Publ. RIMS, Kyoto Univ.
25 (1989), 239-262

Infinite Loop G-Spaces Associated to Monoidal G-Graded Categories

Dedicated to Professor Akio Hattori on his sixtieth birthday

By

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Abstract

We construct a functor K_G which takes each pair of monoidal G-graded categories (D, D') to an infinite loop G-space $K_G(D, D')$. When $D'=D$, its homotopy groups $\pi_n^G K_G(D, D)$ coincide with the equivariant K-groups $K_n \mathrm{Rep} D$ of D. Applications include the simple construction of equivariant infinite deloopings of the maps $BO(G) \to BPL(G) \to BTop(G)$ between equivariant classifying spaces.

§ 0. Introduction

Let G be a finite group. By a (simplicial) *G-graded category* we shall mean a (simplicial) category D equipped with a (simplicial) functor γ from D to G which is regarded as a category with only one object. We often identify a simplicial G-graded category D with its *realization* rD; a topological G-graded category such that $\mathrm{ob}(rD)$ and $\mathrm{mor}(rD)$ are the geometric realizations of the simplicial sets $[k] \mapsto \mathrm{ob}D_k$ and $[k] \mapsto \mathrm{mor}D_k$ respectively.

A G-graded category D is said to be *monoidal* if there exist a functor (over G) $\oplus_D$: $D \times_G D \to D$, a section 0: $G \to D$, and natural isomorphisms $a \oplus_D (b \oplus_D c) \cong (a \oplus_D b) \oplus_D c$, $a \oplus_D b \cong b \oplus_D a$, $0 \oplus_D a \cong a$ (all simplicial in the case D is a simplicial G-graded category) subject to the coherence conditions similar to those for symmetric monoidal categories (cf. [5, 22]). Given a pair (D, D') of a monoidal G-graded category D and its G-graded subcategory D' closed under $\oplus_D$, we define a G-category $B(D, D')$ as follows:

Received May 20, 1988.
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Let EG be the translation category of G considered as a G-graded category via the projection $EG \to EG/G = G$. Take the category $\mathrm{Hom}_G(EG, D)$ whose objects are functors $EG \to D$ over G and whose morphisms are natural transformations. We endow $\mathrm{Hom}_G(EG, D)$ with a G-action

$$(g, f) \mapsto (gf\colon\ x \mapsto f(xg))$$

for every $g \in G$ and $f \in \mathrm{Hom}_G(EG, D)$. Then $B(D, D')$ denotes the G-stable full subcategory of $\mathrm{Hom}_G(EG, D)$ consisting of those functors $EG \to D$ which factors through D'. Observe that if (D, D') is a pair of simplicial G-graded categories then $B(rD, rD')$ is naturally isomorphic to the realization of the simplicial G-category $[k] \mapsto B(D_k, D'_k)$. Throughout the paper we write $B(D, D') = B(rD, rD')$ for any pair of simplicial G-graded categories (D, D').

Clearly $B(D, D')$ has a structure of a symmetric monoidal G-category given by the G-equivariant multiplication $\oplus\colon B(D, D') \times B(D, D') \to B(D, D')$,

$$(f, f') \mapsto (f \oplus f'\colon\ x \mapsto f(x) \oplus_D f'(x))$$

for every $f, f' \in B(D, D')$; hence its classifying space $|B(D, D')|$ becomes a Hopf G-space. The purpose of this paper is to construct a functor K_G which assigns to each pair of simplicial monoidal G-graded categories (D, D') an infinite loop G-space $K_G(D, D')$ having the same G-homotopy type as $|B(D, D')|$ when (and only when) $|B(D, D')|$ is grouplike, i.e., $\pi_0 |B(D, D')|^H$ is a group for every subgroup H of G. To state the results more precisely, we need further definitions.

We use the term *almost Ω-G-spectrum* to mean a system E consisting of based G-spaces E_V indexed on finite dimensional real G-modules V, and basepoint preserving G-maps $e_{V,W}\colon S^V \wedge E_W \to E_{V\oplus W}$ satisfying the following conditions:

(a) $e_{V,V'\oplus W}(1 \wedge e_{V',W}) = e_{V\oplus V',W}$ holds for all G-modules V, V' and W, and

(b) the adjoint $\varepsilon_{V,W}\colon E_W \to \Omega^V E_{V\oplus W}$ of $e_{V,W}$ is a G-homotopy equivalence if $W^G \neq 0$.

Note that any such E gives rise to a G-prespectrum $E_{\mathscr{A}} = \{E_V \mid V \in \mathscr{A}\}$ indexed on any indexing set $\mathscr{A}$ in a G-universe U (cf. [9]). (See also the remark at the end of Section 1.)

By the definition $\Omega E_{\boldsymbol{R}}$ becomes an infinite loop G-space where $\boldsymbol{R}$ denotes the trivial G-module of dimension 1. Moreover, we have fixed point prespectra $E^H=\{E^H_V\}$ indexed on finite dimensional real vector spaces (with trivial H-action). Clearly E^H is an almost Ω-spectrum in the sense that $E^H_W \simeq \Omega^V E^H_{V\oplus W}$ if $W\neq 0$.

Let $\boldsymbol{K}$ denote the functor which takes each simplicial monoidal category C to the prespectrum

$$\boldsymbol{K}C=\boldsymbol{S}|C^{\wedge}|=\boldsymbol{S}(\mathbf{n}\mapsto |C^{\wedge}(\mathbf{n})|)$$

where $C^{\wedge}$ is the special Γ-category constructed from C (cf. [11], [19]) and $\boldsymbol{S}$ is the Segal-Woolfson machine [17, 23] which takes each special Γ-space A to the almost Ω-spectrum $\boldsymbol{S}A=\{A'(S^V)/A'(\infty)\}$ (cf. Section 1). Then the main result of the paper can be stated as follows:

Theorem A. *There is a functor $\boldsymbol{K}_G$ from the pairs of simplicial monoidal G-graded categories to almost Ω-G-spectra equipped with*

(a) *a natural G-homotopy equivalence $\boldsymbol{K}_G(D, D')_0 \to |B(D, D')|$; and*

(b) *natural equivalences of prespectra $\boldsymbol{K}(B(D, D')^H) \to \boldsymbol{K}_G(D, D')^H$ for all subgroups H of G.*

Put $K_G(D, D')=\Omega\boldsymbol{K}_G(D, D')_{\boldsymbol{R}}$. Then there are natural G-maps

$$|B(D, D')| \xleftarrow{\ i\ } \boldsymbol{K}_G(D, D')_0 \xrightarrow{\ \varepsilon\ } K_G(D, D')$$

in which i is a G-homotopy equivalence, and we have

Corollary. $K_G(D, D')$ is an infinite loop G-space, and $|B(D, D')|$ has the same G-homotopy type as $K_G(D, D')$ if and only if $|B(D, D')|$ is grouplike.

Let us consider the particular case $D'=D$ (so that $B(D, D')=\mathrm{Hom}_G(EG, D)$). Suppose D is stable, i.e., given $M\in D$ and $g\in G$, there exists an isomorphism $f\colon M\to N$ of grade $\gamma(f)=g$. Then, for every subgroup H of G, we have an equivalence of categories

$$\mathrm{Hom}_G(EG, D)^H=\mathrm{Hom}_G(EG/H, D)\to \mathrm{Hom}_H(H, D\times_G H)=\mathrm{Rep}(H, D)$$

induced by the inclusion $H=EH/H\to EG/H$. Here $\mathrm{Rep}(H, D)$ is the category of representations of H by automorphisms (of the right grades) of objects of D (cf. [5]). Thus

Proposition. *The coefficient groups* $\pi_n^H \boldsymbol{K}_G(D, D)$ *coincide with the equivariant K-groups* $K_n\mathrm{Rep}(H, D)$ *in the sense of* [5, 22].

(More precisely we can prove that there is a natural isomorphism of Mackey functors $\pi_n^H \boldsymbol{K}_G(D, D) \cong K_n\mathrm{Rep}(H, D)$.)

As we shall see in Section 2, every symmetric monoidal G-category C is accompanied with a monoidal G-graded category $G\int C$ such that $\mathrm{Hom}_G(EG, G\int C)$ is naturally isomorphic to the functor category $\mathbf{Cat}(EG, C)$ having the G-action $(g, F) \mapsto (gF\colon x \mapsto gF(xg))$. Most of interesting examples of monoidal G-graded categories are obtained in this way, and we shall write $\boldsymbol{K}_G(C, C') = \boldsymbol{K}_G(G\int C, G\int C')$ for every pair of symmetric monoidal categories (C, C'). Among the examples, we have

(1) Let $\Sigma = \coprod_{n \geq 0} \Sigma_n$ be the skeletal category of finite sets and isomorphisms with symmetric monoidal structure given by disjoint union. Then $K_0\mathbf{Cat}(EG, \Sigma)^H \cong K_0\mathrm{Rep}(H, \Sigma)$ is the Burnside ring $A(H)$. In fact, each $|\mathbf{Cat}(EG, \Sigma_n)|$ is a classifying space for n-fold G-coverings (cf. Theorem 3. 1), and hence $\boldsymbol{K}_G(\Sigma, \Sigma)$ is equivalent to the sphere G-spectrum.

