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Microhyperbolic Operators in Gevrey Classes

By

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§ 1. Introduction

Kashiwara and Kawai [16] defined microhyperbolicity and proved that the microlocal Cauchy problem for microhyperbolic pseudo-differential operators is well-posed in the framework of microfunctions, which is a microlocalization of the results obtained by Bony and Schapira [3]. In the microlocal studies of pseudo-differential operators, the concept of microhyperbolicity is very useful. From their results one can obtain results on propagation of analytic singularities (propagation of micro-analyticities) of solutions for microhyperbolic operators (see [28]). On the other hand, Bronshtein [5] proved that the hyperbolic Cauchy problem is well-posed in some Gevrey classes which are intermediate spaces between the space of real analytic functions and C^∞ (see, also, [14], [15]). So we can generalize the definition of microhyperbolicity in the framework of some Gevrey classes, to say the least of it. In doing so, we expect to get a clue to a generalization of microhyperbolicity and microlocal studies of microhyperbolic operators in the framework of C^∞.

In this paper we shall consider microhyperbolic operators in Gevrey classes and prove microlocal well-posedness of the microlocal Cauchy problem and theorems on propagation of singularities for microhyperbolic operators. Our aims are to show how one can obtain microlocal results (microlocal well-posedness and, therefore, a microlocal version of Holmgren's uniqueness theorem) from methods to prove well-posedness of the Cauchy problem and to show that theorems on propagation of singularities are immediate consequences of a microlocal version of Holmgren's uniqueness theorem, using

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generalized Hamilton flows. We shall prove microlocal well-posedness, reducing the problems to those in L^2. From this point of view one may assert that consideration in L^2 (or C^∞) are much more important than in Gevrey classes. However, in the framework of Gevrey classes one can easily solve some problems, which seem difficult to be solved in the framework of C^∞, and obtain some conjectures on the problems in the framework of C^∞. We should note that Uchikoshi [27] investigated a related problem.

Let K be a regular compact set in $\boldsymbol{R}^n$, and let $\kappa>1$ and $h>0$. We denote by $\mathscr{E}^{\{\kappa\},h}(K)$ the space of all $f\in C^\infty(K)$ which satisfies, with some constant $C\geq 0$,

$$(1.1)\qquad |D^\alpha f(x)|\leq Ch^{|\alpha|}|\alpha|!^\kappa \text{ for } x\in K \text{ and } |\alpha|=0,1,2,\dots,$$

where $x=(x_1,\dots,x_n)\in\boldsymbol{R}^n$, $D=i^{-1}(\partial/\partial x_1,\dots,\partial/\partial x_n)$, $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index and $|\alpha|=\sum_{j=1}^n\alpha_j$. We also denote by $\mathscr{D}_K^{\{\kappa\},h}$ the space of all $f\in C^\infty(\boldsymbol{R}^n)$ with support in K satisfying (1.1). $\mathscr{E}^{\{\kappa\},h}(K)$ and $\mathscr{D}_K^{\{\kappa\},h}$ are Banach spaces under the norm defined by

$$||f||_{\mathscr{E}^{\{\kappa\},h}(K)}=\sup_{x\in K,\alpha}|D^\alpha f(x)|/(h^{|\alpha|}|\alpha|!^\kappa).$$

Let Ω be an open set in $\boldsymbol{R}^n$. We introduce the following locally convex spaces (Gevrey classes):

$$\mathscr{E}^{(\kappa)}(\Omega)=\varprojlim_{K\Subset\Omega}\mathscr{E}^{(\kappa)}(K),\quad \mathscr{E}^{(\kappa)}(K)=\varprojlim_{h\to 0}\mathscr{E}^{\{\kappa\},h}(K),$$
$$\mathscr{E}^{\{\kappa\}}(\Omega)=\varprojlim_{K\Subset\Omega}\mathscr{E}^{\{\kappa\}}(K),\quad \mathscr{E}^{\{\kappa\}}(K)=\varinjlim_{h\to\infty}\mathscr{E}^{\{\kappa\},h}(K),$$
$$\mathscr{D}^{(\kappa)}(\Omega)=\varinjlim_{K\Subset\Omega}\mathscr{D}_K^{(\kappa)},\quad \mathscr{D}_K^{(\kappa)}=\varprojlim_{h\to 0}\mathscr{D}_K^{\{\kappa\},h},$$
$$\mathscr{D}^{\{\kappa\}}(\Omega)=\varinjlim_{K\Subset\Omega}\mathscr{D}_K^{\{\kappa\}},\quad \mathscr{D}_K^{\{\kappa\}}=\varinjlim_{h\to\infty}\mathscr{D}_K^{\{\kappa\},h},$$

where $A\Subset B$ means that the closure $\bar{A}$ of A is compact and included in the interior $\mathring{B}$ of B. We denote by $\mathscr{D}^{*\prime}(\Omega)$ and $\mathscr{E}^{*\prime}(\Omega)$ the strong dual spaces of $\mathscr{D}^*(\Omega)$ and $\mathscr{E}^*(\Omega)$, respectively, where $*$ denotes (κ) or $\{\kappa\}$. We also write $\mathscr{E}^*,\dots,$ instead of $\mathscr{E}^*(\boldsymbol{R}^n),\dots$ (see, e.g., [18]). Let us define symbol classes S_*^m, where $m\in\boldsymbol{R}$. We say that a symbol $p(x,\xi)$ belongs to $S_{(\kappa)}^m$ (resp. $S_{\{\kappa\}}^m$) if $p(x,\xi)\in C^\infty(T^*\boldsymbol{R}^n)$ and for any compact subset K of $\boldsymbol{R}^n$ and any $A>0$ there is $C\equiv C_{K,A}>0$ (resp. for any compact subset K of $\boldsymbol{R}^n$ there are $A\equiv A_K>0$ and $C\equiv C_K>0$) such that

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq CA^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^{\kappa}\langle\xi\rangle^{m-|\alpha|}$$

for $x \in K$, $\xi=(\xi_1, \ldots, \xi_n) \in \boldsymbol{R}^n$ and any multi-indeces α and β, where $T^*\boldsymbol{R}^n$ is identified with $\boldsymbol{R}^n \times \boldsymbol{R}^n$, $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$. We impose the following conditions:

(A-1) $p(x, \xi) \in S_{*1}^m$, where $*1$ denotes (κ_1) or $\{\kappa_1\}$, and $\kappa_1>1$ and $m \in \boldsymbol{R}$. And $p(x, D)$ is properly supported.

(A-2) There is a symbol $p_m(x, \xi)$, which is positively homogeneous of degree m in ξ, such that $p(x, \xi)-\sigma(\xi) p_m(x, \xi) \in S_{*1}^{m-1}$, $\sigma(\xi) \in \mathscr{E}^{(\kappa_1)}$ and $\sigma(\xi)=1$ for $|\xi| \geq 1$ and $\sigma(\xi)=0$ for $|\xi| \leq 1/2$.

Definition 1.1. Let $z^0=(x^0, \xi^0) \in T^*\boldsymbol{R}^n \backslash 0$ and $\vartheta \in T_{z^0}(T^*\boldsymbol{R}^n) \simeq \boldsymbol{R}^{2n}$. We say that $p(x, \xi)$ (or $p_m(x, \xi)$) is microhyperbolic with respect to ϑ at z^0 if there are a neighborhood $\mathscr{U}$ of z^0 in $T^*\boldsymbol{R}^n \backslash 0$, $l \in \boldsymbol{N} \cup \{0\}$ and positive constants c and t_0 such that

$$\left|\sum_{j=0}^l(-it\vartheta)^j p_m(x, \xi)/j!\right| \geq ct^l \quad \text{for } (x, \xi) \in \mathscr{U} \text{ and } 0 \leq t \leq t_0$$

where $\vartheta=(\vartheta_x, \vartheta_\xi)$ is regarded as a vector field $\vartheta=\vartheta_x \cdot(\partial/\partial x)+\vartheta_\xi \cdot(\partial/\partial \xi)$.

Remark. (i) The above definition coincides with the definition given in [33]. (ii) When $p_m(x, \xi)$ is real analytic, the above definition coincides with the definition of partially microhyperbolicity given by Kashiwara and Kawai [16].

Let Ω be an open conic set in $T^*\boldsymbol{R}^n \backslash 0$. We assume that

(A-3) $p_m(x, \xi)$ is microhyperbolic at each point in Ω.

For $z^0 \in T^*\boldsymbol{R}^n \backslash 0$ we can write

$$p_m(z^0+s\delta z)=s^\mu(p_{mz^0}(\delta z)+o(1)) \quad \text{as } s \rightarrow 0,$$

where $p_{mz^0}(\delta z) \not\equiv 0$ in $\delta z \in T_{z^0}(T^*\boldsymbol{R}^n)$, if there are multi-indeces α and β such that $p_{m(\beta)}^{(\alpha)}(z^0) \neq 0$. $p_{mz^0}(\delta z)$ is called the localization polynomial of $p_m(z)$ at z^0 and $\mu \equiv \mu(z^0)$ is called the multiplicity of $p_m(z)$ at z^0. If $p_m(z)$ is microhyperbolic with respect to ϑ at z^0, then $p_{mz^0}(\delta z)$ is hyperbolic with respect to ϑ, i.e.,

$$p_{mz^0}(\delta z-is\vartheta) \neq 0 \quad \text{for } \delta z \in T_{z^0}(T^*\boldsymbol{R}^n) \text{ and } s>0$$

(see, e.g., [11]). Therefore, we can define $\Gamma(p_{mz^0}, \vartheta)$ as the connected component of the set $\{\delta z\in T_{z^0}(T^*\boldsymbol{R}^n)\,;\ p_{mz^0}(\delta z)\neq 0\}$ which contains ϑ, when $p_m(z)$ is microhyperbolic with respect to ϑ at z^0. For some properties of hyperbolic polynomials and $\Gamma(p_{mz}, \vartheta)$ we refer to Atiyah, Bott and Gårding [2].

Definition 1.2. (i) $t(x, \xi)\in C^1(\Omega)$ is called a time function for p_m in Ω if $t(x, \xi)$ is real-valued and positively homogeneous of degree 0 in ξ, and if $p_m(z)$ is microhyperbolic with respect to $-H_t(z)$ at every $z\in\Omega$, where $H_t(z)=\sum_{j=1}^n\{(\partial t/\partial\xi_j)(z)(\partial/\partial x_j)-(\partial t/\partial x_j)(z)(\partial/\partial\xi_j)\}$. (ii) Let $t(x, \xi)\in C^1(\Omega)$ be a time function for p_m in Ω, and let $z\in\Omega$. We define the generalized Hamilton flows $K^\pm(z;\Omega;t)$ by

$K^\pm(z;\Omega;t)=\{z(s)\in\Omega;\ \pm s\geq 0,$ *and* $\{z(s)\}$ *is a Lipschitz continuous curve in* Ω *satisfying* $(d/ds)z(s)\in\Gamma(p_{mz(s)}, -H_t(z(s)))^\sigma$ $(a.e.\ s)$ *and* $z(0)=z\}$,

where $\Gamma^\sigma=\{(\delta x, \delta\xi)\in T_z(T^*\boldsymbol{R}^n)\,;\ \sigma((\delta y, \delta\eta), (\delta x, \delta\xi))\,(=\delta x\cdot\delta\eta-\delta y\cdot\delta\xi)\geq 0$ for any $(\delta y, \delta\eta)\in\Gamma\}$ for $z\in T^*\boldsymbol{R}^n\backslash 0$ and $\Gamma\subset T_z(T^*\boldsymbol{R}^n)$.

Remark. We should note that Leray [21] and Lascar [20] defined flows similar to $K^\pm(z;\Omega;t)$.

Definition 1.3. Let $\kappa>\kappa_1$ and $f\in\mathscr{D}^{\prime(\kappa_1)}$. $WF_{(\kappa)}(f)$ (resp. $WF_{\{\kappa\}}(f)$) is defined as the complement in $T^*\boldsymbol{R}^n\backslash 0$ of the collection of all (x^0, ξ^0) in $T^*\boldsymbol{R}^n\backslash 0$ such that there are a neighborhood U of x^0 and a conic neighborhood Γ of ξ^0 such that for every $\varphi\in\mathscr{D}^{(\kappa_1)}(U)$ and every $A>0$ there is a positive constant C (resp. for every $\varphi\in\mathscr{D}^{(\kappa_1)}(U)$ there are positive constants A and C) satisfying

$$|\mathscr{F}[\varphi f](\xi)|\leq C\exp[-A|\xi|^{1/\kappa}] \text{ for } \xi\in\Gamma,$$

where $\mathscr{F}[f](\xi)\equiv\hat{f}(\xi)$ denotes the Fourier transform of f (see [10], [28]).

Moreover, we assume that

(A-4) $\mu(\Omega)\equiv\sup_{z\in\Omega}\mu(z)<+\infty$, and $\kappa_1\leq\kappa(\Omega)\equiv\min\{2, \mu(\Omega)/(\mu(\Omega)-1)\}$ if $*1=(\kappa_1)$, and $\kappa_1<\kappa(\Omega)$ if $*1=\{\kappa_1\}$.

Theorem 1.4. *Assume that* (A-1)-(A-4) *are valid, and let* $\vartheta:\Omega\ni z$

$\mapsto \vartheta(z)\in T_z(\Omega)$ *be a continuous vector field such that* $p_m(z)$ *is microhyperbolic with respect to* $\vartheta(z)$ *at each* $z\in\Omega$. *We denote by* $*(\kappa)$ *or* $\{\kappa\}$, *and assume that* $\kappa_1\leq\kappa\leq\kappa(\Omega)$ *and* $*=(\kappa)$ *when* $*1=(\kappa_1)$ *and that* $\kappa_1\leq\kappa<\kappa(\Omega)$ *and* $*=\{\kappa\}$ *when* $*1=\{\kappa_1\}$. *If* $u\in\mathscr{D}^{*1\prime}$, $z^0\in WF_*(u)\cap\Omega$ *and* $WF_*(pu)\cap\Omega=\emptyset$, *then there are* $a\in(-\infty,0)\cup\{-\infty\}$ *and a Lipschitz continuous function* $z(t)$ *defined on* $(a, 0]$ *with values in* Ω *such that* $z(t)\in WF_*(u)$ *for* $t\in(a,0]$, $(d/dt)\,z(t)\in\Gamma(p_{mz(t)}, \vartheta(z(t)))^\sigma\cap\{\delta z;\ |\delta z|=1\}$ *for a. e.* $t\in(a,0]$, *and* $z(0)=z^0$, *and* $\lim_{t\to a+0} z(t)\in\partial\Omega$ *if* $a>-\infty$, *where* $\partial\Omega$ *denotes the boundary of* Ω *in* $T^*\boldsymbol{R}^n$.

