<p$,

$$E|X|^q\leqq (KC)^q+\frac{\sqrt{e}}{\sqrt{e}-1}2K^pC^p\int_{(KC)^q}^{\infty}R^{-p/q}dR$$
$$=(KC)^q\left(1+\frac{2\sqrt{e}}{\sqrt{e}-1}\frac{q}{p-q}\right).$$

This is the desired inequality. □

Now we propose our result in the discrete time case.

Theorem 2.2. *Let* $1\leqq p\leqq 2$. *Assume that there exists a positive constant* C *such that, if* (x_n) *is a finite sequence of disjointly supported elements of* $\boldsymbol{R}_0^{\boldsymbol{N}}$, *the inequality*

$$\|\sum_n x_n\|_{D_X}\leqq C(\sum_n\|x_n\|_{D_X}^p)^{1/p} \tag{2.4}$$

holds. Then, for all q *satisfying* $1\leqq q<p$, *we obtain*

$$\|\sum_k x(k)X_k\|_{L_1(\Omega)}\leqq\|\sum_k x(k)X_k\|_{L_q(\Omega)} \tag{2.5}$$
$$\leqq KCA(p, q)\|\sum_k x(k)X_k\|_{L_1(\Omega)} \qquad \textit{for all } x\in D_X.$$

Proof. We have only to prove the inequality for $x\in\boldsymbol{R}_0^{\boldsymbol{N}}$. Take $x=(x(0), \cdots, x(k-1), 0, \cdots)\in\boldsymbol{R}_0^{\boldsymbol{N}}$. Put

$$Y_0=x(0)X_0+\cdots+x(k-1)X_{k-1},$$
$$\cdots,$$
$$Y_{m-1}=x(0)X_{(m-1)k}+\cdots+x(k-1)X_{mk-1},$$
$$\cdots.$$

Then $\{Y_i\}_{i\in\boldsymbol{N}}$ is a system of independent random variables with identical distributions. Let $\alpha_0, \cdots, \alpha_{m-1}\in\boldsymbol{R}$. Put

$$x_n=(0, \cdots, 0, \underset{\substack{\uparrow\\(nk+1)\text{-th}}}{\alpha_n x(0)}, \alpha_n x(1), \cdots, \underset{\substack{\uparrow\\(n+1)k\text{-th}}}{\alpha_n x(k-1)}, 0, \cdots)\in\boldsymbol{R}_0^{\boldsymbol{N}} \quad (0\leqq n\leqq m-1).$$

The elements x_n's are disjointly supported. From the assumption (2.4), we get

$$\|(\alpha_0 x(0), \cdots, \alpha_0 x(k-1), \cdots, \alpha_{m-1}x(0), \cdots, \alpha_{m-1}x(k-1), 0, \cdots)\|_{D_X}$$

$$\leqq C\Big(\sum_{n=0}^{m-1}\|(0, \cdots, 0, \alpha_n x(0), \cdots, \alpha_n x(k-1), 0, \cdots)\|_{D_X}^p\Big)^{1/p},$$

i.e.,

$$E|\alpha_0 Y_0+\cdots+\alpha_{m-1}Y_{m-1}|\leqq CE|Y_0|\Big(\sum_{n=0}^{m-1}|\alpha_n|^p\Big)^{1/p} \quad \text{for any } m=1, 2, \cdots. \tag{2.6}$$

The Banach space associated with $Y_0, Y_1, \cdots$ is denoted by D_Y (analogously to D_X). Then (2.6) implies that $\|\alpha\|_{D_Y}\leqq CE|Y_0|\|\alpha\|_{l_p}$ holds for all $\alpha\in \boldsymbol{R}_0^N$. Hence, from Proposition 2.1, we obtain

$$(E|Y_0|^q)^{1/q}\leqq KCA(p, q)E|Y_0| \qquad \text{for all } q \text{ satisfying } 0<q<p. \tag{2.7}$$

(2.7) means the right inequality of (2.5). The left one is obvious for $q\geqq 1$. □

Remark 1. The assumption (2.4) of the above theorem holds if D_X is of type p. Conversely, since D_X is a Banach lattice of cotype 2, Maurey's theorem assures that, if $1\leqq p<2$, the assumption (2.4) implies the type p property of D_X ([M1], [P1]). Therefore, if $1\leqq p<2$, we can equivalently replace (2.4) by the assumption that D_X is of type p. We note that Maurey's theorem and our result do not have mutual implication.

Remark 2. In the case $p<2$, we cannot claim $q=p$. In fact, p-stable random variables give a counterexample. In the case $p=2$, however, we can take $q=p=2$. In fact, the inequality (2.6) implies $(E|Y_0|^2)^{1/2}\leqq 3\sqrt{6\log 3}\, CE|Y_0|$ accordingly to the discussion in the case $p=2$ (the inequality (2.1)).

There is a concept of spectrum which describes a geometrical structure of Banach spaces. Here we recall briefly some properties of spectra of Banach spaces. For details, we refer to [S1] and [P1]. The notation l_p^n denotes $\boldsymbol{R}^n$ equipped with the l_p-norm: $\Big(\sum_{k=1}^{n}|x(k)|^p\Big)^{1/p}$ for $x\in\boldsymbol{R}^n$. The Banach-Mazur distance between two Banach spaces E and F is denoted by $d(E, F)$ (i.e. $d(E, F)=\inf\{\|T\|\|T^{-1}\|;\ T$ is a linear isomorphism: $E\to F\}$). A Banach space B is said to contain l_p^n's uniformly if, for any $\varepsilon>0$, there exists a sequence (F_n) of finite dimensional subspaces of B such that $d(l_p^n, F_n)\leqq 1+\varepsilon$ for all n. The spectrum of a Banach space B (denoted by $Sp(B)$) is defined by $Sp(B)=\{1\leqq p\leqq\infty;\ B$ contains l_p^n's uniformly$\}$. A geometrical interpretation of $Sp(B)$ is as follows. Let $p\in Sp(B)$ and F_n be a corresponding n-dimensional subspace of B. Then there exists an isomorphism $T: l_p^n\to F_n$ such that $\|T\|\|T^{-1}\|\leqq 1+\varepsilon$. By an appropriate homothety, we can take $\|T\|=\|T^{-1}\|\leqq(1+\varepsilon)^{1/2}$. Now cut the closed unit ball of B by F_n. Then the section nearly coincides with the closed unit ball of l_p^n (modulo isometry) for all n.

