Endomorphisms of Certain Operator Algebras

By

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Abstract

In this paper we study $C(X)$-endomorphisms of $C(X, \mathfrak{B}(\mathfrak{H}))$, for X a compact space. The local structure of these endomorphisms is analyzed. We prove the existence of a $C(X)$-shift, not all of whose fibers are shifts on $\mathfrak{B}(\mathfrak{H})$.

§ 1. Introduction

In [6] Kadison and Ringrose undertook a penetrating study of automorphisms of certain operator algebras. In particular, they showed that if $\mathfrak{A}$ is a C^*-algebra of operators on a Hilbert space $\mathfrak{H}$, then the condition $\|\alpha - \iota\| < 2$ for an automorphism α implies that α lies in the connected component of the identity, $\Delta(\mathfrak{A})$, in $\mathrm{Aut}(\mathfrak{A})$. Moreover $\Delta(\mathfrak{A})$ is generated as a group by such automorphisms, and each element of $\Delta(\mathfrak{A})$ is permanently weakly inner, or π-inner. This latter property means that α is weakly inner in any (faithful) representation π of $\mathfrak{A}$. If $\mathfrak{A}$ is the C^*-algebra $C(X, \mathfrak{B}(\mathfrak{H}))$, where X is a compact topological space, this property implies that $\alpha(fA) = f\alpha(A)$, for any scalar-valued function f in $\mathfrak{A}$. Following the terminology of more recent papers, e.g., [8], we shall call such an automorphism a $C(X)$-automorphism.

In the case where $\mathfrak{H}$ is finite-dimensional, (i.e., $\mathfrak{B}(\mathfrak{H}) \cong M_n(C)$, for some n) Kadison and Ringrose give an example of a compact space X [6, Example d] for which $\Delta(\mathfrak{A}) \subsetneqq \mathrm{Inn}_{C(X)}(\mathfrak{A}) \subsetneqq \mathrm{Aut}_{C(X)}(\mathfrak{A})$. $\mathrm{Inn}_{C(X)}(\mathfrak{A})$ is the group of inner $C(X)$-automorphisms, i.e., α is inner if there is a (uniformly continuous) unitary-valued function U on X such that $\alpha = Ad(U)$.

Motivated in part by [6], Lance, [7], and M. S. B. Smith, [14], studied the structure of the quotients of the various automorphism groups on $C(X, \mathfrak{B}(\mathfrak{H}))$, equipped with the norm topology, where $\mathfrak{H}$ is an infinite-dimensional Hilbert space. In this setting $\Delta(\mathfrak{A})$ and $\mathrm{Inn}_{C(X)}(\mathfrak{A})$ coincide, [7, Theorem 3.6], and the quotient group $\mathrm{Aut}_{C(X)}(\mathfrak{A})/\Delta(\mathfrak{A})$ is shown to be isomorphic to $H^2(X, \mathbf{Z})$ [7, Theorem 4.3]. A key result obtained by Lance to establish the above isomorphism is that $C(X)$-automorphisms of $C(X, \mathfrak{B}(\mathfrak{H}))$ are induced locally by norm

continuous unitary-valued functions, i.e., for $x \in X$ there is a neighborhood N of x and a unitary function $U \in C(N, \mathfrak{B}(\mathfrak{H}))$ such that $\alpha_{|C(N, \mathfrak{B}(\mathfrak{H}))} = Ad(U)$. More recent papers (viz. [8, 9]) have analyzed the structure of $C(X)$-automorphisms of separable continuous trace algebras; and, even more generally, of C^*-algebras of sections of bundles having fibres which are C^*-algebras.

In this note we introduce the notion of $C(X)$-shifts of $C(X, \mathfrak{B}(\mathfrak{H}))$. (We shall assume throughout that X is separable.) A *-endomorphism σ of $C(X, \mathfrak{B}(\mathfrak{H}))$ which preserves the identity operator I (i.e., $I(x) = I$, for all x) and satisfies $\sigma(fA) = f\sigma(A)$ for $f \in C(X)$, shall be called a $C(X)$-endomorphism. Letting $A = I$ we obtain $\sigma(f) = f$. If σ satisfies the additional range property $\bigcap_{k \geq 1} \sigma^k(C(X, \mathfrak{B}(\mathfrak{H}))$ $= C(X)$, we say that it is a $C(X)$-shift. For x a point, our definition coincides with that given by Powers [10] for a shift on $\mathfrak{B}(\mathfrak{H})$. In this setting, and under a technical assumption, Powers has shown that the Jones' index $[\mathfrak{B}(\mathfrak{H}): \sigma(\mathfrak{B}(\mathfrak{H}))]$ (cf. [5]) is a complete invariant for the outer conjugacy classes of shifts on $\mathfrak{B}(\mathfrak{H})$, [10, Theorem 2.4].

Although the general structure of $C(X)$-shifts on $C(X, \mathfrak{B}(\mathfrak{H}))$ is far from settled, our results here indicate an interesting interplay between the theory of $C(X)$-automorphisms on $C(X, \mathfrak{B}(\mathfrak{H}))$ and the theory of shifts on $\mathfrak{B}(\mathfrak{H})$, (cf. [10, 12, 13]). In Section 2, we show that $[\mathfrak{B}(\mathfrak{H}): \sigma_x(\mathfrak{B}(\mathfrak{H}))]$ is constant on connected component of X, where for $x \in X$, σ_x is the *-endomorphism on $\mathfrak{B}(\mathfrak{H})$ induced by the equation $\sigma(A)(x) = \sigma_x(A(x))$, $A \in C(X, \mathfrak{B}(\mathfrak{H}))$. In this section we also give an analogue of Lance's theorem on locally inner $C(X)$-automorphisms.

In Section 3 we present an example which shows that if σ is a $C(X)$-shift, then not all of the induced endomorphisms, σ_x, of $\mathfrak{B}(\mathfrak{H})$ need to be shifts themselves. On the other hand, if each σ_x is a shift it follows easily that σ must be as well, Lemma 2.6.