Theorem 1.5. *Assume that* (A-1)-(A-4) *are valid and that* $t(z)\in C^1(\Omega)$ *is a time function for* p_m *in* Ω. *Moreover, assume that* $\kappa_1\leq\kappa\leq\kappa(\Omega)$ *and* $*=(\kappa)$ *when* $*1=(\kappa_1)$ *and that* $\kappa_1\leq\kappa<\kappa(\Omega)$ *and* $*=\{\kappa\}$ *when* $*1=\{\kappa_1\}$. (i) *Let* $z^0\in\Omega$ *and* $t_0\in\boldsymbol{R}$ *satisfy* $t_0\leq t(z^0)$, *and assume that* $K^-(z^0;\Omega;t)\cap\{z\in\Omega;\ t(z)\geq t_0\}\Subset\Omega$. *Then* $z^0\notin WF_*(u)$ *if* $u\in\mathscr{D}^{*1\prime}$, $WF_*(pu)\cap K^-(z^0;\Omega;t)\cap\{z\in\Omega;\ t(z)\geq t_0\}=\emptyset$ *and* $WF_*(u)\cap K^-(z^0;\Omega;t)\cap\{z\in\Omega;\ t(z)=t_0\}=\emptyset$. (ii) *Furthermore, assume that* $K^-(z;\Omega;t)\cap\{z^1\in\Omega;\ t(z^1)\geq t_0\}\Subset\Omega$ *for every* $z\in\Omega$. *Then*

$$WF_*(u)\cap\{z\in\Omega;\ t(z)\geq t_0\}\subset$$

$\{z\in\Omega;\ z\in K^+(w;\Omega;t)$ *for some* $w\in(WF_*(pu)\cap\{z\in\Omega;\ t(z)\geq t_0\})\cup(WF_*(u)\cap\{z\in\Omega;\ t(z)=t_0\}))\}$ *for* $u\in\mathscr{D}^{*1\prime}$.

Remark. Theorem 1.5 is an immediate consequence of Theorem 1.4. We note that there do not always exist time functions for p_m even locally (see Proposition 5.1).

The remainder of this paper is organized as follows. In §2 we shall give preliminary lemmas on calculus of pseudo-differential operators. In §3 we shall investigate hypoellipticity in Gevrey classes for operators which satisfy the so-called (H)-condition (see [9]). The microlocal Cauchy problem will be studied and microlocal parametrices will be constructed in §4. We shall give the proof of Theorem 1.4 and some remarks in §5.

§2. Calculus of Pseudo-Differential Operators

Using pseudo-differential operators of infinite order, we can reduce the problem in Gevrey classes to the problem in the Sobolev spaces and prove Theorem 1.4. In doing so, we must establish calculus of pseudo-differential operators of infinite order. By results in this section (Proposition 2.13 below) we can calculate the symbols of the reduced operators. Throughout this paper we denote by $C_{a,b,\cdots}(A, B, \cdots)$ a constant depending on $a, b, \cdots$ and $A, B, \cdots$ which is locally bounded in $A, B, \cdots$. Let $\kappa>1$ and $\varepsilon\in\boldsymbol{R}$, and define

$$\hat{\mathscr{S}}_{\kappa,\varepsilon}=\{v(\xi)\in C^\infty(\boldsymbol{R}^n);\ \exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v(\xi)\in\mathscr{S}\}.$$

We say that $v_j\to v$ in $\hat{\mathscr{S}}_{\kappa,\varepsilon}$ as $j\to\infty$ if $\exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v_j(\xi)\to\exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v(\xi)$ in $\mathscr{S}$ as $j\to\infty$. Since $\mathscr{D}$ is dense in $\hat{\mathscr{S}}_{\kappa,\varepsilon}$, it is obvious that the dual space $\hat{\mathscr{S}}'_{\kappa,\varepsilon}$ of $\hat{\mathscr{S}}_{\kappa,\varepsilon}$ is identified with $\{\exp[\varepsilon\langle\xi\rangle^{1/\kappa}]v(\xi)\in\mathscr{D}';\ v\in\mathscr{S}'\}$. For $\varepsilon\geq 0$ we can define

$$\mathscr{S}_{\kappa,\varepsilon}=\mathscr{F}^{-1}[\hat{\mathscr{S}}_{\kappa,\varepsilon}]\ (=\mathscr{F}[\hat{\mathscr{S}}_{\kappa,\varepsilon}]=\{u\in\mathscr{S};\ \exp[\varepsilon\langle\xi\rangle^{1/\kappa}\hat{u}(\xi)\in\mathscr{S}\}).$$

We introduce the topology in $\mathscr{S}_{\kappa,\varepsilon}$ so that $\mathscr{F}:\ \hat{\mathscr{S}}_{\kappa,\varepsilon}\to\mathscr{S}_{\kappa,\varepsilon}$ is homeomorphic. Denote by $\mathscr{S}'_{\kappa,\varepsilon}$ the dual space of $\mathscr{S}_{\kappa,\varepsilon}$ for $\varepsilon\geq 0$. Then we can define the transposed operators ${}^t\mathscr{F}$ and ${}^t\mathscr{F}^{-1}$ of $\mathscr{F}$ and $\mathscr{F}^{-1}$ which map $\mathscr{S}'_{\kappa,\varepsilon}$ and $\hat{\mathscr{S}}'_{\kappa,\varepsilon}$ onto $\hat{\mathscr{S}}'_{\kappa,\varepsilon}$ and $\mathscr{S}'_{\kappa,\varepsilon}$, respectively. Since $\hat{\mathscr{S}}_{\kappa,-\varepsilon}\subset\hat{\mathscr{S}}'_{\kappa,\varepsilon}$ $(\subset\mathscr{D}')$ for $\varepsilon\geq 0$, we can define $\mathscr{S}_{\kappa,-\varepsilon}={}^t\mathscr{F}^{-1}[\hat{\mathscr{S}}_{\kappa,-\varepsilon}]$ for $\varepsilon\geq 0$. It is easy to see that $\mathscr{S}'_{\kappa,-\varepsilon}=\mathscr{F}[\hat{\mathscr{S}}'_{\kappa,-\varepsilon}]$ is the dual space of $\mathscr{S}_{\kappa,-\varepsilon}$, $\hat{\mathscr{S}}'_{\kappa,-\varepsilon}\subset\mathscr{S}'\subset\hat{\mathscr{S}}'_{\kappa,\varepsilon}$, $\subset\mathscr{S}'_{\kappa,-\varepsilon}\subset\mathscr{S}'\subset\mathscr{S}'_{\kappa,\varepsilon}$ for $\varepsilon\geq 0$ and that $\mathscr{F}={}^t\mathscr{F}$ on $\mathscr{S}'$. So we write ${}^t\mathscr{F}$ as $\mathscr{F}$. Define

$$H^m_{\kappa,\varepsilon}=\{u\in\mathscr{S}'_{\kappa,-\varepsilon};\ \langle\xi\rangle^m\exp[\varepsilon\langle\xi\rangle^{1/\kappa}]\hat{u}(\xi)\in L^2\},\ L^2_{\kappa,\varepsilon}=H^0_{\kappa,\varepsilon},$$

where $m\in\boldsymbol{R}$ and $\varepsilon\in\boldsymbol{R}$.

Lemma 2.1. (i) *$\mathscr{D}^{(\kappa)}$ is a dense subspace of $\mathscr{S}_{\kappa,\varepsilon}$.* (ii) *$\mathscr{D}^{\{\kappa\}}\subset\bigcup_{\varepsilon>0}\mathscr{S}_{\kappa,\varepsilon}$.* (iii) *$\mathscr{E}^{(\kappa)'}\subset\bigcup_{\varepsilon\in\boldsymbol{R}}\mathscr{S}_{\kappa,\varepsilon}$ and $\mathscr{E}^{\{\kappa\}'}\subset\bigcap_{\varepsilon<0}\mathscr{S}_{\kappa,\varepsilon}$.* (iv) *$\mathscr{D}^{(\kappa)}\subset\mathscr{S}_{\kappa,\varepsilon}\subset\mathscr{S}_{\kappa,\varepsilon'}\subset H^m_{\kappa,\varepsilon}\subset H^{m'}_{\kappa,\varepsilon'}\subset H^{m''}_{\kappa,\varepsilon'}\subset\mathscr{S}'_{\kappa,-\varepsilon''}\subset\mathscr{D}^{(\kappa)'}$, where $\varepsilon\geq\varepsilon'>\varepsilon''$, $m\geq m'$ and $m''\in\boldsymbol{R}$.*

Proof. The assertions (ii) and (iii) can be proved by the Paley-Wiener theorem in Gevrey classes (see, e.g., [18]). We can also prove that $w_k(\xi)\equiv\chi(\xi/k)v(\xi)\to v(\xi)$ in $\hat{\mathscr{S}}_{\kappa,\varepsilon}$ as $k\to\infty$ and that $v_{kj}(\xi)\equiv\int\hat{\chi}(\eta)w_k(\xi-\eta/j)đ\eta\to w_k(\xi)$ in $\hat{\mathscr{S}}_{\kappa,\varepsilon}$ as $j\to\infty$, where $\chi\in\mathscr{D}^{(\kappa)}$, $\chi(\xi)=1$ if $|\xi|\leq 1$ and $\chi(\xi)=0$ if $|\xi|\geq 2$, and $đ\eta=(2\pi)^{-n}d\eta$. This proves the assertion (i). **Q.E.D.**

In this paper we shall frequently use the following facts without quoting.

Lemma 2.2. (i) $N!\leq ce^{-N}N^{N+1/2}$ *for* $N\geq 1$, *where* c *is a positive constant.* (ii) *For* $t\geq 1$

$$\inf_{N=0,1,2,\ldots}N!t^{-N}\leq c\ \inf_{N=1,2,\ldots}N^{N+1/2}(et)^{-N}\leq ect^{1/2}e^{-t}.$$

(iii) $|\alpha|!\leq n^{|\alpha|}\alpha!$ *and* $\sum_{|\beta|=l,\beta\leq\alpha}\binom{\alpha}{\beta}=\binom{|\alpha|}{l}$, *where* $\binom{\alpha}{\beta}=\alpha!/(\beta!(\alpha-\beta)!)$. (iv) $\sum_{|\alpha|=N}(\alpha!/N!)^{\kappa-1}\leq c_\kappa^{n-1}$ *if* $\kappa>1$, *where* c_κ *is a constant depending only on* κ. (v) $\sum_{k=0}^{\infty}k!^{1-\kappa}t^k=c_\kappa(t)<+\infty$ *if* $t>0$ *and* $\kappa>1$. (vi) $\langle\xi+\eta\rangle_h\leq\langle\xi\rangle_h+|\eta|$, *where* $\langle\xi\rangle_h=(h^2+|\xi|^2)^{1/2}$. (vii) $|\partial_\xi^\alpha\langle\xi\rangle_h^{1-k}|\leq(1+\sqrt{2})^{|\alpha|}(|\alpha|+[k])!\langle\xi\rangle_h^{1-k-|\alpha|}/[k]!$, *where* $k\geq 0$ *and* $[k]$ *denotes the largest integer* $\leq k$. (viii) *Let* $1\leq\kappa'<\kappa$ *and* $N\in\boldsymbol{N}\cup\{0\}$, *and assume that* $\chi(\xi)\in C^\infty(\boldsymbol{R}^n)$ *satisfies*

$$|\chi^{(\alpha+\beta)}(\xi)|\leq CA^{|\alpha|}B^{|\beta|}N^{|\alpha|}|\beta|!^{\kappa'}\ \textit{for}\ |\alpha|\leq N\ \textit{and any}\ \beta.$$

Then, for any $c>0$, *and* $d>0$ *there is* $C_{A,B,c,d}>0$ *such that*

$$|\partial_\eta^{\alpha+\beta}\partial_\xi^\gamma\chi(\eta\langle\xi\rangle_h^{\pm1})|\leq C_{A,B,c,d}A^{|\alpha|}d^{|\beta|+|\gamma|}N^{|\alpha|}(|\beta|+|\gamma|)!^\kappa\langle\xi\rangle_h^{\pm(|\alpha|+|\beta|)-|\gamma|}$$

for $|\eta|\langle\xi\rangle_h^{\pm1}\leq c$, $h>0$, $|\alpha|\leq N$ *and any* β *and* γ.

Proof. The assertions (i)-(iii), (v) and (vi) are obvious. The assertion (iv) can be proved by induction on the dimension n. The assertion (vii) can be proved by induction on $|\alpha|$. We note that a similar estimate to (vii) can be also obtained by Cauchy's estimates. In order to prove (viii) it suffices to prove that

$$|\partial_\eta^{\alpha+\beta}\partial_\xi^\gamma\chi(\eta\langle\xi\rangle_h^{\pm1})|\leq CA^{|\alpha|}B^{|\beta|}B_1^{|\gamma|}N^{|\alpha|}!(|\beta|+|\gamma|)!^{\kappa'}\langle\xi\rangle_h^{\pm(|\alpha|+|\beta|)-|\gamma|}$$

for $|\eta|\langle\xi\rangle_h^{\pm1}\leq c$, $|\alpha|\leq N$ and any β and γ, which can be proved by induction on $|\gamma|$. Here B_1 depends on A, B and C. **Q.E.D.**

Let $p(\xi,y,\eta)$ be a symbol satisfying

$$|\partial_\xi^\alpha\partial_\eta^\beta D_y^\gamma p(\xi,y,\eta)|\leq C_{\alpha,\beta}A^{|\gamma|}|\gamma|!^\kappa\exp[\delta_1\langle\xi\rangle^{1/\kappa}+\delta_2\langle\eta\rangle^{1/\kappa}]$$

for $(\xi,y,\eta)\in\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ and any multi-indeces α, β and γ, where $A>0$, $\delta_1,\delta_2\in\boldsymbol{R}$ and the positive constants $C_{\alpha,\beta}$ depend on α and β. Define

$$p(D_x,y,D_y)u(x)=\mathscr{F}_\xi^{-1}\Big[\int e^{-iy\cdot\xi}\Big(\int e^{iy\cdot\eta}p(\xi,y,\eta)\hat{u}(\eta)\,d\!\!{}^-\eta\Big)dy\Big](x)$$

for $u\in\mathscr{D}^{(\kappa)}$.