The spectrum of a Banach space B has the following properties:

(1) $Sp(B)$ is a closed subset of $[1, \infty]$

(2) $2 \in Sp(B)$ (This fact is known as Dvoretzky's theorem.)

(3) $1 \leqq p \leqq 2$ and $p \in Sp(B) \Rightarrow [p, 2] \subset Sp(B)$.

The property (3) comes from the fact that, for an arbitrary q such that $q \leqq p \leqq 2$, L_p is isometrically embedded into some L_q.

Moreover the following results due to Maurey-Pisier and Krivine are remarkable:

$$\min Sp(B) = \sup \{1 \leqq p \,;\, B \text{ is of type } p\}, \tag{2.8}$$

$$\max Sp(B) = \inf \{q \leqq \infty \,;\, B \text{ is of cotype } q\}.$$

Now let us return to our discussion. Since D_X is a Banach space of cotype 2, $Sp(D_X)$ is a closed subinterval of $[1, 2]$ whose left end point is the supremum of type indices. We define D_X^p as the completion of $\boldsymbol{R}_0^N$ with respect to the norm: $\|x\|_{D_X^p} = \|\sum_k x(k) X_k\|_{L_p(\Omega)}$. We denote by $T_p(B)$ the type p constant of a Banach space B i.e. the smallest positive number C such that

$$\Big(E\|\sum_n \varepsilon_n x_n\|_B{}^2\Big)^{1/2} \leqq C\Big(\sum_n \|x_n\|_B{}^p\Big)^{1/p}$$

holds for any finite sequence (x_n) in B where (ε_n) is a Bernoulli sequence of independent random variables. Since $\|\sum x_n\|_{D_X} \leqq 2\|\sum \varepsilon_n x_n\|_{D_X}$ holds if x_n's are disjointly supported, we readily obtain from Theorem 2.2 the following

Theorem 2.3. *$Sp(D_X) = [s(X), 2]$ where $s(X) = \sup \{1 \leqq p \leqq 2\,; D_X$ and D_X^p are isomorphic$\}$. Moreover if $1 \leqq p < s(X)$, their Banach-Mazur distance is estimated as*

$$d(D_X, D_X^p) \leqq 2K \inf_{p < r < s(X)} A(r, p) T_r(D_X) \tag{2.9}$$

(*K and $A(r, p)$ are defined in* (1.4) *and* (2.3)).

Thus, if D_X is so small that it cannot contain l_p^n's uniformly, then D_X must be isomorphic to the small subspace (i.e. the closed linear hull of $\{X_0, X_1, \cdots\}$) of $L_p(\Omega, P)$.

§ 3. The Case of Continuous Time

Let $(X(t))_{t \in T}$ be a time homogeneous additive process which is continuous in probability and satisfies $E|X(t)| < \infty$ for all $t \in T$. Throughout this section, we take $T = \boldsymbol{R}_+$. We assume that $X(0) \equiv 0$ constantly and $EX(t) = 0$ for all $t \in T$ (otherwise we have only to consider $X(t) - EX(t) - (X(0) - EX(0))$ instead of $X(t)$).

Moreover we may assume without loss of generality that almost every sample paths are right-continuous and have the left limit at each point, since any additive process which is continuous in probability can be equivalently deformable to a process having this desired property. As for this fact, we refer to [I1]. Then, for a finite subinterval J of T, we get $E\sup_{t\in J}|X(t)|\leqq 8E|X(\sup J)|$ using Ottaviani's inequality and the right-continuity of $X(t)$. Hence, by Lebesgue's convergence theorem, $\lim_{t\to t_0}E|X(t)-X(t_0)|=0$ holds.

Put

$$\mathrm{Step}(T)=\Big\{\sum_k c_k I_{A_k}\ (\text{finite sum});\ c_k\in\boldsymbol{R},\ A_k\text{'s are mutually disjoint finite subintervals of } T\Big\}$$

where I_A is the indicator function of a set A. For $x=\sum_k c_k I_{(t_k,t_{k+1}]}\in\mathrm{Step}(T)$, put

$$\|x\|_{D(X)}=\Big\|\sum_k c_k(X(t_{k+1})-X(t_k))\Big\|_{L_1(\Omega)} \qquad (3.1)$$
$$\Big(=E\Big|\sum_k c_k(X(t_{k+1})-X(t_k))\Big|\Big).$$

Here we note that $E|X(t)|$ is nondecreasing in t and, $X(t)\equiv 0$ a.s. $\Leftrightarrow E|X(t)|$ is degenerate at $t=0$ (i.e. for some $\alpha>0$, $E|X(t)|=0$ for all $t\in[0,\alpha]$). We exclude this trivial case throughout the following discussions. Therefore we assume that $E|X(t)|$ is strictly positive if $t>0$. Then $\|\cdot\|_{D(X)}$ becomes a norm on Step(T).

We prove a simple proposition which estimates strength of $\|\cdot\|_{D(X)}$ roughly.

Proposition 3.1. (1) *$\|\cdot\|_{D(X)}$ is weaker than $\|\cdot\|_{L_\infty(J)}$ on* Step(J) *where J is an arbitrary finite subinterval of T.*

(2) *$\|\cdot\|_{D(X)}$ is stronger than the topology of $L_0(T)$ (which is by definition the topology of convergence in Lebesgue measure) on* Step(T).

