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Pure Hodge Structure of the Harmonic L^2-Forms on Singular Algebraic Surfaces

By

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§1. Introduction

Let X be an algebraic surface (over $\boldsymbol{C}$) embedded in the projective space $\boldsymbol{P}^N(\boldsymbol{C})$ and S be its singularity set. If S is empty, then the harmonic spaces $\mathcal{H}^i(X)=\{\text{smooth } i\text{-form } \alpha \text{ on } X \mid d\alpha=\delta\alpha=0\}$ (or, equivalently, its de Rham cohomology groups) have the pure Hodge structure;

$$\mathcal{H}^i(X)=\bigoplus_{p+q=i}\mathcal{H}_d^{p,q}(X)\,,\quad \mathcal{H}_d^{p,q}(X)=\overline{\mathcal{H}_d^{q,p}(X)}\,, \tag{1.1}$$

where $\mathcal{H}_d^{p,q}(X)=\{\text{smooth } (p,q)\text{-forms } \alpha \text{ on } X \mid d\alpha=\delta\alpha=0\}$. ($\mathcal{H}_d^{p,q}(X)$ is naturally changed into the Dolbeault-type harmonic space $\mathcal{H}_{\bar\partial}^{p,q}(X)$ in this case, but in the case we are going to discuss in this paper, that is, in the case where S is not empty, such a change has a subtle problem (§3) and it seems to be one of the key points not to try to do so.) Also there exists the hard Lefschetz structure compatible with (1.1);

$$\begin{aligned} &L^k\colon \mathcal{H}^{2-k}(X)\cong\mathcal{H}^{2+k}(X)\,,\\ &\mathcal{H}^i(X)=\bigoplus_k L^k\,\mathcal{P}^{i-2k}(X)\,,\quad \mathcal{P}^l(X)=\bigoplus_{p+q=l}\mathcal{P}_d^{p,q}(X)\,, \end{aligned} \tag{1.2}$$

where the operation L means the multiplication by the Kähler form, $\mathcal{P}^l(X)$ is the primitive harmonic space and $\mathcal{P}_d^{p,q}(X)=\mathcal{P}^{p+q}(X)\cap\mathcal{H}_d^{p,q}(X)$. Now the purpose of this paper is to show the following; if S consists of isolated points, hence, if $\mathcal{X}=X-S$ is an incomplete Kähler manifold, then the harmonic L^2-spaces on $\mathcal{X}$ (or, partially, its L^2-cohomology groups) have the similar pure Hodge and hard Lefschetz structures.

Let us explain the above assertion more exactly. In the following S consists of isolated singular points and $\mathcal{X}=X-S$ is endowed with the incomplete

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Kähler metric g induced from the Fubini-Study metric of $\boldsymbol{P}^N(\boldsymbol{C})$. Let $\Lambda^i(\mathcal{X})$ and $\Lambda^{p,q}(\mathcal{X})$ be the spaces consisting of smooth i-forms and smooth (p, q)-forms on $\mathcal{X}$ respectively. Also let $L^2\Lambda^i(\mathcal{X})$ and $L^2\Lambda^{p,q}(\mathcal{X})$ be the spaces consisting of square-integrable i-forms and square-integrable (p, q)-forms on $(\mathcal{X}, g)$ respectively. And let us denote, by d_i and δ_i, the exterior derivative and its formal adjoint (with respect to the metric g) with the following domains;

$$
\begin{aligned}
\operatorname{dom} d_i &= \{\alpha \in \Lambda^i(\mathcal{X}) \cap L^2\Lambda^i \mid d\alpha \in L^2\Lambda^{i+1}\}\,, \\
\operatorname{dom} \delta_i &= \{\alpha \in \Lambda^{i+1}(\mathcal{X}) \cap L^2\Lambda^{i+1} \mid \delta\alpha \in L^2\Lambda^{i}\}\,.
\end{aligned} \tag{1.3}
$$

Moreover, letting $\Lambda^i_c(\mathcal{X})$ be the space of compactly supported smooth i-forms on $\mathcal{X}$, we put $d_{c,i} = d_i \mid \Lambda^i_c(\mathcal{X})$, $\delta_{c,i} = \delta_i \mid \Lambda^{i+1}_c(\mathcal{X})$. Their closures (with respect to the operator norms) are denoted by $\hat{d}_i$, $\hat{\delta}_i$, $\hat{d}_{c,i}$, $\hat{\delta}_{c,i}$. Finally we denote, by $d_{\hat{c},i}$ and $\delta_{\hat{c},i}$, the restrictions of $\hat{d}_{c,i}$ and $\hat{\delta}_{c,i}$ to the following domains;

$$
\begin{aligned}
\operatorname{dom} d_{\hat{c},i} &= \Lambda^i(\mathcal{X}) \cap \operatorname{dom} \hat{d}_{c,i}\,, \\
\operatorname{dom} \delta_{\hat{c},i} &= \Lambda^{i+1}(\mathcal{X}) \cap \operatorname{dom} \hat{\delta}_{c,i}\,.
\end{aligned} \tag{1.4}
$$

Now we define the various harmonic L^2-spaces by

$$
\begin{aligned}
\mathcal{H}^i_{(2)}(\mathcal{X}) &= \operatorname{Ker} d_i \cap \operatorname{Ker} \delta_{i-1}\,, \\
\hat{\mathcal{H}}^i_{(2)}(\mathcal{X}) &= \operatorname{Ker} d_i \cap \operatorname{Ker} \delta_{\hat{c},i-1}\,, \\
\mathcal{H}^{p,q}_{(2)d}(\mathcal{X}) &= \Lambda^{p,q}(\mathcal{X}) \cap \mathcal{H}^{p+q}_{(2)}(\mathcal{X})\,, \\
\hat{\mathcal{H}}^{p,q}_{(2)d}(\mathcal{X}) &= \Lambda^{p,q}(\mathcal{X}) \cap \hat{\mathcal{H}}^{p+q}_{(2)}(\mathcal{X})\,,
\end{aligned} \tag{1.5}
$$

and, moreover, define the L^2-cohomology groups by

$$
\begin{aligned}
H^i_{(2)}(\mathcal{X}) &= \operatorname{Ker} d_i / \operatorname{Range} d_{i-1}\,, \\
H^{p,q}_{(2)d}(\mathcal{X}) &= \Lambda^{p,q}(\mathcal{X}) \cap \operatorname{Ker} d_{p+q} / \Lambda^{p,q}(\mathcal{X}) \cap \operatorname{Range} d_{p+q-1}\,.
\end{aligned} \tag{1.6}
$$

Theorem 1 (*L^2-version of pure Hodge structure*).

(1) $\mathcal{H}^i_{(2)}(\mathcal{X}) = \bigoplus_{p+q=i} \mathcal{H}^{p,q}_{(2)d}(\mathcal{X})$, $\quad \mathcal{H}^{q,p}_{(2)d}(\mathcal{X}) = \overline{\mathcal{H}^{p,q}_{(2)d}(\mathcal{X})}$.

(2) *If $i = p+q \neq 2$, then we have*

$$
\begin{aligned}
\mathcal{H}^i_{(2)}(\mathcal{X}) &= \hat{\mathcal{H}}^i_{(2)}(\mathcal{X}) \cong H^i_{(2)}(\mathcal{X})\,, \\
\mathcal{H}^{p,q}_{(2)d}(\mathcal{X}) &= \hat{\mathcal{H}}^{p,q}_{(2)d}(\mathcal{X}) \cong H^{p,q}_{(2)d}(\mathcal{X})\,,
\end{aligned}
$$

and, hence,

$$
H^i_{(2)}(\mathcal{X}) \cong \bigoplus_{p+q=i} H^{p,q}_{(2)d}(\mathcal{X})\,.
$$

Remark. Refer to §3 and §5 for further investigations of the cases $i=0, 1$.

