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On a Geometric Realization of $\mathcal{A}(2)$

By

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§ 0. Introduction

Let $\mathcal{A}$ be the mod p Steenrod algebra and M be a bounded below left $\mathcal{A}$-module of finite type. M is said to be realizable if there exists some spectrum whose mod p cohomology is isomorphic as $\mathcal{A}$-module to M. For example, $\mathcal{A}$ itself is realized as $\mathcal{A} \cong H^*(HZ/p; Z/p)$. It is a general problem whether or not given M is realizable, but there is no standard method to solve this problem. So we have to try case by case. For many interesting cases, this problem was solved. J. F. Adams [1] showed that there is no spectrum which realizes $M \cong Z/2 \cdot x + Z/2 \cdot Sq^{16}x$. E. H. Brown and S. Gitler [2] constructed certain spectra $B(k)$ such that $H^*B(k) \cong \mathcal{A}/\mathcal{A}\{\chi(Sq^i) | i > k\}$. H. Toda [8] stated that certain algebraic properties of M assure its realizability. In this paper we shall prove that some more conditions give us useful information about the number of the homotopy types of spectra which realize M. (Theorem 1.1)

$\mathcal{A}(n)$ is a sub-Hopf algebra of $\mathcal{A}$ generated by $\beta, \mathcal{P}^1, \cdots, \mathcal{P}^{p^{n-1}}$, with $\mathcal{P}^i = Sq^{2i}$ for $p=2$. S. A. Mitchell [6] proved every $\mathcal{A}(n)$ admits certain left $\mathcal{A}$ module structure extended from its own algebra multiplication and also constructed finite spectra whose cohomologies are $\mathcal{A}(n)$ free. Hence we should ask whether each $\mathcal{A}(n)$ itself is realizable or not, because there exists a non-realizable $\mathcal{A}$-module which is a direct summand of a realizable module.

Independently of Mitchell's work, D. M. Davis and M. Mahowald [3] gave four different module structures on $\mathcal{A}(1)$ ($p=2$) and proved the uniqueness of the homotopy type of spectra which realize each $\mathcal{A}(1)$. For the case of $\mathcal{A}(2)$ ($p=2$), W. H. Lin [4] showed 1600 different $\mathcal{A}$-module structures. Theorem 2.2 gives an affirmative answer to the realization problem for $\mathcal{A}(2)$ ($p=2$) with any possible $\mathcal{A}$-module structure and Theorem 2.4 shows the uniqueness of the homotopy type of spectra which realize $\mathcal{A}(2)$ with the specific $\mathcal{A}$-module structure indicated by Mitchell [6].

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§ 1. The Number of the Homotopy Types

We work in the homotopy category of HZ/p_*-local CW-spectra, because any spectrum X has the same mod p cohomology group as its HZ/p_*-localization $\bar{X}$, cohomology equivalence means (homotopy) equivalence and this category includes usual bounded below p-complete spectra. In larger categories, it might be impossible to count the homotopy types of spectra which realize the same $\mathcal{A}$-module because of the existence of HZ/p_*-acyclic spectra.

Theorem 1.1. *Let* M *be a bounded below* $\mathcal{A}$*-module of finite type with the following properties*:

(1) $M^n \neq 0$ *implies* $\mathrm{Ext}_{\mathcal{A}}^{s+2,s+n}(M, \boldsymbol{F}_p)=0$ *for* $s \geqq 1$,

(2) $M^n \neq 0$ *implies* $\mathrm{Ext}_{\mathcal{A}}^{s+1,s+n}(M, \boldsymbol{F}_p)=0$ *for* $s \geqq 2$.

Then there exists a bounded below HZ/p_**-local spectrum* X *such that* $M \cong H^*X$ *as* $\mathcal{A}$*-module. And let* Σ *be the set of the homotopy types of such spectra, then the following inequalities hold*: (*Here* $|\Sigma|$ *means the number of the elements of* Σ.)

$$|\mathrm{Ext}_{\mathcal{A}}^{2,1}(M, M)| / |\mathrm{Aut}_{\mathcal{A}}(M)| \leqq |\Sigma| \leqq |\mathrm{Ext}_{\mathcal{A}}^{2,1}(M, M)|.$$

Proof. The existence of such a spectrum follows from Toda [8] by only using the condition (1). We recall it for the further proof.