Proof. (1) Let $x=\sum_k c_k I_{(t_k,t_{k+1}]}\in\mathrm{Step}(J)$. Then,

$$\|x\|_{D(X)}\leqq 2\Big(\max_k|c_k|\Big)E\Big|\sum_k(X(t_{k+1})-X(t_k))\Big|\leqq 2\|x\|_{L_\infty(J)}E|X(\sup J)|.$$

(2) Let $x\in\mathrm{Step}(T)$ and $\varepsilon>0$. Express

$$x=\sum_j \alpha_j I_{(s_j,s'_j]}+\sum_k \beta_k I_{(t_k,t'_k]}$$

where $|\alpha_j|\leqq\varepsilon<|\beta_k|$ for all j and k. Then

$$\|x\|_{D(X)}=E\Big|\sum_j\alpha_j(X(s'_j)-X(s_j))+\sum_k\beta_k(X(t'_k)-X(t_k))\Big|$$

$$\geqq E\left|\sum_k \beta_k(X(t'_k)-X(t_k))\right| \geqq \frac{\varepsilon}{2}E\left|\sum_k X(t'_k)-X(t_k)\right|$$

$$=\frac{\varepsilon}{2}E\left|X(\sum_k (t'_k-t_k))\right| = \frac{\varepsilon}{2}E\left|X(m(|x|>\varepsilon))\right|$$

where m is the Lebesgue measure on T. Hence, $\|x\|_{D(X)}\to 0 \Rightarrow E|X(m(|x|>\varepsilon))| \to 0 \Leftrightarrow m(|x|>\varepsilon)\to 0$. □

The completion of Step(T) with respect to the norm $\|\cdot\|_{D(X)}$ is denoted by $D(X)$. We will describe the realization of $D(X)$ as a subspace of $L_0(T)$ in Appendix to § 3.

For $x=\sum_k c_k I_{(t_k, t_{k+1}]} \in$ Step(T), put

$$\int_T x(t)dX(t)=\sum_k c_k(X(t_{k+1})-X(t_k)). \tag{3.2}$$

And for $x\in D(X)$, taking $x_n\in$ Step(T) such that $x_n \underset{n\to\infty}{\longrightarrow} x$ in $D(X)$, we put

$$\int_T x(t)dX(t)=\lim_{n\to\infty}\int_T x_n(t)dX(t) \qquad (L_1(\Omega)\text{-convergence}). \tag{3.3}$$

If $X(t)$ is a p-stable process ($1<p\leqq 2$), then $D(X)$ coincides with $L_p(T)$. In particular, if $X(t)$ is a Brownian motion, then $D(X)$ coincides with $L_2(T)$ and $\int_T x(t)dX(t)$ is Wiener's stochastic integral. Actually $D(X)$ is nothing but the set of those functions whose stochastic integrals $\int_0^t x(s)dX(s)$ can be defined as usual and satisfy $\sup_{t\in T} E\left|\int_0^t x(s)dX(s)\right|<\infty$.

In the case of continuous time also, a similar result to Theorem 2.2 holds as follows.

Theorem 3.2. *Let $1\leqq p\leqq 2$. Assume that there exists a positive constant C such that, if (x_n) is a finite sequence of disjointly supported elements of $D(X)$ (embedded in $L_0(T)$), the inequality*

$$\left\|\sum_n x_n\right\|_{D(X)}\leqq C\Big(\sum_n \|x_n\|_{D(X)}^p\Big)^{1/p} \tag{3.4}$$

holds. Then, for all q satisfying $1\leqq q<p$, we obtain

$$\left\|\int_T x(t)dX(t)\right\|_{L_1(\Omega)}\leqq\left\|\int_T x(t)dX(t)\right\|_{L_q(\Omega)} \tag{3.5}$$

$$\leqq KCA(p,q)\left\|\int_T x(t)dX(t)\right\|_{L_1(\Omega)} \qquad \textit{for all } x\in D(X).$$

In particular, under the above assumption, we get $E|X(t)|^q<\infty$ and

$$\lim_{t \to t_0} E|X(t)-X(t_0)|^q=0 \qquad (for\ 1 \leqq q < p).$$

Proof. Let $\delta>0$ and $\Delta_\delta=\{0=t_0<t_1<t_2<\cdots\}$ be a partition of T such that $t_{k+1}-t_k=\delta$ for each k. Put $F_\delta=\{x \in \mathrm{Step}(T);\ x=\sum_k c_k I_{(t_k, t_{k+1}]},\ c_k \in \boldsymbol{R},\ t_k \in \Delta_\delta\}$ and $Y_k=X(t_{k+1})-X(t_k)$. Y_k's are independent random variables whose distributions coincide with that of $X(\delta)$. For $x=\sum_k c_k I_{(t_k, t_{k+1}]} \in F_\delta$, we have $\|x\|_{D(X)} =\|(c_k)\|_{D_Y}$ (as to the notation D_Y, see the proof of Theorem 2.2). Hence $(F_\delta, \|\cdot\|_{D(X)})$ can be identified with $(\boldsymbol{R}_0^N, \|\cdot\|_{D_Y})$ by $x \leftrightarrow (c_0, c_1, \cdots)$. From the assumption (3.4), we have $\|\sum_n z_n\|_{D_Y} \leqq C\left(\sum_n \|z_n\|_{D_Y}^p\right)^{1/p}$ if z_n's are disjointly supported elements of $\boldsymbol{R}_0^N$. Therefore, from Theorem 2.2, we get, for $1 \leqq q<p$,

$$E|\sum_k z(k)Y_k| \leqq \left(E|\sum_k z(k)Y_k|^q\right)^{1/q} \leqq KCA(p, q)E|\sum_k z(k)Y_k|$$

for all $z \in \boldsymbol{R}_0^N$, and equivalently,

$$(3.6)\qquad \left\|\int_T x(t)dX(t)\right\|_{L_1(\Omega)} \leqq \left\|\int_T x(t)dX(t)\right\|_{L_q(\Omega)}$$
$$\leqq KCA(p, q)\left\|\int_T x(t)dX(t)\right\|_{L_1(\Omega)} \qquad \text{for all } x \in F_\delta.$$

Since $\bigcup_{\delta>0} F_\delta$ is dense in $D(X)$ and the constant in (3.6) is independent of δ, we obtain the desired inequality (3.5). □

Remark. We cannot clain $q=p$ if $p<2$, while, if $p=2$, we have the inequality (3.5) at $p=q=2$ with a constant $3\sqrt{6 \log 3}\, C$ instead of $KCA(p, q)$.