Next, denoting the (1, 1)-form associated to our metric g by ω_g and setting $L_g(\alpha)=\alpha\wedge\omega_g$ for forms α, we define

$$
(1.7)\qquad \begin{aligned} \mathcal{P}^{2-k}_{(2)}(\mathcal{X}) &= \operatorname{Ker}\left(L_g^{k+1}\colon \mathcal{H}^{2-k}_{(2)}(\mathcal{X}) \to \mathcal{H}^{4+k}_{(2)}(\mathcal{X})\right), \\ \mathcal{P}^{p,q}_{(2)d}(\mathcal{X}) &= \mathcal{P}^{p+q}_{(2)}(\mathcal{X}) \cap \mathcal{H}^{p,q}_{(2)d}(\mathcal{X}). \end{aligned}
$$

Then we have

Theorem 2 (*L^2-version of hard Lefschetz structure*).
(a) $L_g^k\colon \mathcal{H}^{2-k}_{(2)}(\mathcal{X}) \cong \mathcal{H}^{2+k}_{(2)}(\mathcal{X})$,
(b) $\mathcal{H}^{i}_{(2)}(\mathcal{X}) = \bigoplus_k L_g^k\, \mathcal{P}^{i-2k}_{(2)}(\mathcal{X})$,
(c) $\mathcal{P}^{i}_{(2)}(\mathcal{X}) = \bigoplus_{p+q=i} \mathcal{P}^{p,q}_{(2)d}(\mathcal{X})$.

Combining Theorem 1(2) and Theorem 2(a), we get

Corollary. $L_g^k\colon H^{2-k}_{(2)}(\mathcal{X}) \cong H^{2+k}_{(2)}(\mathcal{X})$.

Remark. If $i\neq 2$, then we have also the hard Lefschetz decompositions (b) and (c) for the i-th L^2-cohomology groups, which are however just Theorem 1(2).

The author believes that the case $i=2$ omitted in Theorem 1(2) and in the above remark must hold.

Conjecture A. *Theorem* 1(2) *and the hard Lefschetz decomposition of the L^2-cohomology groups hold also in the case $i=p+q=2$.*

Obviously this conjecture is equivalent to a part of it, i.e., $\mathcal{H}^2_{(2)}(\mathcal{X})=\hat{\mathcal{H}}^2_{(2)}(\mathcal{X})$. Moreover it suffices to prove a little bit stronger assertion; $\operatorname{Ker}\delta_1=\operatorname{Ker}\delta_{\hat{c},1}$, which is equivalent to $\operatorname{Ker} d_2=\operatorname{Ker} d_{\hat{c},2}$. Thus Conjecture A can be deduced from the conjecture "$d_{\hat{c},i}=d_i$ for $i=1, 2$" announced in [10].

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§2. Preliminaries

Here we explain some notations and collect the (well-)known facts.

(a) Notations

The Kähler form ω_g is defined to be locally expressed by $\omega_g=2^{-1}\sqrt{-1}\sum$

$\varphi_j \wedge \bar{\varphi}_j$ provided $g=\sum \varphi_j \otimes \bar{\varphi}_j$. For (p, q)-forms $\alpha=\sum \alpha_{I\bar{J}}\, \varphi_I \wedge \bar{\varphi}_J$, $I=(i_1<\cdots<i_p)$, $J=(j_1<\cdots<j_q)$, we define

$$\bar{*}_g\, \alpha = 2^{p+q-2} \sum \varepsilon_{I\bar{J}}\, \bar{\alpha}_{I\bar{J}}\, \varphi_{I^0} \wedge \bar{\varphi}_{J^0}\,, \tag{2.1}$$

where $I^0 = \{1, 2\} - I$, $J^0 = \{1, 2\} - J$ and

$\varepsilon_{I\bar{J}}=(-1)^{(2-p)q} \operatorname{sgn}\begin{pmatrix}1 & 2\\ I & I^0\end{pmatrix} \operatorname{sgn}\begin{pmatrix}1 & 2\\ J & J^0\end{pmatrix}$. Using this complex star operator $\bar{*}_g$, the global inner product is defined by

$$(\alpha, \beta)_g = \int_{\mathcal{X}} \alpha \wedge \bar{*}_g\, \beta\,, \tag{2.2}$$

and $||\alpha||_g=(\alpha, \alpha)_g^{1/2}$ is called the (L^2-)norm of α. The space $L^2 A^i(\mathcal{X})$ is now definitely defined to be the space of the i-forms whose L^2-norms are finite. Besides, the volume element with respect to g is defined by

$$dV_g = \frac{1}{2}\, \omega_g \wedge \omega_g \tag{2.3}$$

and, for a form α, its pointwise (L^2-)norm $|\alpha|_g$ is defined by

$$\alpha \wedge \bar{*}_g\, \alpha = |\alpha|_g^2\, dV_g\,. \tag{2.4}$$

Next, for the continuous functions f_1, f_2 on some space Y, if there exists a constant $K>0$ such that $f_1(y) \leq K f_2(y)$ for all $y \in Y$, then we use the expression $f_1 \lesssim f_2$. Moreover we denote by $f_1 \sim f_2$ the situation that both $f_1 \lesssim f_2$ and $f_1 \gtrsim f_2$ hold. These expressions are naturally generalized also to the relation between the differential forms.

(b) L^2-cohomology $H^i_{(2)}(\mathcal{X})$

The duality of the L^2-cohomology $H^i_{(2)}(\mathcal{X})$ and the intersection homology $IH_i^{\bar{m}}(X)$ with the middle perversity $\bar{m}$ has been already verified in [7] (which has a certain gap) and [10]. That is, there exists the generalized de Rham isomorphism

$$H^i_{(2)}(\mathcal{X}) \cong (IH_i^{\bar{m}}(X))^*\,. \tag{2.5}$$

Letting U be a neighborhood of S in X, there exists the following natural isomorphism:

$$H^i_{(2)}(\mathcal{X}) \cong \begin{cases} H^i_{DR}(X-S) & ;\ i \leq 1\,, \\ \iota^*(H^2_{DR}(X-U, \partial U)) & ;\ i=2\,, \\ H^i_{DR}(X-U, \partial U) & ;\ i \geq 3\,, \end{cases} \tag{2.6}$$

where the map ι means the inclusion map $(X-U, \phi) \to (X-U, \partial U)$. This fact implies that $\dim_C H^i_{(2)}(\mathcal{X})<\infty$ and moreover Range d_{i-1} is closed in $A^i(\mathcal{X}) \cap L^2A^i(\mathcal{X})$. Hence, remembering the Hodge decomposition of $L^2A^i(\mathcal{X})$, we have

$$\hat{\mathcal{H}}^i_{(2)}(\mathcal{X}) \cong H^i_{(2)}(\mathcal{X}) . \tag{2.7}$$

(c) The operators d_i and $d_{\hat{c},i}$ for our $\mathcal{X}$.

If $i\neq 1, 2$, then we have

$$d_{\hat{c},i} = d_i , \quad \delta_{\hat{c},i} = \delta_i . \tag{2.8}$$

They have been verified in [9, Assertion A] and [10, Proposition 1.1].

Lemma 2.1.

(1) *If* $i\neq 2$, *then* Ker $d_{\hat{c},i}$ = Ker d_i.

(2) *If* $i\neq 1$, *then* Ker $\delta_{\hat{c},i}$ = Ker δ_i.