(2) For any ring A we have a symmetric monoidal category $GLA = \coprod_{n \geq 0} GL_nA$ equipped with the trivial G-action. Since $BGL_nA(G) = |\mathbf{Cat}(EG, GL_nA)|$ is a classifying space for G-GL_nA bundles, $\boldsymbol{K}_G(GLA, GLA)$ gives an infinite G-delooping of the G-space $K(A, G) = \Omega B(\coprod_{n \geq 0} BGL_nA(G))$ defining the equivariant K-theory of A in the sense of Fiedorowicz, Hauschild and May [4].

(3) Let k/k_0 be a Galois extension of fields with finite Galois group $G = \mathrm{Gal}(k/k_0)$. Let $V(k)$ be the category of finite dimensional vector spaces over k and isomorphisms. G acts on $V(k)$ via its action on k. Then there is an equivalence of categories $V(k^H) \to \mathbf{Cat}(EH, V(k))^H \simeq \mathbf{Cat}(EG, V(k))^H$ (cf. [21. §5]). Thus $\boldsymbol{K}_G(V(k), V(k))$ contains the (non-equivariant) algebraic K-theory of each intermediate field k^H as the H-fixed point subspectrum.

As another application of the theorem, we will construct, in Section 3, a classifying space $BCAT_n(G)$ for locally linear G-CAT bundles with fibre $\boldsymbol{R}^n$ for $CAT = O$, PL and Top, and show that the G-monoid $\coprod_{n \geq 0} BCAT_n(G)$ can be converted into an infinite loop

G-space $BCAT(G)$ through the group completion map $\coprod_{n\geq 0} BCAT_n(G) \to BCAT(G)$ (determined up to G-homotopy). By the naturality of the constructions, we can also prove that the G-maps $BO(G) \to BPL(G) \to BTop(G)$ can be taken to be maps of infinite loop G-spaces. (In [16] we shall show that $BTop(G) \to BF(G)$ = group completion of $\coprod_{n\geq 0} BF_n(G)$ also becomes an infinite loop G-map, where $BF_n(G)$ is a classifying space for n-dimensional spherical G-fibrations.)

The author wishes to thank M. Murayama for useful discussion about the subject of this work.

§ 1. Γ_G-Spaces

In this section we introduce the notion of a special Γ_G-space and describe the passage from special Γ_G-spaces to G-prespectra following the idea of Segal [18].

Let $\mathscr{W}_G$ denote the category with objects all nondegenerately based G-spaces having the G-homotopy type of a based G-CW complex and morphisms all basepoint preserving maps (not necessarily G-equivariant). Because every element g of G acts on the morphisms of $\mathscr{W}_G$ by conjugation, $\mathscr{W}_G$ can be regarded as a G-category. Denote by Γ_G the full subcategory of all based finite G-sets having the underlying set of the form $\mathbf{n} = \{0, 1, \ldots, n\}$ based at 0. Then every G-equivarint functor from Γ_G to $\mathscr{W}_G$ is called a Γ_G-*space*. (Notice that our $\Gamma = \Gamma_1$ is the opposite of the original Γ of Segal [17].)

As in [23], we associate to every Γ_G-space A and based G-space X a topological G-category $\mathrm{simp}(X, \Gamma_G, A)$ defined as follows:

$$\mathrm{ob}(\mathrm{simp}(X, \Gamma_G, A)) = \coprod_{S\in\Gamma_G} \mathrm{Map}_0(S, X) \times A(S)$$

$$\mathrm{mor}(\mathrm{simp}(X, \Gamma_G, A)) = \coprod_{S,T\in\Gamma_G} \mathrm{Map}_0(T, X) \times \mathrm{Map}_0(S, T) \times A(S).$$

Here each $(x, \xi, a) \in \mathrm{Map}_0(T, X) \times \mathrm{Map}_0(S, T) \times A(S)$ is regarded as a morphism from $(x\xi, a) \in \mathrm{Map}_0(S, X) \times A(S)$ to $(x, A(\xi)a) \in \mathrm{Map}_0(T, X) \times A(T)$; the composition is given by $(y, \eta, A(\xi)a) \circ (y\eta, \xi, a) = (y, \eta\xi, a)$; and every element g of G acts on $\mathrm{simp}(X, \Gamma_G, A)$ by $g(x, \xi, a) = (gxg^{-1}, g\xi g^{-1}, ga)$. Evidently the nerve of $\mathrm{simp}(X, \Gamma_G, A)$ coincides with the two-sided bar construction $B_*(X, \Gamma_G, A)$ in which X is regarded as a contravariant G-functor $S\mapsto \mathrm{Map}_0(S,X)$ from Γ_G to $\mathscr{W}_G$. We shall write $B(X,\Gamma_G,A)$ for the classifying space of simp (X,Γ_G,A). (Woolfson [23] writes $A'(X)=B(X,\Gamma,A)$ when G is the trivial group.)

Because $B_*(X,\Gamma_G,A)$ is a proper simplicial G-space, we can apply the arguments of [10, Appendix] and get

Proposition 1.1. (a) *$B(X,\Gamma_G,A)$ belongs to $\mathscr{W}_G$ if $X\in\mathscr{W}_G$.*

(b) *Let $f\colon X\to X'$ be a G-homotopy equivalence and let $F\colon A\to A'$ be a transformation of Γ_G-spaces such that $F_S\colon A(S)\to A'(S)$ is a G-homotopy equivalence for every object S of Γ_G. Then the induced map $B(f,\Gamma_G,F)\colon B(X,\Gamma_G,A)\to B(X',\Gamma_G,A')$ is a G-homotopy equivalence.*

Given a Γ_G-space A, we have a new Γ_G-space $\sigma A\colon S\mapsto B(S,\Gamma_G,A)$. Then there is a transformation of Γ_G-spaces $\sigma A\to A$ such that, for each $S\in\Gamma_G$, $\sigma A(S)\to A(S)$ is a G-homotopy equivalence induced by the equivalence of G-categories simp $(S,\Gamma_G,A)\to A(S)$ which takes each object (x,a) of simp(S,Γ_G,A) to $A(x)a\in A(S)$ and each arrow $(x,\xi,a)\colon (x\xi,a)\to(x,A(\xi)a)$ to the identity of $A(x\xi)a$. Following [17] let us denote by $X\otimes\sigma A$ the Γ_G-space

$$S\mapsto X\otimes\sigma A(S)=\coprod_{T\in\Gamma_G}\mathrm{Map}_0(T,X)\times\sigma A(S\wedge T)/(x\xi,a)\sim(x,\sigma A(1\wedge\xi)a).$$

Then there is a natural G-homeomorphism $B(X,\Gamma_G,A)\to X\otimes\sigma A(\mathbf{1})$ (cf. the proof of [23, Theorem 1.5]).

Proposition 1.2. (a) There are natural G-homotopy equivalences

$$B(X,\Gamma_G,B(\cdot\wedge Y,\Gamma_G,A))\xrightarrow{\ j\ } B(X\wedge Y,\Gamma_G,A)$$
$$\xleftarrow{\ k\ } B(Y,\Gamma_G,B(X\wedge\cdot,\Gamma_G,A))$$

where $B(\cdot\wedge Y,\Gamma_G,A)$ (resp. $B(X\wedge\cdot,\Gamma_G,A)$) denotes the Γ_G-space $S\mapsto B(S\wedge Y,\Gamma_G,A)$ (resp. $S\mapsto B(X\wedge S,\Gamma_G,A)$).