§ 2. The Structure of $C(X)$-Endomorphisms

Let σ be a $C(X)$-endomorphism of $\mathfrak{A}$. Then σ is an isometry of $\mathfrak{A}$, which fixes the center, $C(X)$. By a trivial adaptation of the proof of either [8, Lemma 1.4] or of [6, Example d], we obtain the following local characterization of σ, which is essentially a restatement of [8, Lemma 1.4] for endomorphisms.

Lemma 2.1. *Let σ be a $C(X)$-endomorphism on $\mathfrak{A} = C(X, \mathfrak{B}(\mathfrak{H}))$, where X is a compact space.*

(i) *If $A_1, A_2 \in \mathfrak{A}$ and $A_1(x) = A_2(x)$ for some $x \in X$, then $\sigma(A_1)(x) = \sigma(A_2)(x)$.*

(ii) *For any compact set Y of X, σ induces a (unit-preserving) *-endomorphism σ_Y of $C(Y, \mathfrak{B}(\mathfrak{H}))$ such that $\sigma_Y(A_{|Y}) = \sigma(A)_{|Y}$ for $A \in \mathfrak{B}$.*

We write σ_x for the unique *-endomorphism of $\mathfrak{B}(\mathfrak{H})$ which satisfies $\sigma(A)(x)$ $= \sigma_x(A(x))$, $x \in X$, $A \in \mathfrak{A}$. We shall write $\sigma = \bigoplus_{x \in X} \sigma_x$. Observe that the *-endomor-

phisms σ_x vary continuously with x in the norm topology. For if $A \in \mathfrak{B}(\mathfrak{H})$, then $\sigma(\underline{A}) \in C(X, \mathfrak{B}(\mathfrak{H}))$, where $\underline{A}$ is the constant function $\underline{A}(y) = A$, for all $y \in X$, so that given $x \in X$ and $\varepsilon > 0$, there is a neighborhood N of x such that $\|\sigma(\underline{A})(y) - \sigma(\underline{A})(x)\| < \varepsilon$, for $y \in N$. But $\|\sigma(\underline{A})(y) - \sigma(\underline{A})(x)\| = \|\sigma_x(A) - \sigma_y(A)\|$, by Lemma 1. Hence we have the following (cf. [8, Lemma 1.6]).

Lemma 2.2. *Let* $\sigma = \bigoplus\limits_{x \in X} \sigma_x$ *be a* *-endomorphism on* $C(X, \mathfrak{B}(\mathfrak{H}))$. *Then the map* $x \to \sigma_x$ *is a continuous map of* X *into the set of* *-endomorphisms of* $\mathfrak{B}(\mathfrak{H})$ *in the topology of pointwise norm convergence.*

Now suppose α is a *-endomorphism of $\mathfrak{B}(\mathfrak{H})$ which preserves the identity. Then $\alpha(\mathfrak{B}(\mathfrak{H}))$ is a subfactor of $\mathfrak{B}(\mathfrak{H})$, isomorphic to $\mathfrak{B}(\mathfrak{H})$. Moreover, $\alpha(\mathfrak{B}(\mathfrak{H}))'$ is a type I_n subfactor N of $\mathfrak{B}(\mathfrak{H})$, for some n (possibly ∞) and $\mathfrak{B}(\mathfrak{H})$ is generated by N and $\alpha(\mathfrak{B}(\mathfrak{H}))$ [1, I.8.2, Corollary 3]. Using the identification $\mathfrak{B}(\mathfrak{H}) \cong N \otimes \alpha(\mathfrak{B}(\mathfrak{H}))$, it is clear that if $e \in \mathfrak{B}(\mathfrak{H})$ is a rank one projection, then $\alpha(e)$ is a projection in $\mathfrak{B}(\mathfrak{H})$ of rank n. Another way to characterize the "size" of $\alpha(\mathfrak{B}(\mathfrak{H}))$ in $\mathfrak{B}(\mathfrak{H})$ is to observe that $n^2 = [\mathfrak{B}(\mathfrak{H}) : \alpha(\mathfrak{B}(\mathfrak{H}))]$, the Jones index (cf. [5]).

Definition 2.1. Let α be a *-endomorphism of $\mathfrak{B}(\mathfrak{H})$. Then α has index n^2 if $\alpha(\mathfrak{B}(\mathfrak{H}))'$ is a type I_n factor.

Proposition 2.3. *Let* $\sigma = \bigoplus \sigma_x$ *be a* $C(X)$-*endomorphism of* $\mathfrak{A}$. *The index of the endomorphisms* σ_x *is constant on connected components of* X.

Proof. Let $e \in \mathfrak{B}(\mathfrak{H})$ be a rank one projection, then the mapping $x \to \sigma(\underline{e})(x) = \sigma_x(e)$ is uniformly continuous. Given $x \in X$, let N be a neighborhood of x such that for $y \in N$, $\|\sigma_y(e) - \sigma_x(e)\| < 1$. Then [3, Lemma 1.8] there is a partial isometry with support projection $\sigma_x(e)$ and range projection $\sigma_y(e)$, so the projections $\sigma_y(e)$, $\sigma_x(e)$ have the same rank. Hence $x \to \mathrm{rank}(\sigma_x(e))$ is a continuous function from X to $N \cup \{\infty\}$, and the result follows. ∎

In what follows we shall make the assumption that X is connected. Fix $x \in X$; then for any $y \in X$, σ_x, σ_y have the same index. We shall also assume that this index is finite. We may then construct a unitary operator $U_y \in \mathfrak{B}(\mathfrak{H})$ such that $Ad(U_y) \circ \sigma_x = \sigma_y$. In fact, choosing matrix units $\{e_{ij} : i, j \in N\}$ for $\mathfrak{B}(\mathfrak{H})$, we set

$$U_y = \sum_{i=1}^{\infty} \sigma_y(e_{i1}) V_y \sigma_x(e_{1i}), \tag{1}$$

as in [10, Theorem 2.4], where $V_y \in \mathfrak{B}(\mathfrak{H})$ is a partial isometry from $\sigma_x(e_{11})$ to $\sigma_y(e_{11})$. Observe that if U_y' is any other unitary such that $Ad(U_y') \circ \sigma_x = \sigma_y$, then $U_y^* U_y' \in \sigma_x(\mathfrak{B}(\mathfrak{h}))'$.