Proposition 2.3. *$p(D_x, y, D_y)$ maps continuously $\mathscr{S}_{\kappa,\varepsilon_2}$ to $\mathscr{S}_{\kappa,\varepsilon_1}$ if $\delta_2-\kappa(nA)^{-1/\kappa}<\varepsilon_2$, $\varepsilon_1\leq\varepsilon_2-\delta_1-\delta_2$ and $\varepsilon_1<\kappa(nA)^{-1/\kappa}-\delta_1$. In particular, $p(D_x, y, D_y)$ maps continuously $\mathscr{S}_{\kappa,\varepsilon}$ to $\mathscr{S}_{\kappa,\varepsilon-\delta_1-\delta_2}$ if $|\varepsilon-\delta_2|<\kappa(nA)^{-1/\kappa}$.*

Proof. Let $u\in\mathscr{D}^{(\kappa)}$ and write

$$\langle\xi\rangle^j D_\xi^\alpha\{\exp[\varepsilon_1\langle\xi\rangle^{1/\kappa}]\mathscr{F}[p(D_x, y, D_y)u(x)](\xi)\}=\int F(\xi,\eta)\,d\!\!\bar{}\,\eta,$$

where $F(\xi,\eta)=\int e^{-iy\cdot(\xi-\eta)}\langle\xi\rangle^j f(\xi,\eta,y)\,dy$, $f(\xi,\eta,y)=\sum_{\alpha^1+\alpha^2=\alpha}\alpha!(\alpha^1!\alpha^2!)^{-1}$ $(-y)^{\alpha^1}\langle y\rangle^{-2N}D_\xi^{\alpha^2}\langle D_\eta\rangle^{2N}\{\exp[\varepsilon_1\langle\xi\rangle^{1/\kappa}]p(\xi,y,\eta)\hat{u}(\eta)\}$ and $N=[(n+|\alpha|)/2]+1$. Then we have

$$|D_y^\beta f(\xi,\eta,y)|\leq C_\alpha A^{|\beta|}|\beta|!^\kappa\langle y\rangle^{-2N+|\alpha|}\langle\eta\rangle^{-M}$$
$$\times\exp[(\varepsilon_1+\delta_1)\langle\xi\rangle^{1/\kappa}+(\delta_2-\varepsilon_2)\langle\eta\rangle^{1/\kappa}]\,|u|_{\mathscr{S}_{\kappa,\varepsilon_2},M+2N},$$

where $|u|_{\mathscr{S}_{\kappa,\varepsilon},l}=\sup_{\xi\in\mathbf{R}^n,\,j+|\alpha|\leq l}|\langle\xi\rangle^j D_\xi^\alpha(\exp[\varepsilon\langle\xi\rangle^{1/\kappa}]\hat{u}(\xi))|$. Since $\langle\xi+\eta\rangle\leq\sqrt{2}\langle\xi\rangle\langle\eta\rangle$, it follows from Lemma 2.2 that

$$|F(\xi,\eta)|\leq C_{A',\alpha,j}\langle\eta\rangle^{-M+j}$$
$$\times\exp[(\varepsilon_1+\delta_1)\langle\xi\rangle^{1/\kappa}+(\delta_2-\varepsilon_2)\langle\eta\rangle^{1/\kappa}-\kappa(nA')^{-1/\kappa}|\xi-\eta|^{1/\kappa}]$$
$$\times|u|_{\mathscr{S}_{\kappa,\varepsilon_2},M+2N},$$

where $A'>A$. Noting that $\pm\langle\eta\rangle^{1/\kappa}-|\xi-\eta|^{1/\kappa}\leq\pm\langle\xi\rangle^{1/\kappa}$, we have

$$\left|\int F(\xi,\eta)\,d\!\!\bar{}\,\eta\right|\leq C'_{A',\alpha,j}$$
$$\times\exp[(\varepsilon_1+\delta_1+\max\{\delta_2-\varepsilon_2,\ -\kappa(nA')^{-1/\kappa}\})\langle\xi\rangle^{1/\kappa}]\,|u|_{\mathscr{S}_{\kappa,\varepsilon_2},M+2N}$$

if $A'>A$, $\delta_2-\varepsilon_2<\kappa(nA)^{-1/\kappa}$ and $M>j+n$. This proves the proposition. **Q.E.D.**

Corollary. *$p(D_x, y, D_y)$ maps continuously $\mathscr{S}'_{\kappa,-\varepsilon_2}$ to $\mathscr{S}'_{\kappa,-\varepsilon_1}$ if $\delta_2-\kappa(nA)^{-1/\kappa}<\varepsilon_2$, $\varepsilon_1\leq\varepsilon_2-\delta_1-\delta_2$ and $\varepsilon_1<\kappa(nA)^{-1/\kappa}-\delta_1$.*

Let $\{\varphi_j^R(\xi)\}\subset\mathscr{E}^{(\kappa)}$ satisfy the following conditions; $0\leq\varphi_j^R(\xi)\leq1$, $\varphi_j^R(\xi)=1$ if $\langle\xi\rangle_h^{1/\kappa}>2Rj$, $\varphi_j^R(\xi)=0$ if $\langle\xi\rangle_h^{1/\kappa}<Rj$, and $|\varphi_j^{R(\alpha)}(\xi)|\leq C_d d^{|\alpha|}|\alpha|!^\kappa\langle\xi\rangle_h^{-|\alpha|}$ for any $d>0$, where $R>0$, $j=0,1,2,\ldots$, and C_d is a positive constant depending on d. For example, $\varphi_0^R(\xi)\equiv1$ and $\varphi_j^R(\xi)\equiv\chi(\langle\xi\rangle_h/(Rj)^\kappa)$ $(j=1,2,\cdots)$ satisfy the above conditions if $1<\kappa'<\kappa$, $\chi\in\mathscr{E}^{(\kappa')}(\mathbf{R}^1)$, $0\leq\chi(t)\leq1$, $\chi(t)=1$ if $t\geq2^\kappa$ and $\chi(t)=0$ if

$t\le 1$. A simple calculation gives the following

Lemma 2.4. *Let $R_0>0$, $\kappa'>0$ and $h>0$. If*

$$|q_{j(\beta)}^{(\alpha)}(x,\xi)|\le C_\alpha A^{|\beta|}B^j|\beta|!^\kappa j!^{\kappa'}\langle\xi\rangle_h^{m-|\alpha|-j\kappa'/\kappa}\exp[\delta\langle\xi\rangle_h^{1/\kappa}]$$

for any α, β, $j=0,1,2,\cdots$ and $\langle\xi\rangle_h^{1/\kappa}\ge R_0 j$, then $q^R(x,\xi)=\sum_{j=0}^\infty\varphi_j^R(\xi)$ $q_j(x,\xi)$ is well-defined and satisfies

$$|q_{(\beta)}^{R(\alpha)}(x,\xi)|\le C(|\alpha|,\ \{C_\gamma\}_{|\gamma|\le|\alpha|})A^{|\beta|}|\beta|!^\kappa\langle\xi\rangle_h^{m-|\alpha|}\exp[\delta\langle\xi\rangle_h^{1/\kappa}]$$

for $R\ge\max(R_0,\ 2e^{-1}B^{1/\kappa'})$. Moreover, if

$$|q_{j(\beta)}^{(\alpha)}(x,\xi)|\le CA^{|\alpha|+|\beta|}B^j(|\alpha|+|\beta|)!^\kappa j!^{\kappa'}\langle\xi\rangle_h^{m-|\alpha|-j\kappa'/\kappa}\exp[\delta\langle\xi\rangle_h^{1/\kappa}]$$

for any α, β, $j=0,1,2,\cdots$ and $\langle\xi\rangle_h^{1/\kappa}\ge R_0 j$, then

$$|q_{(\beta)}^{R(\alpha)}(x,\xi)|\le C_A(C)A^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa\langle\xi\rangle_h^{m-|\alpha|}\exp[\delta\langle\xi\rangle_h^{1/\kappa}]$$

for $R\ge\max(R_0,2e^{-1}B^{1/\kappa'})$.

Let $h\le 1$ and m_1, $m_2\in\boldsymbol{R}$, and let $p(x,\xi,y,\eta)$ be a symbol satisfying

$$\begin{aligned}(2.1)\qquad &|\partial_\xi^\alpha D_x^\beta\partial_\eta^\gamma D_y^\delta p(x,\xi,y,\eta)|\,(\equiv|p_{(\beta)(\delta)}^{(\alpha)(\gamma)}(x,\xi,y,\eta)|)\\ &\le L_{|\gamma|,A}A^{|\beta|+|\gamma|}A_1^{|\alpha|}A_2^{|\delta|}|\alpha|!^{\kappa'}|\beta|!^\kappa|\gamma|!^\kappa|\delta|!^{\kappa''}\\ &\quad\times\langle\xi\rangle_h^{m_1-|\alpha|}\langle\eta\rangle_h^{m_2-|\gamma|}\exp[\delta_1\langle\xi\rangle_h^{1/\kappa}+\delta_2\langle\eta\rangle_h^{1/\kappa}],\end{aligned}$$

where $L_{k,A}\equiv C$ or $L_{k,A}A^k k!^\kappa\equiv C_k$. We set $L_j=C$ if $L_{k,A}\equiv C$ and $L_j=\max_{0\le i\le j}C_i$ if $L_{k,A}A^k k!^\kappa\equiv C_k$. We consider only the cases where "$\kappa'=\kappa$ and $\kappa''=1$" or "$\kappa'=1$ and $\kappa''=\kappa$". For $u\in\mathscr{D}^{(\kappa)}$ we can define

$$p(x,D_x,y,D_y)u(x)=\int e^{ix\cdot\xi}\Bigl(\int e^{-iy\cdot\xi}\Bigl(\int e^{iy\cdot\eta}p(x,\xi,y,\eta)\hat{u}(\eta)\,d\!\!{}^-\eta\Bigr)dy\Bigr)d\!\!{}^-\xi$$

if $\delta_1<\kappa(nA_2)^{-1/\kappa}$ when $\kappa'=1$. Here we have applied the same argument as in the proof of Proposition 2.3. Put

$$(2.2)\quad q_j(x,\xi)=\sum_{|\alpha|=j}\alpha!^{-1}\partial_\xi^\alpha D_y^\alpha p(x,\xi,y,\eta)|_{y=x,\eta=\xi},\ j=0,1,2,\cdots.$$

Then we have

$$\begin{aligned}&|q_{j(\beta)}^{(\alpha)}(x,\xi)|\le(\max_{0\le k\le|\alpha|}L_{k,A})(|\alpha|+|\beta|+j)!^\kappa\langle\xi\rangle_h^{m_1+m_2-|\alpha|-j}\\ &\times\exp[(\delta_1+\delta_2)\langle\xi\rangle_h^{1/\kappa}]\sum_{\mu=0}^{|\alpha|}\sum_{\nu=0}^{|\beta|}I(\mu,\nu)(nA_1A_2)^jA^{|\alpha|-\mu+\nu}A_1^\mu A_2^{|\beta|-\nu},\end{aligned}$$

where $I(\mu,\nu)=|\alpha|!\,|\beta|!\,(j+\mu)!^{\kappa'}\nu!^\kappa(|\alpha|-\mu)!^\kappa(|\beta|+j-\nu)!^{\kappa''}\{j!\mu!(|\alpha|-\mu)!\nu!(|\beta|-\nu)!(|\alpha|+|\beta|+j)!^\kappa\}^{-1}$. It is easy to see that

$$I(\mu, \nu) \leq \begin{cases} \binom{|\alpha|}{\mu}^{1-\kappa} (|\beta|-\nu)!^{1-\kappa} & \text{if } \kappa'=\kappa, \\ \binom{|\beta|}{\nu}^{1-\kappa} \mu!^{1-\kappa} & \text{if } \kappa'=1. \end{cases}$$

Applying Lemma 2. 2, we have

$$|q_{j(\beta)}^{(\alpha)}(x, \xi)| \leq C(\hat{A}/A)(\max_{0\leq k\leq|\alpha|} L_{k,A}) \tilde{A}^{|\alpha|+|\beta|}\tilde{B}^j(|\alpha|+|\beta|)!^{\kappa} \times j!^{\kappa}\langle\xi\rangle_h^{m_1+m_2-|\alpha|-j}\exp[(\delta_1 \oplus \delta_2)\langle\xi\rangle_h^{1/\kappa}],$$

where

(2. 3)
$$\tilde{A}=2^{\kappa}\max(A, A_1) \text{ and } \hat{A}=A_2 \text{ if } \kappa'=\kappa,$$
$$\tilde{A}=2^{\kappa}\max(A, A_2) \text{ and } \hat{A}=A_1 \text{ if } \kappa'=1,$$

and $\tilde{B}=2^{\kappa}nA_1A_2$. By Lemma 2. 4

(2. 4)
$$q^R(x, \xi)=\sum_{j=0}^{\infty}\varphi_j^R(\xi)q_j(x, \xi)$$

can be defined for $R\geq 4e^{-1}(nA_1A_2)^{1/\kappa}$ and satisfies

(2. 5)
$$|q_{(\beta)}^{R(\alpha)}(x, \xi)| \leq C_{\tilde{A}'}(|\alpha|, L_{|\alpha|}, \tilde{A}, A^{-1}, \hat{A}/A)\tilde{A}'^{|\beta|}|\beta|!^{\kappa} \times\langle\xi\rangle_h^{m_1+m_2-|\alpha|}\exp[(\delta_1+\delta_2)\langle\xi\rangle_h^{1/\kappa}],$$

if $R\geq 4e^{-1}(nA_1A_2)^{1/\kappa}$, $\tilde{A}'>\tilde{A}$ and $L_{k,A}A^kk!^{\kappa}\equiv C_k$, and

(2. 6)
$$|q_{(\beta)}^{R(\alpha)}(x, \xi)| \leq C_{\tilde{A}}(C, \hat{A}/A)\tilde{A}^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^{\kappa} \times\langle\xi\rangle_h^{m_1+m_2-|\alpha|}\exp[(\delta_1+\delta_2)\langle\xi\rangle_h^{1/\kappa}],$$

if $R\geq 4e^{-1}(nA_1A_2)^{1/\kappa}$ and $L_{k,A}\equiv C$. Therefore, Proposition 2. 3 shows that $q^R(x, D)$ maps continuously $\mathscr{S}_{\kappa,\varepsilon}$ to $\mathscr{S}_{\kappa,\varepsilon-\delta_1-\delta_2}$ when $|\varepsilon-\delta_1-\delta_2| <\kappa(n\tilde{A})^{-1/\kappa}$.

Lemma 2. 5. *Let $\chi(x)$ be a function in $\mathscr{D}^{(\kappa)}$ such that $0\leq\chi(x)\leq 1$ and $\chi(x)=1$ near the origin. Then,*

$$\sigma(p(x, D_x, y, D_y))(x, \xi)$$
$$=\lim_{j\to\infty}\int e^{-iy\cdot\eta}p(x, \xi+\eta, x+y, \xi)\chi(\eta/j)\chi(y/j)dy\bar{d}\eta$$
$$(=Os-\int e^{-iy\cdot\eta}p(x, \xi+\eta, x+y, \xi)dy\bar{d}\eta)$$

if $\delta_1<\kappa(nA_2)^{-1/\kappa}$ when $\kappa''=\kappa$. Here $\sigma(p(x, D_x, y, D_y))(x, \xi)$ denotes the simplified symbol of p, that is, $p(x, D_x, y, D_y)u(x)=\sigma(p(x, D_x, y, D_y))(x, D)u(x)$ for $u\in\mathscr{D}^{(\kappa)}$.