Characterization of the spectrum of $D(X)$ has the same form as in the discrete case. We denote by $D^p(X)$ the completion of Step (T) with respect to the norm: $\|x\|_{D^p(X)}=\left\|\int_T x(t)dX(t)\right\|_{L_p(\Omega)}$ and by $T_p(D(X))$ the type p constant.

Theorem 3.3. *$Sp(D(X))=[s(X), 2]$ where $s(X)=\sup\{1 \leqq p \leqq 2;\ D(X)$ and $D^p(X)$ are isomorphic$\}$. Moreover if $1 \leqq p < s(X)$, their Banach-Mazur distance is estimated as*

$$(3.7)\qquad d(D(X), D^p(X)) \leqq 2K \inf_{p<r<s(X)} A(r, p)T_r(D(X)).$$

Appendix to § 3

We realize $D(X)$ as a subspace of $L_0(T)$. The essentials in this appendix were obtained through discussions with Y. Yamasaki.

Proposition A.1. *Let J be a bounded subinterval of T and c a positive number. $D(X)$-topology and $L_0(T)$-topology are equivalent on a set* $S_J^c=\{x\in \mathrm{Step}(T);\ |x|\leqq c$ *and* $x=0$ *outside* $J\}$.

Proof. We proved in Proposition 3.1 (2) that $D(X)$-topology is stronger than $L_0(T)$-topology. Now we prove the converse implication on S_J^c. Let $x\in S_J^c$ and $\varepsilon>0$. Express $x=\sum_j \alpha_j I_{(s_j, s_j']}+\sum_k \beta_k I_{(t_k, t_k']}$ where $|\alpha_j|\leqq\varepsilon<|\beta_k|$ for all j and k. Then

$$\|x\|_{D(X)}\leqq E|\sum_j \alpha_j(X(s_j')-X(s_j))|+E|\sum_k \beta_k(X(t_k')-X(t_k))|$$

$$\leqq 2\varepsilon E|X(\sup J)|+2cE|X(m(|x|>\varepsilon))|$$

where m is the Lebesgue measure on T. This inequality implies the desired result. □

For a bounded subinterval J of T and a positive number c, put

(A.3.1) $$L_J^c=\{x\in L_0(T);\ |x|\leqq c \text{ a.e. and } x=0 \text{ a.e. outside } J\}.$$

From Proposition 3.1 and Proposition A.1, we see $L_J^c\subset D(X)$ and $D(X)$-topology coincides with $L_0(T)$-topology on L_J^c. We note that $\int_T x(t)dX(t)$ and $\int_T y(t)dX(t)$ are independent random variables if x and y are disjointly supported elements of L_J^c.

Remark. Put $M(A)=\int_T I_A(t)dX(t)$ for a sum A of subintervals of J. Then M can be extended to an $L_1(\Omega)$-valued σ-additive measure on J (see [D-U1], Chapter 1). And

(A.3.2) $$\int_T x(t)dX(t)=\int_J x(t)dM(t) \qquad \text{for } x\in L_J^c.$$

(The right hand side is an integration with respect to a measure M).

We define truncations of an element x of $L_0(T)$. Put, for $k\in N$,

(A.3.3) $$x^{(k)}(t)=\begin{cases} x(t) & \text{if } 0\leqq t\leqq k \text{ and } |x(t)|\leqq k \\ 0 & \text{if } t>k \text{ or } |x(t)|>k \end{cases},$$

$$x^{[k]}(t)=\begin{cases} x(t) & \text{if } 0\leqq t\leqq k \text{ and } |x(t)|\leqq k \\ k \operatorname{sgn} x(t) & \text{if } 0\leqq t\leqq k \text{ and } |x(t)|>k\,. \\ 0 & \text{if } t>k \end{cases}$$

We put

(A.3.4) $$N(x)=\sup_{k\in N}\|x^{(k)}\|_{D(X)} \quad \text{for } x\in L_0(T)$$

and

$$D=\{x\in L_0(T)\,;\ N(x)<\infty\}.$$

D is a topological vector space with a fundamental system of neighborhoods of zero: $\{\{x\in D;\ N(x)<\varepsilon\}\,;\ \varepsilon>0\}$. Actually, as we see from Theorem A.2 below, N is a norm on D therefore D is a normed space. But here we need not give a direct proof (which is not quite trivial). Clearly D-topology coincides with $D(X)$-topology on L_J^c (in (A.3.1)) and $\|x\|_{D(X)}=N(x)$ for $x\in L_J^c$.

Theorem A.2. *D is complete and contains* Step(T) *densely. Therefore we obtain $D(X)=D$ and*

(A.3.5) $$\|x\|_{D(X)}=N(x) \qquad \textit{for } x\in D(X)\,.$$

Lemma. *The following hold for $x\in D$:*

(1) $N(x-x^{(k)}) \xrightarrow[k\to\infty]{} 0$

(2) $N(x)=\sup_{k\in N}\|x^{[k]}\|_{D(X)}$

(3) $N(x-x^{[k]}) \xrightarrow[k\to\infty]{} 0.$

Proof. Since $(x^{(j)}-x^{(j-1)})$ is a sequence of disjointly supported functions,

(A.3.6) $$\int_T x^{(k)}(t)dX(t)=\sum_{j=1}^{k}\int_T(x^{(j)}(t)-x^{(j-1)}(t))dX(t)$$

is a sum of independent random variables. Hence (A.3.6) converges in $L_1(\Omega)$ as $k\to\infty$ if $N(x)<\infty$. This implies the property (1).

From the estimation

$$\|x^{[k]}-x^{(k)}\|_{D(X)}=E\left|\int_T x^{[k]}(t)I_{\{|x|>k\}}(t)dX(t)\right|$$

$$\leqq kE\left|\int_T I_{\{|x|>l\}\cap[-k,k]}(t)dX(t)\right|+E\left|\int_T x(t)I_{\{k<|x|\leqq l\}\cap[-k,k]}(t)dX(t)\right|$$

$$\leqq \sup_{l>k}\|x^{(l)}-x^{(k)}\|_{D(X)}\,,$$

we obtain

(A.3.7) $$\|x^{[k]}\|_{D(X)}\leqq\|x^{(k)}\|_{D(X)}+\sup_{l>k}\|x^{(l)}-x^{(k)}\|_{D(X)}\,.$$

The property (2) follows immediately from (1) and (A.3.7).