Fix a minimal resolution of M as $\mathcal{A}$-module:

$$0 \longleftarrow M \overset{\varepsilon}{\longleftarrow} C^0 \overset{\delta^1}{\longleftarrow} C^1 \overset{\delta^2}{\longleftarrow} C^2 \overset{\delta^3}{\longleftarrow} \cdots, \quad \text{where } C^s \cong \mathcal{A} \otimes \mathrm{Ext}_{\mathcal{A}}^{s,*}(M, \boldsymbol{F}_p).$$

C^s is realized by a generalized Eilenberg-MacLane spectrum $\boldsymbol{W}_s$. And starting from $\boldsymbol{X}_0 = \boldsymbol{W}_0$, we can construct a sequence of spectra $\{\boldsymbol{X}_s\}$ satisfying the following conditions:

a) There are fibrations $\Sigma^{-s}\boldsymbol{W}_s \overset{k_s}{\to} \boldsymbol{X}_s \overset{i_s}{\to} \boldsymbol{X}_{s-1} \overset{\pi_s}{\to} \Sigma^{-s+1}\boldsymbol{W}_s$ which induce exact sequences:

$$C^{s,s+n} \overset{k_s^*}{\longleftarrow} H^n \boldsymbol{X}_s \overset{i_s^*}{\longleftarrow} H^n \boldsymbol{X}_{s-1} \overset{\pi_s^*}{\longleftarrow} C^{s,s+n-1}.$$

b) $\delta^{s+1} = k_s^* \circ \pi_{s+1}^* : C^{s+1} \to H^*\boldsymbol{X}_s \to C^s$.

c) There are split short exact sequences:

$$0 \longrightarrow M \longrightarrow H^*\boldsymbol{X}_s \overset{k_s^*}{\longrightarrow} \delta^{s+1}(C^{s+1}) \longrightarrow 0.$$

Then the spectrum $\boldsymbol{X} = \lim \boldsymbol{X}_s$ realizes M.

$$
(*)\qquad
\begin{array}{ccccccccc}
W_0=X_0 & \xleftarrow{i_1} & X_1 & \xleftarrow{i_2} & X_2 & \xleftarrow{i_3} & X_3 & \longleftarrow \cdots\cdots \longleftarrow & \lim X_s=X \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & & & \\
W_1 & & \Sigma^{-1}W_2 & & \Sigma^{-2}W_3 & & & &
\end{array}
$$

Next we prove any spectrum realizing M is homotopy equivalent to some spectrum obtained by the above method. Let Y be such a spectrum, and $g_0: Y \to X_0$ be a map realizing $\varepsilon: \mathrm{C}^0=\mathrm{H}^*X_0 \to \mathrm{M} \cong \mathrm{H}^*Y$. Since $\pi_1 \circ g_0 \simeq 0$, g_0 has a lift $g_1: Y \to X_1$. Moreover there exists a map $\alpha: X_0 \to \Sigma^{-1}W_2$ such that $\alpha \circ g_0 \simeq \pi_2 \circ g_1$, because ε is surjective. Even if we replace π_2 by $\pi_2' = \pi_2 - \alpha \circ i_1$, we can proceed the construction of another sequence of spectra $\{X_s'\}$ from which $X' = \lim X_s'$ also realizes M, because each π_s is required only to satisfy the above condition b).

$$
(**)\qquad
\begin{array}{ccccccccc}
 & & Y & & & & & & \\
 & {\scriptstyle g_0}\swarrow & \downarrow{\scriptstyle g_1} & & & & & & \\
W_0=X_0 & \xleftarrow{i_1} & X_1 & \xleftarrow{i_2'} & X_2' & \xleftarrow{i_3'} & X_3' & \longleftarrow \cdots\cdots \longleftarrow & \lim X_s'=X' \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2'} & & \downarrow{\scriptstyle \pi_3'} & & & & \\
W_1 & & \Sigma^{-1}W_2 & & \Sigma^{-2}W_3 & & & &
\end{array}
$$

Now consider the Adams spectral sequence associated to this tower $(**)$:

$$\mathrm{Ext}_{\mathcal{A}}^{**}(\mathrm{H}^*X', \mathrm{H}^*Y) \Longrightarrow [Y, X']^*.$$