(b) *If X and $A(\mathbf{0})$ are G-connected (i.e., $\pi_0X^H=\pi_0A(\mathbf{0})^H=0$ for every subgroup H of G), so is $B(X,\Gamma_G,A)$.*

(c) *If X has the trivial G-action, then the natural map $i\colon B(X,\Gamma,A)\to B(X,\Gamma_G,A)$, induced by the evident inclusion $\Gamma\subset\Gamma_G$, is a G-homotopy equivalence; that is, $i^H\colon B(X,\Gamma,A^H)\to B(X,\Gamma_G,A)^H$ is a homotopy* *equivalence for every subgroup H of G.*

Proof. Because $B(\cdot\wedge Y, \Gamma_G, A) = Y\otimes\sigma A$, we can define j to be the canonical G-map $X\otimes\sigma(Y\otimes\sigma A)(\mathbf{1}) \to X\otimes(Y\otimes\sigma A)(\mathbf{1}) = (X\wedge Y)\otimes\sigma A(\mathbf{1})$ (cf. [17, Lemma 3.7]). To see that j is a G-homotopy equivalence, let us consider the diagram

$$\begin{array}{ccc} B(X, \Gamma_G, B(\cdot\wedge Y, \Gamma_G, A)) & \xrightarrow{j} & B(X\wedge Y, \Gamma_G, A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle d} \\ B(X, \Gamma_G, S\mapsto B(Y, \Gamma_G, A(S\wedge\cdot))) & = & B((X\wedge Y)\circ\wedge, \Gamma_G\times\Gamma_G, A\circ\wedge) \end{array}$$

in which $f = B(1, \Gamma_G, |F|)$ is induced by the map of Γ_G-spaces $|F|: |\mathrm{simp}(S\wedge Y, \Gamma_G, A)| \to |\mathrm{simp}(Y, \Gamma_G, A(S\wedge\cdot))|$ $(S\in\Gamma_G)$;

$$F(S\wedge Y \xleftarrow{(s,y)} T, a\in A(T)) = (Y \xleftarrow{y} T, A((s, 1))a\in A(S\wedge T))$$

and $d = |\Delta| : |\mathrm{simp}\,(X\wedge Y, \Gamma_G, A)| \to |\mathrm{simp}\,((X\wedge Y)\circ\wedge, \Gamma_G\times\Gamma_G, A\circ\wedge)|$ is given by

$$\Delta(X\wedge Y \xleftarrow{(x,y)} T, a\in A(T)) = (X\wedge Y \xleftarrow{x\wedge y} T\wedge T, A((1, 1))a\in A(T\wedge T)).$$

Then it is easy to see that f and d are G-homotopy equivalences, and that there is a G-homotopy $dj \simeq_G f$. Therefore j becomes a G homotopy equivalence. The second arrow k in (a) can be constructed similarly.

(b) follows from the fact that $\mathrm{Map}_0(S, X)$ is G-connected for all $S\in\Gamma_G$ provided X is G-connected.

We now prove (c). The G-map $i: B(X, \Gamma, A) \to B(X, \Gamma_G, A)$ is induced by the inclusion $\iota: \mathrm{simp}(X, \Gamma, A) \to \mathrm{simp}(X, \Gamma_G, A)$. Hence we have only to prove that $\iota^H: \mathrm{simp}(X, \Gamma, A)^H = \mathrm{simp}(X, \Gamma, A^H) \to \mathrm{simp}(X, \Gamma_G, A)^H$ is an equivalence of categories for every subgroup H of G. Because X has the trivial G-action, every H-map $x\in\mathrm{Map}_0(S, X)^H$ can be written as a composite

$$S \xrightarrow{q_s} H\backslash S \xrightarrow{x'} X$$

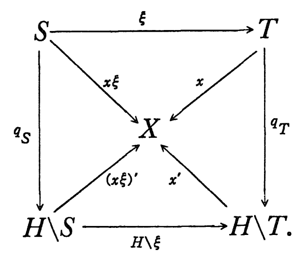
with x' in Γ. We now define a functor $\rho: \mathrm{simp}(X, \Gamma_G, A)^H \to \mathrm{simp}(X, \Gamma, A^H)$ by

$$\rho(x, a) = (x', A(q_S)a) \in \mathrm{Map}_0(H\backslash S, X)\times A(H\backslash S)^H$$

for each object $(x, a) \in \mathrm{Map}_0(S, X)^H \times A(S)^H$, and

$$\rho(x, \xi, a) = (x', H\backslash\xi, A(q_S)a)$$

for every arrow $(x, \xi, a) : (x\xi, a) \to (x, A(\xi)a)$ in $\mathrm{simp}(X, \Gamma_G, A)^H$. Note that there is a commutative diagram



Clearly $\rho\iota^H = \mathrm{Id}$ and there is a natural transformation $\mathrm{Id} \to \iota^H\rho$ given by $(x', q_S, a) : (x, a) \mapsto (x', A(q_S)a)$ for each $(x, a) \in \mathrm{simp}(X, \Gamma_G, A)^H$. This proves that ι^H is an equivalence of categories, and completes the proof of the proposition.

Definition 1.3. A Γ_G-space A is said to be *special* if

(a) $A(\mathbf{0})$ is G-contractible; and

(b) for every object S of Γ_G, the adjoint $P_S\colon A(S) \to \mathrm{Map}_0(S, A(\mathbf{1}))$ of the based G-map $S \wedge A(S) \to A(\mathbf{1})$, $(s, a) \mapsto A(p_s)a$ is a G-homotopy equivalence. Here $p_s\colon S \to \mathbf{1}$ denotes the based map such that $p_s(s) = 1$ and $p_s(S - \{s\}) = 0$.

Given a Γ_G-space A and a finite dimensional real G-module V, we put

$$\boldsymbol{S}_G A_V = B(S^V, \Gamma_G, A)/B(\infty, \Gamma_G, A) = S^V \otimes \sigma A(\mathbf{1})/\sigma A(\mathbf{0})$$

where S^V denotes the onepoint compactification of V based at ∞. Because $\sigma A(\mathbf{0})$ is G-contractible and the inclusion $B(\infty, \Gamma_G, A) \to B(S^V, \Gamma_G, A)$ is a G-cofibration, the projection $B(S^V, \Gamma_G, A) \to \boldsymbol{S}_G A_V$ is a G-homotopy equivalence. Furthermore it is easily checked that the inclusion $S^V \times (S^W \otimes \sigma A)(\mathbf{1}) \to S^V \times (S^W \otimes \sigma A)(\mathbf{1}) = (S^V \wedge S^W) \otimes \sigma A(\mathbf{1}) = S^{V \oplus W} \otimes \sigma A(\mathbf{1})$ (cf. Proposition 1.2 (a)) induces a based G-map

$$e_{V,W}\colon S^V \wedge \boldsymbol{S}_G A_W \to \boldsymbol{S}_G A_{V \oplus W}$$

such that the equality $e_{V,V' \oplus W}(1 \wedge e_{V',W}) = e_{V \oplus V',W}$ holds. Thus we have a G-prespectrum $\boldsymbol{S}_G A = \{\boldsymbol{S}_G A_V\}$ such that

$$S_G A_0 = \sigma A(\mathbf{1})/\sigma A(\mathbf{0}) \simeq_G A(\mathbf{1}).$$

Moreover by Proposition 1.2 (c), there are natural equivalences of prespectra

$$f_H\colon\ S(A^H) \to (S_G A)^H$$

where $S(A^H)$ denotes the prespectrum $\{B(S^V, \Gamma, A^H)/B(\infty, \Gamma, A^H)\}$ constructed from the special Γ-space $A^H\colon\ \mathbf{n} \mapsto A(\mathbf{n})^H$ by the method of Woolfson [23]. (Compare the remark at the end of this section.)

The following theorem is essentially due to Segal [18].

Theorem B. *Let A be a special Γ_G-space. Then $S_G A$ is an almost Ω-G-spectrum, that is, the maps $\varepsilon_{V,W}\colon S_G A_W \to \Omega^V S_G A_{V\oplus W}$ are G-homotopy equivalences whenever $W^G \neq 0$. Moreover $\varepsilon\colon S_G A_0 \to \Omega S_G A_{\mathbf{R}}$ is a G-homotopy equivalence if and only if $A(\mathbf{1})$ is grouplike.*

We now sketch a proof of this theorem and explain why the condition (b) of Definition 1.3 is required. (The situation was not clear in the original proof of [18, Theorem A].)

For simplicity of notation, we shall write

$$EA(X) = B(X, \Gamma_G, A)/B(*, \Gamma_G, A)$$

for every $X \in \mathscr{W}_G$; in particular $S_G A_V = EA(S^V)$. Because the inclusion $B(*, \Gamma_G, A) \to B(X, \Gamma_G, A)$ is a G-cofibration, $EA(X)$ has the same G-homotopy type as $B(X, \Gamma_G, A)$. Let us regard $EA\colon X \mapsto EA(X)$ as a G-equivariant functor from $\mathscr{F}\mathscr{W}_G$ to $\mathscr{W}_G$ where $\mathscr{F}\mathscr{W}_G$ denotes the G-stable full subcategory of $\mathscr{W}_G$ consisting of all compact G-ANR's. (Compare [14, Theorem 1].)

Lemma 1.4. *Let A be a special Γ_G-space. Then EA enjoys the following properties:*

P1. *For every $X \in \mathscr{F}\mathscr{W}_G$ and $S \in \Gamma_G$, the G-map $P_{S,X}\colon EA(S \wedge X) \to \mathrm{Map}_0(S, EA(X))$, induced by $S \wedge EA(S \wedge X) \to EA(X)$, $(s, x) \mapsto EA(p_S \wedge 1)x$, is a G-homotopy equivalence.*

P2. *If $Y \to X$ is a G-cofibration and $EA(Y)$ is grouplike under the G-equivariant multiplication $EA(Y) \times EA(Y) \simeq_G EA(Y \wedge \mathbf{2}) \to EA(Y)$, then $EA(Y) \to EA(X) \to EA(X/Y)$ is a G-fibration sequence.*

Notice that **P1** implies the speciality of the Γ_G-space $S \mapsto EA(S \wedge X)$ for every $X \in \mathscr{F}\mathscr{W}_G$.