Since $|x-x^{[k]}|\leqq|x-x^{(k)}|$ holds, we obtain from (2) and (1)

$$N(x-x^{[k]})=\sup_{l\in N}\|(x-x^{[k]})^{[l]}\|_{D(X)}\leqq\sup_{l\in N}\|(x-x^{(k)})^{[l]}\|_{D(X)}$$
$$=N(x-x^{(k)})\xrightarrow[k\to\infty]{}0\,.$$

This completes the proof of Lemma. □

Proof of Theorem A.2. Lemma (1) implies that Step(T) is dense in D. We prove the completeness of D. Let (x_n) be a Cauchy sequence of D. x_n converges to some element x in $L_0(T)$. Denoting by '$\lesssim$' an inequality which holds except a constant factor, we have

$$N(x-x_n)\lesssim N(x-x^{[k]})+N(x^{[k]}-x_n^{[k]})+N(x_n^{[k]}-x_m^{[k]})$$
$$+N(x_m^{[k]}-x_m)+N(x_m-x_n)\ (m<n)\,.$$

The 3rd and 5th terms are arbitrarily small if m is sufficiently large and $m<n$ (Note that $N(x_n^{[k]}-x_m^{[k]})\leqq N(x_n-x_m)$ by Lemma (2)). So are the 1st and 4th terms for fixed m and sufficiently large k by Lemma (3). Lastly the 2nd term can be arbitrarily small if k is fixed and n is sufficiently large since D-topology coincides with $L_0(T)$-topology on $L^k_{[-k,k]}$. Thus x_n converges to x in D. □

Remark. Theorem A.2 shows an ideal property of a Banach space $D(X)$: $x\in D(X)$, $y\in L_0(T)$, $|y|\leqq|x|\Rightarrow y\in D(X)$. As for the values of $D(X)$-norm, however, we have only $\|y\|_{D(X)}\leqq 2\|x\|_{D(X)}$ from $|y|\leqq|x|$.

§4. Relation to Differentiable Shifts for Measures on R^N

As we noted in the introduction, we state in this section the relation of the spaces investigated in the preceding sections to the differentiable shifts for measures. First we briefly recall the definition and some properties of differentiable shifts. For details, we refer to [B1], [Y-H1] and [Sh1].

Let $(V, \mathcal{B})$ be a measurable vector space and μ a finite real measure on $(V, \mathcal{B})$. We say that an element a of V is a differentiable shift of μ if the limit of $t^{-1}\{\mu(A+ta)-\mu(A)\}$ (as $t\to 0$) exists for all $A\in\mathcal{B}$. The limit is denoted by $\partial_a\mu(A)$. The set function $\partial_a\mu$ is also a real measure on V. The set of all differentiable shifts of μ is denoted by D_μ. We can prove that $a\in D_\mu$ holds if and only if $t^{-1}(\mu_{ta}-\mu)$ is of Cauchy as $t\to 0$ in the total variation norm $\|\cdot\|_{var}$ of measures (where we put $\mu_{ta}(A)=\mu(A+ta)$). The derivative $\partial_a\mu$ of μ is absolutely continuous with respect to μ, whose density $(d\partial_a\mu/d\mu)(x)$ is called a logarithmic derivative of μ and denoted by $l_\mu^a(x)$ or $l_\mu(a\,;x)$. Put $\|a\|_{D_\mu}=\|\partial_a\mu\|_{var}$ for $a\in D_\mu$. Clearly $\|a\|_{D_\mu}=\|l_\mu^a\|_{L_1(V,\mu)}$ holds for $a\in D_\mu$. Let Φ be a set of $\mathcal{B}$-measurable linear functionals on V which separates V. For $\xi\in\Phi$ and $a\in D_\mu$, we have $\xi\circ\partial_a\mu=\langle a,\xi\rangle\partial_1(\xi\circ\mu)$ therefore $|\langle a,\xi\rangle|\,\|\partial_1(\xi\circ\mu)\|_{var}\leqq\|\partial_a\mu\|_{var}=\|a\|_{D_\mu}$. Thus if V is sequentially complete in the weak topology $\sigma(V,\Phi)$, [Y-H1]

Corollary to Theorem 3.1 shows that D_μ is a Banach space with respect to $\|\cdot\|_{D_\mu}$.

Now we consider the case where $V=\boldsymbol{R}^N$ and μ is a product measure of a one-dimensional differentiable probability ν (we write $\mu=\nu^\infty$). We write l_ν instead of l_ν^1 for simplicity. For $a=(a(k))_{k\in N}\in D_\mu$, we have

$$l_\mu^a(x)=\sum_{k=0}^{\infty} a(k)l_\nu(x(k)), \qquad x=(x(k))_{k\in N}\in \boldsymbol{R}^N, \tag{4.1}$$

$$\|a\|_{D_\mu}=\|l_\mu^a\|_{L_1(\mu)}=\sup_{n\in N}\|\sum_{k=0}^{n} a(k)l_\nu(x(k))\|_{L_1(\mu)}. \tag{4.2}$$

Here $\{l_\nu(x(k))\}_{k\in N}$ is a system of independent random variables on $(\boldsymbol{R}^N, \mu)$ with the identical distribution and $\int_{\boldsymbol{R}}|l_\nu(t)|d\nu(t)<\infty$, $\int_{\boldsymbol{R}} l_\nu(t)d\nu(t)=0$ hold. Thus, in the case where μ is a product measure of one-dimensional identical probabilities, the space D_μ is one of those Banach spaces investigated in §2 (the case of discrete time).

Conversely we will prove that such D_μ's cover all of D_X's. That is, given an arbitray real-valued random variable X satisfying $E|X|<\infty$ and $EX=0$, we can find a probability ν on $\boldsymbol{R}$ whose logarithmic derivative l_ν has the same distribution (w.r.t. ν) as that of X. We must exclude, however, the trivial case where $X=0$ (a.s.) i.e. the distribution of X coincides with the Dirac measure δ_0 since $l_\nu=0$ (ν-a.s.) would imply $\nu=0$ ([Y-H1] Lemma 3.1).