Proof. (2.8) implies (1), (2) provided $i\neq 1, 2$. Moreover, (1) with $i=1$ and (2) with $i=2$ are equivalent. Hence we will prove (1) with $i=1$. (2.8) with $i=0$ implies that Range $d_{\hat{c},0}$ is equal to Range d_0 and is closed in $A^1(\mathcal{X}) \cap L^2A^1(\mathcal{X})$. Therefore we have the decomposition:

$$\begin{aligned} \text{Ker}\, d_{\hat{c},1} &= \overline{\text{Range}\, d_{\hat{c},0}} \oplus (\text{Ker}\, d_{\hat{c},1}/\overline{\text{Range}\, d_{\hat{c},0}}) \\ &= \text{Range}\, d_0 \oplus (\text{Ker}\, d_{\hat{c},1}/\text{Range}\, d_{\hat{c},0}) . \end{aligned} \tag{2.9}$$

Now take a closed neighborhood W of S and set $W^*=W-S$. And, for $(\mathcal{X}, W^*)$, we consider two kinds of long exact sequences, that is, with respect to the $\{d_i\}$-type L^2-cohomology $H^*_{(2)}$ and the $\{d_{\hat{c},i}\}$-type L^2-cohomology $\hat{H}^*_{(2)}$. Note that the intrinsic operator $d_{c,i}$ on W^* is acting on the smooth i-forms which are identically zero near the singularity S (and may not be zero on ∂W).

$$\begin{array}{ccccccccc} H^0_{(2)}(W^*) & \to & H^1_{(2)}(\mathcal{X}, W^*) & \to & H^1_{(2)}(\mathcal{X}) & \to & H^1_{(2)}(W^*) & \to & H^2_{(2)}(\mathcal{X}, W^*) \\ \wr\wr & & \| & & \uparrow & & \wr\wr & & \| \\ \hat{H}^0_{(2)}(W^*) & \to & \hat{H}^1_{(2)}(\mathcal{X}, W^*) & \to & \hat{H}^1_{(2)}(\mathcal{X}) & \to & \hat{H}^1_{(2)}(W^*) & \to & \hat{H}^2_{(2)}(\mathcal{X}, W^*) \end{array} \tag{2.10}$$

Here the isomorphism $\hat{H}^1_{(2)}(W^*)\cong H^1_{(2)}(W^*)$ is given by the results [10, Proposition 3.1 for $i=1$ and Assertion C(2)]. Thus the so-called five lemma implies $\hat{H}^1_{(2)}(\mathcal{X})\cong H^1_{(2)}(\mathcal{X})$. Hence, combined with (2.9), we get

$$\begin{aligned} \text{Ker}\, d_{\hat{c},1} &= \text{Range}\, d_0 \oplus (\text{Ker}\, d_1/\text{Range}\, d_0) \\ &= \text{Ker}\, d_1 . \end{aligned}$$

Q.E.D.

(d) Complete Kähler manifolds.

Let our $\mathcal{X}=X-S$ be endowed with a complete Kähler metric h. Then, in a sense, $\mathcal{X}(h)=(\mathcal{X}, h)$ has the quite similar properties as the compact Kähler case, some of which we will enumerate here.

Lemma 2.2.

(1) *On* $\mathcal{X}(h)$, $d_{\hat{c},i}=d_i$ *and* $\delta_{\hat{c},i}=\delta_i$.

(2) $\mathcal{H}^i_{(2)}(\mathcal{X}(h))=\bigoplus_{p+q=i}\mathcal{H}^{p,q}_{(2)d}(\mathcal{X}(h))=\bigoplus_{p+q=i}\mathcal{H}^{p,q}_{(2)\bar{\partial}}(\mathcal{X}(h))$.

(3) $H^i_{(2)}(\mathcal{X}(h))\cong\bigoplus_{p+q=i}H^{p,q}_{(2)d}(\mathcal{X}(h))=\bigoplus_{p+q=i}H^{p,q}_{(2)\bar{\partial}}(\mathcal{X}(h))$.

Here $\mathcal{X}^{p,q}_{(2)\bar{\partial}}(\mathcal{X}(h))$ and $H^{p,q}_{(2)\bar{\partial}}(\mathcal{X}(h))$ are of Dolbeault-types: see §3. Refer to [1], [14] for further details of the properties of the complete Kähler case. As for (1), we can assert more strongly that, for $\psi\in\text{dom } d_i\cap\text{dom } \delta_{i-1}$, we can find a sequence $\psi_j\in A^i_c(\mathcal{X})$ satisfying $\lim_{j\to\infty}\psi_j=\psi$, $\lim_{j\to\infty}d\psi_j=d\psi$ and $\lim_{j\to\infty}\delta\psi_j=\delta\psi$ in the L^2-sense. This is a quite nice property. As for $\mathcal{X}(g)$, though it has the property (1) provided $i\neq 1, 2$, we do not know if there exists such a good sequence (i.e., a sequence which converges with respect to the operator norm $\|\cdot\|+\|d\cdot\|+\|\delta\cdot\|$) for the element of dom $d_i\cap$ dom δ_{i-1} ($i\neq 1, 2$).

§3. $\mathcal{H}^{p,q}_{(2)d}(\mathcal{X})$ and $\mathcal{H}^{p,q}_{(2)\bar{\partial}}(\mathcal{X})$

Let us decompose the exterior derivative into $d=\partial+\bar{\partial}$ and its formal adjoint into $\delta=\delta'+\delta''$. (The formal adjoint of ∂ is denoted by δ'.) In the same way as (1.3), we specify their domains and denote them by $\partial_{(p,q)}$, $\bar{\partial}_{(p,q)}$, $\delta'_{(p,q)}$, $\delta''_{(p,q)}$; for examples,

$$
\begin{aligned}
\text{dom } \partial_{(p,q)} &= \{\alpha\in A^{p,q}(\mathcal{X})\cap L^2A^{p,q}\,|\,\partial\alpha\in L^2A^{p+1,q}\}\,, \\
\text{dom } \delta'_{(p,q)} &= \{\alpha\in A^{p+1,q}(\mathcal{X})\cap L^2A^{p+1,q}\,|\,\delta'\alpha\in L^2A^{p,q}\}\,.
\end{aligned}
\tag{3.1}
$$

Similarly we set $\partial_{c,(p,q)}=\partial_{(p,q)}\,|\,A^{p,q}_c(\mathcal{X})$, etc. and $\partial_{\hat{c},(p,q)}=\hat{\partial}_{c,(p,q)}\,|\,A^{p,q}(\mathcal{X})\cap\text{dom }\hat{\partial}_{c,(p,q)})$, etc. Moreover we define $\mathcal{H}^{p,q}_{(2)\bar{\partial}}(\mathcal{X})$, $H^{p,q}_{(2)\bar{\partial}}(\mathcal{X})$, etc. in the same way as (1.5) and (1.6). Then the relations between $\mathcal{H}^{p,q}_{(2)d}(\mathcal{X})$ and $\mathcal{H}^{p,q}_{(2)\bar{\partial}}(\mathcal{X})$ and among others should be investigated. However little is known about them. Only the cases $i=0$, 4 can be studied now.

Proposition 3.1.

(1) $\text{Ker }\partial_{\hat{c},(0,0)}=\text{Ker }\bar{\partial}_{\hat{c},(0,0)}=\text{Ker } d_{\hat{c},0}=\text{Ker } d_0$
$=\{$*constant functions on* $\mathcal{X}\}$.

(2) $\text{Ker }\delta'_{\hat{c},(1,2)}=\text{Ker }\delta''_{\hat{c},(2,1)}=\text{Ker }\delta_{\hat{c},3}=\text{Ker }\delta_3$
$=\{adV_g\,|\,a\in\boldsymbol{C}\}$.

Proof. Due to the duality, we have only to prove (1). And, because of (2.6) for $i=0$ and Lemma 2.1(1) for $i=0$, it suffices to prove the first two equalities at (1). Moreover, because Ker $\partial_{\hat{c},(0,0)}$ and Ker $\bar{\partial}_{\hat{c},(0,0)}$ are conjugate to each other, it suffices to prove a part of it, i.e., Ker $\partial_{\hat{c},(0,0)}=$Ker $d_{\hat{c},0}$. Let us prove the non-trivial implication Ker $\partial_{\hat{c},(0,0)}\subset$Ker $d_{\hat{c},0}$. Take $f\in$Ker $\partial_{\hat{c},(0,0)}$. Then there exists a sequence $f_j\in A_c^0(\mathcal{X})=\mathrm{dom}\,\partial_{c,(0,0)}$ satisfying $\lim_{j\to\infty} f_j=f$ and $\lim_{j\to\infty}\partial f_j=0$ in the L^2-sense. Since we have the Hodge identity $\Delta_d=2\Delta_\partial$ on $A_c^0(\mathcal{X})$,

$$\lim_{j\to\infty}(df_j, df_j)=\lim_{j\to\infty}(\delta df_j, f_j)=2\lim_{j\to\infty}(\delta'\partial f_j, f_j)$$
$$=2\lim_{j\to\infty}(\partial f_j, \partial f_j)=0\,.$$

Thus f belongs to Ker $d_{\hat{c},0}$. Q.E.D.