Proof. By Proposition 1.2 (a) and the definition of EA, we have a commutative square

$$\begin{array}{ccc} EA(S\wedge X)\simeq_G E(T\mapsto \sigma A(S\wedge T))(X) & \to & E(T\mapsto A(S\wedge T))(X) \\ \downarrow{\scriptstyle P_{S,X}} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Map}_0(S, EA(X))\simeq_G E(T\mapsto \sigma A(T)^S)(X) & \to & E(T\mapsto A(T)^S)(X) \end{array}$$

in which the horizontal arrows are induced by the natural transformation $\sigma A\to A$ and π is induced by the G-homotopy equivalences $P_{S,T}\colon A(S\wedge T)\to A(T)^S=\mathrm{Map}_0(S, A(T))$, $T\in\Gamma_G$ (cf. Definition 1.3(b)). By Proposition 1.1 (b), all the arrows except for $P_{S,X}$ are G-homotopy equivalences. Hence $P_{S,X}$ becomes a G-homotopy equivalence. This shows that **P1** holds.

Next, by the arguments quite similar to [23, Theorem 1.7], we see that

$$B(Y,\Gamma_G,A)\to B(X,\Gamma_G,A)\to B(X\cup CY,\Gamma_G,A)$$

is a G-fibration sequence if $B(Y,\Gamma_G,A)$ is grouplike. This implies that **P2** holds. (Observe that in the proof of Theorem 1.7 of [23] the connectivity of Y is only used to ensure that $A'(Y)\to\Omega A'(SY)$ is a homotopy equivalence. Of course this follows from the weaker condition that $A'(Y)$ is grouplike. See also [17, p. 296].)

Now suppose we are given a based G-map

$$\mu\colon\ X\to\mathrm{Map}_0(Y,Z).$$

Then, by fuctoriality, we get a G-map

$$\mu'\colon\ X\to\mathrm{Map}_0(EA(Y),EA(Z)).$$

Because $EA(\text{point})=\text{point}$, μ' preserves basepoints; and so defines, by adjunction, a based G-map

$$D_\mu\colon\ EA(Y)\to\mathrm{Map}_0(X,EA(Z)).$$

For example, if S is a based finite G-set and μ is a based G-map $S\to\mathrm{Map}_0(S\wedge X, X)$, $s\mapsto(p_s\wedge 1\colon S\wedge X\to\mathbb{1}\wedge X=X)$, then D_μ coincides with $P_{S,X}\colon EA(S\wedge X)\to\mathrm{Map}_0(S,EA(X))$; and if $\mu\colon S^V\to\mathrm{Map}_0(S^W, S^{V\oplus W})$ is the adjoint of the identity map $S^V\wedge S^W\to S^{V\oplus W}$, then $D_\mu=\varepsilon_{V,W}\colon \boldsymbol{S}_GA_W\to\Omega^V\boldsymbol{S}_GA_{V\oplus W}$.

Let M be a compact G-stable subset of a real G-module V, and let M_ε be the ε-neighborhood of M in V. Then there is a G-map

$M \to \mathrm{Map}(O_\varepsilon, M_\varepsilon)$ which takes each element m of M to the map $x \mapsto m+x$ from the ε-neighborhood of the origin to M_ε. By the Pontryagin-Thom construction we get a based G-map

$$\mu\colon M_+ \to \mathrm{Map}_0(M_\varepsilon^c, O_\varepsilon^c) \cong \mathrm{Map}_0(M_\varepsilon^c, S^V).$$

Here, for every open subset X of V, X^c denotes the onepoint compactification of X based at ∞; i. e., $X^c = V/V-X$. Consequently we get a based G-map

$$D_M = D_\mu\colon EA(M_\varepsilon^c) \to \mathrm{Map}_0(M_+, EA(S^V)).$$

Lemma 1.5. ([18, Proposition (2.2)]). *Let M be the unit sphere of V. Suppose $\Gamma^G \neq 0$ or $EA(X)$ is G-connected for every $X \in \mathscr{F}\mathscr{W}_G$ (e.g., $A = Z \otimes \sigma A'$ for some special Γ_G-space A' and a G-connected space $Z \in \mathscr{F}\mathscr{W}_G$). Then D_M is a G-homotopy equivalence.*

Proof. Choose an equivariant triangulation of M (cf. [6]), and let $\{C_\lambda\}_{\lambda \in \Lambda}$ be the covering by the open stars of open simplexes. We identify the indexing set Λ with the G-set of the barycenters of open simplexes. Moreover, by taking a refinement of the triangulation if necessary, we may assume that each C_λ either coincides with or is disjoint from its translate by elements of G.

Suppose ε is small compared with the minimum of the radii of the simplexes of positive dimension. Let $\pi\colon M_\varepsilon \to M$ be the radial projection, and let $X = \cup_{\lambda \in T} C_\lambda$ be a G-stable union of some of the C_λ. Let us denote $\hat{X} = \pi^{-1}(X)$ and $\check{X} = X - (M-X)_\varepsilon$. Then the G-map $M \to \mathrm{Map}(O_\varepsilon, M_\varepsilon)$ restricts to $\check{X} \to \mathrm{Map}(O_\varepsilon, \hat{X})$, so that we have a based G-map

$$D_X\colon EA(\hat{X}^c) \to \mathrm{Map}_0(\check{X}_+, EA(S^V)).$$

We will show, by induction on the cardinal of the orbits contained in T, that this is a G-homotopy equivalence.

If T, is a single orbit, then the closed embedding $T = \cup_{\lambda \in T}$ (barycenter of λ) $\to \cup_{\lambda \in T} C_\lambda$ induces G-homotopy equivalences $T_+ \simeq_G \check{X}_+$ and $T_+ \wedge S^V \simeq_G \hat{X}^c$. Therefore D_X is identified with P_{T_+, S^V}: $EA(T_+ \wedge S^V) \to \mathrm{Map}_0(T_+, EA(S^V))$ which is a G-homotopy equivalence by **P1**.

Next let X_1 and X_2 be two G-stable unions of C_λ's, and let $X =$

$X_1 \cup X_2$, $X_{12}=X_1 \cap X_2$. Then we have a diagram

$$\begin{array}{ccccc} EA((\hat{X}-\hat{X}_1)^c) & \longrightarrow & EA(\hat{X}^c) & \longrightarrow & EA(\hat{X}_1^c) \\ \| & & \downarrow & & \downarrow \\ EA((\hat{X}_2-\hat{X}_{12})^c) & \longrightarrow & EA(\hat{X}^c) & \longrightarrow & EA(\hat{X}_{12}^c) \end{array}$$

induced by the cofibration sequences $(\hat{X}-\hat{X}_1)^c \to \hat{X}^c \to \hat{X}_1^c = \hat{X}^c/(\hat{X}-\hat{X}_1)^c$ and $(\hat{X}_2-\hat{X}_{12})^c \to \hat{X}_2^c \to \hat{X}_{12}^c$. (Notice that $\hat{X}-\hat{X}_1 \to \hat{X}$ and $\hat{X}_2-\hat{X}_{12} \to \hat{X}_2$ are closed G-embeddings.) Because $EA((\hat{X}-\hat{X}_1)^c)$ is G-connected by the assumption, the horizontal sequences in the above diagram are G-fibration sequences by **P2**. Therefore the square

$$\begin{array}{ccc} EA(\hat{X}^c) & \longrightarrow & EA(\hat{X}_1^c) \\ \downarrow & & \downarrow \\ EA(\hat{X}_2^c) & \longrightarrow & EA(\hat{X}_{12}^c) \end{array}$$

is G-homotopy cartesian. Moreover the corresponding square

$$\begin{array}{ccc} \mathrm{Map}_0(\check{X}_+, EA(S^V)) & \longrightarrow & \mathrm{Map}_0(\check{X}_{1+}, EA(S^V)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_0(\check{X}_{2+}, EA(S^V)) & \longrightarrow & \mathrm{Map}_0(\check{X}_{12+}, EA(S^V)) \end{array}$$

is also G-homotopy cartesian. Hence we can prove inductively that D_X, and consequently D_M, too, is a G-homotopy equivalence.

Proof of Theorem B. We will show that $EA(S^0) \to \Omega^V EA(S^V)$ is a G-homotopy equivalence if $A(\mathbb{1})$ is grouplike and $V^G \neq 0$, or if $EA(X)$ is G-connected for every $X \in \mathscr{F}\mathscr{W}_G$. When $W^G \neq 0$, $EA(\cdot \wedge S^W) = E(S^W \otimes \sigma A)(\cdot)$ satisfies the latter condition; and so $\varepsilon_{V,W}$: $EA(S^W) \to \Omega^V EA(S^{V\oplus W})$ is a G-homotopy equivalence for any V.