We begin with preliminary considerations. First let ϕ be a given strictly positive function on $\boldsymbol{R}$. We consider an equation

$$\int_{\{s\in \boldsymbol{R};\ \frac{f'(s)}{f(s)}\geq t\}} f(s)ds=\int_{[t,\infty)}\phi(s)ds. \tag{4.3}$$

Let $f(s)>0$ and ν be the probability which has the density f. Then (4.3) implies that the distribution of l_ν with respect to ν has the density ϕ. Now we put $\varphi=f'/f$ and assume that φ is strictly decreasing. Integrating (4.3) on $[u, \infty)$ and changing the orders of integration, we get

$$f(\varphi^{-1}(u))=\int_u^\infty s\phi(s)ds. \tag{4.4}$$

Denote by $h(u)$ the right hand side of (4.4). Differentiating (4.4) and using $\dfrac{f'(\varphi^{-1}(u))}{f(\varphi^{-1}(u))}=u$, we get $(\varphi^{-1})'(u)=-\dfrac{\phi(u)}{h(u)}$, hence

$$\varphi^{-1}(t)=-\int_0^t \frac{\phi(\tau)}{h(\tau)}d\tau+\text{constant}. \tag{4.5}$$

If f is a solution of (4.3), so is an arbitrarily shifted function $f(\cdot+\text{constant})$. This corresponds to the constant summand in (4.5). Thus we find a solution f of (4.3):

(4.6) $$f(\varphi^{-1}(t))=h(t), \qquad \varphi^{-1}(t)=-\int_0^t \frac{\psi(\tau)}{h(\tau)}d\tau\,.$$

Next let the distribution of X be a purely atomic measure whose support is a finite set. Then the equation we should deal with is

(4.3)′ $$\int_{\{s\in \boldsymbol{R};\ \frac{f'(s)}{f(s)}=a_k\}} f(s)ds=P(X=a_k)$$

(where $\{a_1, \cdots, a_n\}=\mathrm{supp}\, P^X$). Noting that $d/ds \exp(a_k s)/\exp(a_k s)=a_k$, we can construct a solution f of (4.3)′ by patching the functions of the form: $\exp(a_k s)$.

Under these considerations, we prove the following

Theorem 4.1. *Let X be a nontrivial real-valued random variable satisfying $E|X|<\infty$ and $EX=0$. Then there exists a probability measure ν on $\boldsymbol{R}$ whose density f satisfies $f, f'\in L_1(\boldsymbol{R})$ and whose logarithmic derivative l_ν has the same distribution with respect to ν as that of X.*

Note. The author's original proofs to Theorem 4.1 were accomplished in the case where (i) the distribution of X is absolutely continuous with respect to the Lebesgue measure or (ii) the support of the distribution of X is a finite set. Afterwards Y. Yamasaki completed the proof in a general situation unifying these two proofs.

Proof of Theorem 4.1. Denote by ρ the distribution of X. We divide the proof into four steps.

(*STEP* 1) We define a function f which will prove to be one of the desired solution. Put

(4.7) $$h(s)=\int_{[s,\infty)} \sigma\, d\rho(\sigma)\,.$$

h is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$ and left-continuous. Moreover h satisfies $h(s)\geqq 0$, $h(-\infty)=0$, $h(\infty)=0$, $h(0)>0$ and $\lim_{s\to 0+0} h(s)>0$. Put

(4.8) $$\alpha=\inf\{s\,;\, h(s)>0\}, \qquad \beta=\sup\{s\,;\, h(s)>0\}$$

$(-\infty\leqq\alpha<0<\beta\leqq\infty)$. Denote by $\{s_n\}_{n\in \boldsymbol{N}}$ the set of atomic points of ρ. Then $\alpha\leqq s_n\leqq\beta$ and $h(s_n)>|s_n|\rho(s_n)$ for $s_n\neq\alpha, \beta$. We express ρ as

(4.9) $$\rho=\rho_1+\sum_n \rho(s_n)\delta_{s_n}$$

where ρ_1 has no atomic points and δ_{s_n} is the Dirac measure supported by s_n. Modifying (4.9), we put

(4.10) $$\rho'=\rho_1+\sum_n m_n\delta_{s_n}\,,$$

$$m_n=\begin{cases} -\dfrac{h(s_n)}{s_n}\log\left(1-\dfrac{s_n\rho(s_n)}{h(s_n)}\right) & \text{if } s_n\neq\alpha,\,0,\,\beta \\ \rho(s_n) & \text{if } s_n=0 \\ 0 & \text{if } s_n=\alpha,\,\beta \end{cases}.$$

And using this, we define

$$\gamma(s)=\begin{cases} \displaystyle\int_{[0,s)}\frac{d\rho'(t)}{h(t)} & \text{if } 0<s<\beta \\ 0 & \text{if } s=0 \\ \displaystyle-\int_{[s,0)}\frac{d\rho'(t)}{h(t)} & \text{if } \alpha<s<0 \end{cases} \tag{4.11}$$

($-\gamma$ corresponds to φ^{-1} in the preceding consideration before the theorem). γ is left-continuous and increasing on (α, β). Put

$$\gamma_1=\lim_{s\to\alpha}\gamma(s), \qquad \gamma_2=\lim_{s\to\beta}\gamma(s) \tag{4.12}$$

($-\infty\leqq\gamma_1\leqq 0\leqq\gamma_2\leqq\infty$). The case $\gamma_1=\gamma_2=0$ occurs if and only if ρ is a convex sum of δ_α and δ_β. In this case we can immediately verify the assertion of the theorem (see around (4.3)′), so we shall assume $\gamma_1<0$ or $\gamma_2>0$.