On the other hand, T. Ohsawa ([12, Theorem 1]) got the following result: for any n-dimensional compact Kähler space Y with isolated singular points, its smooth part Y^* has the Hodge structure $H^i_{DR}(Y^*)\cong\bigoplus_{p+q=i}H^{p,q}_{\bar{\partial}}(Y^*)$ provided $i<n-1$. Here H^i_{DR} and $H^{p,q}_{\bar{\partial}}$ mean the de Rham and Dolbeault cohomology groups (with no L^2-condition). Applying it to our $\mathcal{X}$, we get

$$H^0_{DR}(\mathcal{X})=H^{0,0}_{\bar{\partial}}(\mathcal{X})\,. \tag{3.2}$$

Since $H^0_{DR}(\mathcal{X})=\{$constant functions on $\mathcal{X}\}$, Proposition 3.1(1) implies Ker $\partial_{\hat{c},(0,0)}=H^0_{DR}(\mathcal{X})$: remark that the domain of $\partial_{\hat{c},(0,0)}$ has been specified as in (3.1). Thus Ker $\partial_{\hat{c}\,(0,0)}\supset$Ker $\partial_{(0,0)}$. The converse implication is trivial and we get Ker $\partial_{\hat{c},(0,0)}=$Ker $\partial_{(0,0)}$. By the conjugation and the duality, finally we have the following.

Proposition 3.2.

$$\mathrm{Ker}\,\partial_{\hat{c},(0,0)}=\mathrm{Ker}\,\partial_{(0,0)}\,,\quad \mathrm{Ker}\,\bar{\partial}_{\hat{c},(0,0)}=\mathrm{Ker}\,\bar{\partial}_{(0,0)}\,,$$
$$\mathrm{Ker}\,\delta'_{\hat{c},(1,2)}=\mathrm{Ker}\,\delta'_{(1,2)}\,,\quad \mathrm{Ker}\,\delta''_{\hat{c},(2,1)}=\mathrm{Ker}\,\delta''_{(2,1)}\,.$$

§4. The Relation between Two Kinds of Kähler Metrics on $\mathcal{X}$

In this section, we first investigate the quasi-isometric classes of

(*a*) the incomplete Kähler metric g on $\mathcal{X}$,

(*b*) the incomplete Kähler metric ds^2 on $\mathcal{X}$: the restriction of a Kähler metric of a smooth compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$.

In order to do so near S, we will make a very good resolution

$$\pi: \tilde{X} \to X. \tag{4.1}$$

That is, we make a resolution with the following properties (I), (II):

(I) $\pi^{-1}(S)$ has normal crossings with smooth irreducible components $\{D_j\}_{j=1}^{m}$ and $D_i \cap D_j$ is empty or a single point for $i \neq j$.

(II) Take any point $p \in S$. For simplicity, we regard $p=[1, 0, \cdots, 0] \in \mathbb{P}^N(\mathbb{C})$ and take the local coordinates $[w_0, w_1, \cdots, w_N] \mapsto (z_1, \cdots, z_N)=(w_1/w_0, \cdots, w_N/w_0)$ around the point. Then, for any point $x \in \pi^{-1}(p)$, there exists a local coordinate neighborhood $(U, (u, v))$ around x and a permutation σ such that the π can be expressed on U as follows;

$$\begin{aligned}
z_{\sigma(1)} &= u^{n_1} v^{m_1}, \\
z_{\sigma(2)} &= f_2(z_{\sigma(1)}) + u^{n_2} v^{m_2} g_2(u, v), \quad \det\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} \neq 0, \\
&\ \vdots \\
z_{\sigma(l)} &= f_l(z_{\sigma(1)}) + u^{n_l} v^{m_l} g_l(u, v), \quad \det\begin{pmatrix} n_1 & m_1 \\ n_l & m_l \end{pmatrix} \neq 0, \\
z_{\sigma(l+1)} &= f_{l+1}(z_{\sigma(1)}), \\
&\ \vdots \\
z_{\sigma(N)} &= f_N(z_{\sigma(1)}),
\end{aligned} \tag{4.2}$$

where $f_j(z) = \sum a_{jn} z^{\varepsilon_n}$ with $\varepsilon_n \geq 1$, $g_j(0, 0) \neq 0$ and moreover

$$\begin{aligned} &n_1 \leq n_2 \leq \cdots \leq n_l, \\ &m_1 \leq m_2 \leq \cdots \leq m_l. \end{aligned} \tag{4.3}$$

Such a very good resolution can be made by first making a resolution and then performing blowing-ups as many times as we need ([7, III], [9, §2], [10, §2]). In the following, through the map π, we identify

$$\mathcal{X} = \tilde{X} - \pi^{-1}(S). \tag{4.4}$$

On the neighborhood $(U, (u, v))$ around the point $x \in \pi^{-1}(p)$, we set

$$\begin{cases} \rho = |u|, \\ \phi = \arg u, \end{cases} \quad \begin{cases} \tau = |v|, \\ \psi = \arg v, \end{cases} \tag{4.5}$$

and study the Kähler metrics (a) and (b) on $U - \pi^{-1}(p)$. Referring to [9, (2.4,) (2.11), (2.13)], we set

$$\begin{aligned} \text{Case } (-)\text{:} \quad & U \cap \pi^{-1}(p) = \text{“}u\text{-axis”} \quad (\text{hence, } n_1 = 0, \\ & n_2 = 1 \quad \text{and} \quad 1 \leq m_1 \leq m_2), \end{aligned}$$

(4.6) Case (+): $U\cap\pi^{-1}(p)=$ "$uv=0$" and $\det\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2\end{pmatrix}>0$ (hence $1\leq n_1, m_1$).

First we can describe explicitly the quasi-isometric class of the incomplete Kähler metric g (or its underlying Riemannian metric) on $\mathscr{X}$: see (*a*).

Proposition 4.1 ([9, Corollary 2.2]).

(1) *In Case* $(-)$, *on* $U-\pi^{-1}(p)$,
$$g\sim\tau^{2m_2}(d\rho^2+\rho^2d\phi^2)+\tau^{2(m_1-1)}(d\tau^2+\tau^2d\psi^2).$$
(2) *In Case* $(+)$, *on* $U-\pi^{-1}(p)$,
$$g\sim\{\rho^{n_1}\tau^{m_1-1}(n_1\tau d\rho+m_1\rho d\tau)\}^2+\{\rho^{n_2}\tau^{m_2-1}d\tau\}^2$$
$$+\{\rho^{n_2}\tau^{m_2}d\phi\}^2+\{\rho^{n_1}\tau^{m_1}(n_1d\phi+m_1d\psi)\}^2.$$

Remark. We have the similar representation also in the other cases, i.e., $U\cap\pi^{-1}(p)=$ "v-axis", or, $U\cap\pi^{-1}(p)=$ "$uv=0$" and $\det\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2\end{pmatrix}<0$.

Second, let ds^2 be the restriction (to $\mathscr{X}$) of a Kähler metric $d\tilde{s}^2$ on $\overline{\mathscr{X}}=\tilde{X}$.

Proposition 4.2. *On* $U-\pi^{-1}(p)$,
$$ds^2\sim d\rho^2+\rho^2d\phi^2+d\tau^2+\tau^2d\psi^2.$$

Next, let us investigate the relationship between the square-integrabilities on $\mathscr{X}(g)=(\mathscr{X}, g)$ and $\mathscr{X}(ds^2)=(\mathscr{X}, ds^2)$. We set, on $U-\pi^{-1}(p)$,

$$dV=d\rho\wedge d\phi\wedge d\tau\wedge d\psi. \tag{4.7}$$

Then the volume elements with respect to g and ds^2 can be written quasi-isometrically as follows: on $U-\pi^{-1}(p)$,

$$dV_g\sim\begin{cases}\rho\,\tau^{2(m_1+m_2)-1}\,dV & ;\ \text{Case }(-),\\ \rho^{2(n_1+n_2)-1}\,\tau^{2(m_1+m_2)-1}\,dV; & \text{Case }(+),\end{cases} \tag{4.8}$$

$$dV_s\sim\rho\tau dV. \tag{4.9}$$

Moreover let us denote the pointwise norms of a form α (with respect to the above metrics) by $|\alpha|_g$ and $|\alpha|_s$: see (2.4). Then we have

Lemma 4.3. *On* $U-\pi^{-1}(p)$,

(1) $dV_g\lesssim dV_s$,

(2) $|d\rho|_g^2\,dV_g\lesssim|d\rho|_s^2\,dV_s,\cdots,\ |d\psi|_g^2\,dV_g\lesssim|d\psi|_s^2\,dV_s$.