Let B_r denote the closed disk of radius r in V and S_r its boundary sphere. Because $EA(S^0) \simeq_G A(\mathbb{1})$ is grouplike, the horizontal sequences in the diagram

$$\begin{array}{ccccc} EA(B_{1-\varepsilon} \cup S_{1+\varepsilon}/S_{1+\varepsilon}) & \to & EA(B_{1+\varepsilon}/S_{1+\varepsilon}) & \to & EA(B_{1+\varepsilon}/B_{1-\varepsilon} \cup S_{1+\varepsilon}) \\ \downarrow & & \downarrow D_{B_1} & & \downarrow D_{S_1} \\ \mathrm{Map}_0(B_1/S_1, EA(S^V)) & \to & \mathrm{Map}_0(B_{1+}, EA(S^V)) & \to & \mathrm{Map}_0(S_{1+}, EA(S^V)) \end{array}$$

are G-fibration sequences. By Lemma 1.5, D_{S_1} is a G-homotopy equivalence and D_{B_1} is trivially a G-homotopy equivalence. Therefore the induced map $EA(S^0) \simeq_G EA(B_{1-\varepsilon} \cup S_{1+\varepsilon}/S_{1+\varepsilon}) \to \Omega^V EA(S^V)$ is a G-homotopy equivalence. This completes the proof of Theorem B.

Remark. (Cf. [9, Chapters I and II].) Let $G\mathscr{P}\mathscr{A}$ (resp. $G\mathscr{S}\mathscr{A}$) denote the category of G-prespectra (resp. G-spectra) indexed on a indexing set $\mathscr{A}$ in some G-universe $\boldsymbol{U}$. Then our $\boldsymbol{S}_G A$ canonically defines a G-prespectrum $\boldsymbol{S}_G A_{\mathscr{A}} = \{EA(S^V) \mid V \in \mathscr{A}\} \in G\mathscr{S}\mathscr{A}$ with the structure maps $S^{W-V} EA(S^V) \to EA(S^{(W-V)\oplus V}) \cong EA(S^W)$, and also the associated G-spectrum $L\boldsymbol{S}_G A \in G\mathscr{S}\mathscr{A}$. By [9, Chapter II] any G-linear isometry $f\colon \boldsymbol{U} \to \boldsymbol{U}'$ between G-universes induces an equivalence $f^*\colon G\mathscr{S}\boldsymbol{U}' \to G\mathscr{S}\boldsymbol{U}$ and hence $G\mathscr{S}\mathscr{A} \cong G\mathscr{S}\boldsymbol{U}$ is equivalent to $G\mathscr{S}\mathscr{A}'$ for another indexing set $\mathscr{A}'$ in $\boldsymbol{U}'$. In particular we see that the prespectrum $(\boldsymbol{S}_G A)^H_{\mathscr{A}} = \boldsymbol{S}(A^H)_{\mathscr{A}}$ indexed on any $\mathscr{A}$ in a H-trivial universe $\boldsymbol{U}^H$ becomes equivalent, upon passage to stable category, to the usual prespectrum $\{E(A^H)(S^n)\}$ indexed on the standard n-spaces $\boldsymbol{R}^n \subset \boldsymbol{R}^\infty$.

§2. Proof of Theorem A

We now prove Theorem A. Thanks to Theorem B, it suffices to construct a functor which assigns to every (D, D') a special Γ_G-space such that the associated H-fixed point Γ-space coincides with the Γ-space arising from $B(D, D')^H$.

First recall the passage from symmetric monoidal categories to special Γ-categories (cf. [11], [19]). Given a monoidal category C, we have a Γ-category $C^\wedge$ such that, for each $\mathbf{n} \in \Gamma$, the objects of $C^\wedge(\mathbf{n})$ are of the form $\langle a_U; \alpha_{U,V}\rangle$ in which a_U is an object of C for every based subset U of $\mathbf{n}$, and $\alpha_{U,V}$ is an isomorphism $a_{U\vee V} \to a_U \oplus a_V$ for every pair of subsets U, $V \subset \mathbf{n}$ with $U \cap V = \{0\}$. Here $a_{\{0\}} = 0 \in C$ and the evident coherence conditions between $\alpha_{U,V}$'s (i.e., associativity, commutativity, and unit axioms) must be satisfied. When C is a symmetric monoidal G-category, the above construction of $C^\wedge$ can be extended to give a Γ_G*-category*, i.e., a G-equivariant functor from Γ_G to the category $\mathbf{Cat}_G$ of based G-categories and basepoint preserving functors: For every finite G-set S with underlying set $\mathbf{n}$, $C^\wedge(S)$ is defined to be the category $C^\wedge(\mathbf{n})$ equipped with a G-action

$$g\langle a_U;\ \alpha_{U,V}\rangle = \langle g a_{g^{-1}U};\ g\alpha_{g^{-1}U,g^{-1}V}\rangle.$$

Then, for every $f\colon S \to T$ in Γ_G, we have $A(gfg^{-1}) = gA(f)g^{-1}$ where $A(f)$ denotes the functor $C^\wedge(S) \to C^\wedge(T)$, $\langle a_U, \alpha_{U,V}\rangle \mapsto \langle a_{f^*U};\ \alpha_{f^*U,f^*V}\rangle$ ($f^*U = \{0\} \cup f^{-1}(U - \{0\})$) induced by f.

Note that, if C is the realization of a simplicial monoidal G-category, then $|C^\wedge(S)| \in \mathscr{W}_G$ because $C^\wedge(S)$ is obtained as the realization of the simplicial G-category $[k] \mapsto C_k^\wedge(S)$. Thus we have a Γ_G-space $S \mapsto |C^\wedge(S)|$ such that the associated Γ-spaces $\mathbf{n} \mapsto |C^\wedge(\mathbf{n})|^H = |C^\wedge(\mathbf{n})^H|$ coincide with the Γ-spaces $|(C^H)^\wedge|$ arising from the (simplicial) monoidal categories C^H. However we do not know, in general, whether this $|C^\wedge|$ is special or not.

Definition 2.1. A Γ_G-category $F\colon \Gamma_G \to \mathbf{Cat}_G$ is said to be *special* if F is obtained as the realization of a simplicial Γ_G-category, and satisfies the following conditions

(a) $F(\mathbf{0}) = \text{point}$; and

(b) for every $S \in \Gamma_G$, the G-functor $P_S\colon F(S) \to F(\mathbf{1})^S = \mathbf{Cat}_G(S, F(\mathbf{1}))$ induced by $S \wedge F(S) \mapsto F(\mathbf{1})$, $(s, x) \mapsto F(p_s)x$ is an equivalence of G-categories. (Compare Definition 1.3.)

If F is a special Γ_G-category, then $|F|\colon S \mapsto |F(S)|$ is a special Γ_G-space; and so we have an almost Ω-G-spectrum $\boldsymbol{S}_G|F|$.

Proposition 2.2. *Let (D, D') be a pair of simplicial monoidal G-graded categories. Then $B(D, D')^\wedge$ is a special Γ_G-category.*

Of course Theorem A follows from this proposition: We define $\boldsymbol{K}_G(D, D') = \boldsymbol{S}_G|B(D, D')^\wedge|$. Then there are a natural G-homotopy equivalence

$$\boldsymbol{K}_G(D, D')_0 \to |B(D, D')^\wedge(\mathbf{1})| = |B(D, D')|$$

and natural equivalences of prespectra

$$\boldsymbol{K}(B(D, D')^H) = \boldsymbol{S}|(B(D, D')^H)^\wedge| \to (\boldsymbol{S}_G|B(D, D')^\wedge|)^H = \boldsymbol{K}_G(D, D')^H$$

for all subgroups H of G.

Proof of Proposition 2.2. For simplicity, write $C = B(D, D')$ and

$\oplus=\oplus_D$. We define an adjoint $T_S\colon C^S=C^\wedge(\mathbf{1})^S\to C^\wedge(S)$ of P_S as follows.