Proof. Assume that $\delta_1<\kappa(nA_2)^{-1/\kappa}$ when $\kappa''=\kappa$. By the same argument as in the proof of Proposition 2. 3, we have

$$p(x, D_x, y, D_y)u(x) = \int G(x, \eta)\,d\!\!\bar{}\,\eta \quad \text{for } u \in \mathscr{D}^{(\kappa)},$$

where $G(x, \eta) = \int(\int e^{ix\cdot\xi}e^{i(x-y)\cdot(\eta-\xi)}p(x, \eta, y, \xi)\,\hat{u}(\xi)\,d\!\!\bar{}\,\xi)dy$ is integrable in η. And we have also

$$G(x, \eta) = \lim_{j\to\infty} G_j(x, \eta),$$

where $G_j(x, \eta) = \int e^{ix\cdot\xi}e^{i(x-y)\cdot(\eta-\xi)}\chi((\eta-\xi)/j)\,\chi((y-x)/j)\,p(x, \eta, y, \xi) \times \hat{u}(\xi)\,d\!\!\bar{}\,\xi dy$. Moreover, from the same argument as in the proof of Proposition 2.3, it follows that there is a function $F(x, \eta)$ integrable in η satisfying $|G_j(x, \eta)| \leq F(x, \eta)$ $(j = 1, 2, \cdots)$. Therefore, applying Lebesgue's theorem and Fubini's theorem, we have

$$p(x, D_x, y, D_y)u(x) = \lim_{j\to\infty} \int e^{ix\cdot\xi}(\int e^{-iy\cdot\eta} \times p(x, \xi+\eta, x+y, \xi)\,\chi(\eta/j)\,\chi(y/j)\,dy\,d\!\!\bar{}\,\eta)\,\hat{u}(\xi)\,d\!\!\bar{}\,\xi. \tag{2.7}$$

Similarly, there is a function $F_1(x, \xi)$ integrable in ξ such that

$$\begin{aligned}&|\hat{u}(\xi)\int e^{-iy\cdot\eta}p(x, \xi+\eta, x+y, \xi)\,\chi(\eta/j)\,\chi(y/j)\,dy\,d\!\!\bar{}\,\eta| \\ &= |\hat{u}(\xi)\int(\int e^{-iy\cdot\eta}\langle y\rangle^{-2M}\langle D_\eta\rangle^{2M}\{p(x, \xi+\eta, x+y, \xi) \\ &\quad \times \chi(\eta/j)\,\chi(y/j)\}dy)\,d\!\!\bar{}\,\eta| \leq F_1(x, \xi),\end{aligned}$$

where $M = [n/2]+1$. So we can apply Lebesgue's theorem to (2.7), which proves the lemma. **Q. E. D.**

Let $1 < \tilde{\kappa} < \kappa$, and let $\{\psi_N\}_{N=0,1,2,\dots}$ be a sequence in $\mathscr{D}^{(\tilde{\kappa})}$ such that $\psi_N(\xi) = 1$ if $|\xi| \leq 1/4$, $\psi_N(\xi) = 0$ if $|\xi| \geq 1/2$, and

$$|\psi_N^{(\alpha+\beta)}(\xi)| \leq C(A_3(N+1)/2)^{|\alpha|}B^{|\beta|}|\beta|!^{\tilde{\kappa}} \quad \text{for } |\alpha| \leq N+1, \tag{2.8}$$

where A_3, B and C are positive constants. By Lemma 2.2, for any $d > 0$ there is $C_d > 0$ such that

$$|\partial_\eta^{\alpha+\beta}\partial_\xi^\gamma \psi_N(\eta/\langle\xi\rangle_h)| \leq \begin{cases} C_d 2^N A_3^{|\alpha|} d^{|\beta|+|\gamma|}|\alpha|!(|\beta|+|\gamma|)!^\kappa \\ \times\langle\xi\rangle_h^{-(|\alpha|+|\beta|+|\gamma|)} \quad \text{for } |\alpha| \leq N+1, \\ C_d d^{|\beta|+|\gamma|}(|\beta|+|\gamma|)!^\kappa \\ \times\langle\xi\rangle_h^{-(|\beta|+|\gamma|)} \quad \text{if } \alpha = 0,\end{cases}$$

since $(N+1)^{|\alpha|} \leq (N+|\alpha|)!/N! \leq 2^{N+|\alpha|}|\alpha|!$. Define for $R \geq 4e^{-1}(nA_1A_2)^{1/\kappa}$

$$\begin{aligned}&r^R(x, D) = p(x, D_x, y, D_y) - q^R(x, D), \\ &r_{1N}^R(x, \xi) = (\varphi_N^R(\xi) - \varphi_{N+1}^R(\xi))\{Os - \int e^{-iy\cdot\eta}p(x, \xi+\eta, x+y, \xi) \\ &\quad \times\psi_N(\eta/\langle\xi\rangle_h)\,dy\,d\!\!\bar{}\,\eta - \textstyle\sum_{k=0}^N q_k(x, \xi)\},\end{aligned}$$

$$r_{2N}^{R}(x, \xi) = (\varphi_N^R(\xi) - \varphi_{N+1}^R(\xi))\,\{Os-\int e^{-iy\cdot\eta}p(x, \xi+\eta, x+y, \xi) \times (1-\psi_N(\eta/\langle\xi\rangle_h))\,dyd\!\!\bar{}\,\eta\}.$$

Then it is obvious that

$$r^R(x, \xi) \ \ (=\sigma(r^R(x, D))(x, \xi)) = \sum_{N=0}^{\infty}\{r_{1N}^R(x, \xi) + r_{2N}^R(x, \xi)\}.$$

First consider $r_{1N}^R(x, \xi)$. We can write

$$r_{1N}^R(x, \xi) = (\varphi_N^R(\xi) - \varphi_{N+1}^R(\xi))\int_0^1 \sum_{|\gamma|=N+1}(N+1)\gamma!^{-1}(1-\theta)^N \times (\int e^{-iy\cdot\eta}\langle y\rangle^{-2M} r_{1N}^{\gamma}(x, \xi, \theta y, \eta)\,dyd\!\!\bar{}\,\eta)\,d\theta,$$

where $r_{1N}^{\gamma}(x, \xi, \theta y, \eta) = \langle D_\eta\rangle^{2M}\partial_\eta^\gamma\{\psi_N(\eta/\langle\xi\rangle_h)\ (D_y^\gamma p)\ (x, \xi+\eta, x+\theta y, \xi)\}$ and $M=[n/2]+1$.

Lemma 2.6. *Put*

$$\delta = \begin{cases} (3/2)^{1/\kappa}\delta_1 + \delta_2 & \text{if } \delta_1 \geq 0, \\ 2^{-1/\kappa}\delta_1 + \delta_2 & \text{if } \delta_1 < 0, \end{cases}$$

(2.9) $$A' = \begin{cases} A & \text{if } \kappa' = \kappa, \\ \max\,(A, A_2) & \text{if } \kappa' = 1, \end{cases}$$

(2.10) $$\hat{A}_1 = \begin{cases} A_1 & \text{if } \kappa' = \kappa, \\ \max\,(A_1, A_3/3) & \text{if } \kappa' = 1. \end{cases}$$

Then,

(2.11) $$|r_{1N(\beta)}^{R(\alpha)}(x, \xi)| \leq C(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_3/A_1)\,(2^\kappa A')^{|\beta|} \times |\beta|!^\kappa \langle\xi\rangle_h^{m_1+m_2-|\alpha|+n}\exp[(\delta-\kappa/(2R))\langle\xi\rangle_h^{1/\kappa}]\,(N+1)^{\kappa/2}\rho^{N+1},$$

where $\rho = 7\cdot 2^\kappa n\hat{A}_1 A_2 R^{-\kappa}$.

Proof. For $|\gamma|=N+1$ we have

$$|\partial_\xi^\alpha D_x^\beta r_{1N}^\gamma(x, \xi, \theta y, \eta)| \leq 2^N C(|\alpha|, L_{|\alpha|}, A, A_1)\,(N+1)!\,(N+1+|\beta|)!^\kappa \times \langle\xi\rangle_h^{m_1+m_2-|\alpha|-N-1}\exp[\delta\langle\xi\rangle_h^{1/\kappa}] \times \sum_{\mu=0}^{|\beta|}\sum_{\nu=0}^{N+1} I(\mu, \nu)\,A^\mu(3A_1)^\nu A_2^{N+1+|\beta|-\mu}A_3^{N+1-\nu},$$

where $I(\mu, \nu) = |\beta|!\nu!^{\kappa'}\mu!^\kappa(N+1+|\beta|-\mu)!^{\kappa'}\{\mu!(|\beta|-\mu)!\nu!(N+1+|\beta|)!^\kappa\}^{-1}$. Here we have used the facts that $(j+k)! \leq C_\varepsilon(j)(1+\varepsilon)^k\times k!$ for $\varepsilon>0$ and that $\langle\xi\rangle_h/2 \leq \langle\xi+\eta\rangle_h \leq \langle\xi\rangle_h + |\eta| \leq 3\langle\xi\rangle_h/2$ if $\psi_N(\eta/\langle\xi\rangle_h) \neq 0$. It is obvious that

$$I(\mu, \nu) \leq \begin{cases} (|\beta|-\mu)!^{1-\kappa}(N+1-\nu)!^{1-\kappa} & \text{if } \kappa'=\kappa, \\ \binom{|\beta|}{\mu}^{1-\kappa} & \text{if } \kappa'=1. \end{cases}$$

Therefore, applying Lemma 2.2, we have

$$\begin{aligned} |\partial_\xi^\alpha D_x^\beta r_{1N}^\gamma(x, \xi, \theta y, \eta)| \leq & C(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_3/A_1) A'^{|\beta|} \\ & \times (7\hat{A}_1 A_2)^{N+1}(N+1)!(N+1+|\beta|)!^\kappa \\ & \times \langle\xi\rangle_h^{m_1+m_2-|\alpha|-N-1}\exp[\delta\langle\xi\rangle_h^{1/\kappa}]. \end{aligned}$$

This gives

$$\begin{aligned} |r_{1N(\beta)}^{R(\alpha)}(x, \xi)| \leq & C'(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_3/A_1) A'^{|\beta|} \\ & \times (7n\hat{A}_1 A_2)^{N+1}(N+1+|\beta|)!^\kappa\langle\xi\rangle_h^{m_1+m_2-|\alpha|-N-1+n} \\ & \times \exp[\delta\langle\xi\rangle_h^{1/\kappa}]\Phi_N^R(\xi), \end{aligned}$$

where $\Phi_N^R(\xi)$ is the characteristic function of $\{\xi\in\boldsymbol{R}^n;\ RN\leq\langle\xi\rangle_h^{1/\kappa}\leq 2R(N+1)\}$. From Lemma 2.2 it follows that

$$\begin{aligned} (2.12) \qquad |r_{1N(\beta)}^{R(\alpha)}(x, \xi)| \leq & C''(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_3/A_1) \\ & \times (2^\kappa A')^{|\beta|}|\beta|!^\kappa\langle\xi\rangle_h^{m_1+m_2-|\alpha|+n}\exp[(\delta-\tilde{\delta})\langle\xi\rangle_h^{1/\kappa}] \\ & \times (1+1/N)^{(N+1)\kappa}(N+1)^{\kappa/2}(7\cdot 2^\kappa n e^{2R\tilde{\delta}-\kappa}\hat{A}_1 A_2 R^{-\kappa})^{N+1} \end{aligned}$$

for $\tilde{\delta}\leq 0$. (2.12) with $\tilde{\delta}=\kappa/(2R)$ shows (2.11). **Q.E.D.**

Lemma 2.6. implies that

$$\begin{aligned} |\textstyle\sum_{N=0}^\infty r_{1N(\beta)}^{R(\alpha)}(x, \xi)| \leq & C(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_3/A_1)(2^\kappa A')^{|\beta|} \\ & \times |\beta|!^\kappa\langle\xi\rangle_h^{m_1+m_2-|\alpha|+n}\exp[(\delta-\kappa/(2R))\langle\xi\rangle_h^{1/\kappa}] \end{aligned}$$

for $R\geq 2^{1+3/\kappa}(n\hat{A}_1A_2)^{1/\kappa}$. Next let us estimate $r_{2N}^R(x, \xi)$. We can write

$$r_{2N}^R(x, \xi) = (\varphi_N^R(\xi) - \varphi_{N+1}^R(\xi))\int(\int e^{-iy\cdot\eta}r_{2N}(x, \xi, y, \eta)\,dy)\,đ\eta,$$

where $r_{2N}(x, \xi, y, \eta) = \langle y\rangle^{-2M}\langle D_\eta\rangle^{2M}\{p(x, \xi+\eta, x+y, \xi)(1-\psi_N(\eta/\langle\xi\rangle_h))\}$ and $M=[n/2]+1$.

Lemma 2.7. *Let A' be defined in Lemma* 2.6, and let $B>0$ if $\kappa'=\kappa$ *and* $B=A_2$ *if* $\kappa'=1$. *Then,*

$$\begin{aligned} |r_{2N(\beta)}^{R(\alpha)}(x, \xi)| \leq & C(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_2/B, 1/B) \\ & \times (2^\kappa A')^{|\beta|}|\beta|!^\kappa\exp[\delta'\langle\xi\rangle_h^{1/\kappa}](4R^{-2}+1)(N+1)^{-2} \end{aligned}$$

if $|\delta_1|\leq 2^{-3}\kappa(nB)^{-1/\kappa}$ *and* $\delta'=\delta_1+\delta_2-4^{-1-1/\kappa}\kappa(nB)^{-1/\kappa}$.