Since we fixed a map $g_0: Y \to X_0$ realizing $\varepsilon: \mathrm{C}^0 \to \mathrm{M}$, there is one and only one isomorphism $\beta: \mathrm{H}^*X' \to \mathrm{H}^*Y$ such that $g_0^* = \beta \circ f_0'^*$. ($f_s'$ is a composition of maps $X' \to \cdots \to X_{s+1}' \to X_s'$) And $\beta \in \mathrm{E}_2^{0,0} = \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^*X', \mathrm{H}^*Y)$ is represented by g_0 in the spectral sequence. Since $\pi_2' \circ g_1 = \pi_2 \circ g_1 - \alpha \circ i_1 \circ g_1 = \pi_2 \circ g_1 - \alpha \circ g_0 = 0 \in [Y, \Sigma^{-1}W_2]$, we get $d_2(\beta) = 0 \in \mathrm{E}_2^{2,1}$. And the condition (2) implies:

$$\mathrm{Hom}_{\mathcal{A}}^s(\mathrm{C}^{s+1}, \mathrm{M}) \cong \mathrm{Hom}_{\mathcal{A}}^s(\mathcal{A} \otimes \mathrm{Ext}_{\mathcal{A}}^{s+1,*}(\mathrm{M}, \boldsymbol{F}_p), \mathrm{M}) \cong \mathrm{Hom}^s(\mathrm{Ext}_{\mathcal{A}}^{s+1,*}(\mathrm{M}, \boldsymbol{F}_p), \mathrm{M}) \cong 0$$

for $s \geqq 2$. So $\mathrm{Ext}_{\mathcal{A}}^{s+1,s}(\mathrm{M}, \mathrm{M}) \cong 0$, that is, $d_{s+1}(\beta) = 0$. Thus there exists a map $g: Y \to X'$ realizing β. Since X' and Y are HZ/p_*-local spectra, g is a homotopy equivalence.

Next we construct a set function Φ which has Σ as its domain and the set of subsets of $\mathrm{Ext}_{\mathcal{A}}^{2,1}(\mathrm{M}, \mathrm{M})$ as its target. As studied above, the isomorphism $\beta': \mathrm{M} = \mathrm{H}^*X \to \mathrm{H}^*X'$ is uniquely determined for any $X' \in \Sigma$ and any map $f_0': X' \to W_0$ realizing ε. We consider the Adams spectral sequence $\mathrm{E}_2^{**} \cong \mathrm{Ext}_{\mathcal{A}}^{**}(\mathrm{H}^*X, \mathrm{H}^*X') \Rightarrow [X', X]^*$ associated to the tower $(*)$, and put

$$\Phi(X', f_0')=\beta_*'^{-1}(d_2(\beta'))\in \mathrm{Ext}_{\mathcal{A}}^{2,1}(\mathrm{M}, \mathrm{M}).$$

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^*X, \mathrm{H}^*X') & \xrightarrow{d_2} & \mathrm{Ext}_{\mathcal{A}}^{2,1}(\mathrm{H}^*X, \mathrm{H}^*X') & \xrightarrow{\beta_*'^{-1}} & \mathrm{Ext}_{\mathcal{A}}^{2,1}(\mathrm{M}, \mathrm{M}) \\ \Cup & & \Cup & & \Cup \\ \beta' & \longmapsto & d_2(\beta') & \longmapsto & \Phi(X', f_0') \end{array}$$

Let $\Phi(X')=\{\Phi(X', f_0') \mid \text{for all possible } f_0'\text{'s}\}$, then Φ has a property such that $\Phi(X')\cap\Phi(X'')\neq\phi$ implies $X'\cong X''$. To see this, suppose $\Phi(X', f_0')=\Phi(X'', f_0'')$, then from the above diagram,

$$\beta_*'\circ\beta_*''^{-1}[\pi_2\circ f_1'']=[\pi_2\circ f_1']\in \mathrm{Ext}_{\mathcal{A}}^{2,1}(\mathrm{H}^*X, \mathrm{H}^*X').$$

$\mathrm{Ext}^{2,1}$ is defined as the (co)homology of sequence:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{A}}^1(\mathrm{C}^1, \mathrm{H}^*X') & \xrightarrow{\delta^{2*}} & \mathrm{Hom}_{\mathcal{A}}^1(\mathrm{C}^2, \mathrm{H}^*X') & \xrightarrow{\delta^{3*}} & \mathrm{Hom}_{\mathcal{A}}^1(\mathrm{C}^3, \mathrm{H}^*X')\cong 0, \\ \uparrow\cong & & \uparrow\cong & & \\ [X', \Sigma^{-1}W_1] & & [X', \Sigma^{-1}W_2] & & \end{array}$$

where the last isomorphism is due to the condition (1), and the vertical isomorphisms mean sending a map between spectra to its induced map between their cohomologies.