Let $a=(a_s)$ be an object of C^S. Each a_s is a functor $EG\to D'$ over G and particularly $a_0\colon EG\to G\to D'$ has value $0\in D'$. For each $x\in G$ and every based ordered subset $U=\{0, u_1, \dots, u_r\}\subset S$, $0<u_1<\cdots<u_r$, we write

$$a'_U(x)=0\oplus a_{x^{-1}v_1}(x)\oplus\cdots\oplus a_{x^{-1}v_{r-1}}(x)\oplus a_{x^{-1}v_r}(x)$$
$$\overset{\text{def.}}{=}0\oplus(a_{x^{-1}v_1}(x)\oplus(\cdots\oplus(a_{x^{-1}v_{r-1}}(x)\oplus a_{x^{-1}v_r}(x))\cdots))$$

where $\{0, v_1, \dots, v_r\}=xU\subset S$, $0<v_1<\cdots<v_r$. Since D is a monoidal G-graded category, there is an isomorphism (of grade 1)

$$\rho_U=\rho_U(x)\colon\ a'_U(x)\to 0\oplus a_{u_1}(x)\oplus\cdots\oplus a_{u_r}(x).$$

uniquely determined by the permutation of $U-\{0\}$, $u_j\mapsto x^{-1}v_j(1\leqq j\leqq r)$. Then, for every U, $V\subset\mathbf{n}$ with $U\cap V=\{0\}$, we have an isomorphism

$$\alpha'_{U,V}(x)\colon\ a'_{U\vee V}(x)\to a'_U(x)\oplus a'_V(x)$$

such that $(\rho_U\oplus\rho_V)\alpha'_{U,V}(x)\rho_{U\vee V}^{-1}$ coincides with the uniquely determined isomorphism

$$0\oplus a_{w_1}(x)\oplus\cdots\oplus a_{w_{r+s}}(x)\to$$
$$(0\oplus a_{u_1}(x)\oplus\cdots\oplus a_{u_r}(x))\oplus(0\oplus a_{v_1}(x)\oplus\cdots\oplus a_{v_s}(x))$$

where $U=\{0, u_1, \dots, u_r\}$, $V=\{0, v_1, \dots, v_s\}$ and $U\vee V=\{0, w_1, \dots, w_{r+s}\}$ $(0<u_1<\cdots<u_r,\ 0<v_1<\cdots<v_s,\ 0<w_1<\cdots<w_{r+s})$.

Similarly for every arrow $f\colon x\to y$ in EG, $a'_U(f)\colon a'_U(x)\to a'_U(y)$ of the same grade as f, is uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} a'_U(x) & \xrightarrow{\rho_U} & 0\oplus a_{u_1}(x)\oplus\cdots\oplus a_{u_r}(x) \\ {\scriptstyle a'_U(f)}\downarrow & & \downarrow{\scriptstyle 0\oplus a_{u_1}(f)\oplus\cdots\oplus a_{u_r}(f)} \\ a'_U(y) & \xrightarrow{\rho_U} & 0\oplus a_{u_1}(y)\oplus\cdots\oplus a_{u_r}(y). \end{array}$$

It is a routine exercise to show that $a'_U\colon x\mapsto a'_U(x)$ is an object of C and $\alpha'_{U,V}(x)\colon a'_{U\vee V}(x)\to a'_U(x)\oplus a'_V(x)$ is natural in x. Thus we have an object

$$T_S a=\langle a'_U;\ \alpha'_{U,V}\rangle\in C^\wedge(S).$$

Clearly the construction of $T_S a$ is natural in a, and we get a fuctor T_S: $C^S \to C^\wedge(S)$.

We will show that T_S is G-equivariant. Let g be an element of G. Then

$$g(T_S a) = \langle g a'_{g^{-1}U};\ g\alpha'_{g^{-1}U, g^{-1}V} \rangle,$$

and we have

$$\begin{aligned} g a'_{g^{-1}U}(x) &= a'_{g^{-1}U}(xg) \\ &= 0 \oplus a_{g^{-1}x^{-1}v_1}(xg) \oplus \cdots \oplus a_{g^{-1}x^{-1}v_r}(xg) \\ &= 0 \oplus (ga)_{x^{-1}v_1}(x) \oplus \cdots \oplus (ga)_{x^{-1}v_r}(x) \\ &= (ga)'_U(x) \end{aligned}$$

because $(xg)(g^{-1}U) = xU = \{0, v_1, \ldots, v_r\}$. Moreover it is easily checked that $g\alpha'_{g^{-1}U, g^{-1}V}$ coincides with $(ga)'_{U \vee V} \to (ga)'_U \oplus (ga)'_V$. Therefore the functor T_S is G-equivariant.

Evidently $P_S T_S$ is the identity of C^S. On the other hand, the natural transformation $\langle t_U \rangle : \langle a_U;\ \alpha_{U,V} \rangle \to T_S P_S \langle a_U;\ \alpha_{U,V} \rangle$ given by the composite isomorphisms

$$\begin{aligned} t_U(x):\ a_U(x) &\xrightarrow{\alpha} 0 \oplus a_{u_1}(x) \oplus \cdots \oplus a_{u_r}(x) \\ &\xrightarrow{\rho_U^{-1}} 0 \oplus a_{x^{-1}v_1}(x) \oplus \cdots \oplus a_{x^{-1}v_r}(x), \end{aligned}$$

where a_u denotes $a_{\{0,u\}}$ for every $u \in U$, is compatible with the G-action on $C^\wedge(S)$. It follows that P_S is an equivalence of G-categories.

We now state, in view of future applications, an immediate consequence of Theorem A.

Recall that the Grothendieck construction (cf. [20]) converts a (simplicial) monoidal G-category C into a (simplicial) monoidal G-graded category $G\int C$ which has

(a) the same objects as C;

(b) the pairs $\langle g, f \rangle$ with f: $ga \to b$ in C as morphisms $a \to b$ of grade g; and

(c) the unique multiplication $\oplus_{G\int C}$: $G\int C \times_G G\int C \to G\int C$ such that, for every pair of morphisms $\langle g, f \rangle$: $a \to b$ and $\langle g, f' \rangle$: $a' \to b'$ of the same grade g, the following holds:

$$\langle g, f \rangle \oplus_{G\int C} \langle g, f' \rangle = \langle g, f \oplus_C f' \rangle:\ a \oplus_{G\int C} a' \to b \oplus_{G\int C} b'.$$

Observe that $G\int C$ becomes the product G-graded category $G\times C$ if C has the trivial G-action.

Given a pair of monoidal G-categories (C, C'), let us denote by $B_G(C, C')$ the full subcategory of the functor category $\mathbf{Cat}(EG, C)$ consisting of the objects $EG\to C$ which factors through C'. Then $B_G(C, C')$ equipped with the G-action

$$(g, f)\mapsto (gf\colon x\mapsto gf(xg)),\qquad g\in G$$

and the G-equivariant multiplication

$$(f, f')\mapsto (f\oplus f'\colon x\mapsto f(x)\oplus f'(x))$$

is naturally isomorphic to $B(G\int C, G\int C')$ under the monoidal G-functor Φ: $\mathbf{Cat}(EG, C)\to \mathrm{Hom}_G(EG, G\int C)$ which takes each $f\colon EG\to C$ to Φf: $EG\to G\int C$;

$$\begin{aligned}&\Phi f(x)=xf(x)\qquad (x\in \mathrm{ob}EG=G),\\&\Phi f(x\to gx)=\langle g, gxf(x\to gx)\rangle\colon\ \Phi f(x)\to \Phi f(gx).\end{aligned}$$

Hence we have

Theorem A′. *$\boldsymbol{K}_G$ restricts, via the Grothendieck construction, to a functor from the pairs of simplicial monoidal G-categories to almost Ω-G-spectra equipped with*

(a) *a natural G-homotopy equivalence $\boldsymbol{K}_G(C, C')_0\to |B_G(C, C')|$, and*

(b) *natural equivalences of prespectra $\boldsymbol{K}(B_G(C, C')^H)\to \boldsymbol{K}_G(C, C')^H$ for all subgroups H of G.*

$|B_G(C, C')|$ has the same G-homotopy type as the infinite loop G-space $K_G(C, C')=\Omega\boldsymbol{K}_G(C, C')_{\boldsymbol{R}}$ if and only if $|B_G(C, C')|$ is grouplike.

Remark. Our approach to Theorem A was based on Γ_G-spaces. There is another approach based on E_∞ G-operads [9].

Let $\tilde{\mathscr{D}}_j=\mathrm{Hom}_G(EG, G\times E\Sigma_j)=\mathbf{Cat}(EG, E\Sigma_j)$ and let $\mathscr{D}_j$ be the G-space $|\tilde{\mathscr{D}}_j|$. By Theorem 3. 1 (see also [15])

$$\mathscr{D}_j\to \mathscr{D}_j/\Sigma_j=|\mathbf{Cat}(EG, \Sigma_j)|$$

is a universal G-Σ_j bundle, and there are G-maps

$$\gamma\colon\ \mathscr{D}_k\times\mathscr{D}_{j_1}\times\cdots\times\mathscr{D}_{j_k}\to\mathscr{D}_j,\qquad j=j_1+\cdots+j_k$$

induced by the functors $\tilde{\gamma}$: $E\Sigma_k \times E\Sigma_{j_1} \times \cdots \times E\Sigma_{j_k} \to E\Sigma_j$;

$$\tilde{\gamma}(\sigma; \tau_1, \ldots, \tau_k) = \tau_{\sigma^{-1}(1)} \oplus \cdots \oplus \tau_{\sigma^{-1}(k)}.$$

(Compare [10, Lemma 4.4].) Thus we have an E_∞ G-operad $\mathscr{D}$.