Proof. All is verified by straightforward computations. First, in Case

(−),

$$
\begin{aligned}
& dV_g/dV_s \sim \tau^{2(m_1+m_2-1)} , \\
& |d\rho|_g^2\, dV_g/|d\rho|_s^2\, dV_s \sim |d\phi|_g^2\, dV_g/|d\phi|_s^2\, dV_s \sim \tau^{2(m_1-1)} , \\
& |d\tau|_g^2\, dV_g/|d\tau|_s^2\, dV_s \sim |d\psi|_g^2\, dV_g/|d\psi|_s^2\, dV_s \sim \tau^{2m_2} .
\end{aligned}
$$

Next, in Case (+),

$$
\begin{aligned}
& dV_g/dV_s \sim \rho^{2(n_1+n_2-1)}\, \tau^{2(m_1+m_2-1)} , \\
& |d\rho|_g^2\, dV_g/|d\rho|_s^2\, dV_s \sim |d\phi|_g^2\, dV_g/|d\phi|_s^2\, dV_s \sim \rho^{2n_2}\, \tau^{2(m_2-1)} , \\
& |d\tau|_g^2\, dV_g/|d\tau|_s^2\, dV_s \sim |d\psi|_g^2\, dV_g/|d\psi|_s^2\, dV_s \sim \rho^{2(n_2-1)}\, \tau^{2m_2} .
\end{aligned}
$$

Q.E.D.

The lemma implies

Proposition 4.4. *Assume $i \leq 1$. Then the identity map on $A^i(\mathscr{X})$ induces the bounded inclusion map*

$$L^2A^i(\mathscr{X}(ds^2)) \to L^2A^i(\mathscr{X}(g)) .$$

Now, as for the L^2-cohomology groups, we can prove

Corollary 4.5. *Assume $i \leq 1$. The identity map on $A^i(\mathscr{X})$ induces the isomorphism*

$$H^i_{(2)}(\mathscr{X}(ds^2)) \cong H^i_{(2)}(\mathscr{X}(g)) .$$

Proof. Take a closed neighborhood W of S in X and set $W^* = W - S$. Then we consider the L^2-versions of long exact sequences for $(\mathscr{X}(ds^2), W^*(ds^2))$ and $(\mathscr{X}(g), W^*(g))$. If the inclusion map given at Proposition 4.4 induces the isomorphisms $H^i_{(2)}(W^*(ds^2)) \cong H^i_{(2)}(W^*(g))$ for $i \leq 1$, then the five lemma implies the corollary. And the above isomorphisms for $i \leq 1$ are certainly induced by the assertion that the restriction from W^* to $\partial W^* (= \partial W)$ gives the isomorphisms

$$
\begin{aligned}
& H^i_{(2)}(W^*(ds^2)) \cong H^i_{DR}(\partial W^*) , \\
& H^i_{(2)}(W^*(g)) \cong H^i_{DR}(\partial W^*) ,
\end{aligned}
\tag{4.10}
$$

for $i \leq 1$. The second isomorphism has been already shown in [10, Proposition 3.1]. The first isomorphism is essentially due to [2, (3.36)]: let us explain it a little bit more. Let $ds^2_{\boldsymbol{C}}$ be the standard metric on $\boldsymbol{C}$ and let us set $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$. Then our $(\mathscr{X}, ds^2)$ is quasi-isometric to $(\boldsymbol{C}^*, ds^2_{\boldsymbol{C}}) \times (\boldsymbol{C}^*, ds^2_{\boldsymbol{C}})$ near the intersections of the divisors and quasi-isometric to $(\boldsymbol{C}, ds^2_{\boldsymbol{C}}) \times (\boldsymbol{C}^*, ds^2_{\boldsymbol{C}})$ near the other points on the divisors. Hence, near any point of $\pi^{-1}(S)$, the local L^2-cohomology (with respect to our metric ds^2) can be easily computed using [2, (3.36)] (in the metric cone case). Now the argument using the L^2-version of Mayer-Vietoris exact sequence (or the derived category argument ([10, Proposition 3.1])) implies the first isomorphism. Q.E.D.

§5. Proof of Theorem 1 for $i \leq 1$ and Some Remarks

In this section, we will prove Theorem 1 for $i \leq 1$ and then discuss a little bit the so-called (L^2-, usual, logarithmic) irregularities.

Since the case $i=0$ is trivial (see §3), we will treat the case $i=1$ in the following.

Lemma 5.1. *On our* $\mathcal{X}(g)=(\mathcal{X}, g)$,

$$\operatorname{dom}\partial_{(0,0)} \cap \operatorname{Ker}\bar{\partial}_{(0,0)} = \operatorname{Ker}\partial_{\hat{c},(0,0)},$$
$$\operatorname{Ker}\partial_{(0,0)} \cap \operatorname{dom}\bar{\partial}_{(0,0)} = \operatorname{Ker}\bar{\partial}_{\hat{c},(0,0)}.$$

Proof. Let us prove the first equality. Since $\operatorname{Ker}\partial_{\hat{c},(0,0)} = \operatorname{Ker} d_0$ (by Proposition 3.1(1)), the right hand side is contained in the left hand side. Conversely, because the element f of the left hand side belongs to $\operatorname{dom} d_0 = \operatorname{dom} d_{\hat{c},0}$, there exists a sequence $f_j \in \operatorname{dom} d_{c,0}$ such that $\lim_{j\to\infty} f_j = f$ and $\lim_{j\to\infty} df_j = df$. Since $df = \partial f$, it also satisfies $\lim_{j\to\infty} \bar{\partial} f_j = 0$. That is, f belongs to the right hand side. Q.E.D.

Corollary 5.2. *Assume* $p+q=1$, *then we have*

$$H^{p,q}_{(2)d}(\mathcal{X}(g)) = \mathcal{H}^{p,q}_{(2)d}(\mathcal{X}(g)) = \hat{\mathcal{H}}^{p,q}_{(2)d}(\mathcal{X}(g)).$$

Proof. We prove only the case $(p, q)=(1,0)$. Since

$$A^{1,0}(\mathcal{X}) \cap \operatorname{Range} d_0 = \{\partial f \mid f \in \operatorname{dom}\partial_{(0,0)} \cap \operatorname{Ker}\bar{\partial}_{(0,0)}\},$$

Lemma 5.1 implies $A^{1,0}(\mathcal{X}) \cap \operatorname{Range} d_0 = \{0\}$. Hence $H^{1,0}_{(2)d}(\mathcal{X}(g)) = A^{1,0}(\mathcal{X}) \cap \operatorname{Ker} d_1$. Moreover its element φ satisfies $*_g\varphi = \sqrt{-1}\,\bar{\varphi}\wedge\omega_g$, $\delta\varphi = -*_g d *_g \varphi = -\sqrt{-1}\, *_g d(\bar{\varphi}\wedge\omega_g) = 0$. That is, $H^{1,0}_{(2)d}(\mathcal{X}(g)) = \mathcal{H}^{1,0}_{(2)d}(\mathcal{X}(g))$. The remained equality $\mathcal{H}^{1,0}_{(2)d}(\mathcal{X}(g)) = \hat{\mathcal{H}}^{1,0}_{(2)d}(\mathcal{X}(g))$ is implied by Lemma 2.1(2). Q.E.D.

Lemma 5.3. *The restriction of the forms on* $\tilde{X}$ *to* $\mathcal{X}$ *induces the isomorphism*

$$H^i_{DR}(\tilde{X}) \cong H^i_{(2)}(\mathcal{X}(ds^2)).$$

Proof. More strongly we can assert that the Stokes' theorem in the L^2-sense holds for $\mathcal{X}(g)$, that is, we have $d_{\hat{c},i} = d_i$ for all i; refer to [2, Theorem 2.2

for metric cones]. Hence the restriction induces the isomorphism

$$H^i_{(2)}(\tilde{X}(d\tilde{s}^2)) \cong H^i_{(2)}(\mathscr{X}(ds^2)) . \tag{5.1}$$

This and the obvious one $H^i_{DR}(\tilde{X}) \cong H^i_{(2)}(\tilde{X}(d\tilde{s}^2))$ imply the lemma. Q.E.D.