Lemma 2.4. *Let* N *be a neighborhood of* $x \in X$, *where* X *is connected. Suppose there is a uniformly continuous mapping* $y \to V_y$ *from* N *to partial isometries* V_y

with support projection $\sigma_x(e_{11})$ and range projection $\sigma_y(e_{11})$. *Then there is a strongly continuous mapping $y \to U_y$ on N where the U_y are unitary operators satisfying* (1).

Proof. Suppose $\{f_i : i \in N\}$ is an orthonormal basis for $\mathfrak{H}$ such that for each $i, j \in N$, either $\sigma_x(e_{jj})f_i = 0$, or f_i. Let $U_y \in \mathfrak{B}(\mathfrak{H})$ be defined as in (1). Since $\|U_y\| = 1$, it suffices to show that $y \to U_y f$ is continuous for f a finite linear combination of elements of the basis; for the same reason, we may assume that $f = f_i$, for some $i \in N$. Choose j such that $\sigma_x(e_{jj})f = f$; then $U_y f = \sigma_y(e_{jj})V_y \varphi$, where $\varphi = \sigma_x(e_{11})f$. But $y \to \sigma_y(e_{jj})$ and $y \to V_y$ are uniformly continuous for $y \in N$, so the mapping $y \to \sigma_y(e_{jj})V_y \varphi$ is also continuous on N. ■

Proposition 2.5. *Let $x \in X$. There is a neighborhood N of x such that a uniformly continuous mapping $y \to V_y$ exists from N to partial isometries V_y with support projection $\sigma_x(e_{11})$ and range projection $\sigma_y(e_{11})$.*

Proof. For any γ such that $0 < \gamma < 1$, let N be a neighborhood of x such that $\|\sigma_y(e_{11}) - \sigma_x(e_{11})\| < \gamma$, for all $y \in N$. Let P (respectively, Q) be the projection given by $P(y) = \sigma_x(e_{11})$ (resp., $Q(y) = \sigma_y(e_{11})$) for all $y \in \overline{N}$. Then $\|P - Q\| < 1$, so by [3, Lemma 1.8] there exists a unitary $U \in C(\overline{N}, \mathfrak{B}(\mathfrak{H}))$ such that $Q = UPU^*$. Setting $V = UP$, $y \to V(y)$ gives the desired mapping. ■

Corollary. *There are, for each $x \in X$, a neighborhood N of x, and unitary operators $U_y \in \mathfrak{B}(\mathfrak{H})$ such that the mapping $y \to U_y$ is strongly continuous for $y \in N$.*

In what follows we shall show that under some restrictions the preceding result may be strengthened in the sense that the U_y may be chosen to vary continuously in norm for $y \in N$, (Theorem 2.8). This result is inspired by a similar result obtained by Lance, [7], for $C(X)$-automorphisms, which was described in Section 1. We have also adapted some of the techniques of that proof.

Definition 2.2. Let $\sigma = \oplus \sigma_x$ be a $C(X)$-endomorphism of $C(X, \mathfrak{B}(\mathfrak{H}))$, and let ρ be a state of $\mathfrak{B}(\mathfrak{H})$. We say that ρ is σ-invariant if $\rho \circ \sigma_x = \rho$ for all $x \in X$.

Throughout the remainder of this section we assume the existence of a pure normal state ρ on $\mathfrak{B}(\mathfrak{H})$ which is σ-invariant. We shall also assume that σ_x is a shift on $\mathfrak{B}(\mathfrak{H})$, for each $x \in X$. In this situation, we have the following structure theorem of Powers, [10, Theorem 2.3].

Theorem P. *Suppose α is a shift of $\mathfrak{B}(\mathfrak{H})$. If there is an α-invariant pure normal state on $\mathfrak{B}(\mathfrak{H})$, then there are mutually commuting, isomorphic type I subfactors $B_1, B_2, \cdots$ of $\mathfrak{B}(\mathfrak{H})$ such that*
  (i) *$\mathfrak{B}(\mathfrak{H})$ is the weak closure of $B_1 \otimes B_2 \otimes \cdots$,*
  (ii) *$\alpha(B_k) = B_{k+1}$ for $k \in N$, and*

(iii) ρ is a symmetric product state with respect to the factorization in (i).

Lemma 2.6. *Suppose* $\sigma=\oplus\sigma_x$ *is a* $C(X)$-*endomorphism such that* σ_x *is a shift of* $\mathfrak{B}(\mathfrak{H})$, *for all* $x\in X$. *Then* σ *is a* $C(X)$-*shift.*

Proof. Let $A\in C(X, \mathfrak{B}(\mathfrak{H}))$ lie in the range of σ^k, for all $k\in N$. For each k, let A_k satisfy $\sigma^k(A_k)=A$. If $x\in X$, $\sigma_x^k(A_k(x))=\sigma^k(A_k)(x)=A(x)$, so $A(x)\in\bigcap_{k\geq 1}\sigma_x^k(\mathfrak{B}(\mathfrak{H}))$ is a scalar multiple of the identity, and therefore $A\in C(X)$. Hence σ is a shift. ■

Remark. The converse of the lemma is not true, in general. This is the principal result of the following section.

The following technical lemma will be helpful in proving the main result of this section, Theorem 2.8, (cf. [7, Lemma 3.2]).

Lemma 2.7. *Suppose, for some* $x\in X$ *there are a neighborhood N of x and unitary operators U_y for each $y\in N$ such that $U_y\to I$ strongly, but not uniformly, as $y\to x$. Then there are an orthonormal sequence $\{h_m\}$ of vectors of $\mathfrak{H}$, a sequence $\{y_m\}$ in N, and a $\delta>0$ such that*
(i) $y_m\to x$
(ii) $\|U_{y_n}h_m-h_m\|<\delta/(n2^{n+1})$, *for* $m<n$,
(iii) $\|U_{y_m}h_m-h_m\|>\delta/2.$

Proof. Under the hypotheses, there are a sequence $\{\varphi_m\}$ of unit vectors in $\mathfrak{H}$, elements $x_n\in N$, and $\delta>0$ such that (a) $\|U_{x_n}\varphi_m-\varphi_m\|<\delta/(n2^{n+1})$, for $m<n$, and (b) $\|U_{x_n}\varphi_n-\varphi_n\|>\delta$. Fix n, and suppose orthonormal vectors h_1, h_2, $\cdots$, h_n have been chosen in the subspace of $\mathfrak{H}$ spanned by the φ_m's, and elements y_1, y_2, $\cdots$, y_n have been selected from the sequence $\{x_m\}$ such that

(1) $\|U_{y_p}h_m-h_m\|<\delta/(p2^{p+1})$,   for $1\leq m<p\leq n$,

(2) $\|U_{y_p}h_p-h_p\|>\delta/2$,   for $1\leq p\leq n$.