We define $f(t)$ on $(-\gamma_2, -\gamma_1)$ as follows: (i) If $t=-\gamma(s)$ for some $s\in(\alpha, \beta)$, we put

$$f(t)=h(s) \quad (>0). \tag{4.13}$$

We can easily check the consistency of this definition. (ii) If there exists no s satisfying $t=-\gamma(s)$, we put

$$s_t=\sup\{s\in(\alpha, \beta)\,;\ t<-\gamma(s)\} \tag{4.14}$$

(then $\alpha<s_t<\beta$ and $t<-\gamma(s_t)$) and

$$f(t)=h(s_t)\exp s_t(t+\gamma(s_t)). \tag{4.15}$$

(*STEP* 2) We prove continuity of f on $(-\gamma_2, \gamma_1)$ and extend f continuously on $\boldsymbol{R}$. To estimate $|f(t)-f(t')|\,(-\gamma_2<t<t'<-\gamma_1)$, we consider four possible cases.

(i) If $t=-\gamma(s)$ and $t'=-\gamma(s')$ for some $s, s'\in(\alpha, \beta)$, we have

$$f(t)-f(t')=h(s)-h(s')=-\int_{[s',s)}\sigma\,d\rho(\sigma),$$

$$t-t'=-(\gamma(s)-\gamma(s'))=-\int_{[s',s)}\frac{d\rho'(\sigma)}{h(\sigma)}.$$

Since $\rho\sim\rho'$ and $h(\sigma)>0$ on (α, β), we see $\lim_{t'-t\to 0}|f(t)-f(t')|=0$.

(ii) If $t\neq-\gamma(s)$ for any $s\in(\alpha, \beta)$ and $t'=-\gamma(s')$ for some $s'\in(\alpha, \beta)$, taking

s_t of (4.14), we have $t<-\gamma(s_t)\leqq t'$ and

$$|f(t)-f(t')|\leqq|f(t)-f(-\gamma(s_t))|+|f(-\gamma(s_t))-f(t')|$$
$$\leqq h(s_t)|\exp s_t(t+\gamma(s_t))-1|+|f(-\gamma(s_t))-f(t')|$$
$$\xrightarrow[t'-t\to 0]{} 0$$

(where $\lim|f(-\gamma(s_t))-f(t')|=0$ follows from (i)).

(iii) If $t=-\gamma(s)$ for some $s\in(\alpha, \beta)$ and $t'\neq-\gamma(s')$ for any $s'\in(\alpha, \beta)$, taking $s_{t'}$ as above, we have $t\leqq-\gamma(s_{t'}+0)\leqq t'$ and

$$|f(t')-f(-\gamma(s_{t'}+0))|\leqq h(s_{t'})|\exp s_{t'}(t'+\gamma(s_{t'}))-$$
$$\exp s_{t'}(-\gamma(s_{t'}+0)+\gamma(s_{t'}))|\xrightarrow[t'-t\to 0]{} 0.$$

Moreover, noting $f(-\gamma(s_{t'}+0))=h(s_{t'})-s_{t'}\rho(s_{t'})=h(s_{t'}+0)$ since $-\gamma(s_{t'}+0)+\gamma(s_{t'})=-\dfrac{\rho'(s_{t'})}{h(s_{t'})}=-\dfrac{1}{s_{t'}}\log\Big(1-\dfrac{s_{t'}\rho(s_{t'})}{h(s_{t'})}\Big)$, we have $f(-\gamma(s_{t'}+0))-f(t)=\int_{(s_{t'},s)}\sigma d\rho(\sigma)$. And since $-\gamma(s_{t'}+0)-t=\int_{(s_{t'},s)}\dfrac{d\rho'(\sigma)}{h(\sigma)}$, the same argument as in (i) shows $\lim_{t'-t\to 0}|f(-\gamma(s_{t'}+0))-f(t)|=0$.

(iv) If $t\neq-\gamma(s)$ and $t'\neq-\gamma(s')$ for any $s, s'\in(\alpha, \beta)$, taking s_t and $s_{t'}$, we have, if $s_{t'}<s_t$, $t<-\gamma(s_t)\leqq\gamma(s_{t'}+0)\leqq t'$ while the case $s_{t'}=s_t$ is trivial. Then we get $\lim_{t'-t\to 0}|f(t')-f(t)|=0$ from (ii) and (iii). This completes the proof of continuity of f on $(-\gamma_2, -\gamma_1)$.

Now we extend f continuously on $\boldsymbol{R}$. Since $\rho(\alpha)=0$ implies $f(-\gamma_1-0)=h(\alpha+0)=0$, we put

(4.16) $$f(t)=0 \quad \text{on } (-\gamma_1, \infty) \quad \text{if } \rho(\alpha)=0.$$

But $\rho(\alpha)>0$ implies $f(-\gamma_1-0)=h(\alpha+0)=-\alpha\rho(\alpha)$. In this case we put

(4.16)′ $$f(t)=-\alpha\rho(\alpha)\exp\alpha(t+\gamma_1) \quad \text{on } (-\gamma_1, \infty) \quad \text{if } \rho(\alpha)>0.$$

Similarly we extend on $(-\infty, -\gamma_2)$ as

(4.17) $$f(t)=0 \quad \text{on } (-\infty, -\gamma_2) \quad \text{if } \rho(\beta)=0$$

(4.17)′ $$f(t)=\beta\rho(\beta)\exp\beta(t+\gamma_2) \quad \text{on } (-\infty, -\gamma_2) \quad \text{if } \rho(\beta)>0.$$

Thus we obtain a continuous positive function f on $\boldsymbol{R}$.

(*STEP* 3) We show that f is differentiable except at most countable points on $\boldsymbol{R}$. The exceptional set is $\{t\in(-\gamma_2, -\gamma_1);\ t=-\gamma(s_1)=-\gamma(s_2)$ for distinct $s_1, s_2\in(\alpha, \beta)\}\cup\{-\gamma(s), -\gamma(s+0);\ \gamma(s)<\gamma(s+0),\ s\in(\alpha, \beta)\}\cup\{-\gamma_2, -\gamma_1\}$. The differentiability of f outside $(-\gamma_2, -\gamma_1)$ and at t satisfying $-\gamma(s+0)<t<-\gamma(s)$ is obvious. And we have

(4.18)
$$f'(t)=\begin{cases} sf(t) & \text{if } -\gamma(s+0)<t<-\gamma(s) \\ \alpha f(t) & \text{if } t>-\gamma_1 \\ \beta f(t) & \text{if } t<-\gamma_2 \end{cases}.$$

Hence the remaining case is the one where $t=-\gamma(s)$ holds for some $s\in(\alpha, \beta)$ and $-\gamma$ is strictly decreasing near s. To estimate $f(t')-f(t)/t'-t$, we consider two possible cases.