Hence, combined with Corollary 4.5, it implies

Corollary 5.4. *Assume $i \leq 1$. Then the restriction induces the isomorphism*

$$H^i_{DR}(\tilde{X}) \cong H^i_{(2)}(\mathscr{X}(g)) .$$

Since $H^i_{DR}(\tilde{X})$ has the pure Hodge structure, we can prove Theorem 1 for $i=1$.

Proof of Theorem 1 *for* $i=1$. First there exist the natural isomorphisms $\mathscr{H}^1_{(2)}(\mathscr{X}(g)) \cong \hat{\mathscr{H}}^1_{(2)}(\mathscr{X}(g)) \cong H^1_{(2)}(\mathscr{X}(g))$ because of Lemma 2.1(2) and (2.7). Also we have Corollary 5.2. Hence what is remained is to verify the pure Hodge decomposition of $\mathscr{H}^1_{(2)}(\mathscr{X}(g))$ (or $H^1_{(2)}(\mathscr{X}(g))$). To do so, consider the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{p+q=1} \mathscr{H}^{p,q}_{(2)\bar{\partial}}(\mathscr{X}(g)) & \xrightarrow[\iota_{\mathscr{H}}]{} & \mathscr{H}^1_{(2)}(\mathscr{X}(g)) \\
\| \; r_{p,q} & & \wr\| \downarrow r_1 \\
\bigoplus_{p+q=1} H^{p,q}_{(2)\bar{\partial}}(\mathscr{X}(g)) & \xrightarrow[\iota]{} & H^1_{(2)}(\mathscr{X}(g)) \\
\uparrow \alpha & & \uparrow \wr\| \; \beta \\
\bigoplus_{p+q=1} H^{p,q}_{\bar{\partial}}(\tilde{X}) & \xrightarrow[\tilde{\iota}]{\simeq} & H^1_{DR}(\tilde{X})
\end{array} \tag{5.2}$$

Here the isomorphisms r_1, β, $\tilde{\iota}$ and the equality $r_{p,q}$ have been already given. Then, since the map $\iota_{\mathscr{H}}$ is injective, the map ι is also injective. Moreover the lower part of the diagram implies the surjectivity of ι. Thus the map ι is isomorphic and hence the maps $\iota_{\mathscr{H}}$, α are also isomorphic. Q.E.D.

Now, because the maps at (5.2) are all isomorphic, the L^2-irregularity $q_{(2)}(\mathscr{X}(g)) = \dim_{\boldsymbol{C}} H^{1,0}_{(2)\bar{\partial}}(\mathscr{X}(g))$ is equal to the (usual) irregularity $q(\tilde{X}) = \dim_{\boldsymbol{C}} H^{1,0}_{\bar{\partial}}(\tilde{X})$. Moreover, since Corollary 5.4 (i.e., the isomorphism β at (5.2)) and (2.6) for $i=1$ give the natural isomorphism

$$H^1_{DR}(\tilde{X}) \cong H^1_{DR}(\mathscr{X}) , \tag{5.3}$$

the pure Hodge decomposition of $H^1_{DR}(\tilde{X})$ is naturally equivalent to the algebraic de Rham decomposition ([4])

$$H^1_{DR}(\mathscr{X}) \cong H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log \pi^{-1}(S))) \oplus H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) , \tag{5.4}$$

where $\Omega^1_{\tilde{X}}(\log \pi^{-1}(S))$ is the sheaf of germs of logarithmic 1-forms (possibly) with logarithmic poles along $\pi^{-1}(S)$. That is, $q(\tilde{X})$ is equal to the logarithmic irregularity $\bar{q}(\mathcal{X})=\dim_{\mathbf{C}} H^0(\tilde{X}, \Omega^1_{\tilde{X}}(\log \pi^{-1}(S)))$. Thus we got

$$q_{(2)}(\mathcal{X}(g)) = q(\tilde{X}) = \bar{q}(\mathcal{X}) . \tag{5.5}$$

The equality $q(\tilde{X})=\bar{q}(\mathcal{X})$ is well-known in itself: note that (4.1) is the point modification at S. Here we emphasize that thus it can be also verified only by using the L^2-cohomology theory.

§6. Proof of Theorem 1 for $i\geq 3$

Since the case $i=4$ is trivial (see §3), we will treat the case $i=3$ in the following.

Letting the complex star operator $\bar{*}_g$ operate on the decomposition of Theorem 1(1) for $i=1$, we get Theorem 1(1) for $i=3$. Lemma 2.1(2) implies $\mathcal{H}^3_{(2)}(\mathcal{X}(g))=\hat{\mathcal{H}}^3_{(2)}(\mathcal{X}(g))$ and $\mathcal{H}^{p,q}_{(2)d}(\mathcal{X}(g))=\hat{\mathcal{H}}^{p,q}_{(2)d}(\mathcal{X}(g))$ for $p+q=3$. Moreover, remembering (2.7), we have the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{p+q=3} \mathcal{H}^{p,q}_{(2)d}(\mathcal{X}(g)) & = & \mathcal{H}^3_{(2)}(\mathcal{X}(g)) \\ \downarrow r_{p,q} & & \wr\!\| \\ \bigoplus_{p+q=3} H^{p,q}_{(2)d}(\mathcal{X}(g)) & \underset{\iota}{\longrightarrow} & H^3_{(2)}(\mathcal{X}(g)) \end{array} \tag{6.1}$$

Hence, in order to finish the proof of Theorem 1 for $i=3$, it suffices to prove that the map $r_{p,q}$, which is certainly injective, is surjective.

In the following, let us prove the surjectivity of $r_{1,2}$. (The proof for $r_{2,1}$ is similar.) Consider the (Hodge) decomposition

$$\begin{aligned} \operatorname{Ker} d_3 &= \operatorname{Range} d_2 \oplus \mathcal{H}^3_{(2)}(\mathcal{X}(g)) \\ &= \operatorname{Range} d_2 \oplus \mathcal{H}^{1,2}_{(2)d}(\mathcal{X}(g)) \oplus \mathcal{H}^{2,1}_{(2)d}(\mathcal{X}(g)) \end{aligned}$$

and decompose $\varphi\in A^{1,2}(\mathcal{X})\cap \operatorname{Ker} d_3$ accordingly;

$$\varphi = d\psi+\varphi^{1,2}_{\mathcal{H}}+\varphi^{2,1}_{\mathcal{H}} . \tag{6.2}$$

Moreover, decompose ψ into $\psi^{0,2}+\psi^{1,1}+\psi^{2,0}$, where $\psi^{p,q}$ is the (p, q)-part of ψ. Then (6.2) can be rewritten as follows;

$$\begin{aligned} \varphi &= \partial\psi^{0,2}+\bar{\partial}\psi^{1,1}+\varphi^{1,2}_{\mathcal{H}} , \\ \partial\psi^{1,1}+\bar{\partial}\psi^{2,0}+\varphi^{2,1}_{\mathcal{H}} &= 0 . \end{aligned} \tag{6.3}$$

Lemma 6.1. $\partial\psi^{1,1}+\bar{\partial}\psi^{2,0} = 0$.