Choose an element $y\in\{x_m\}$ such that $\|U_y h_p-h_p\|<\delta/((n+1)2^{n+1})$, for $1\leq p\leq n$. Writing $\varphi'=\varphi_y$, then $\|U_y\varphi'-\varphi'\|>\delta$. Let $\varphi=\sum_{k=1}^{n}(\varphi', h_k)h_k$, and let $\psi=\varphi-\varphi'$. Then $\psi\perp h_k$, for $k\leq n$, and

$$\delta<\|U_y\varphi'-\varphi'\|\leq\|U_y\psi-\psi\|+\|U_y\varphi-\varphi\|$$

$$\leq\|U_y\psi-\psi\|+\left\|\sum_{k=1}^{n}(\varphi', h_k)(U_y h_k-h_k)\right\|$$

$$\leq\|U_y\psi-\psi\|+[\max_{1\leq k\leq n}\{|(\varphi', h_k)|\}](n\delta/((n+1)2^{n+1}))$$

$$\leq\|U_y\psi-\psi\|+\delta/(2^{n+1}).$$

Hence $\delta/2 < \delta - (\delta/2^{n+1}) < \|U_y\phi - \phi\| \leq \|U_y\phi - \phi\|/\|\phi\|$. Setting $y_{n+1} = y$, and $h_{n+1} = \phi/\|\phi\|$, (i) and (ii) hold for $k = 1, 2, \cdots, n+1$. The result now follows by induction. ∎

Using the preceding results we may now improve Corollary 2.5 as follows. The main idea of the proof (i.e., the construction of the operator V), is inspired by a similar construction in [7, Lemma 3.2].

Theorem 2.8. *Let* $\sigma = \bigoplus_{x \in X} \sigma_x$ *be a* $C(X)$-*shift of* $C(X, B(\mathfrak{H}))$ *on a connected space* X, *for which each* σ_x *is a shift of finite index on* $\mathfrak{B}(\mathfrak{H})$. *Suppose there is a* σ-*invariant pure normal state* ρ *on* $\mathfrak{B}(\mathfrak{H})$. *Then for each* $x \in X$, *there is a neighborhood* N *of* x *such that*
 (i) $\sigma_y = Ad(U_y) \circ \sigma_x$, *for* $y \in N$, *and*
 (ii) *the* U_y *vary continuously in norm for* $y \in N$.

Proof. From Corollary 2.5 we may choose for each $x \in X$ a neighborhood N of x and unitary operators V_y for $y \in N$, which satisfy (i) and which vary continuously in the strong topology. For $A \in \mathfrak{B}(\mathfrak{H})$, $\sigma_x(A) = V_x\sigma_x(A)V_x^*$, so $V_x \in \sigma_x(\mathfrak{B}(\mathfrak{H}))' = B_1$, where we use the notation of Theorem P. Taking $U_y = V_yV_x^*$, (i) still holds and $U_x = I$.

From Theorem P we may identify $\mathfrak{B}(\mathfrak{H})$ as the weak closure of the tensor product $B_1 \otimes B_2 \otimes \cdots$. By making the identification $B_k = B_j$, for all k, j each B_k may be viewed as acting on the same n-dimensional Hilbert space $\mathfrak{H}_0$ with orthonormal basis $\{f_1, \cdots, f_n\}$. If E_k is the minimal projection of B_k for which $\rho(E_k) = 1$, then $\sigma_x(E_k) = E_{k+1}$, and we may assume $E_kf_1 = f_1$, for all k. We then view $\mathfrak{H}$ as the Hilbert space tensor product $\otimes \mathfrak{H}_0$, having orthonormal basis the vectors $f_{j_1} \otimes f_{j_2} \otimes \cdots$ for which all but finitely many indices j_k are 1(see [4]).

If $\|U_y - I\| \not\to 0$ as $y \to x$ in N, there is an orthonormal sequence $\{h_m\}$ in $\mathfrak{H}$ which satisfies the conditions of Lemma 2.7. From the finite-dimensionality of $\mathfrak{H}_0$ we may assume that there is a unit vector $f \in \mathfrak{H}_0$ such that each h_i has the form $h_i = f \otimes \varphi_i$, where the φ_i's are orthonormal. Viewing the φ_i as elements of $\mathfrak{H}$, let e_i be the rank one projections satisfying $e_i\varphi_i = \varphi_i$. Define $e_{i+1} \otimes e_i$ to be the rank one operator $(e_{i+1} \otimes e_i)\varphi = (\varphi, \varphi_i)\varphi_{i+1}$. Then the sum $\sum_{i=1}^{\infty} e_{i+1} \otimes e_i$ converges strongly to a partial isometry V in $\mathfrak{B}(\mathfrak{H})$. Moreover, for a vector of the form $f \otimes \varphi \in \mathfrak{H}$, $\sigma_x(V)(f \otimes \varphi) = f \otimes V\varphi$.