Proof. The same calculation as in the proof of Lemma 2.6 yields

$$|\partial_\xi^\alpha D_x^\beta D_y^\gamma r_{2N}(x, \xi, y, \eta)| \le C(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_2/B, 1/B) \times A'^{|\beta|} B^{|\gamma|}(|\beta|+|\gamma|)!^\kappa \langle y\rangle^{-2M} |\eta|^{|m_1|+|m_2|} \times \exp[\delta_1\langle\xi+\eta\rangle_h^{1/\kappa}+\delta_2\langle\xi\rangle_h^{1/\kappa}] \Psi(\eta/\langle\xi\rangle_h),$$

where $\Psi(\xi)$ is the characteristic function of $\{\xi\in\mathbb{R}^n;\ |\xi|\ge 1/4\}$. Thus we have

$$\begin{aligned} &|\partial_\xi^\alpha D_x^\beta \int e^{-iy\cdot\eta} r_{2N}(x, \xi, y, \eta)\,dy| \\ &\le C_{B'}(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_2/B, 1/B)(2^\kappa A')^{|\beta|}|\beta|!^\kappa \langle\eta\rangle^{-n-1} \\ &\quad\times\exp[-2^{-1}\kappa(nB')^{-1/\kappa}|\eta|^{1/k}+\delta_1\langle\xi+\eta\rangle_h^{1/\kappa}+\delta_2\langle\xi\rangle_h^{1/\kappa}] \\ &\quad\times\Psi(\eta/\langle\xi\rangle_h)\langle\xi\rangle_h^{-2/\kappa} \\ &\le C_{B'}(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_2/B, 1/B)(2^\kappa A')^{|\beta|}|\beta|!^\kappa\langle\eta\rangle^{-n-1} \\ &\quad\times\exp[\delta''\langle\xi\rangle_h^{1/\kappa}]\langle\xi\rangle_h^{-2/\kappa} \end{aligned}$$

if $B'>B$, $|\delta_1|\le 2^{-1}\kappa(nB')^{-1/\kappa}$ and $\delta''=\delta_1+\delta_2+4^{-1/\kappa}(|\delta_1|-2^{-1}\kappa(nB')^{-1/\kappa})$, which proves the lemma. **Q. E. D.**

Lemma 2.7. implies that

$$|\sum_{N=0}^\infty r_{2N(\beta)}^{R(\alpha)}(x, \xi)| \le C(|\alpha|, L_{|\alpha|}, A, A_1, A_2/A, A_2/B, 1/B, 1/R) \times(2^\kappa A')^{|\beta|}|\beta|!^\kappa \exp[\delta'\langle\xi\rangle_h^{1/\kappa}]$$

if $|\delta_1|\le 2^{-3}\kappa(nB)^{-1/\kappa}$. So we have the following

Proposition 2.8. *Let $p(x, \xi, y, \eta)$ satisfy* (2.1). *Then there are $r_0>0$ and $\delta(1/\hat{A}_1, 1/A_2)>0$ such that $\delta(1/\hat{A}_1, 1/A_2)=\delta(1/A_2)A_1^{-1/\kappa}$ if $\kappa'=\kappa$, $\delta(1/\hat{A}_1, 1/A_2)=\delta'(1/\hat{A}_1)A_2^{-1/\kappa}$ if $\kappa'=1$, and the following estimates hold if $a\ge 1$ and $R=ar_0\hat{A}_1^{1/\kappa}A_2^{1/\kappa}$:*

$$\begin{aligned} &|q_{(\beta)}^{R(\alpha)}(x, \xi)| \le C_{\tilde{A}'}(|\alpha|, L_{|\alpha|}, \tilde{A}, A^{-1}, \hat{A}/A)\tilde{A}'^{|\beta|}|\beta|!^\kappa \\ &\qquad\times\langle\xi\rangle_h^{m_1+m_2-|\alpha|}\exp[(\delta_1+\delta_2)\langle\xi\rangle_h^{1/\kappa}] \\ &\qquad \textit{if } \tilde{A}'>\tilde{A} \textit{ and } L_{k,A}A^k k!^\kappa \equiv C_k, \\ &|q_{(\beta)}^{R(\alpha)}(x, \xi)| \le C_{\tilde{A}}(C, \hat{A}/A)\tilde{A}^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa \\ &\qquad\times\langle\xi\rangle_h^{m_1+m_2-|\alpha|}\exp[(\delta_1+\delta_2)\langle\xi\rangle_h^{1/\kappa}] \\ &\qquad \textit{if } L_{k,A}\equiv C, \\ &|r_{(\beta)}^{R(\alpha)}(x, \xi)| \le C_{A_1, A_2}(|\alpha|, L_{|\alpha|}, A, A_2/A)(2^\kappa A')^{|\beta|}|\beta|!^\kappa \\ &\qquad\times\exp[(\delta_1+\delta_2-\delta(1/\hat{A}_1, 1/A_2)/a)\langle\xi\rangle_h^{1/\kappa}] \\ &\qquad \textit{if } |\delta_1|\le\delta(1/\hat{A}_1, 1/A_2)/a, \end{aligned}$$

where $q^R(x, \xi)$ is the symbol defined by (2.2) *and* (2.4), *$r^R(x, \xi)=\sigma(p(x, D_x, y, D_y))(x, \xi)-q^R(x, \xi)$, and $\tilde{A}$ and $\hat{A}$, $\hat{A}_1$ and A' are defined by* (2.3), (2.10) *and* (2.9), *respectively.*

Proof. If, for example, we choose $r_0=2^{1+3/\kappa}n^{1/\kappa}$ and

$$\delta(1/\hat{A}_1,\ 1/A_2)=\begin{cases}2^{-3-3/\kappa}\kappa(n\hat{A}_1A_2)^{-1/\kappa} & when\ \kappa'=\kappa,\\ 2^{-3-3\kappa}\kappa(nA_2)^{-1/\kappa}\min(\hat{A}_1^{-1/\kappa},\ 2^{1/\kappa}) & when\ \kappa'=1,\end{cases}$$

then the proposition easily follows from (2.5), (2.6) and Lemmas 2.6 and 2.7. **Q.E.D.**

Let $\Lambda(x,\xi)$ be a symbol satisfying

$$|\Lambda_{(\beta)}^{(\alpha)}(x,\xi)|\le C_0A_0^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\langle\xi\rangle_h^{1/\kappa-|\alpha|},\tag{2.13}$$

and set $\omega_\beta^\alpha(\Lambda;\ x,\xi)=e^{-\Lambda(x,\xi)}(e^{\Lambda(x,\xi)})_{(\beta)}^{(\alpha)}$.

Lemma 2.9. *If $A_1>A_0$, $\rho>0$ and $A_1/A_2+C_0A_0\rho^{-1}A_2^{-1}(1-A_0/A_1)^{-1}\le 1$, then*

$$|\omega_{\beta(\delta)}^{\alpha(\gamma)}(\Lambda;\ x,\xi)|\le A_1^{|\gamma|+|\delta|}A_2^{|\alpha|+|\beta|}(|\alpha|+|\beta|+|\gamma|+|\delta|)!\times\langle\xi\rangle_h^{-|\alpha|-|\gamma|}\textstyle\sum_{k=0}^{|\alpha|+|\beta|}\rho^k\langle\xi\rangle_h^{k/\kappa}/k!.\tag{2.14}$$

In particular, we can take $A_1=(1+(C_0/\rho)^{1/2})A_0$ and $A_2=(1+(C_0/\rho)^{1/2})^2A_0$ for $\rho>0$.

Proof. It is obvious that (2.14) holds for $|\alpha|+|\beta|=0$. Assume that (2.14) holds for $|\alpha|+|\beta|\le N$. Let $|\alpha|+|\beta|=N$ and $|e|+|e'|=1$. Then

$$\begin{aligned}&|\omega_{\beta+e'(\delta)}^{\alpha+e(\gamma)}(\Lambda;x,\xi)|=\\&|\omega_{\beta(\delta+e')}^{\alpha(\gamma+e)}(\Lambda;x,\xi)+(\Lambda_{(e')}^{(e)}(x,\xi)\omega_\beta^\alpha(\Lambda;x,\xi))_{(\delta)}^{(\gamma)}|\\&\quad\le A_1^{|\gamma|+|\delta|}A_2^{N+1}(N+|\gamma|+|\delta|+1)!\langle\xi\rangle_h^{-|\alpha|-|\gamma|-|e|}\\&\quad\times\textstyle\sum_{k=0}^{N+1}\rho^k\langle\xi\rangle_h^{k/\kappa}/k!\,\{A_1/A_2+\sum_{\mu=0}^{|\gamma|+|\delta|}\binom{|\gamma|+|\delta|}{\mu}\\&\quad\times\binom{N+|\gamma|+|\delta|+1}{\mu+1}^{-1}(N+1)C_0A_0\rho^{-1}A_2^{-1}(A_0/A_1)^\mu\},\end{aligned}$$

which proves the lemma. **Q.E.D.**

Corollary. *For $\rho>0$,*

$$|(e^{\Lambda(x,\xi)})_{(\beta)}^{(\alpha)}|\le\{(1+(C_0/\rho)^{1/2})^2A_0\}^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\times\langle\xi\rangle_h^{-|\alpha|}\exp[\rho\langle\xi\rangle_h^{1/\kappa}+\mathrm{Re}\ \Lambda(x,\xi)].$$

Lemma 2.10. *Let $p(x,\xi)$ be a symbol satisfying*

$$|p_{(\beta)}^{(\alpha)}(x,\xi)|\le L_{|\alpha|,A}A^{|\alpha|+|\beta|}|\alpha|!^\kappa|\beta|!^\kappa\langle\xi\rangle_h^{m-|\alpha|}\exp[\delta\langle\xi\rangle_h^{1|\kappa}],\tag{2.15}$$

where $m, \delta\in \boldsymbol{R}$ *and* $L_{k,A}A^k k!^\kappa\equiv C_k$ *or* $L_{k,A}\equiv C$, *and set* $\lambda_0=\inf_{L>0}\sup_{x\in \boldsymbol{R}^n, |\xi|\geq L}\mathrm{Re}\, \Lambda(x, \xi)\langle\xi\rangle_h^{-1/\kappa}$. *Then* $(e^\Lambda)(x, D)p(x, D)$ *maps continuously* $\mathscr{S}_{\kappa,\varepsilon}$ *to* $\mathscr{S}_{\kappa,\varepsilon-\rho}$ *if* $\rho>\lambda_0+\delta$ *and* $|\varepsilon-\delta|<\kappa(nA)^{-1/\kappa}$. *Moreover there are* $r(A_0)>0$ *and* $\delta_{A_0}>0$ *such that* $q^R(x, \xi)=\sum_{j=0}^\infty\varphi_j^R(\xi)q_j(x, \xi)$ *is well-defined, and* $q^R(x, \xi)$ *and* $r^R(x, \xi)\equiv\sigma((e^\Lambda)(x, D)p(x, D))(x, \xi)-q^R(x, \xi)$ *satisfy the following estimates if* $a\geq 1$, $R=ar(A_0)A^{1/\kappa}$ *and* $\rho>\lambda_0+C_0+\delta$, *where* $q_j(x, \xi)=\sum_{|\alpha|=j}\alpha!^{-1}\omega^\alpha(\Lambda; x, \xi)p_{(\alpha)}(x, \xi)e^{\Lambda(x,\xi)}$:

$$(2.16)\qquad |q_{(\beta)}^{R(\alpha)}(x, \xi)|\leq C_{\rho,A'}(|\alpha|, L_{|\alpha|}, A, A^{-1}, A_0/A)(2^\kappa A')^{|\beta|} \times|\beta|!^\kappa\langle\xi\rangle_h^{m-|\alpha|}\exp[\rho\langle\xi\rangle_h^{1/\kappa}] \quad \text{if } A'>A \text{ and } L_{K,A}A^k k!^\kappa\equiv C_k,$$

$$(2.17)\qquad |q_{(\beta)}^{R(\alpha)}(x, \xi)|\leq C_{\rho,A}(C, A_0/A)(2^\kappa A)^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa \times\langle\xi\rangle_h^{m-|\alpha|}\exp[\rho\langle\xi\rangle_h^{1/\kappa}] \quad \text{if } L_{k,A}\equiv C,$$

$$(2.18)\qquad |r_{(\beta)}^{R(\alpha)}(x, \xi)|\leq C_{\rho,A,A_0}(|\alpha|, L_{|\alpha|})(2^\kappa A)^{|\beta|} \times|\beta|!^\kappa\exp[(\rho-a^{-1}\delta_{A_0}A^{-1/\kappa})\langle\xi\rangle_h^{1/\kappa}] \quad \text{if } |\lambda_0+C_0|<a^{-1}\delta_{A_0}A^{-1/\kappa}.$$

Remark. (i) If $|\varepsilon-\delta|<\kappa(nA)^{-1/\kappa}$ and $|\varepsilon-\lambda_0-C_0-\delta|<2^{-1}\kappa(nA)^{-1/\kappa}$, then $(e^\Lambda)(x, D)p(x, D)=q^R(x, D)+r^R(x, D)$ on $\mathscr{S}_{\kappa,\varepsilon}$. (ii) For example, one can take $r(A_0)=2^{1+3/\kappa}(n\hat{A}_1)^{1/\kappa}$ and $\delta_{A_0}=2^{-3-3/\kappa}\kappa n^{-1/\kappa}\min(\hat{A}_1^{-1/\kappa}, 2^{1/\kappa})$, where $\hat{A}_1=\max(8A_0, A_3/3)$ and A_3 is the constant in (2.8).

Proof. From the corollary of Lemma 2.9 it follows that

$$|(e^{\Lambda(x,\xi)}p(y, \eta))_{(\beta)(\delta)}^{(\alpha)(\gamma)}|\leq C_\rho(A_0/A)L_{|\gamma|,A}A^{|\beta|+|\gamma|} \times(8A_0)^{|\alpha|}A^{|\delta|}|\alpha|!|\beta|!^\kappa|\gamma|!^\kappa|\delta|!^\kappa\langle\xi\rangle_h^{-|\alpha|}\langle\eta\rangle_h^{m-|\gamma|} \times\exp[\rho\langle\xi\rangle_h^{1/\kappa}+\delta\langle\eta\rangle_h^{1/\kappa}]$$

if $\rho>\lambda_0+C_0$. Therefore, the lemma immediately follows from Propositions 2.3 and 2.8. **Q.E.D.**

Lemma 2.11 ([6], [7], [17], [19]). *Let* $0\leq\rho<1$ *and* $m\in\boldsymbol{R}$. *Then, for each* $s\in\boldsymbol{R}$ *there are* $C_s>0$ *and a non-negative integer* $\hat{N}_s$ *such that*

$$\|\langle D\rangle_h^s a(x, D)u\|_{L^2}\leq C_s M\|\langle D\rangle_h^{s+m}u\|_{L^2} \quad \text{for } u\in H^{s+m}$$

if $|a_{(\beta)}^{(\alpha)}(x, \xi)|\leq M\langle\xi\rangle_h^{m+(|\beta|-|\alpha|)\rho}$ *for* $(x, \xi)\in T^*\boldsymbol{R}^n$, $h\geq 1$, $|\alpha|\leq\hat{N}_s$ *and* $|\beta|\leq\hat{N}_s$, *where* H^s *denotes the Sobolev space of order s.*

Proof. Make a change of variables: $y=h^{\rho}x$. Taking $\lambda(\xi)=h^{\rho-1}\times\langle\xi\rangle_{h^{1-\rho}}$ as a basic weight function, Theorem 1.6 in Chapter 7 of [19] gives the lemma. **Q. E. D.**

Proposition 2.12. *There is* $\varepsilon_0>0$ *such that* $p(x, D)$ *maps continuously* $H^s_{\kappa,\varepsilon}$ *to* $H^{s-m}_{\kappa,\varepsilon-\delta}$ *if* $p(x,\xi)$ *satisfies* (2.15) *and* $|\varepsilon-\delta|<\varepsilon_0A^{-1/\kappa}$.

Remark. Proposition 2.12 was proved in [13] and [24] when $\delta=0$.