If D is a monoidal G-graded category (with strictly associative multiplication $\oplus = \oplus_D$), there is a Σ_j-equivariant functor over G

$$(G \times E\Sigma_j) \times_G D^{[j]} \to D \qquad (D^{[j]} = D \times_G \cdots \times_G D)$$

which takes each object $(\tau; x_1, \ldots, x_j)$ of $(G \times E\Sigma_j) \times_G D^{[j]}$ to $x_{\tau^{-1}(1)} \oplus \cdots \oplus x_{\tau^{-1}(j)} \in D$ (cf. [10, Lemma 4.3]). This induces a Σ_j-equivariant G-functor $\tilde{\mathscr{D}}_j \times B(D, D')^j \to B(D, D')$ for every pair of monoidal G-graded categories (D, D'), and hence we have a natural action

$$\mathscr{D}_j \times |B(D, D')|^j \to |B(D, D')|$$

of the E_∞ G-operad $\mathscr{D}$ on $|B(D, D')|$.

§3. Equivariant Classifying Spaces

We now apply our theorems to deloop the maps $BO(G) \to BPL(G) \to BTop(G)$ equivariantly and infinitely.

To begin with, we shall describe a functorial construction of the classifying space for equivariant bundles. Let A be a topological group, and let $\eta = \eta_A$: $UA \to BA$ be a universal principal A-bundle. We assume here that $(A, 1)$ is a strong NDR (e.g., the realization of a simplicial group) and take $|EA| \to |A|$ as our universal bundle unless otherwise stated. Then there is a new bundle $\langle \eta, \eta \rangle : \langle UA, UA \rangle \to BA \times BA$ whose fibre $\langle \eta, \eta \rangle^{-1}(x, y)$ over $(x, y) \in BA \times BA$ consists of all admissible maps $\eta^{-1}(x) \to \eta^{-1}(y)$; so that $\langle \eta, \eta \rangle^{-1}(x, y) \cong A$. (Compare [2] as well as [13].) It is easy to see that the maps

$$s = pr_1 \circ \langle \eta, \eta \rangle, \ t = pr_2 \circ \langle \eta, \eta \rangle \ : \ \langle UA, UA \rangle \to BA$$

and

$$i \ : \ BA \to \langle UA, UA \rangle, \ i(x) = \mathrm{id}_x \in \langle \eta, \eta \rangle^{-1}(x, x)$$

together with the evident composition

$$\circ \ : \ \langle UA, UA \rangle \times_{BA} \langle UA, UA \rangle \to \langle UA, UA \rangle$$

define a topological category (with trivial G-action) $\mathscr{G}A$ such that $\mathrm{ob}\,\mathscr{G}A = BA$ and $\mathrm{mor}\,\mathscr{G}A = \langle UA, UA \rangle$.

Theorem 3.1. *Let A be the realization of a simplicial group and A' a subgroup of A. Then, for any compact Lie group G, $|B_G(\mathscr{G}A, \mathscr{G}A')|$ is a classifying space for G-(A, A') bundles in the sense of* [8]. *If G is a finite group, $|B_G(A, A')| = |B_G(EA/A, EA'/A')|$ is also a classifying space for G-(A, A') bundles.*

(For a generalization of this theorem, see [15].)

Proof. There is a category $\mathscr{S}A$ with

$$\mathrm{ob}\mathscr{S}A = UA, \quad \mathrm{mor}\mathscr{S}A = \langle UA, UA\rangle \times_{BA} UA = \{(\psi, a) \mid s(\psi) = \eta(a)\},$$

and with structure maps $s(\psi, a) = a$, $t(\psi, a) = \psi(a)$, $i(a) = (\mathrm{id}_{\eta(a)}, a)$, $(\phi, \psi(a)) \circ (\psi, a) = (\phi\psi, a)$. Let π: $\mathbf{Cat}(EG, \mathscr{S}A) \to \mathbf{Cat}(EG, \mathscr{G}A)$ denote the G-functor induced by the projection $\mathscr{S}A \to \mathscr{G}A = \mathscr{S}A/A$, $(\psi, a) \mapsto \psi$. We will show that $|\pi^{-1}B_G(\mathscr{G}A, \mathscr{G}A')| \to |B_G(\mathscr{G}A, \mathscr{G}A')|$ is a universal G-(A, A') bundle. Observe that $\pi^{-1}B_G(\mathscr{G}A, \mathscr{G}A')$ coincides with $B_G(\mathscr{S}A, \mathscr{S}A' \times_{A'} A)$. For simplicity of notation, write $E = \pi^{-1}B_G(\mathscr{G}A, \mathscr{G}A')$ and $B = B_G(\mathscr{G}A, \mathscr{G}A')$. Then, for every element $f = (f_n \leftarrow \cdots \leftarrow f_1 \leftarrow f_0)$ of N_nB, we have a representation $\alpha(f): H \to A'$ ($H = G_f$) defined by

$$\alpha(f)(h) = (f_0(1 \to h) : f_0(1) \to f_0(h) = f_0(1)) \\ \in \langle \eta_{A'}, \eta_{A'}\rangle^{-1}(f_0(1), f_0(1)) = A',$$

and $\pi^{-1}Gf \to Gf = G/H$ is G-A equivalent to the trivial $G-A$ bundle $G \times_H A_{\alpha(f)} \to G/H$. Clearly this extends to a local trivialization of $N_nE \to N_nB$, and in fact we can prove that $|E| \to |B|$ is a numerable G-(A, A') bundle. (For details see [15].)

We now prove that $|E| \to |B|$ satisfies the condition (1) and (2) of [8, Theorem 6] for every representation ρ: $H \to A'$. Let us consider $|E|$ as an H-space under the action $a \mapsto ha\rho(h)^{-1}$. Since $UA \to BA$ is a universal A-bundle, there exists a bundle map $(\tilde{f}, f)$: $(G \times_H A_\rho, G/H) \to (UA, BA)$ and the G-action on $G \times_H A_\rho$ determines a functor F: $EG \to \mathscr{G}A$; $F(x) = f(xH)$, $F(x \to gx) = (g$: $\eta^{-1}(f(xH)) \to \eta^{-1}(f(gxH))) \in \langle UA, UA\rangle$. Evidently F belongs to $B_G(\mathscr{G}A, \mathscr{G}A')$ and the lift $\tilde{F}$: $EG \to \mathscr{S}A$ of F given by $\tilde{F}(x) = \tilde{f}[x, 1]$ and $\tilde{F}(x \to gx) = (F(x \to gx), \tilde{f}[x, 1]) \in \langle UA, UA\rangle \times_{BA} UA$ is invariant under the H-action on E. Hence $|E|^H \neq \emptyset$. Moreover, since $\mathscr{S}A$ has a unique morphism between each pair of its objects, $|E|$ is H-contractible to any vertex of $|E|^H$. Hence, by [8, Theorem 6], $|E| \to |B|$ becomes a universal G-(A, A') bundle.

When G is finite, every trivial G-A bundle $G \times_H A_\rho \to G/H$ is in fact a trivial A-bundle, and so classified by the constant map $G/H \to *$. It follows that $|\pi^{-1}B_G(A, A')| \to |B_G(A, A')|$ is also a universal G-(A, A') bundle. Here π denotes the G-functor $\mathbf{Cat}(EG, EA) \to \mathbf{Cat}(EG, A)$ induced by the projection $EA \to EA/A = A$ and $\pi^{-1}B_G(A, A')$ is identical with $B_G(EA, EA' \times_{A'} A)$.

In particular, take the simplicial group $CAT_n = O_n$ or PL_n or Top_n as A, and the discrete group GL_n as A'. (Compare [7]. Note that GL_n is the 0-skeleton of O_n, and in fact $GL_n = O_n \cap PL_n$ in Top_n; cf. [1, p. 216].) Then we have a classifying space

$$BCAT_n(G) = |B_G(\mathscr{G}CAT_n, \mathscr{G}GL_n)|$$

for locally linear G-CAT bundles with fibre $\mathbb{R}^n$. However, from the viewpoint of smoothing theory, there is a need to construct a G-fibration $BO_n(G) \to BPL_n(G)$, and $|B_G(\mathscr{G}O_n, \mathscr{G}GL_n)|$ is not adequate for this purpose. Therefore we replace $|B_G(\mathscr{G}O_n, \mathscr{G}GL_n)|$ by an equivalent G-space defined as follows: (Compare [7, §3].)