Let $\varepsilon = \delta/4$ (for δ as in Lemma 2.7). By the lemma there is an $m \in N$ and a $y = y_{m+1} \in N$ such that $\|U_yh_m - h_m\| < \varepsilon/2$ and $\|U_yh_{m+1} - h_{m+1}\| > 2\varepsilon$. Then $\|\sigma_x(V)U_yh_m - \sigma_x(V)h_m\| < \varepsilon/2$, and

$$\sigma_x(V)h_m = \sigma_x(V)(f \otimes \varphi_m)$$

$$= f \otimes V\varphi_m$$

$$= f \otimes \varphi_{m+1}.$$

On the other hand, $\|U_y \sigma_x(V) h_m - f \otimes \varphi_{m+1}\| = \|U_y(f \otimes \varphi_{m+1}) - f \otimes \varphi_{m+1}\| > 2\varepsilon$, so by combining we have $\|\sigma_x(V)U_y h_m - U_y \sigma_x(V) h_m\| > \varepsilon$. But

$$\|\sigma_y(V) - \sigma_x(V)\| = \|U_y \sigma_x(V)U_y^* - \sigma_x(V)\|$$

$$= \|U_y \sigma_x(V) - \sigma_x(V)U_y\|$$

$$\geq \|U_y \sigma_x(V) h_m - \sigma_x(V)U_y h_m\| > \varepsilon,$$

so that $\sigma_y(V) \not\to \sigma_x(V)$ in norm for y in the sequence $\{y_m\}$, a contradiction, since $\sigma(\underline{V}) \in C(X, \mathfrak{B}(\mathfrak{H}))$, where $\underline{V}(y) = V$, for all y. By the contradiction, the U_y vary continuously in norm in N, and we are done. ∎

Remark. If X is connected, fix an element $x \in X$, and let $\{N_i\}_{i \in \mathcal{J}}$ be a finite open cover of X with uniformly continuous functions $\{U_i\}_{i \in \mathcal{J}}$ implementing σ in the sense that for $y \in N_i$, $\sigma_y = U_i(y)\sigma_x(\ )U_i(y)^*$. On $N_{ij} = N_i \cap N_j$, $\sigma_y = Ad(U_i(y)) \circ \sigma_x = Ad(U_j(y)) \circ \sigma_x$, so $U_i(y)U_j(y)^* \in \sigma_x(\mathfrak{B}(\mathfrak{H}))' = B_1$. If γ denotes the determinant on B_1, then let $\lambda_{ij}(y) = \gamma(U_i(y)U_j(y)^*)$. Then on $N_{ijk} = N_i \cap N_j \cap N_k$, $\lambda_{ij}\lambda_{jk} = \gamma(U_iU_j^*)\gamma(U_jU_k^*) = \gamma(U_iU_k^*) = \lambda_{ik}$, so the λ_{ij}'s define a one-cocycle of $\{N_i\}$ whose values lie in the sheaf of germs of continuous functions from X to Π^1 (cf. [7, Lemma 4.1]). Arguing as in [8], if $\{M_p\}$ is another cover, with associated unitary functions $V_p(y)$ on M_p, then passing to a common refinement of $\{N_i\}$ and $\{M_p\}$ we may assume that the two covers coincide. On N_i, let Δ_i be the continuous function determined by $\gamma(V_i) = \Delta_i \gamma(U_i)$. If $\{\mu_{ij}\}$ define the one-cocycle of $\{N_i\}$ for $\{V_i\}$ in the sense that $\mu_{ij}(y) = \gamma(V_i(y)V_j(y)^*)$, then

$$\mu_{ij} = \gamma(V_iV_j^*) = \Delta_i\Delta_j^{-1}\gamma(U_iU_j^*) = \Delta_i\Delta_j^{-1}\lambda_{ij},$$

so that μ_{ij}, λ_{ij} differ by a coboundary. Hence they define the same element of $H^1(X, \Pi^1)$. It would be interesting to pursue these results, which are analogous to the automorphism case, to determine more information about the structure of the unitaries $\{U_i\}$ implementing σ.

§3. An Example of a Shift

In this section we present a counterexample to the converse of Lemma 2.6. We verify the existence of a $C(X)$-shift $\sigma = \bigoplus_{x \in X} \sigma_x$, where X is a closed interval in R, for which each σ_x is a *-endomorphism of $\mathfrak{B}(\mathfrak{H})$, of index $n^2 < \infty$, but not all σ_x's are shifts on $\mathfrak{B}(\mathfrak{H})$.

We fix some notation. As above, let B_k, $k \in N \cup \{0\}$, be copies of $M_n(C)$ acting on C^n, with orthonormal basis $\{\Omega_1, \cdots, \Omega_n\}$. Let $\{e_{ij}^k : 1 \leq i, j \leq n\}$ be matrix units for B_k satisfying $e_{ij}^k \Omega_p = \delta_{jp}\Omega_i$. For convenience, write $\Omega = \Omega_1$. $\mathfrak{A}$ is the UHF algebra $\bigotimes_{k \geq 0} B_k$ of type n^∞. This algebra acts on the Hilbert

space $\mathfrak{H}$ having an orthonormal basis consisting of all vectors of the form $\Omega_j \otimes \Omega_{i_1} \otimes \Omega_{i_2} \otimes \cdots$, where all but finitely many of the Ω_{i_k} are Ω. Under this action, $B(\mathfrak{H})$ is the weak operator closure of $\mathfrak{A}$. Let ω be the vector state ω_Ω, on $\mathfrak{A}$, where $\Omega' = \Omega \otimes \Omega \otimes \cdots$. Then ω is a pure symmetric product state, $\underset{k \geq 0}{\otimes} \omega_k$, on $\mathfrak{A}$, where

$$\omega_k(e_{ij}^k) = \delta_{ij}\delta_{11}, \qquad k \in N \cup \{0\}. \qquad (2)$$

Let α be the shift on $(\underset{k \geq 1}{\otimes} B_k)''$ given by $\alpha(e_{ij}^k) = e_{ij}^{k+1}$. Define a *-endomorphism $\sigma_0 = \iota \otimes \alpha$ on $\mathfrak{B}(\mathfrak{H})$, i.e., $\sigma_0(A) = A$ for $A \in B_0$, and $\sigma_0(A) = \alpha(A)$ for $A \in (\underset{k \geq 1}{\otimes} B_k)''$. It is straightforward to verify that $n^2 = [\mathfrak{B}(\mathfrak{H}): \sigma_0(\mathfrak{B}(\mathfrak{H}))]$. Moreover σ_0 is not a shift on $\mathfrak{B}(\mathfrak{H})$; in fact, it is easy to verify that $B_0 = \underset{k \geq 1}{\cap} \sigma_0^k(\mathfrak{B}(\mathfrak{H}))$.