Proof. It suffices to show that $\exp[(\varepsilon-\delta)\langle D\rangle_h^{1/\kappa}]p(x,D)\exp[-\varepsilon\times\langle D\rangle_h^{1/\kappa}]$ maps continuously H^s to H^{s-m}. By Lemma 2.10 and its remark we can write

$$\exp[(\varepsilon-\delta)\langle D\rangle_h^{1/\kappa}]p(x,D)=q(x,D)+r(x,D),$$

where $q(x,\xi)=\sum_{j=0}^{\infty}\varphi_j^R(\xi)q_j(x,\xi)$, $q_j(x,\xi)=\sum_{|\alpha|=j}\alpha!^{-1}\omega^\alpha(\xi)p_{(\alpha)}(x,\xi)$ $\exp[(\varepsilon-\delta)\langle\xi\rangle_h^{1/\kappa}]$, $R=2^{1+3/\kappa}(n\hat{A}_1A)^{1/\kappa}$, $\hat{A}_1=\max(8A_0, A_3/3)$, $A_0=1+\sqrt{2}$ and $\omega^\alpha(\xi)\equiv\omega^\alpha((\varepsilon-\delta)\langle\xi\rangle_h^{1/\kappa};x,\xi)$. Moreover, we have

$$\begin{aligned}|r^{(\alpha)}_{(\beta)}(x,\xi)|\le{}&C_{\rho,A}(|\alpha|,L_{|\alpha|})(2^\kappa A)^{|\beta|}|\beta|!^\kappa\\&\times\exp[(\rho-2\varepsilon_0A^{-1/\kappa})\langle\xi\rangle_h^{1/\kappa}]\end{aligned}$$

if $\rho>\varepsilon+|\varepsilon-\delta|$ and $|\varepsilon-\delta|+\varepsilon-\delta<2\varepsilon_0A^{-1/\kappa}$, where $\varepsilon_0=2^{-4-3/\kappa}\kappa(n\hat{A}_1)^{-1/\kappa}$. Therefore, we have

$$|(r(x,\xi)\exp[-\varepsilon\langle\xi\rangle_h^{1/\kappa}])^{(\alpha)}_{(\beta)}|\le C_A(|\alpha|,|\beta|,L_{|\alpha|})\langle\xi\rangle_h^{m-|\alpha|}\tag{2.19}$$

if $|\varepsilon-\delta|<\varepsilon_0A^{-1/\kappa}$. On the other hand, a simple calculation yields

$$\begin{aligned}&|(q_j(x,\xi)\exp[-\varepsilon\langle\xi\rangle_h^{1/\kappa}])^{(\alpha)}_{(\beta)}|\\&\le C(|\alpha|,|\beta|,L_{|\alpha|},A)(nAA_0')^j j!^\kappa\langle\xi\rangle_h^{m-(1-1/\kappa)|\alpha|-j}\\&\quad\times\sum_{k=0}^{j}C_0^k\langle\xi\rangle_h^{k/\kappa}/k!\\&\le C'(|\alpha|,|\beta|,L_{|\alpha|},A)\langle\xi\rangle_h^{m-(1-1/\kappa)|\alpha|}j^{(\kappa-1)/2}(j+1)\\&\quad\times\{e^{1-\kappa}nAA_0'R^{1-\kappa}\max(C_0,R^{-1})\}^j\ \text{if}\ \langle\xi\rangle_h^{1/\kappa}\ge Rj,\end{aligned}$$

where $C_0=|\varepsilon-\delta|$ and $A_0'=8A_0$. Since

$$e^{1-\kappa}nAA_0'R^{1-\kappa}\max(C_0,R^{-1})<1\ \text{when}\ C_0<\varepsilon_0A^{-1/\kappa},$$

we have

$$\begin{aligned}&|(q(x,\xi)\exp[-\varepsilon\langle\xi\rangle_h^{1/\kappa}])^{(\alpha)}_{(\beta)}|\\&\le C''(|\alpha|,|\beta|,L_{|\alpha|},A)\langle\xi\rangle_h^{m-(1-1/\kappa)|\alpha|}\end{aligned}\tag{2.20}$$

if $|\varepsilon-\delta|<\varepsilon_0A^{-1/\kappa}$. Thus (2.19), (2.20) and Lemma 2.11 show that

$\exp[(\varepsilon-\delta)\langle D\rangle_h^{1/\kappa}]p(x, D)\exp[-\varepsilon\langle D\rangle_h^{1/\kappa}]$ maps continuously H^s to H^{s-m} if $|\varepsilon-\delta|<\varepsilon_0 A^{-1/\kappa}$. **Q. E. D.**

Proposition 2.13. *Assume that $\Lambda(x, \xi)$ satisfies* (2.13) *and that $p(x, \xi)$ is a symbol satisfying* (2.15) *with $L_{k,A}\equiv C$. Then $(e^{\Lambda})(x, D)p(x, D)$ ${}^R(e^{-\Lambda})(x, D)$ maps continuously $\mathscr{S}_{\kappa,\varepsilon}$ to $\mathscr{S}_{\kappa,\varepsilon-\rho}$ if $\rho>\lambda_0+\lambda_1+\delta$ and $|\varepsilon-\lambda_1-\delta|<\kappa(nA)^{-1/\kappa}$, and $H^s_{\kappa,\varepsilon}$ to $H^{s-m}_{\kappa,\varepsilon-\rho}$ if $\rho>\lambda_0+\lambda_1+\delta$, $|\varepsilon-\lambda_1-\delta|<\varepsilon_0 A^{-1/\kappa}$ and $s\in\boldsymbol{R}$, where ε_0 and λ_0 are the constants defined in Proposition* 2.12 *and Lemma* 2.10, *respectively, and $\lambda_1=\inf_{L>0}\sup_{x\in R^n,|\xi|\geq L} -\mathrm{Re}\,\Lambda(x, \xi)\langle\xi\rangle_h^{-1/\kappa}$. Here ${}^R(e^{-\Lambda})(x, D)$ denotes the transposed operator of $(e^{-\Lambda})(x, -D)$. Moreover there is $c_{A_0}>0$ such that there are symbols $p_\Lambda(x, \xi)$ and $r_\Lambda(x, \xi)$ satisfying the following properties if $C_0<c_{A_0}A^{-1/\kappa}$ and $|\delta|<c_{A_0}A^{-1/\kappa}$:*

$$
\begin{aligned}
&(e^{\Lambda})(x, D)p(x, D)\,{}^R(e^{-\Lambda})(x, D)=p_\Lambda(x, D)+r_\Lambda(x, D),\\
(2.21)\qquad &|\{p_\Lambda(x, \xi)-\textstyle\sum_{|\alpha|+|\beta|<N}(\alpha!\beta!)^{-1}(p_{(\beta)}(x, \xi)\omega^\beta(\Lambda; x, \xi)\\
&\quad\times\omega_\alpha(-\Lambda; x, \xi))^{(\alpha)}\}^{(\tilde\alpha)}_{(\tilde\beta)}|\leq C_{A,A_0}(C, N)(2^\kappa A)^{|\tilde\alpha|+|\tilde\beta|}\\
&\quad\times(|\tilde\alpha|+|\tilde\beta|)!^\kappa\langle\xi\rangle_h^{m-|\tilde\alpha|-(1-1/\kappa)N}\exp[\delta\langle\xi\rangle_h^{1/\kappa}],\\
&\qquad\qquad N=0, 1, 2, \dots,
\end{aligned}
$$

$$
\begin{aligned}
(2.22)\qquad &|r_{\Lambda(\beta)}^{(\alpha)}(x, \xi)|\leq C_{A,A_0}(|\alpha|, C)(2^{3\kappa}A)^{|\beta|}|\beta|!^\kappa\\
&\quad\times\exp[(\delta-c_{A_0}A^{-1/\kappa})\langle\xi\rangle_h^{1/\kappa}],
\end{aligned}
$$

$$
\begin{aligned}
(2.23)\qquad &r_\Lambda:\ \mathscr{S}_{\kappa,\varepsilon}\to\mathscr{S}_{\kappa,\varepsilon-\rho}\ \text{continuously if}\ \rho=\delta-c_{A_0}A^{-1/\kappa}\ \text{and}\\
&\quad|\varepsilon-\delta|<2^{-4}\kappa(nA)^{-1/\kappa},
\end{aligned}
$$

$$
\begin{aligned}
(2.24)\qquad &r_\Lambda:\ H^s_{\kappa,\varepsilon}\to H^{s'}_{\kappa,\varepsilon-\rho}\ \text{continuously if}\ \rho=\delta-c_{A_0}A^{-1/\kappa},\\
&\quad s, s'\in\boldsymbol{R}\ \text{and}\ |\varepsilon-\delta|<2^{-4}\varepsilon_0A^{-1/\kappa},
\end{aligned}
$$

Proof. We set $q(x, \xi)=\sum_{j=0}^\infty\varphi_j^R(\xi)q_j(x, \xi)$ and $r(x, \xi)=\sigma((e^{\Lambda})(x, D)p(x, D))(x, \xi)-q(x, \xi)$, where $R=ar(A_0)A^{1/\kappa}$, $a\geq 1$, $q_j(x, \xi)=\sum_{|\alpha|=j}\alpha!^{-1}\omega^\alpha(\Lambda; x, \xi)p_{(\alpha)}(x, \xi)e^{\Lambda(x,\xi)}$ and $r(A_0)$ is the constant in Lemma 2.10. Lemma 2.10 implies that $q(x, \xi)$ and $r(x, \xi)$ satisfy (2.17) and (2.18), respectively. The symbol $r'_\Lambda(x, \xi)\equiv\sigma(r(x, D)\,{}^R(e^{-\Lambda})(x, D))$ can be written as

$$r'_\Lambda(x, \xi)=Os-\int e^{-iy\cdot\eta}r(x, \xi+\eta)e^{-\Lambda(x+y,\xi+\eta)}dy\,d\!\!\bar{}\,\eta$$

if $|\lambda_0+\lambda_1+C_0+\delta|<a^{-1}\delta_{A_0}A^{-1/\kappa}$ and $|\lambda_0+C_0|<a^{-1}\delta_{A_0}A^{-1/\kappa}$, where δ_{A_0} is the constant in Lemma 2.10. Then we have

$$r'^{(\alpha)}_{A(\beta)}(x,\xi)=\int f_{\alpha,\beta}(x,\xi,\eta)\,d\!\!{}^-\eta,$$
$$f_{\alpha,\beta}(x,\xi,\eta)$$
$$=\int e^{-iy\cdot\eta}\langle y\rangle^{-2M}\partial_\xi^\alpha D_x^\beta\langle D_\eta\rangle^{2M}\{r(x,\xi+\eta)e^{-A(x+y,\xi+\eta)}\}\,dy,$$

where $M=[n/2]+1$. A simple calculation gives

$$|\eta^\gamma f_{\alpha,\beta}(x,\xi,\eta)|\le C_{A,A_0,d}(|\alpha|,C)(2^\kappa A)^{|\beta|}|\beta|!^\kappa|\gamma|!^\kappa d^{|\gamma|}$$
$$\times\exp[(\delta-c_{A_0}A^{-1/\kappa})\langle\xi+\eta\rangle_h^{1/\kappa}]$$

if $d>0$, $|\lambda_0+\lambda_1+2C_0|<a^{-1}\delta_{A_0}A^{-1/\kappa}/4$, $c_{A_0}\le a^{-1}\delta_{A_0}/2$, $|\lambda_0+\lambda_1+C_0+\delta|<a^{-1}\delta_{A_0}A^{-1/\kappa}$ and $|\lambda_0+C_0|<a^{-1}\delta_{A_0}A^{-1/\kappa}$. This gives

$$|f_{\alpha,\beta}(x,\xi,\eta)|\le C_{A,A_0}(|\alpha|,C)(2^\kappa A)^{|\beta|}|\beta|!^\kappa$$
$$\times\exp[(\delta-c_{A_0}A^{-1/\kappa})\langle\xi\rangle_h^{1/\kappa}-|\eta|^{1/\kappa}],$$

(2.25) $$|r'^{(\alpha)}_{A(\beta)}(x,\xi)|\le C'_{A,A_0}(|\alpha|,C)(2^\kappa A)^{|\beta|}|\beta|!^\kappa$$
$$\times\exp[(\delta-c_{A_0}A^{-1/\kappa})\langle\xi\rangle_h^{1/\kappa}],$$

if $|\delta|<a^{-1}\delta_{A_0}A^{-1/\kappa}/2$, $C_0<2^{-4}a^{-1}\delta_{A_0}A^{-1/\kappa}$ and $c_{A_0}\le a^{-1}\delta_{A_0}/2$. Put

$$p_A(x,\xi)=\sum_{j=0}^\infty\varphi_j^R(\xi)\sum_{|\alpha|=j}\alpha!^{-1}\{q(x,\xi)(e^{-A(x,\xi)})_{(\alpha)}\}^{(\alpha)},$$
$$r''(x,\xi)=\sigma(q(x,D)^R(e^{-A})(x,D))(x,\xi)-p_A(x,\xi),$$

where $R=ar(A_0)A^{1/\kappa}$, $a\ge a_0(\ge 2)$, a_0 is a constant satisfying $a_0r(A_0)\ge 2^{2+3/\kappa}r_0A_0^{1/\kappa}$ and r_0 is the constant in Proposition 2.8. Then it follows from Proposition 2.8 that

(2.26) $$|r''^{(\alpha)}_{A(\beta)}(x,\xi)|\le C_{\rho,A,A_0}(|\alpha|,C)(2^{3\kappa}A)^{|\beta|}|\beta|!^\kappa$$
$$\times\exp[(\rho-a^{-1}\delta'_{A_0}A^{-1/\kappa})\langle\xi\rangle_h^{1/\kappa}]$$

if $\rho>\lambda_0+\lambda_1+2C_0+\delta$ and $|\lambda_0+\lambda_1+2C_0+\delta|<a^{-1}\delta'_{A_0}A^{-1/\kappa}$, where $\delta'_{A_0}=2^{-2}a_0\delta(2^{-3}A_0^{-1})$ and $\delta(\cdot)$ is the constant in Proposition 2.8. In fact,

$$|\{q(x,\xi)e^{-A(y,\xi)}\}^{(\alpha)}_{(\beta)(\gamma)}|\le C_{\rho,A}(C,A_0/A)(2^{2\kappa}A)^{|\beta|}$$
$$\times(2^{2\kappa}A)^{|\alpha|}(8A_0)^{|\gamma|}|\alpha|!^\kappa|\beta|!^\kappa|\gamma|!\langle\xi\rangle_h^{m-|\alpha|}\exp[\rho\langle\xi\rangle_h^{1/\kappa}]$$

if $\rho>\lambda_0+\lambda_1+2C_0+\delta$. (2.25), (2.26) and Propositions 2.3 and 2.12 imply that $r_A(x,\xi)\equiv r'_A(x,\xi)+r''_A(x,\xi)$ satisfies (2.22)-(2.24) if $c_{A_0}\le\min(2^{-4}a^{-1}\delta_{A_0},a^{-1}\delta'_{A_0}/6,2^{-4}\kappa n^{-1/\kappa},2^{-4}\varepsilon_0)$, $C_0<c_{A_0}A^{-1/\kappa}$, $|\delta|<c_{A_0}A^{-1/\kappa}$ and $a\ge a_0$. A simple calculation yields

$$|\textstyle\sum_{|\alpha|=j,|\beta|=k}g^{(\alpha+\tilde\alpha)}_{\alpha,\beta(\tilde\beta)}(x,\xi)|$$
$$\le C(C,A_0/A)(8nA_0A)^{j+k}A^{|\tilde\alpha|+|\tilde\beta|}(|\tilde\alpha|+|\tilde\beta|+j+k)!^\kappa\langle\xi\rangle_h^{m-|\tilde\alpha|-j-k}$$
$$\times\exp[\delta\langle\xi\rangle_h^{1/\kappa}](\textstyle\sum_{l=0}^jC_0^l\langle\xi\rangle_h^{l/\kappa}/l!)(\sum_{l=0}^kC_0^l\langle\xi\rangle_h^{l/\kappa}/l!),$$

where $g_{\alpha,\beta}(x,\xi)=(\alpha!\beta!)^{-1}p_{(\beta)}(x,\xi)\omega^{\beta}(\Lambda;x,\xi)\omega_{\alpha}(-\Lambda;x,\xi)$. Here we have used the inequalities that

$$\begin{aligned}&\sum_{\mu_1+\mu_2+\mu_3=|\tilde\alpha|+j,\nu_1+\nu_2+\nu_3=|\tilde\beta|}(|\tilde\alpha|+j)!|\tilde\beta|!\mu_1!^{\kappa}(\nu_1+k)!^{\kappa}\\ &\quad\times(\mu_2+\nu_2+k)!(\mu_3+\nu_3+j)!\{j!k!\mu_1!\mu_2!\mu_3!\nu_1!\nu_2!\nu_3!\\ &\quad\times(|\tilde\alpha|+|\tilde\beta|+j+k)!^{\kappa}\}^{-1}(2A_0/A)^{\mu_2+\mu_3+\nu_2+\nu_3}\\ &\le 2^{j+k}\sum_{\mu_1+\mu_2+\mu_3+\mu_4=|\tilde\alpha|+|\tilde\beta|+j}\mu_3!^{1-\kappa}\mu_4!^{1-\kappa}\binom{\mu_1+\mu_2}{\mu_1}^{1-\kappa}\\ &\quad\times(4A_0/A)^{\mu_3+\mu_4}\le 2^{j+k}C(A_0/A).\end{aligned}$$