So there exists a map $h: X'\to\Sigma^{-1}W_1$ such that

$$f_1'^*\circ\pi_2^*-\beta'\circ\beta''^{-1}\circ f_1''^*\circ\pi_2^*=h^*\circ\delta^2 \qquad \text{in} \quad \mathrm{Hom}_{\mathcal{A}}^1(\mathrm{C}^2, \mathrm{H}^*X').$$

And by taking maps $\alpha', \alpha'': X_0\to\Sigma^{-1}W_2$ such that

$$\pi_2\circ f_1'=\alpha'\circ f_0', \qquad \pi_2\circ f_1''=\alpha''\circ f_0',$$

we get the following equation:

$$f_0'^*\circ(\alpha'^*-\alpha''^*)=h^*\circ\delta^2.$$

Again consider the Adams spectral sequence associated to the tower (**):

$$\mathrm{E}_2^{**}\cong \mathrm{Ext}_{\mathcal{A}}^{**}(\mathrm{H}^*X'', \mathrm{H}^*X')\Longrightarrow [X', X'']^*.$$

$$d_2(\beta'\circ\beta''^{-1})=[(\pi_2''\circ f_1')^*]=[f_0'^*\circ(\alpha'^*-\alpha''^*)]=[h^*\circ\delta^2]=0.$$

Therefore $\beta'\circ\beta''^{-1}$ is realizable, namely, $X'\simeq X''$.

On the other hand, for any $[\alpha\circ f_0]\in \mathrm{Ext}_{\mathcal{A}}^{2,1}(\mathrm{M}, \mathrm{M})$, we can construct a spectrum X' from $\pi_2'=\pi_2-\alpha\circ i_1$. Then $\Phi(X', f_0')=\beta^{-1}{}_*[\pi_2\circ f_1']=[\alpha\circ f_0]$. Thus we can conclude:

$$\coprod_{X\in\Sigma}\Phi(X)=\mathrm{Ext}_{\mathcal{A}}^{2,1}(\mathrm{M}, \mathrm{M}),$$

where $\coprod$ means disjoint union.

Suppose the following situation: $\mathrm{H}^*X'\overset{\cong}{\underset{\beta_2'}{\longleftarrow}}\mathrm{M}\overset{\cong}{\underset{\beta_1'}{\longrightarrow}}\mathrm{H}^*X'$. As studied above,

$\beta_1'^{-1}{}_*d_2(\beta_1')=\beta_2'^{-1}{}_*d_2(\beta_2')$ iff $\beta_1'\circ\beta_2'^{-1}$ is realizable. Thus there is a one-to-one correspondence between $\Phi(X')$ and $\mathrm{Heq}(X')\backslash\mathrm{Aut}_{\mathcal{A}}(\mathrm{H}^*X')$, where $\mathrm{Heq}(X')$ is a subgroup of $\mathrm{Aut}_{\mathcal{A}}(\mathrm{H}^*X')$ whose elements are induced from self homotopy equivalences on X'. Then the inequalities in the theorem follow easily. (Q. E. D.)

Remark. It is very hard to calculate $|\mathrm{Heq}(X')|$ and, what is worse, $|\mathrm{Heq}(X')|$ may be different for each X'. So I had to find satisfaction in these inequalities, regrettably.

§ 2. Realization of $\mathcal{A}(2)$ ($p=2$)

Milnor basis of $\mathcal{A}$ is an $\boldsymbol{F}_2$ vector space basis of $\mathcal{A}$ written as $\{Sq(r_1, r_2, r_3, \cdots) \mid r_i\geqq 0\}$. See J. Milnor [5] for the further structures. Using this notation, we can define $\mathcal{A}(n)$ as a vector subspace of $\mathcal{A}$ whose basis is $\{Sq(r_1, r_2, \cdots, r_{n+1}) \mid 0\leqq r_i<2^{n+2-i}\}$. P_t^s is defined as $Sq(0, \cdots, 0, 2^s)$, where 2^s is occured in the t-th entry.