As in [11] we define an action on $\mathfrak{A}$ of the discrete group $S(\infty)$ of permutations of $N \cup \{0\}$ fixing all but finitely many elements. We define this action by setting $(rs)e_{pq}^k(rs) = e_{pq}^k$, if $k \neq r$ and $k \neq s$, and $(rs)e_{pq}^r(rs) = e_{pq}^s$, for transpositions (rs) in $S(\infty)$. It is straightforward to show that (rs) is inner; in fact, we may take (rs) to be the (self-adjoint) unitary element $\sum_{p=1} \sum_{q=1} e_{pq}^r e_{qp}^s$ (cf. [11]).

Definition 3.1. Let T be the transposition (01). For $t \in R$, let U_t be the unitary operator $e^{-it} \exp(itT)$. More generally for $k \in N$, let $T_k = (0k)$, and $U_{k,t} = e^{-it} \exp(itT_k)$.

Lemma 3.1. *The following conditions hold for the operators defined as above:*

(i) $T_k = \sigma_0^k(T)$, *for* $k \in N \cup \{0\}$,
(ii) $U_{k,t} = e^{-it}\{\cos(t) + i\sin(t)T_k\}$,
(iii) $\omega \circ Ad(U_{k,t}) = \omega$
(iv) $\omega \circ \sigma_0 = \omega$
(v) $\omega \circ Ad(U_{k,t}) \circ \sigma_0 = \omega$.

Proof. (i) and (ii) are obvious. To show (iii) observe that $T_k \Omega' = \Omega'$, so $\exp(itT_k) = e^{it}\Omega'$. Hence $\omega(U_{k,t}AU_{k,t}^*) = \langle U_{k,t}AU_{k,t}^*\Omega', \Omega' \rangle = \langle A\Omega', \Omega' \rangle = \omega(A)$, $A \in \mathfrak{B}(\mathfrak{H})$. To verify (iv) observe that for $k \in N$, $\omega(\sigma_0(e_{ij}^k)) = \omega(e_{ij}^{k+1}) = \omega(e_{ij}^k)$, by (2). Sicne ω is a product state the preceding equation shows that ω is σ_0-invariant. (v) follows from (iii) and (iv). ∎

Now let $x = [-\pi/4, \pi/4]$, and define $\sigma' = \underset{t \in x}{\oplus} \sigma_t'$, where $\sigma_t' = \sigma_0$ for all t. Clearly, σ' is a $C(X)$-endomorphism of $C(X, \mathfrak{B}(\mathfrak{H}))$. Define $\sigma = \underset{t \in x}{\oplus} \sigma_t$, where $\sigma_t = Ad(U_t) \circ \sigma_t'$. Since $U = \oplus U_t \in C(X, \mathfrak{B}(\mathfrak{H}))$, $\sigma = Ad(U) \circ \sigma'$ is also a $C(X)$-endomorphism. We shall show that σ is in fact a $C(X)$-shift. This is achieved through the series of lemmas below.

Definition 3.2. For $k \in N \cup \{0\}$, and $t \in R$, let $\Gamma_{k,t}$ be the unitary operator

$$U_t \sigma_0(U_t) \cdots \sigma_0^k(U_t) = U_{1,t} U_{2,t} \cdots U_{k+1,t}.$$

It is straightforward to show that since $\sigma_t = Ad(U_t) \circ \sigma_0$,

$$\sigma_t^{k+1}(A) = \Gamma_{k,t} \sigma_0^{k+1}(A) \Gamma_{k,t}^*, \tag{3}$$

for $A \in \mathfrak{B}(\mathfrak{H})$.

Lemma 3.2. *For $t \in X$, and for $k \in \boldsymbol{N} \cup \{0\}$, $[\sigma_t^{k+1}(\mathfrak{B}(\mathfrak{H}))]' = \Gamma_{k,t} \{B_1, \cdots, B_{k+1}\}'' \Gamma_{k,t}^*.$*

Proof. Since $\sigma_0 = \iota \otimes \alpha$, and since $[\alpha^k(\bigotimes_{j \geq 1} B_j)]' \cap (\bigotimes_{j \geq 1} B_j) = \{B_1, \cdots, B_k\}''$, by [10, Lemma 2.1], $\sigma_0^k(\mathfrak{B}(\mathfrak{H}))' = \{B_1, \cdots, B_k\}''$. Then by (3), $[\sigma_t^{k+1}(\mathfrak{B}(\mathfrak{H}))]' = (\Gamma_{k,t} \sigma_0^{k+1}(\mathfrak{B}(\mathfrak{H})) \Gamma_{k,t}^*)' = \Gamma_{k,t} [\sigma_0^{k+1}(\mathfrak{B}(\mathfrak{H}))]' \Gamma_{k,t}^* = \Gamma_{k,t} \{B_1, \cdots, B_{k+1}\}'' \Gamma_{k,t}^*.$ ∎

Lemma 3.3. *For $t \in X$, and for $k \in \boldsymbol{N}$, $\sigma_t^{k+1}(\mathfrak{B}(\mathfrak{H})) = \{\Gamma_{k,t} B_0 \Gamma_{k,t}^*, B_{k+2}, B_{k+3}, \cdots\}''.$*

Proof. Clearly $\sigma_0^{k+1}(\mathfrak{B}(\mathfrak{H})) = \{B_0, B_{k+2}, \cdots\}''$. Then $\sigma_t^{k+1}(\mathfrak{B}(\mathfrak{H})) = \{\Gamma_{k,t} B_0 \Gamma_{k,t}^*, \Gamma_{k,t} B_{k+2} \Gamma_{k,t}^*, \cdots\}''$. But $\Gamma_{k,t} \in \{B_0, B_1, \cdots, B_{k+1}\}''$, so commutes with B_{k+j}, for $j > 1$, so $\Gamma_{k,t} B_{k+j} \Gamma_{k,t}^* = B_{k+j}$. ∎

By Φ_k, $k \in \boldsymbol{N}$, we shall denote the conditional expectation of the vector state ω onto $\bigotimes_{j=0}^{k} B_j$. We may define Φ_k as follows (cf. [2, Lemma 2]). Let $\{E_{pq}\}$ be matrix units for $\bigotimes_{j=0}^{k} B_j$. Each $A \in \mathfrak{B}(\mathfrak{H})$ has a unique decomposition of the form $\Sigma E_{pq} A_{pq}$, where $A_{pq} \in \{B_{k+1}, \cdots\}''$. Then $\Sigma \omega(A_{pq}) E_{pq} = \Phi_k(A)$.