Since $\langle\xi\rangle_h^{1/\kappa}\ge 2(N-1)R$ if $j+k<N$, $\varphi_j^R(\xi)=1$ and $\varphi_k^R(\xi)=1$, we have

$$\begin{aligned}&|\sum_{|\alpha|+|\beta|<N}\{\varphi_{|\alpha|}^R(\xi)(\varphi_{|\beta|}^R(\xi)g_{\alpha,\beta}(x,\xi))^{(\alpha)}-g_{\alpha,\beta}^{(\alpha)}(x,\xi)\}_{(\tilde\beta)}^{(\tilde\alpha)}|\\ &\le C(C,A_0/A,N,C_0,R,A_0A)(2^{\kappa}A)^{|\tilde\alpha|+|\tilde\beta|}(|\tilde\alpha|+|\tilde\beta|)!^{\kappa}\\ &\quad\times\langle\xi\rangle_h^{m-|\tilde\alpha|-(1-1/\kappa)N}\exp[\delta\langle\xi\rangle_h^{1/\kappa}].\end{aligned}$$

Moreover, we have

$$\begin{aligned}&|\sum_{|\alpha|+|\beta|\ge N}\{\varphi_{|\alpha|}^R(\xi)(\varphi_{|\beta|}^R(\xi)g_{\alpha,\beta}(x,\xi))^{(\alpha)}\}_{(\tilde\beta)}^{(\tilde\alpha)}|\\ &\le C(C,A_0/A)(2^{\kappa}A)^{|\tilde\alpha|+|\tilde\beta|}(|\tilde\alpha|+|\tilde\beta|)!^{\kappa}\langle\xi\rangle_h^{m-|\tilde\alpha|-(1-1/\kappa)N}\\ &\quad\times\exp[\delta\langle\xi\rangle_h^{1/\kappa}]\sum_{j=N}^{\infty}\sum_{k=0}^{j}(2^{3+\kappa}nA_0A)^j j!^{\kappa}(jR/2)^{(\kappa-1)(N-j)}\\ &\quad\times\{\max(C_0,1/R)\}^j(k!(j-k)!)^{-1}(j/2+1)^2\\ &\le C'(C,A_0/A,N,R)(2^{\kappa}A)^{|\tilde\alpha|+|\tilde\beta|}(|\tilde\alpha|+|\tilde\beta|)!^{\kappa}\\ &\quad\times\langle\xi\rangle_h^{m-|\tilde\alpha|-(1-1/\kappa)N}\exp[\delta\langle\xi\rangle_h^{1/\kappa}]\end{aligned}$$

if $2^{4+2\kappa}e^{1-\kappa}nA_0AR^{1-\kappa}\max(C_0,1/R)\le 1$. Thus we obtain (2.21) if a is chosen large enough and if c_{A_0} is chosen small enough.

Q. E. D.

Lemma 2.14. *There are symbols $q(x,\xi)$, $\tilde q(x,\xi)$, $r(x,\xi)$ and $\tilde r(x,\xi)$ such that*

$$\begin{aligned}&{}^R(e^{-\Lambda})(x,D)(e^{\Lambda})(x,D)=1+q(x,D)+r(x,D),\\ &(e^{\Lambda})(x,D)\,{}^R(e^{-\Lambda})(x,D)=1+\tilde q(x,D)+\tilde r(x,D),\end{aligned}$$

where $\sigma(1)(x,\xi)=1$,

(2.27) $\quad |q_{(\beta)}^{(\alpha)}(x,\xi)|\le C_{A_0,d}(C_0)d^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^{\kappa}\langle\xi\rangle_h^{1/\kappa-1-|\alpha|}$,

(2.28) $\quad |r_{(\beta)}^{(\alpha)}(x,\xi)|\le C_{\rho,A_0,d}(|\alpha|,C_0)d^{|\beta|}|\beta|!^{\kappa}\exp[-\rho\langle\xi\rangle_h^{1/\kappa}]$

if $d>0$ and $\rho\in\boldsymbol{R}$, and $\tilde q(x,\xi)$ and $\tilde r(x,\xi)$ satisfy the same estimates as (2.27) *and* (2.28), *respectively. Moreover we have*

$$(2.29)\qquad (1+q(x,D)+r(x,D))\,{}^{R}(e^{-\Lambda})(x,D) = {}^{R}(e^{-\Lambda})(x,D)(1+\tilde{q}(x,D)+\tilde{r}(x,D)).$$

Remark. With obvious notations, we have $q(x,\xi)\sim\sum_{\alpha>0}\omega^{\alpha}_{(\alpha)}(-\Lambda;x,\xi)/\alpha!$ and $\tilde{q}(x,\xi)\sim\sum_{\alpha>0}\omega^{(\alpha)}_{\alpha}(-\Lambda;x,\xi)/\alpha!$, and we can define $q(x,\xi)$, $\tilde{q}(x,\xi)$, $r(x,\xi)$ and $\tilde{r}(x,\xi)$ as analytic symbols (see [26]).

Proof. From Proposition 2.8 and Lemmas 2.4 and 2.9 the lemma easily follows. **Q. E. D.**

Lemma 2.15. *Assume that a symbol $p(x,\xi.\,y,\eta)$ satisfies*

$$|p^{(\alpha)(\gamma)}_{(\beta)(\delta)}(x,\xi,y,\eta)|)\le C_\gamma A^{|\alpha|+|\beta|+|\delta|}|\alpha|!^{\kappa}|\beta|!^{\kappa}|\delta|!^{\kappa}$$
$$\times\langle\xi\rangle^{-|\alpha|}\exp[\delta_1\langle\xi\rangle^{1/\kappa}+\delta_2\langle\eta\rangle^{1/\kappa}]\quad \textit{for}\ (x,\xi),\,(y,\eta)\in T^*\boldsymbol{R}^n,$$
$$|p^{(\alpha)(\gamma)}_{(\beta((\delta)}(x,\xi+\eta,x+y,\eta)|\le C_{\alpha,\gamma}(2^{\kappa}A)^{|\beta|+|\delta|}|\beta|!^{\kappa}|\delta|!^{\kappa}$$
$$\times\exp[-a\langle\xi\rangle^{1/\kappa}]\ \textit{if}\quad |\eta|\le c_1\langle\xi\rangle\ \textit{and}\ |y|\le c_2,$$

where $a\in\boldsymbol{R}$, c_1 and c_2 are positive constants. Then there are $d_0>0$ and $d_1>0$ such that $p(x,D_x,y,D_y)$ maps continuously $\mathscr{S}_{\kappa,\varepsilon}$ to $\mathscr{S}_{\kappa,\varepsilon+\rho}$ for $|\varepsilon+\rho|<\kappa(nA)^{-1/\kappa}/2$ and $H^s_{\kappa,\varepsilon}$ to $H^s_{\kappa,\varepsilon+\rho}$ for $|\varepsilon+\rho|<\varepsilon_0A^{-1/\kappa}/2$ if $|\delta_1|<d_0A^{-1/\kappa}$ and $\rho=\min(a,d_1A^{-1/\kappa}-\delta_1-\delta_2)$, where ε_0 is the constant in Proposition 2.12.

Proof. By the same argument as in the proof of Lemma 2.5, we have

$$p(x,D_x,y,D_y)u(x)=q(x,D)u(x)\qquad\text{for } u\in\mathscr{D}^{(\kappa)}$$

if $\delta_1<\kappa(nA)^{-1/\kappa}$, where

$$q(x,\xi)=Os-\int e^{-iy\cdot\eta}p(x,\xi+\eta,x+y,\xi)\,dy\,đ\eta.$$

We may assume that $0<c_1<1$. Choose $\chi(\xi)\in\mathscr{D}^{(\tilde{\kappa})}(1<\tilde{\kappa}<\kappa)$ so that $\chi(\xi)=1$ for $|\xi|\le1/2$ and $\chi(\xi)=0$ for $|\xi|\ge1$. Put

$$q_1(x,\xi)=\int e^{-iy\cdot\eta}p(x,\xi+\eta,x+y,\xi)\chi(\eta/(c_1\langle\xi\rangle))\chi(y/c_2)\,dy\,đ\eta,$$
$$q_2(x,\xi)=Os-\int e^{-iy\cdot\eta}p(x,\xi+\eta,x+y,\xi)\chi(\eta/(c_1\langle\xi\rangle))\times(1-\chi(y/c_2))\,dy\,đ\eta,$$
$$q_3(x,\xi)=q(x,\xi)-q_1(x,\xi)-q_2(x,\xi).$$

Applying the same arguments as in the proofs of Lemmas 2.6 and 2.7, we have

$$
\begin{aligned}
&|q^{(\alpha)}_{1(\beta)}(x,\xi)|\le C'_\alpha(2^\kappa A)^{|\beta|}|\beta|!^\kappa\langle\xi\rangle^n\exp[-a\langle\xi\rangle^{1/\kappa}],\\
&|q^{(\alpha)}_{2(\beta)}(x,\xi)|\le\inf_{N=0,1,2,\dots}|\int(\int e^{-iy\cdot\eta}|y|^{-2N}\langle y\rangle^{-2M}\\
&\quad\times(y\cdot D_\eta)^N\partial_\xi^\alpha D_x^\beta\langle D_\eta\rangle^{2M}\{p(x,\xi+\eta,x+y,\xi)\chi(\eta/(c_1\langle\xi\rangle))\\
&\quad\times(1-\chi(y/c_2))\}dy)\,d\eta\le C_{\alpha,\rho}(A)A^{|\beta|}|\beta|!^\kappa\exp[\rho\langle\xi\rangle^{1/\kappa}]\\
&\qquad\text{if }\rho>\delta_1+\delta_2+c_1^{1/\kappa}|\delta_1|-\kappa(1-c_1)^{1/\kappa}c_2^{1/\kappa}(2nA)^{-1/\kappa},
\end{aligned}
$$

where $M=[n/2]+1$. Similarly, we have

$$
|q^{(\alpha)}_{3(\beta)}(x,\xi)|\ge C_{\alpha,A,\rho}(2^\kappa A)^{|\beta|}|\beta|!^\kappa\exp[\rho\langle\xi\rangle^{1/\kappa}]
$$

if $\rho>\delta_1+\delta_2-(c_1/2)^{1/\kappa}(\kappa(nA)^{-1/\kappa}/2-|\delta_1|)$ and $|\delta_1|<\kappa(nA)^{-1/\kappa}/2$. In fact,

$$
\begin{aligned}
&|\int e^{-iy\cdot\eta}\langle y\rangle^{-2M}\partial_\xi^\alpha D_x^\beta\langle D_\eta\rangle^{2M}\\
&\quad\times\{p(x,\xi+\eta,x+y,\xi)(1-\chi(\eta/(c_1\langle\xi\rangle)))\}dy|\\
&\ge C'_{\alpha,A,\rho}(2^\kappa A)^{|\beta|}|\beta|!^\kappa\exp[\rho|\eta|^{1/\kappa}+\delta_1\langle\xi+\eta\rangle^{1/\kappa}+\delta_2\langle\xi\rangle^{1/\kappa}]\\
&\quad\times\psi(\eta/(c_1\langle\xi\rangle))\qquad\text{if }\rho>-\kappa(nA)^{-1/\kappa}/2,
\end{aligned}
$$

where $\psi(\xi)$ is the characteristic function of $\{\xi\in\boldsymbol{R};\ |\xi|\ge 1/2\}$. Therefore, taking $d_0=\min(2^{-1-1/\kappa}\kappa\{(1-c_1)c_2/(nc_1)\}^{1/\kappa},\ \kappa n^{-1/\kappa}/4)$ and $d_1=\min(2^{-1-1/\kappa}\kappa\{(1-c_1)c_2/n\}^{1/\kappa},\ 2^{-2-1/\kappa}\kappa(c_1/n)^{1/\kappa})$, the lemma follows from Propositions 2.3 and 2.12. **Q. E. D.**

Corollary 1. *Let* $1<\kappa_1\le\kappa$, *and assume that* $p(x,\xi)$ *satisfies* (A-1). *Then we have*

$$
WF_*(p(x,D)u)\subset WF_*(u)\qquad\textit{for }u\in\mathscr{D}^{*1\prime},
$$

where $*=(\kappa)$ *if* $*1=(\kappa_1)$ and $*=\{\kappa\}$ if $*1=\{\kappa_1\}$.