We consider the reindexed version of the May spectral sequence for $\mathrm{Ext}_{\mathcal{A}}^{**}(\mathcal{A}(2), \boldsymbol{F}_2)$ according to D. C. Ravenel [7]:

$$\mathrm{E}_1^{s,t,u}\cong\mathrm{Ext}_{\mathrm{E}_0\mathcal{A}}^{s,t,u}(\mathrm{E}_0\mathcal{A}(2), \boldsymbol{F}_2)\Longrightarrow\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A}(2), \boldsymbol{F}_2),$$
$$d_r:\mathrm{E}_r^{s,t,u}\longrightarrow\mathrm{E}_r^{s+1,t,u+1-2r}.$$

Here, $\mathrm{E}_0\mathcal{A}$ is the bigraded Hopf algebra associated with a decreasing filtration on $\mathcal{A}$ defined by setting $|\mathrm{P}_t^s|=2t-1$. In fact:

$$\mathrm{E}_0\mathcal{A}\cong\mathrm{E}(\mathrm{P}_t^s\,;\,t>0,\ s\geqq 0),\qquad \mathrm{P}_t^s: \text{primitive}.$$

Lemma 2.1. *$\mathcal{A}(2)$ has 1600 different $\mathcal{A}$-module structures (Lin [4]), but every $\mathrm{E}_0\mathcal{A}(2)$ has the same $\mathrm{E}_0\mathcal{A}$ module structure such that:*

$$\mathrm{E}_0\mathcal{A}(2)\cong\mathrm{E}(\mathrm{P}_t^s\,;\,t>0,\ s\geqq 0,\ t+s\leqq 3).$$

Proof. $\mathcal{A}$ has a free $\mathcal{A}(2)$ basis $\{\mathrm{P}_1^3, \mathrm{P}_2^2, \mathrm{P}_3^1, \mathrm{P}_4^0, \mathrm{P}_1^4\}$ up to degree 23, the maximal degree of $\mathcal{A}(2)$. So weh ave only to show these are mapped into higher filtration when they are applied to ι, the fundamental class of $\mathcal{A}(2)$. But this is immediate because $\mathcal{A}(2)$ has only such higher filtration degree elements in the degree of P_t^s.

$|\mathrm{P}_1^3|=1$, $|Sq(5, 1)|=5$, $|Sq(2, 2)|=4$, $|Sq(1, 0, 1)|=6$,

$|\mathrm{P}_2^2|=3$, $|Sq(6, 2)|=5$, $|Sq(3, 3)|=8$, $|Sq(5, 0, 1)|=7$, $|Sq(2, 1, 1)|=9$,

$|\mathrm{P}_3^1|=5$, $|Sq(5, 3)|=8$, $|Sq(7, 0, 1)|=8$, $|Sq(4, 1, 1)|=9$, $|Sq(1, 2, 1)|=9$,

$|\mathrm{P}_4^0|=7$, $|Sq(6, 3)|=8$, $|Sq(5, 1, 1)|=10$, $|Sq(2, 2, 1)|=9$,

$|\mathrm{P}_1^4|=1$, $|Sq(7, 3)|=9$, $|Sq(6, 1, 1)|=10$, $|Sq(3, 2, 1)|=10$, $|Sq(0, 3, 1)|=11$.

(Q. E. D.)

Theorem 2.2. *For any $\mathcal{A}$-module structure, $\mathcal{A}(2)$ is realizable.*

Proof.
$$\mathrm{Ext}^{***}_{E_0\mathcal{A}}(\boldsymbol{F}_2, \boldsymbol{F}_2) \cong P(h_{i,j}\,;\, i>0,\ j\geqq 0),$$

where $\deg h_{i,j}=(1, 2^j(2^i-1), 2i-1)$. Each $h_{i,j}$ is represented by $[P_i^j]$ in the bar complex. The above lemma and change-of-rings isomorphism induce:

$$\mathrm{Ext}^{***}_{E_0\mathcal{A}}(E_0\mathcal{A}(2), \boldsymbol{F}_2) \cong \mathrm{Ext}^{***}_{E}(\boldsymbol{F}_2, \boldsymbol{F}_2)$$
$$\cong P(h'_{i,j}\,;\, i>0,\ j\geqq 0,\ i+j>3),$$

where $E=E(P_t^s\,;\, t>0,\ s\geqq 0,\ t+s>3)$ and $h'_{i,j}$ is an image of $h_{i,j}$ through the map $E_0\mathcal{A}(2)\to \boldsymbol{F}_2$.