Lemma 3.4. *The sequence $\{\Phi_k(A) : k \in \boldsymbol{N}\}$ converges weakly to A, for all $A \in \mathfrak{B}(\mathfrak{H})$.*

Proof. Since $\|A\| \geq \|\Phi_k(A)\|$ for all $k \in \boldsymbol{N}$, it suffices to show $\langle \Phi_k(A) \xi, \eta \rangle$ converges to $\langle A\xi, \eta \rangle$ for basis vectors $\xi = \bigotimes_{i \geq 0} \Omega_{p_i}$, $\eta = \bigotimes_{i \geq 0} \Omega_{q_i}$. For N sufficiently large, $\Omega_{p_k} = \Omega_{q_k} = \Omega$, if $k > N$. Recalling that ω is a product state with respect to the B_j, and choosing matrix units $\{E_{rs}\}$ for $\{B_0, B_1, \cdots, B_k\}''$, where $k > N$, we have

$$\langle A\xi, \eta \rangle = \Sigma \langle E_{rs} A_{rs} \xi, \eta \rangle$$

$$= \Sigma \langle E_{rs}(\Omega_{p_0} \otimes \cdots \otimes \Omega_{p_k}), \Omega_{q_0} \otimes \cdots \otimes \Omega_{q_k} \rangle \langle A_{rs}(\Omega \otimes \Omega \otimes \cdots), \Omega \otimes \Omega \cdots \rangle$$

$$= \Sigma \langle E_{rs}(\Omega_{p_0} \otimes \cdots \otimes \Omega_{p_k}), \Omega_{q_0} \otimes \cdots \otimes \Omega_{q_k} \rangle \omega(A_{rs})$$

$$= \Sigma \langle \Phi_k(A)(\Omega_{p_0} \otimes \cdots \otimes \Omega_{p_k}), \Omega_{q_0} \otimes \cdots \otimes \Omega_{q_k} \rangle$$

$$= \langle \Phi_k(A) \xi, \eta \rangle,$$

and we are done. ∎

Theorem 3.5. *If $t \neq 0$ is an element of $[-\pi/4, \pi/4]$, then σ_t is a shift of $\mathfrak{B}(\mathfrak{H})$.*

Proof. Suppose $A \in \sigma_t^k(\mathfrak{B}(\mathfrak{H}))$ for all k. Then $A \in \{\Gamma_{k,t} B_0 \Gamma_{k,t}^*, B_{k+2}, \cdots\}''$ for all k, by Lemma 3.3. Since $\Gamma_{k,t} B_0 \Gamma_{k,t}^* \subseteq \{B_0, B_1, \cdots, B_{k+1}\}''$, $A_k = \Phi_{k+1}(A) \in \Gamma_{k,t} B_0 \Gamma_{k,t}^*$. Hence there is a sequence $y_k \in B_0$ such that $A = \text{wk-}\lim_{k \to \infty} \Gamma_{k,t} y_k \Gamma_{k,t}^*$, by the preceding lemma.

We shall show in the proposition below that for each $t \in [-\pi/4, \pi/4]$, the sequence $\{\Gamma_{k,t}^*\}$ converges strongly to an isometry Γ^* in $\mathfrak{B}(\mathfrak{H})$: moreover, if $\xi = \bigotimes_i \Omega_{p_i}$ is a basis vector in $\mathfrak{H}$, then $\Gamma^*\xi$ is a vector of the form $\Omega \otimes \Lambda_t$. If also $\eta = \bigotimes_i \Omega_{q_i}$ and e_{ij}^0 is a matrix unit of B_0, observe that

$$\lim_k \langle \Gamma_{k,t} e_{ij}^0 \Gamma_{k,t}^* \xi, \eta \rangle = \lim_k \langle e_{ij}^0 \Gamma_{k,t}^* \xi, \Gamma_{k,t}^* \eta \rangle$$

$$= \langle e_{ij}^0 \Gamma^* \xi, \Gamma^* \eta \rangle \qquad (4)$$

$$= \delta_{i1} \delta_{j1} \langle \xi, \eta \rangle.$$

Let $y_k = \sum a_{ij}^k e_{ij}^0$ for scalars a_{ij}^k. Then $\|y_k\| = \|\Gamma_{k,t} y_k \Gamma_{k,t}^*\| = \|A_k\| \leq \|A\|$, so that all $|a_{ij}^k|$ are bounded by $\|A\|$. From (4), $\lim_{k \to \infty} \langle \Gamma_{k,t} a_{ij}^k e_{ij}^0 \Gamma_{k,t}^* \xi, \eta \rangle = 0$ for $i \neq 1$ or $j \neq 1$, so

$$\langle A\xi, \eta \rangle = \lim_{k \to \infty} \langle A_k \xi, \eta \rangle$$

$$= \lim_{k \to \infty} \langle \Gamma_{k,t} y_k \Gamma_{k,t}^* \xi, \eta \rangle$$

$$= \lim_{k \to \infty} \langle \Gamma_{k,t} a_{11}^k e_{11}^0 \Gamma_{k,t}^* \xi, \eta \rangle$$

$$= \lim_{k \to \infty} a_{11}^k \langle \xi, \eta \rangle.$$

Hence A is the scalar operator λI, where $\lambda = \lim_{k \to \infty} a_{11}^k$. ∎

Corollary. *σ is a $C(X)$-shift on $C(X, \mathfrak{B}(\mathfrak{H}))$.*

Proof. Let $A \in \cap \sigma^k(C(X), \mathfrak{B}(\mathfrak{H}))$. Then $A(t)$ is a scalar operator for $t \neq 0$, by the theorem above. Hence, by continuity, $A(0)$ is a scalar operator, so $A \in C(X)$. ∎

Proposition 3.6. *For $t \in [-\pi/4, \pi/4]$, the operators $\{\Gamma_{k,t}^* : k \in N\}$ converge strongly to an isometry Γ_t^* on $\mathfrak{B}(\mathfrak{H})$. For any vector $\varphi \in \mathfrak{H}$, $\Gamma_t^*\varphi$ has the form $\Omega \otimes \varphi'$.*