Proof. It suffices to prove $\varphi_{\mathcal{H}}^{2,1}=0$. Since $\varphi_{\mathcal{H}}^{2,1}\in \operatorname{Ker}\delta_2=\operatorname{Ker}\delta_{\hat{c},2}$ (refer to Lemma 2.1(2)), there exists a sequence $\phi_j\in A_c^3(\mathcal{X})$ satisfying $\lim_{j\to\infty}\phi_j=\varphi_{\mathcal{H}}^{2,1}$ and $\lim_{j\to\infty}\delta\phi_j=0$. Decompose ϕ_j into $\phi_j^{1,2}+\phi_j^{2,1}$ with $\phi_j^{p,q}$ the (p,q)-part of ϕ_j, then we have

$$\lim_{j\to\infty}\phi_j^{1,2}=0\,,\quad \lim_{j\to\infty}\phi_j^{2,1}=\varphi_{\mathcal{H}}^{2,1}\,,$$
$$\lim_{j\to\infty}\delta'\phi_j^{1,2}=\lim_{j\to\infty}(\delta''\phi_j^{1,2}+\delta'\phi_j^{2,1})=\lim_{j\to\infty}\delta''\phi_j^{2,1}=0\,.$$

Therefore we have

$$\begin{aligned}(\varphi_{\mathcal{H}}^{2,1},\varphi_{\mathcal{H}}^{2,1})_g &= -\lim_{j\to\infty}(\phi_j^{2,1},\partial\psi^{1,1}+\bar\partial\psi^{2,0})\\ &= -\lim_{j\to\infty}\{(\delta'\phi_j^{2,1},\psi^{1,1})+(\delta''\phi_j^{2,1},\psi^{2,0})\}\\ &= -\lim_{j\to\infty}(\delta'\phi_j^{2,1},\psi^{1,1})=\lim_{j\to\infty}(\delta''\phi_j^{1,2},\psi^{1,1})\\ &= \lim_{j\to\infty}(\phi_j^{1,2},\bar\partial\psi^{1,1})=-\lim_{j\to\infty}(\phi_j^{1,2},\partial\psi^{0,2})\\ &= -\lim_{j\to\infty}(\delta'\phi_j^{1,2},\psi^{0,2})=0\,.\end{aligned}$$

Q.E.D.

Hence, $\partial\psi^{0,2}+\bar\partial\psi^{1,1}\in A^{1,2}(\mathcal{X})\cap\text{Range } d_2$ at (6.3). That is, $[\varphi]=[\varphi_{\mathcal{H}}^{1,2}]$ as the elements of $H_{(2)d}^{1,2}(\mathcal{X}(g))$, which means that the map $r_{1,2}$ is surjective.

§7. Proofs of Theorem 1 for $i=2$ and Theorem 2

Because of the duality of $\mathcal{H}_{(2)}^*(\mathcal{X}(g))$ with respect to the complex star operator $\bar{*}_g$ and of the facts

$$\bar{*}_g 1=\frac{1}{2}L_g^2(1)\,,\qquad (7.1)$$
$$\bar{*}_g\varphi=\sqrt{-1}\,L_g(\bar\varphi)\,,\quad \varphi\in A^1(\mathcal{X})\,,$$

certainly Theorem 2(*a*) holds. Moreover (*b*) and (*c*) of Theorem 2 for $i\neq 2$ are just Theorem 1 for $i\neq 2$. Hence we have only to prove Theorems 1, 2 for $i=2$. In the following we will prove them.

Since

$$\bar{*}_g\,\omega_g=\omega_g\,,\qquad (7.2)$$

we have $d\omega_g=\delta\omega_g=0$, i.e., $\omega_g\in\mathcal{H}_{(2)d}^{1,1}(\mathcal{X}(g))$. Moreover $\mathcal{H}_{(2)}^4(\mathcal{X}(g))$ $(\simeq\boldsymbol{C})$ is generated by $\omega_g\wedge\omega_g$. Therefore we have the decomposition

$$\mathcal{H}_{(2)}^2(\mathcal{X}(g))=L_g\,\mathcal{H}_{(2)}^0(\mathcal{X}(g))\oplus\operatorname{Ker}L_g\,,\qquad (7.3)$$

where Ker L_g is the kernel of the map

$$L_g\colon \mathcal{H}^2_{(2)}(\mathcal{X}(g)) \to \mathcal{H}^4_{(2)}(\mathcal{X}(g))\,. \tag{7.4}$$

Now, in order to finish the proofs of Theorems 1, 2, it suffices to prove the following.

Proposition 7.1.

$$\operatorname{Ker} L_g = \mathcal{H}^{0,2}_{(2)d}(\mathcal{X}(g)) \oplus (\mathcal{H}^{1,1}_{(2)d}(\mathcal{X}(g)) \cap \operatorname{Ker} L_g) \oplus \mathcal{H}^{2,0}_{(2)d}(\mathcal{X}(g))\,. \tag{7.5}$$

Proof. If $\varphi \in \operatorname{Ker} L_g$, then $\bar{\varphi} \in \operatorname{Ker} L_g$. Hence it suffices to show that $\varphi = \bar{\varphi} \in \operatorname{Ker} L_g$ can be decomposed accordingly to the right hand side of (7.5). Let us now decompose $\varphi = \varphi^{0,2} + \varphi^{1,1} + \varphi^{2,0}$, where $\varphi^{p,q}$ is the (p, q)-part of φ. The conditions say

$$\varphi^{1,1} = \overline{\varphi^{1,1}}\,, \quad \varphi^{1,1} \wedge \omega_g = 0\,, \quad \varphi^{2,0} = \overline{\varphi^{0,2}}\,. \tag{7.6}$$

Since $d\varphi = \delta\varphi = 0$, we also have

$$\partial\varphi^{0,2} + \bar{\partial}\varphi^{1,1} = 0\,, \quad \delta''\varphi^{0,2} + \delta'\varphi^{1,1} = 0\,. \tag{7.7}$$

Here $\bar{*}_g\, \varphi^{0,2} = \overline{\varphi^{0,2}}$ and (because of the second condition at (7.6)) $\bar{*}_g\, \varphi^{1,1} = -\overline{\varphi^{1,1}}$. Hence the second condition at (7.7) implies

$$\partial\varphi^{0,2} - \bar{\partial}\varphi^{1,1} = 0\,. \tag{7.8}$$

This and the first condition at (7.7) imply $\partial\varphi^{0,2} = \bar{\partial}\varphi^{1,1} = 0$. That is, we get $d\varphi^{0,2} = 0$, $d\varphi^{1,1} = \partial\varphi^{1,1} + \bar{\partial}\varphi^{1,1} = \overline{\bar{\partial}\varphi^{1,1}} + \bar{\partial}\varphi^{1,1} = 0$, $d\varphi^{2,0} = \overline{d\varphi^{0,2}} = 0$, $\delta\varphi^{0,2} = -\bar{*}_g d\bar{*}_g \varphi^{0,2} = -\bar{*}_g \overline{d\varphi^{0,2}} = 0$, $\delta\varphi^{1,1} = \bar{*}_g \overline{d\varphi^{1,1}} = 0$, $\delta\varphi^{2,0} = \overline{\delta\varphi^{0,2}} = 0$. Hence the decomposition $\varphi = \varphi^{0,2} + \varphi^{1,1} + \varphi^{2,0}$ is the one according to the right hand side of (7.5). Q.E.D.

§8. Appendix

We have finished the verification of the pure Hodge and hard Lefschetz structures for the incomplete Kähler metric g. Here we study such structures for other Kähler metrics and investigate their relationships.

(a) On the L^2-cohomology $H^i_{(2)}(\mathcal{X}(ds^2))$.

Let us consider the metric ds^2 explained at §4(*b*). The following is not trivial in itself. However, now we can easily verify

Proposition 8.1. *The L^2-cohomology $H^i_{(2)}(\mathcal{X}(ds^2))$ has the pure Hodge and hard Lefschetz structures in the sense of Theorems* 1, 2. *Moreover Conjecture A for $H^i_{(2)}(\mathcal{X}(ds^2))$ holds.*

Proof. Obviously Corollary 5.2 holds also for $\mathscr{X}(ds^2)$. Moreover we have Lemma 5.3. Hence the same arguments as in §5–§7 imply the first part. Moreover the second part is valid because the Stokes' theorem in the L^2-sense holds for $\mathscr{X}(ds^2)$: refer to the proof of Lemma 5.3. Q.E.D.

(b) The relation between the metric g and a certain complete Kähler metric.