$$\begin{array}{ccc}
\mathrm{Ext}^{***}_{E_0\mathcal{A}}(\boldsymbol{F}_2, \boldsymbol{F}_2) & \longrightarrow & \mathrm{Ext}^{***}_{E_0\mathcal{A}}(E_0\mathcal{A}(2), \boldsymbol{F}_2) \\
\Cup & & \Cup \\
h_{i,j} & \longmapsto & h'_{i,j}
\end{array}$$

In the E_1-term, there is one element which might survive in the E_∞-term and give some non-zero element of $\mathrm{Ext}^{s+2,s+n}_{\mathcal{A}}(\mathcal{A}(2), \boldsymbol{F}_2)$ for $s\geqq 1$, $0\leqq n\leqq 23$. To say precisely, since $\deg h'_{1,3}=(1, 8, 1)$, $h'^{s+2}_{1,3}$ is the lowest degree element in $\{x \mid \deg x=(s+2, *, *)\}$. But $\deg h'^{s+2}_{1,3}=8(s+2)>s+23$ for $s>1$, so the element mentioned above is $h'^{3}_{1,3}$.

$d_2(h'^{2}_{2,2})=h'^{3}_{1,3}$, however, because the corresponding differential in the May spectral sequence for $\mathrm{Ext}^{**}_{\mathcal{A}}(\boldsymbol{F}_2, \boldsymbol{F}_2)$ is

$$d_2(h_{2,2}^2)=h_{1,2}^2 h_{1,4}+h_{1,3}^3.$$

$$\begin{array}{ccc}
\mathrm{Ext}^{2,24,6}_{E_0\mathcal{A}}(E_0\mathcal{A}(2), \boldsymbol{F}_2) & \xrightarrow{\ d_2\ } & \mathrm{Ext}^{3,24,3}_{E_0\mathcal{A}}(E_0\mathcal{A}(2), \boldsymbol{F}_2) \\
\Cup & & \Cup \\
h'^{2}_{2,2} & \longmapsto & h'^{3}_{1,3} \\
\uparrow & & \uparrow \\
\mathrm{Ext}^{2,24,6}_{E_0\mathcal{A}}(\boldsymbol{F}_2, \boldsymbol{F}_2) & \xrightarrow{\ d_2\ } & \mathrm{Ext}^{3,24,3}_{E_0\mathcal{A}}(\boldsymbol{F}_2, \boldsymbol{F}_2) \\
\Cup & & \Cup \\
h_{2,2}^2 & \longmapsto & h_{1,2}^2\cdot h_{1,4}+h_{1,3}^3 .
\end{array}$$

Therefore for any $\mathcal{A}$-module structure on $\mathcal{A}(2)$, we can conclude $\mathrm{Ext}^{s+2,s+n}_{\mathcal{A}}(\mathcal{A}(2), \boldsymbol{F}_2)\equiv 0$ for $s\geqq 1$, $0\leqq n\leqq 23$, that is, $\mathcal{A}(2)$ is realizable. (Q. E. D.)

Note. We cannot proceed the same approach as the above for the realization of $\mathcal{A}(n)$ $(n>2)$, because there might survive many elements in the E_∞-term so as to generate obstructions in $\mathrm{Ext}^{s+2,s+m}_{\mathcal{A}}(\mathcal{A}(n), \boldsymbol{F}_2)$ for $s\geqq 1$, $0\leqq m\leqq \mathrm{maxdeg}\ \mathcal{A}(n)$. For example, $h'^{2}_{1,4}\cdot h'_{5,0}\in \mathrm{Ext}^{3,63,11}_{E_0\mathcal{A}}(E_0\mathcal{A}(3), \boldsymbol{F}_2)$ is a permanent cycle, because there exists no element in $E_1^{2,63,*}$ whose filtration degree is greater than 11, and $E_1^{4,t,*}\equiv 0$ for $t<64$.

Next we will prove the uniqueness of the homotopy type of spectra which realize $\mathcal{A}(2)$ with the specific $\mathcal{A}$-module structure indicated by Mitchell [6]. I calculated in my master thesis its explicit presentation form as follows.