Proof. Fix $t \neq 0$ in $[-\pi/4, \pi/4]$. Since $\{\Gamma_{k,t} : k \in N\}$ is a uniformly bounded sequence, the strong continuity will follow upon verifying that $\lim_{k \to \infty} \Gamma_{k,t}^* \xi$ exists for basis vectors $\xi = \Omega_{i_0} \otimes \Omega_{i_1} \otimes \Omega_{i_2} \otimes \cdots$. For $p \in N$ and $\eta = \Omega_{j_0} \otimes \Omega_{j_1} \otimes \cdots$, note that

$$U^*_{p,t}\eta = e^{it}\{\cos(t) - i\sin(t)T_p\}\eta$$

$$= e^{it}[\cos(t)\eta - i\sin(t)(\Omega'_{j_0}\otimes\Omega'_{j_1}\otimes\cdots)],$$

(5)

where $\Omega'_{j_q} = \Omega_{j_q}$, if $q \neq 0$, or p; $\Omega'_{j_0} = \Omega_{jp}$; and $\Omega'_{j_p} = \Omega_{j_0}$. For $k \in N$, write $\Gamma^*_{k,t}\xi = \sum_{j=1}^{n}\xi_{k,j}$, where $\xi_{k,j}$ is the sum of all tensors in $\Gamma^*_{k,t}\xi$ beginning with Ω_j. Then the $\xi_{k,j}$ are orthogonal, for $1 \leq j \leq n$, and $1 = \sum_{j=1}^{n}\|\xi_{k,j}\|^2$.

Now choose N so that $\Omega_{j_k} = \Omega$, for $k > N$. An application of (5) shows that for such k, $\xi_{k,j}$ is a linear combination of tensors of the form $\bigotimes_{s=0}^{\infty}\Omega_{j_s}$, where $\Omega_{j_s} = \Omega$ for $s > k+1$. Then, from (5) again,

$$U^*_{k+1,t}\xi_{k,1} = \xi_{k,1},$$

$$U^*_{k+1,t}\xi_{k,j} = e^{it}\{\cos(t)\xi_{k,j} - i\sin(t)T_{k+1}\xi_{k,j}\}$$

(6)

$$= e^{it}\{\cos(t(\xi_{k,j} - i\sin(t)\xi'_{k,j}\},$$

where $\xi'_{k,j} = T_{k+1}\xi_{k,j}$. Each summand of $\xi'_{k,j}$ has Ω in the first tensorand, and has Ω_j in the $(k+1)$st tensorand.

It is now straightforwad to verify that

$$\xi_{k+1,j} = e^{it}\cos(t)\xi_{k,j}, \qquad 1 < j \leq n, \quad \text{and}$$

(7.1)

$$\xi_{k+1,1} = \xi_{k,1} - i\sin(t)\sum_{j=2}^{n}\xi'_{k,j},$$

(7.2)

where the n terms in (7.2) are mutually orthogonal (since the summands of each have Ω_j in the $(k+1)$st tensorand). Moreover, for $j \neq 1$, $\lim\|\xi_{k+r,j}\| = 0$, by (7.1), so that

$$\lim_{k\to\infty}\|\xi_{k,1}\| = 1.$$

(8)

Given $\varepsilon > 0$, choose $k > N$ sufficiently large so that

$$S = \sup_{p\in N}[\max_{1<j\leq n}\{\|\xi_{k+p,1}\|\}]$$

$$< \min\{\varepsilon/2\sqrt{2}(n-1), \ \varepsilon(1-\cos(t))/(n-1)|\sin(t)|\sqrt{2}\}.$$

Then for $r > s \geq 1$,

$$\|\Gamma^*_{k+r,t}\xi - \Gamma^*_{k+s,t}\xi\|^2 = \left\|\sum_{j=1}^{n}\xi_{k+r,j} - \xi_{k+s,j}\right\|^2$$

$$= \|\xi_{k+r,1} - \xi_{k+s,1}\|^2 + \sum_{j=2}^{n}\|\xi_{k+r,j} - \xi_{k+s,j}\|^2$$

$$\leq \|\xi_{k+r,1} - \xi_{k+s,1}\|^2 + \sum_{j=2}^{n}(\|\xi_{k+r,j}\| + \|\xi_{k+s,j}\|)^2$$

$$< \|\xi_{k+r,1} - \xi_{k+s,1}\|^2 + \varepsilon^2/2.$$

But using (7.2) repeatedly,

$$\xi_{k+r,1} - \xi_{k+s,1} = -ie^{it}\sin(t)\sum_{j=2}^{n} T_{k+s+1}\xi_{k+s,j} - \cdots$$

$$-ie^{it}\sin(t)\sum_{j=2}^{n} T_{k+r}\xi_{k+r-1,j}.$$

Now using (7.1) repeatedly,

$$\|\xi_{k+r,1} - \xi_{k+s,1}\| \leqq |\sin(t)| \left\{ \sum_{j=2}^{n} \|\xi_{k+s,j}\| + \cdots + \sum_{j=2}^{n} \|\xi_{k+r-1,j}\| \right\}$$

$$\leqq (n-1)|\sin(t)| \{S + S\cos(t) + \cdots + S\cos^{r-s}(t)\}$$

$$< (n-1)|\sin(t)|(S)\{1/1-\cos(t)\}$$

$$< \varepsilon/\sqrt{2}.$$

Combining our estimates, we have $\|\Gamma^*_{k+r,\iota}\xi - \Gamma^*_{k+s,\iota}\xi\|^2 < (\varepsilon/\sqrt{2})^2 + \varepsilon^2/2 = \varepsilon^2$, so that $\{\Gamma^*_{k,\iota}\}$ converges strongly on the basis vectors.

Now, if ξ, η are arbitrary (orthogonal) basis vectors, $\langle \Gamma^*_\iota\xi, \Gamma^*_\iota\eta \rangle = \lim_{k\to\infty} \langle \Gamma^*_{k,\iota}\xi, \Gamma^*_{k,\iota}\eta \rangle = \langle \xi, \eta \rangle$. Using this identity it is easy to show that Γ^*_ι is an isometry. Moreover, an application of (8) establishes the last statement of the proposition. ■

Acknowledgements

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