L. Saper ([13, §2]) (and T. Ohsawa ([11, §4]) previously to him) introduced a certain complete Kähler metric h on $\mathscr{X}$, which possesses a noteworthy property. That is, it implies the generalized de Rham isomorphism ([13, Theorem 0.1(i)])

$$H^i_{(2)}(\mathscr{X}(h)) \cong (IH^{\bar{m}}_i(X))^* . \tag{8.1}$$

By Lemma 2.2 and this identification, the middle intersection homology of X turns out to have the pure Hodge structure. This way of approach to such a structure (i.e., to find appropriate complete Kähler metrics) seems to have been widely adopted and have produced fruitful results (by S. Zucker, L. Saper, T. Ohsawa, M. Kashiwara and others). Here we first explain the metric h according to [13].

Let us use the notations in §4. For the intersection matrix $(a_{ij})=(D_i\cdot D_j)$, which is negative definite, we can take two kinds of sequences consisting of positive integers, $(a_1, \cdots, a_m)$ and $(a'_1, \cdots, a'_m)$, satisfying $\sum_i a_i a_{ij}<0$ and $\sum_i a'_i a_{ij}<0$ for any j and $a_i a'_j \neq a_j a'_i$ for any $i\neq j$ ([13, Lemma 2.1]). Then, let us consider the line bundles on $\tilde{X}$,

$$L = \sum a_i [D_i] , \quad L' = \sum a'_i [D_i] . \tag{8.2}$$

Since $L\cdot D_i<0$ and $L'\cdot D_i<0$ for any i, there exist the hermitian metrics ds^2_L, $ds^2_{L'}$ on L, L' whose curvature forms are negative definite near $\pi^{-1}(S)$ ([13, Proposition 2.2]). We now take the holomorphic sections s and s' of L and L' with $L=[(s)]$ and $L'=[(s')]$ respectively. Then, for a neighborhood $\tilde{W}$ of $\pi^{-1}(S)$, the following (1, 1)-form on $\tilde{W}-\pi^{-1}(S)$ is positive definite ([13, (2.1)]);

$$\tilde{\omega} = -\sqrt{-1}\ \{\partial\bar{\partial} \log (\log |s|^2)^2+\partial\bar{\partial} \log (\log |s'|^2)^2\} , \tag{8.3}$$

where $|s|$, $|s'|$ mean the pointwise norms defined by ds^2_L, $ds^2_{L'}$. Set $P(x)=-\log (\log |s|^2)^2-\log (\log |s'|^2)^2$, called the potential function of $\tilde{\omega}$. And, take a smooth function φ on $\tilde{X}$ satisfying $\varphi\equiv 1$ near $\pi^{-1}(S)$ and $\varphi\equiv 0$ on the complement of $\tilde{W}$, and, moreover, take a sufficiently large constant $K>0$. Then the following (1, 1)-form on $\mathscr{X}$ is positive definite ([13, Proposition 2.9]);

$$\omega_h = \omega_g + \sqrt{-1}\,\partial\bar{\partial}\,(K^{-1}\varphi P)\,, \tag{8.4}$$

where ω_g is the (1, 1)-form associated to our metric g. Now the associated Kähler metric h is complete and of finite volume. Since $\omega_h \sim \tilde{\omega}$ near $\pi^{-1}(S)$, the metric h has the following quasi-isometric representation near $\pi^{-1}(S)$.

Proposition 8.2 ([13, Proposition 2.4]).

(1) *In Case* $(-)$, *on* $U-\pi^{-1}(p)$,

$$h \sim |\log \tau^2|^{-1}(d\rho^2+\rho^2 d\phi^2)+\tau^{-2}|\log \tau^2|^{-2}(d\tau^2+\tau^2 d\psi^2)\,.$$

(2) *In Case* $(+)$, *on* $\{(u, v)\in U-\pi^{-1}(p)\,|\,\rho\leq\tau\}$,

$$h \sim \rho^{-2}|\log \rho^2|^{-2}(d\rho^2+\rho^2 d\phi^2) + (|\log \rho^2|^{-1}+\tau^{-2}|\log \rho^2|^{-2})\,(d\tau^2+\tau^2 d\psi^2)\,.$$

Remark. For the other cases, there are also similar representations. Besides, obviously the above results say that the quasi-isometric class of the metric h does not depend on the positive integers $\{a_i\}$, $\{a_i'\}$, nor on the metrics $|s|$, $|s'|$. However it certainly depends on the choice of the resolution (4.1).

Hence the volume element with respect to h can be written on $U-\pi^{-1}(p)$ as follows.

$$dV_h \sim \begin{cases} \rho\tau^{-1}|\log \tau^2|^{-3}\,dV & ;\ \text{Case } (-)\,, \\ \rho^{-1}\,\tau\,|\log \rho^2|^{-4}(|\log \rho^2|+\tau^{-2})\,dV; & \text{Case } (+) \text{ with } \rho\leq\tau\,. \end{cases} \tag{8.5}$$

Similarly to Lemma 4.3, let us denote the pointwise norm of a form α with respect to h by $|\alpha|_h$. Then we have

Lemma 8.3. *On* $U-\pi^{-1}(p)$,

(1) $dV_g \lesssim dV_h$,

(2) $|d\rho|_g^2\,dV_g \lesssim |d\rho|_h^2\,dV_h, \cdots, |d\psi|_g^2\,dV_g \lesssim |d\psi|_h^2\,dV_h$.

Proof. In Case $(-)$,

$$dV_g/dV_h \sim \tau^{2(m_1+m_2)}|\log \tau^2|^3\,,$$
$$|d\rho|_g^2\,dV_g/|d\rho|_h^2\,dV_h \sim |d\phi|_g^2\,dV_g/|d\phi|_h^2\,dV_h \sim \tau^{2m_1}|\log \tau^2|^2\,,$$
$$|d\tau|_g^2\,dV_g/|d\tau|_h^2\,dV_h \sim |d\psi|_g^2\,dV_g/|d\psi|_h^2\,dV_h \sim \tau^{2m_2}|\log \tau^2|\,.$$

Next, in Case $(+)$ with $\rho\leq\tau$,

$$dV_g/dV_h \sim \rho^{2(n_1+n_2)}\,\tau^{2(m_1+m_2-1)}|\log \rho^2|^4(|\log \rho^2|+\tau^{-2})^{-1}\,,$$
$$|d\rho|_g^2\,dV_g/|d\rho|_h^2\,dV_h \sim |d\phi|_g^2\,dV_g/|d\phi|_h^2\,dV_h$$
$$\sim \rho^{2n_2}\,\tau^{2(m_2-1)}|\log \rho^2|^2(|\log \rho^2|+\tau^{-2})^{-1}\,,$$

$$|d\tau|_g^2\, dV_g/|d\tau|_h^2\, dV_h \sim |d\psi|_g^2\, dV_g/|d\psi|_h^2\, dV_h \sim \rho^{2n_2}\tau^{2m_2}|\log\rho^2|^2 .$$

Q.E.D.

The lemma implies

Proposition 8.4. *Assume $i\leq 1$. Then the identity map on $A^i(\mathcal{X})$ induces the bounded inclusion map*

$$L^2A^i(\mathcal{X}(h)) \to L^2A^i(\mathcal{X}(g)) .$$

Therefore, similarly to Corollary 4.5, we have

Corollary 8.5. *Assume $i\leq 1$. Then the identity map on $A^i(\mathcal{X})$ induces the isomorphism*

$$H^i_{(2)}(\mathcal{X}(h)) \cong H^i_{(2)}(\mathcal{X}(g)) .$$

Proof. Take W^* as in the proof of Corollary 4.5. Then we have only to show

$$H^i_{(2)}(W^*(h)) \cong H^i_{DR}(\partial W^*) , \tag{8.6}$$

for $i\leq 1$. It is due to [13, the assertion following (1.3)]. Q.E.D.

(c) On the relation between the pure Hodge structures for the various Kähler metrics.

Now, similarly to the proof of Theorem 1 for $i=1$ (§5) and using the dual argument, we can easily verify

Theorem 8.6. *Assume $i\neq 2$. Then the pure Hodge and hard Lefschetz structures for $\mathcal{X}(g)$, $\mathcal{X}(h)$, $\mathcal{X}(ds^2)$ and $\tilde{X}(=\tilde{X}(d\tilde{s}^2))$ can be all identified through the restriction maps, the inclusion maps and the dual maps (of the inclusion maps).*

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