Proposition 2.3. $0\leftarrow\mathcal{A}(2)\overset{\varepsilon}{\leftarrow}\mathcal{A}\overset{\delta 1}{\leftarrow}C^1\overset{\delta 2}{\leftarrow}C^2\leftarrow 0$ *is an exact sequence up to degree 27, where* C^1 *and* C^2 *are free* $\mathcal{A}$*-modules whose bases are* $\{b_1, b_2, b_3, b_4, b_5\}$ *and* $\{e_1, e_2, e_3, e_4\}$, *with their degrees*:

$$\deg b_1=8,\quad \deg b_2=12,\quad \deg b_3=14,\quad \deg b_4=15,\quad \deg b_5=16,$$
$$\deg e_1=16,\quad \deg e_2=20,\quad \deg e_3=22,\quad \deg e_4=23.$$

δ^1 *and* δ^2 *are defined as follows*:

$$\delta^1(b_1)=Sq(8)+Sq(5,1)+Sq(2,2)+Sq(1,0,1),$$
$$\delta^1(b_2)=Sq(0,4)+Sq(6,2)+Sq(5,0,1)+Sq(2,1,1),$$
$$\delta^1(b_3)=Sq(0,0,2)+Sq(5,3)+Sq(7,0,1)+Sq(4,1,1)+Sq(1,2,1),$$
$$\delta^1(b_4)=Sq(0,0,0,1),$$
$$\delta^1(b_5)=Sq(16);$$
$$\delta^2(e_1)=\{Sq(8)+Sq(5,1)+Sq(2,2)+Sq(1,0,1)\}b_1$$
$$+\{Sq(4)+Sq(1,1)\}b_2+Sq(2)b_3+Sq(1)b_4,$$
$$\delta^2(e_2)=\{Sq(0,4)+Sq(6,2)+Sq(5,0,1)+Sq(2,1,1)\}b_1$$
$$+\{Sq(8)+Sq(5,1)\}b_2+Sq(3,1)b_3,$$
$$\delta^2(e_3)=\{Sq(0,0,2)+Sq(5,3)+Sq(7,0,1)+Sq(4,1,1)+Sq(1,2,1)\}b_1$$
$$+\{Sq(1,3)+Sq(3,0,1)+Sq(0,1,1)\}b_2+\{Sq(8)+Sq(2,2)\}b_3,$$
$$\delta^2(e_4)=Sq(0,0,0,1)b_1+\{Sq(8)+Sq(5,1)+Sq(2,2)+Sq(1,0,1)\}b_4.$$

Proof. We can get them by a routine calculation. (Q.E.D.)

Theorem 2.4. *There is one and only one homotopy type of spectra which realize* $\mathcal{A}(2)$ *with the* $\mathcal{A}$*-module structure indicated by Mitchell* [6].

Proof. We proved $\mathrm{Ext}_{\mathcal{A}}^{s+2,s+n}(\mathcal{A}(2), \boldsymbol{F}_2)\equiv 0$ for $s\geqq 1$, $0\leqq n\leqq 23$. But the fact that $d_2(h'_{2,2})=h'_{1,3}$ also implies:

$$\mathrm{Ext}_{\mathcal{A}}^{s+1,s+n}(\mathcal{A}(2), \boldsymbol{F}_2)\equiv 0 \qquad \text{for} \quad s\geqq 2,\ 0\leqq n\leqq 23.$$

So we must indicate $\mathrm{Ext}_{\mathcal{A}}^{2,1}(\mathcal{A}(2), \mathcal{A}(2))\cong 0$, in other words,

$$\delta^{2*}: \mathrm{Hom}_{\mathcal{A}}^1(C^1, \mathcal{A}(2)) \longrightarrow \mathrm{Hom}_{\mathcal{A}}^1(C^2, \mathcal{A}(2))$$

is surjective.

$$\mathrm{Hom}_{\mathcal{A}}^1(C^1, \mathcal{A}(2))\cong \mathcal{A}(2)^7\oplus\mathcal{A}(2)^{11}\oplus\mathcal{A}(2)^{13}\oplus\mathcal{A}(2)^{14}\oplus\mathcal{A}(2)^{15}$$
$$\mathrm{Hom}_{\mathcal{A}}^1(C^2, \mathcal{A}(2))\cong \mathcal{A}(2)^{15}\oplus\mathcal{A}(2)^{19}\oplus\mathcal{A}(2)^{21}\oplus\mathcal{A}(2)^{22}$$

An easy calculation concerning δ^2 verifies this statement. (Q. E. D.)

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