(i) If $t'=-\gamma(s')$ for some $s'\in(\alpha, \beta)$, we have

$$\frac{f(t')-f(t)}{t'-t}=\frac{h(s')-h(s)}{\gamma(s)-\gamma(s')}=\frac{\int_{[s',s)}\sigma d\rho(\sigma)}{\int_{[s',s)}\frac{d\rho'(\sigma)}{h(\sigma)}}\xrightarrow[t'\to t]{} sh(s),$$

where we use also the fact that, for a fixed s, the difference between ρ and ρ' are arbitrarily small in a sufficiently small neighborhood of s.

(ii) If $t'\neq-\gamma(s')$ for any $s'\in(\alpha, \beta)$ and $t<t'$, taking $s_{t'}$ (see (4.14)), we have $t\leqq-\gamma(s_{t'}+0)\leqq t'<-\gamma(s_{t'})$ and

$$\frac{f(-\gamma(s_{t'}+0))-f(t)}{-\gamma(s_{t'}+0)-t}=\frac{h(s_{t'}+0)-h(s)}{\gamma(s)-\gamma(s_{t'}+0)}\xrightarrow[t'\to t]{} sh(s),$$

$$\frac{f(-\gamma(s_{t'}))-f(t)}{-\gamma(s_{t'})-t}=\frac{h(s_{t'})-h(s)}{\gamma(s)-\gamma(s_{t'})}\xrightarrow[t'\to t]{} sh(s),$$

$$\frac{f(-\gamma(s_{t'}))-f(-\gamma(s_{t'}+0))}{-\gamma(s_{t'})-(-\gamma(s_{t'}+0))}=s_{t'}h(s_{t'})\xrightarrow[t'\to t]{} sh(s).$$

These imply $\frac{f(t')-f(t)}{t'-t}\xrightarrow[t'\to t]{} sh(s)$. Quite similar is the case $t'<t$. Thus f is differentiable at t and satisfies

(4.19)
$$f'(t)=sf(t).$$

(*STEP* 4) Let ν be the measure on $\boldsymbol{R}$ with the density f. We show that the distribution of f'/f with respect to ν coincides with ρ. If s is an atomic point of ρ', we have from (4.18) and (4.15)

(4.20)
$$\nu\Big(\frac{f'}{f}=s\Big)=\int_{-\gamma(s+0)}^{-\gamma(s)}f(t)dt=\int_{-\gamma(s+0)}^{-\gamma(s)}h(s)e^{s(t+\gamma(s))}dt$$
$$=\rho(s).$$

If α is an atomic point of ρ, we have from (4.18) and (4.16)′

(4.21)
$$\nu\Big(\frac{f'}{f}=\alpha\Big)=\int_{-\gamma_1}^{\infty}f(t)dt=\int_{-\gamma_1}^{\infty}(-\alpha)\rho(\alpha)e^{\alpha(t+\gamma_1)}dt=\rho(\alpha).$$

Similarly, if β is an atomic point of ρ, we have from (4.18) and (4.17)′

$$(4.22)\qquad \nu\Big(\frac{f'}{f}=\beta\Big)=\rho(\beta).$$

If s is not an atomic point of ρ, we see from (4.19) that $\frac{f'(t)}{f(t)}=s$ is equivalent to $t=-\gamma(s)$. Hence, for a Borel set A containing no atomic points of ρ, we get

$$(4.23)\qquad \nu\Big(\frac{f'}{f}\in A\Big)=\int_{-\gamma(A)}f(t)dt=\int_A h(s)d\gamma(s)=\rho'(A)=\rho(A).$$

Thus (4.20), (4.21), (4.22) and (4.23) shows the desired result:

$$(4.24)\qquad \nu\Big(\frac{f'}{f}\in B\Big)=\rho(B)\qquad \text{for any Borel set } B.$$

We see from (4.24) $f,\ f'\in L_1(\boldsymbol{R})$ noting that ρ is a probability on $\boldsymbol{R}$ satisfying $\int_{\boldsymbol{R}}|\sigma|d\rho(\sigma)<\infty$. □

Remark 1. Let $1<p\leqq 2$. A p-stable random variable X satisfies the assumption of Theorem 4.1. So we can construct a corresponding probability ν on $\boldsymbol{R}$. If X is normalized to be $E|X|=1$, $\|\cdot\|_{D_X}=\|\cdot\|_{l_p}$ holds on $\boldsymbol{R}_0^N$. Thus we obtain a stationary product measure $\mu=\nu^\infty$ such that D_μ coincides with l_p isometrically.

Remark 2. We consider the case of continuous time. Let μ be a probability on a space of functions on $T=\boldsymbol{R}_+$. If Step$(T)\subset D_\mu$ holds, then, for $x=\sum c_k I_{(t_k,t_{k+1}]}\in$ Step(T), we have

$$l_\mu^x(y)=\sum_k c_k l_\mu(I_{(t_k,t_{k+1}]};y)=\sum_k c_k(l_\mu(I_{(0,t_{k+1}]};y)-l_\mu(I_{(0,t_k]};y)).$$

So, in this case, the problem is: how wide class of stochastic processes on T does $\{l_\mu(I_{(0,t]};y)\}_\mu$ cover? We have no satisfactory answers to this question.

Acknowledgement

The author is grateful to Prof. Y. Yamasaki for valuable comments and advices especially concerning Appendix to § 3 and Theorem 4.1.

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Added in proof. A remark about Theorem 4.1 and its note: After submitting the revised version of this paper, the author was informed that H. Shimomura had succeeded independently to complete the theorem in a general situation. His proof, which is simpler than ours, will be soon published with some additional facts.