Let PD_n be the simplicial set whose k-simplexes are fibre preserving $p. d.$ homeomorphisms $\Delta_k \times (\mathbb{R}^n, 0) \to \Delta_k \times (\mathbb{R}^n, 0)$. Then PD_n admits a left free PL_n-action $(h, f) \mapsto fh^{-1}$, $(h, f) \in PL_n \times PD_n$, and a right free O_n-action $(f, k) \mapsto k^{-1}f$, $(f, k) \in PD_n \times O_n$. Now consider the G-map $UPL_n(G) \times_{PL_n} PD_n \to UPL_n(G) \times_{PL_n} PD_n/O_n$ induced by the projection $PD_n \to PD_n/O_n$ where $UPL_n(G) = |B_G(\mathscr{S}PL_n, \mathscr{S}GL_n \times_{GL_n} PL_n)|$ is the total space of the universal G-(PL_n, GL_n) bundle we have constructed in the proof of Theorem 1. Because the inclusion $PL_n \to PD_n$ is a homotopy equivalence, $UPL_n(G) \times_{PL_n} PD_n \simeq_G UPL_n(G)$ becomes a total space of a universal G-(O_n, GL_n) bundle over $UPL_n(G) \times_{PL_n} PD_n/O_n$. From now on we write $BO_n(G) = UPL_n(G) \times_{PL_n} PD_n/O_n = |B_G(\mathscr{G}'O_n, \mathscr{G}'GL_n)|$ where

$$\mathscr{G}'O_n = \mathscr{S}PL_n \times_{PL_n} PD_n/O_n \text{ and } \mathscr{G}'GL_n = \mathscr{S}GL_n \times_{GL_n} PD_n/O_n.$$

Then there are G-fibrations $BO_n(G) \to BPL_n(G)$ induced by the projection $(\mathscr{S}PL_n \times_{PL_n} PD_n/O_n, \mathscr{S}GL_n \times_{GL_n} PD_n/O_n) \to (\mathscr{G}PL_n, \mathscr{G}GL_n)$ and

$BPL_n(G) \to BTop_n(G)$ induced by the evident inclusion $(\mathscr{G}PL_n, \mathscr{G}GL_n) \to (\mathscr{G}Top_n, \mathscr{G}GL_n)$.

Remark. Since G is a finite group, we can take much smaller G-space $|B_G(CAT_n, GL_n)|$ (or $|B_G(EPL_n \times_{PL_n} PD_n/O_n, EGL_n \times_{GL_n} PD_n/O_n)|$ when $CAT_n = O_n$) as our $BCAT_n(G)$ (cf. Theorem 3. 1). All the arguments below are valid for this choice of $BCAT_n(G)$ with $\mathscr{S}CAT_n$ and $\mathscr{G}CAT_n$ replaced by $ECAT_n$ and CAT_n respectively.

Let us denote by (CAT, GL) the pair of simplicial categories

$$(\coprod_{n\geqq 0} \mathscr{G}'O_n, \coprod_{n\geqq 0} \mathscr{G}'GL_n) \quad \text{if } CAT=O,$$
$$(\coprod_{n\geqq 0} \mathscr{G}CAT_n, \coprod_{n\geqq 0} \mathscr{G}GL_n) \quad \text{if } CAT=PL \text{ or } Top.$$

We make (CAT, GL) into a pair of symmetric monoidal categories by defining the multiplication $CAT \times CAT \to CAT$ as follows:

For every pair of integers m and n, there is a simplicial map $\oplus$: $Top_m \times Top_n \to Top_{m+n}$ which assigns to every $(x, y) \in Top_m \times Top_n$ the Whitney sum $x \oplus y \in Top_{m+n}$. Here we use the standard identification $\boldsymbol{R}^m \times \boldsymbol{R}^n = \boldsymbol{R}^{m+n}$. Clearly $\oplus$ restricts to $PD_m \times PD_n \to PD_{m+n}$, $PL_m \times PL_n \to PL_{m+n}$, $O_m \times O_n \to O_{m+n}$, $GL_m \times GL_n \to GL_{m+n}$, and also induces $PD_m/O_m \times PD_n/O_n \to PD_{m+n}/O_{m+n}$ (cf. [7, §4]). Since $\mathscr{S}$ (and hence $\mathscr{G}$) is compatible with the product of bundles, we get the functors

$$\mathscr{G}'O_m \times \mathscr{G}'O_n \to \mathscr{G}'O_{m+n} \text{ and } \mathscr{G}CAT_m \times \mathscr{G}CAT_n \to \mathscr{G}CAT_{m+n}$$

($CAT=PL$ or Top) which restrics to $\mathscr{G}'GL_m \times \mathscr{G}'GL_n \to \mathscr{G}'GL_{m+n}$ and $\mathscr{G}GL_m \times \mathscr{G}GL_n \to \mathscr{G}GL_{m+n}$ respectively. Therefore we have a multiplication $CAT \times CAT \to CAT$ with respect to which (CAT, GL) can be regarded as a pair of symmetric monoidal categories (with strictly associative multiplication).

Now apply Theorem A′ to (CAT, GL) and we get an infinite loop G-space

$$BCAT(G) = K_G(CAT, GL) \quad (CAT=O, PL \text{ or } Top).$$

Moreover the functors $(O, GL) \to (PL, GL)$ and $(PL, GL) \to (Top, GL)$, given by the projections $(\mathscr{S}PL_n \times_{PL_n} PD_n/O_n, \mathscr{S}GL_n \times_{GL_n} PD_n/O_n) \to (\mathscr{G}PL_n, \mathscr{G}GL_n)$ and inclusions $(\mathscr{G}PL_n, \mathscr{G}GL_n) \to (\mathscr{G}Top_n, \mathscr{G}GL_n)$ respectively, are compatible with the multiplications. Thus we have

Proposition 3.2. There exist maps of infinite loop G-spaces

$$BO(G) \to BPL(G) \to BTop(G).$$

We finally show that the G-map

$$\varepsilon i^{-1}: \coprod_{n\geq 0} BCAT_n(G) \to BCAT(G)$$

is in fact an equivariant group completion, where i^{-1} is a G-homotopy inverse of $i: K_G(CAT, GL)_0 \to |B_G(CAT, GL)| = \coprod_{n\geq 0} BCAT_n(G)$. Let M denote the G-monoid $\coprod_{n\geq 0} BCAT_n(G)$ and identify $BCAT(G)$ with ΩBM. If $\xi: EG \to \mathscr{G}GL_1$ is the constant functor with value $*\in BGL_1$, $\xi\in BCAT_1(G)^G$ and we get G-maps $\oplus\xi: BCAT_n(G) \to BCAT_{n+1}(G)$. (Equivalently $\oplus\xi$ is the G-map induced by the inclusion $BCAT_n(G) \subset BCAT_{n+1}(G)$.) Let M_∞ be the telescope formed from the sequence

$$M \xrightarrow{\oplus\xi} M \xrightarrow{\oplus\xi} M \xrightarrow{\oplus\xi} \cdots\cdot.$$

Then M acts on M_∞ and we get a G-map $p: X = EM\times_M M_\infty \to BM$ with fibre M_∞ at the basepoint b. Because $(M_\infty)^H = (M^H)_\infty$ for every subgroup H of G, p restricts to a homology fibration $X^H \to BM^H$ with X^H contractible and with fibre $(M_\infty)^H$ at the basepoint. (Compare [12, Proposition 2].) Therefore the natural map $(M_\infty)^H \to F(p, b)^H \simeq \Omega BM^H$ is a homology equivalence, and $H_*(\Omega BM^H) \cong H_*(M^H)[\pi^{-1}]$ $(\pi = \pi_0(M^H))$. This implies that

Proposition 3.3. $\varepsilon i^{-1}: \coprod_{n\geq 0} BCAT_n(G) \to BCAT(G)$ is an equivariant group completion map.

Remark. In [16] we shall show that a classifying space $BF_n(G)$ for n-dimensional (locally linear) spherical G-fibrations can be constructed as follows:

Let $B'_G(\mathscr{G}F_n, \mathscr{G}GL_n)$ be an O_G-subcategory of $B_G(\mathscr{G}F_n, \mathscr{G}GL_n)$ (which is considered as an O_G-category $G/H \mapsto B_G(\mathscr{G}F_n, \mathscr{G}GL_n)^H$) such that, for every G-orbit $G/H\in O_G$, $B'_G(\mathscr{G}F_n, \mathscr{G}GL_n)(G/H)$ has the same objects as $B_G(\mathscr{G}F_n, \mathscr{G}GL_n)^H$ and morphisms all natural transformations $f\to f'$ in $B_G(\mathscr{G}F_n, \mathscr{G}GL_n)^H$ which induces an H-homotopy equivalence $S^n_{\alpha(f)} \to S^n_{\alpha(f')}$. Here $\alpha(f)$ denotes the representation $H\to GL_n$ associated to $f\in \mathrm{Funct}(EG, \mathscr{G}GL_n)^H \cong \mathrm{Funct}(EG/H,$

$\mathscr{G}GL_n$). (Compare the proof of Theorem 3. 1.) Let C be the Elmendorf's functor [3] which converts O_G-spaces to G-spaces. Then we can show that the G-space

$$BF_n(G) = C\,|B'_G(\mathscr{G}F_n,\ \mathscr{G}GL_n)|$$

classifies n-dimensional spherical G-fibrations. Moreover, with minor modifications of the arguments of Sections 2 and 3, we can prove that there exist an equivariant group completion map $\coprod_{n\geq 0} BF_n(G) \to BF(G)$ and also an infinite loop G-map $BTop(G) \to BF(G)$. Thus we get a sequence of infinite loop G-maps $BO(G) \to BPL(G) \to BTop(G) \to BF(G)$. Details will appear in [16].

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