Proof. Assume that $(x^0,\xi^0)\notin WF_*(u)$. Then there are $\chi(x)\in\mathscr{D}^{*1}$ and $\phi(\xi)\in\mathscr{E}^{*1}$ such that $\chi(x)=1$ near x^0, $\phi(\xi)$ is positively homogeneous of degree 0 for $|\xi|\ge 1$, $\phi(\xi)=1$ if $|\xi|\ge 1$ and ξ belongs to a conic neighborhood of ξ^0, and $\phi(D)\chi(x)u\in\mathscr{S}_{\kappa,a}$ for any $a>0$ when $*=(\kappa)$ and for some $a>0$ when $*=\{\kappa\}$. So we have $p(x,D)\phi(D)\times\chi(x)u\in\mathscr{E}^*$. Let $\chi_1(x)\in\mathscr{D}^{*1}$ satisfy supp $\chi_1\Subset\{x\in\boldsymbol{R}^n;\ \chi(x)=1\}$, and let $\phi_1(\xi)\in\mathscr{E}^{*1}$ be a positively homogeneous function of degree 0 for $|\xi|\ge 2$ such that supp $\phi_1\cap(\xi\in\boldsymbol{R}^n;\ |\xi|=2\}\Subset\{\xi\in\boldsymbol{R}^n;\ \phi(\xi)=1\}$. Then Lemma 2.15 implies that $\chi_1(x)p(x,D)\phi(D)(1-\chi(x))u\in\mathscr{E}^{*1}$ and $\phi_1(D)\chi_1(D)p(x,D)(1-\phi(D))u\in\mathscr{E}^{*1}$. Thus we have $\phi_1(D)\chi_1(x)p(x,D)u\in\mathscr{E}^*$, which proves $(x^0,\xi^0)\notin WF_*(p(x,D)u)$. **Q. E. D.**

Corollary 2. *Let $\mathscr{C}$ be a conic subset of $T^*\boldsymbol{R}^n\backslash 0$, and let $\chi(x,\xi)\in\mathscr{E}^{*1}$ be a positively homogeneous function of degree 0 for $|\xi|\geq 1$ such that $\chi(x,\xi)=1$ near $\mathscr{C}\cap\{|\xi|\geq 1\}$ and $\{x\in\boldsymbol{R}^n;\ (x,\xi)\in\operatorname{supp}\chi$ for some $\xi\in\boldsymbol{R}^n\}$ is compact. Then, $WF_*(u)\cap\mathscr{C}=\emptyset$ if $\chi(x,D)u\in\mathscr{E}^*$ and $u\in\mathscr{D}^{*1\prime}$.*

Proof. Let $\chi_1(x)\in\mathscr{D}^{*1}$ and $\psi_1(\xi)\in\mathscr{E}^{*1}$ be functions such that $\chi_1(x)\psi_1(\xi)\Subset\chi(x,\xi)$, i.e., $\chi_1(x)\psi_1(\xi)$ is positively homogeneous of degree 0 for $|\xi|\geq M$ and $\operatorname{supp}\chi_1(x)\psi_1(\xi)\cap\{|\xi|=M\}\Subset\{(x,\xi)\in T^*\boldsymbol{R}^n;\ \chi(x,\xi)=1\}$ for a sufficiently large M. Since $\psi_1(D)\chi_1(x)u=\psi_1(D)\chi_1(x)\chi(x,D)u-\psi_1(D)\chi_1(x)(\chi(x,D)-1)u$ and $\psi_1(D)\chi_1(x)(\chi(x,D)-1)u\in\mathscr{E}^{*1}$, we have $\psi_1(D)\chi_1(x)u\in\mathscr{E}^*$ if $\chi u\in\mathscr{E}^*$. This proves the lemma. **Q. E. D.**

Corollary 3. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be conic subsets of $T^*\boldsymbol{R}^n\backslash 0$ such that $\mathscr{C}_1\Subset\mathscr{C}_2$, i.e., $\mathscr{C}_1\cap\{|\xi|=1\}\Subset\mathscr{C}_2\cap\{|\xi|=1\}$, and let $\chi(x,\xi)\in\mathscr{E}^{(\kappa)}$ be a function such that $\chi(x,\xi)$ is positively homogeneous of degree 0 in ξ for $|\xi|\geq 1$, $\chi(x,\xi)=1$ near $\mathscr{C}_1\cap\{|\xi|\geq 1\}$ and $\operatorname{supp}\chi(x,\xi)\cap\{|\xi|\geq 1\}\Subset\mathscr{C}_2$. Assume that symbols $p(x,\xi)$ and $q(x,\xi)$ satisfy*

$$|p_{(\beta)}^{(\alpha)}(x,\xi)|\leq CA^{|\alpha|+|\beta|}|\alpha|!^{\kappa}|\beta|!^{\kappa}\exp[\delta\langle\xi\rangle^{1/\kappa}],$$
$$|q_{(\beta)}^{(\alpha)}(x,\xi)|\leq C'A^{|\alpha|+|\beta|}|\alpha|!^{\kappa}|\beta|!^{\kappa}\exp[\delta'\langle\xi\rangle^{1/\kappa}],$$

and $\operatorname{supp}q(x,\xi)\cap\{|\xi|\geq 1\}(\cap\mathscr{C}_2\backslash\mathscr{C}_1)=\emptyset$. Then there is $\hat{\varepsilon}_1>0$ such that $q(x,D)[p(x,D),\chi(x,D)]f\in L^2_{\kappa,\varepsilon_1}$ for $f\in L^2_{\kappa,\varepsilon}$ if $|\varepsilon|\leq\varepsilon_1\equiv\hat{\varepsilon}_1A^{-1/\kappa}$ and $|\delta|+|\delta'|\leq\varepsilon_1$, where $[A,B]=AB-BA$.

Proof. We can write

$$q(x,D)[p,\chi]=q(1-\chi)p\chi-q\chi p(1-\chi).$$

Let $\chi_j(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ $(j=1,2)$ satisfy $\chi_1(x,\xi)\Subset\chi(x,\xi)\Subset\chi_2(x,\xi)$, $\chi_1(x,\xi)=1$ near $\mathscr{C}_1\cap\{|\xi|\geq 1\}$ and $\operatorname{supp}\chi_2(x,\xi)\cap\{|\xi|\geq 1\}\Subset\mathscr{C}_2$, and put $q_1(x,\xi)=q(x,\xi)$, $\chi_1(x,\xi)$, $p_1(x,\xi)=p(x,\xi)\chi_2(x,\xi)$, $q_2(x,\xi)=q(x,\xi)-q_1(x,\xi)$ and $p_2(x,\xi)=p(x,\xi)-p_1(x,\xi)$. Then it follows from Lemma 2.15 that $q_1(x,D)(1-\chi(x,D))f$, $q_2(x,D)\chi(x,D)f$, $q_2(x,D)p_1(x,D)f$ and $p_2(x,D)\chi(x,D)f$ belong to $L^2_{\kappa,\varepsilon_1'}$ if $f\in L^2_{\kappa,\varepsilon}$, $|\varepsilon|\leq\varepsilon_1'\equiv\hat{\varepsilon}_1'A^{-1/\kappa}$ and $|\delta|+|\delta'|\leq\varepsilon_1'$, where $\hat{\varepsilon}_1'>0$. This proves that $q(1-\chi)p\chi f\in L^2_{\kappa,\varepsilon_1}$ if $f\in L^2_{\kappa,\varepsilon}$, $|\varepsilon|\leq\varepsilon_1\equiv\varepsilon_1A^{-1/\kappa}$ and $|\delta|+|\delta'|\leq\varepsilon_1$, where $\hat{\varepsilon}_1>0$. We can also apply the same argument to $q\chi p(1-\chi)$ and prove the assertion. **Q. E. D.**

To end this section we have to remark that calculus of pseudo-differential operators in the space of real analytic functions and Gevrey classes has been studied by many authors (see [1], [4], [8], [22-26]).

§ 3. Hypoellipticity

To prove Theorem 1.4 we shall prepare several lemmas on construction of parametrices in this section. As a consequence of the lemmas, we shall prove that operators satisfying the so-called (H)-condition are hypoelliptic in some Gevrey classes, which was essentially proved by Taniguchi [25].

Let $0<\delta\leq 1-1/\kappa$, $1/\kappa\leq\rho<1$, $h\geq 1$ and $m, m'\in\boldsymbol{R}$. We say that a symbol $p(x, \xi)$ satisfies the condition $(H; C, A, d_0, d_1, B, N_0)$, where C, A, d_0, d_1, $B>0$ and N_0 is a non-negative integer, if

$$|p_{(\beta)}^{(\alpha)}(x, \xi)|\leq CA^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^{\kappa}\langle\xi\rangle_h^{m-|\alpha|} \text{ for any } \alpha \text{ and } \beta,$$
$$|p(x, \xi)|\geq d_0\langle\xi\rangle_h^{m'},$$
$$|p_{(\beta)}^{(\alpha)}(x, \xi)/p(x, \xi)|\leq d_1B^{|\alpha|+|\beta|}\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|} \text{ for } |\alpha|,\ |\beta|\leq N_0.$$

Lemma 3.1. *Assume that $p(x, \xi)$ satisfies the condition $(H; C, A, d_0, d_1, B, N_0)$ and that $\Lambda(x, \xi)$ satisfies (2.13). Then there are positive constants $\hat{\varepsilon}_\Lambda$, a_Λ, $h_{p,\Lambda}(a, 1/d_0, d_1, B, N_0)$, c_{d_1}, $C_{A,A_0}(C)$ and $C(d_1)$ and symbols $p_\Lambda^a(x, \xi)$ and $r_\Lambda^a(x, \xi)$ for $0\leq a\leq a_\Lambda A^{-1/\kappa}$ such that*

$$(e^{a\Lambda})(x, D)p(x, D)\,{}^R(e^{-a\Lambda})(x, D)=p_\Lambda^a(x, D)+r_\Lambda^a(x, D),$$
$$|r_{\Lambda(\beta)}^{a(\alpha)}(x, \xi)|\leq C_{A,A_0}(|\alpha|, C)(2^{4\kappa}A)^{|\beta|}|\beta|!^{\kappa}\exp[-3\varepsilon_0\langle\xi\rangle_h^{1/\kappa}],$$

$r_\Lambda^a(x, D)$ maps continuously $L^2_{\kappa,\varepsilon}$ to $L^2_{\kappa,2\varepsilon_0}$ if $|\varepsilon|\leq\varepsilon_0\equiv\hat{\varepsilon}_\Lambda A^{-1/\kappa}$, and $p_\Lambda^a(x, \xi)$ satisfies the condition $(H; C_{A,A_0}(C), 2^{2\kappa}A, d_0/2, C(d_1), B, N_0-r)$ if $h\geq h_{p,\Lambda}(a, 1/d_0, d_1, B, N_0)$ and if $aC_0A_0B\leq c_{d_1}$ when $\delta=1-1/\kappa$ or $\rho=1/\kappa$, where $r=[(m-m'+1)/(1-1/\kappa)]$. Here a_Λ is a constant depending on A_0 and C_0, and $h_{p,\Lambda}(\cdots)$ is a constant depending on A, C, A_0, C_0, $\cdots$.

Remark. When $\delta=0$ or $\rho=1$, we can also obtain similar results, which is not necessary in this paper.

Proof. Applying Proposition 2. 13 with $\Lambda(x, \xi)$ replaced by $a\Lambda(x, \xi)$, we obtain $p^a_\Lambda(x, \xi)$ and $r^a_\Lambda(x, \xi)$. It is obvious that $r^a_\Lambda(x, \xi)$ has the properties in the lemma if $\hat{\varepsilon}_\Lambda$ and a_Λ are chosen suitably. $p^a_\Lambda(x, \xi)$ can be written as

$$\begin{aligned} p^a_\Lambda(x, \xi) &= \textstyle\sum_{|\alpha|+|\beta|\le r} (\alpha!\beta!)^{-1} \{p_{(\beta)}(x, \xi)\omega^\beta(a\Lambda; x, \xi) \\ &\quad \times \omega_\alpha(-a\Lambda; x, \xi)\}^{(\alpha)} + \hat{p}^{a(\alpha)}_{\Lambda(\beta)}(x, \xi), \\ |\hat{p}^{a(\alpha)}_{\Lambda(\beta)}(x, \xi)| &\le C_{A, A_0}(C)(2^{2\kappa}A)^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa \\ &\quad \times \langle\xi\rangle_h^{m-|\alpha|-(1-1/\kappa)(r+1)}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &|\{\omega^{\tilde{\alpha}}(a\Lambda; x, \xi) - (a\nabla_\xi \Lambda(x, \xi))^{\tilde{\alpha}}\}^{(\alpha)}_{(\beta)}| \\ &\le C(C_0, A_0, \mathrm{a}, \alpha, \beta, \tilde{\alpha})\langle\xi\rangle_h^{-(1-1/\kappa)|\tilde{\alpha}|-1/\kappa-|\alpha|}, \\ &|\{\omega_{\tilde{\beta}}(-a\Lambda; x, \xi) - (ia\nabla_x \Lambda(x, \xi))^{\tilde{\beta}}\}^{(\alpha)}_{(\beta)}| \\ &\le C(C_0, A_0, a, \alpha, \beta, \tilde{\beta})\langle\xi\rangle_h^{|\tilde{\beta}|/\kappa-1/\kappa-|\alpha|}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (3.1) \qquad &|p^{a(\alpha)}_{\Lambda(\beta)}(x, \xi)/p(x, \xi) - p^{(\alpha)}_{(\beta)}(x, \xi)/p(x, \xi)| \\ &\le \{\textstyle\sum_{j=1}^r \sum_{|\tilde{\alpha}|+|\tilde{\beta}|=j} d_1 B^{j+|\alpha|+|\beta|}(aC_0A_0)^j \\ &\quad \times \langle\xi\rangle_h^{(\delta+1/\kappa-1)|\tilde{\beta}|+(1/\kappa-\rho)|\tilde{\alpha}|}/(\tilde{\alpha}!\tilde{\beta}!) \\ &\quad + C_{p,\Lambda}(a, 1/d_0, d_1, B, N_0)(\langle\xi\rangle_h^{-1/\kappa} + \langle\xi\rangle_h^{-\delta} + \langle\xi\rangle_h^{-1+\rho})\} \\ &\quad \times \langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|} \quad \text{if } |\alpha|, \ |\beta| \le N_0 - r. \end{aligned}$$

This shows that there are positive constants $h_{p,\Lambda}(a, 1/d_0, d_1, B, N_0)$ and c_{d_1} such that $|p^a_\Lambda(x, \xi)| \ge |p(x, \xi)|/2$ if $h \ge h_{p,\Lambda}(a, 1/d_0, d_1, B, N_0)$ and if $aC_0A_0B \le c_{d_1}$ when $\delta = 1-1/\kappa$ or $\rho = 1/\kappa$. (3. 1) also gives

$$|p^{a(\alpha)}_{\Lambda(\beta)}(x, \xi)/p_\Lambda(x, \xi)| \le 2(1+d_1)B^{|\alpha|+|\beta|}\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|}$$

if $h \ge h_{p,\Lambda}(a, 1/d_0, d_1, B, N_0)$ and $|\alpha|$, $|\beta| \le N_0 - r$, and if $aC_0A_0B \le c_{d1}$ when $\delta = 1-1/\kappa$ or $\rho = 1/\kappa$, modifying $h_{p,\Lambda}(a, 1/d_0, d_1, B, N_0)$. This proves the lemma. **Q. E. D.**

Lemma 3. 2. *Assume that* $p(x, \xi)$ *satisfies the condition* $(H; C, A, d_0, d_1, B, N_0)$. *Then there is* $C(d_1, N_0) > 0$ *such that*

$$|(1/p(x, \xi))^{(\alpha)}_{(\beta)}| \le C(d_1, N_0)B^{|\alpha|+|\beta|}\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|}/|p(x, \xi)| \quad \textit{for } |\alpha|, |\beta| \le N_0.$$

Proof. It is sufficient to show that

$$|(1/p(x,\xi))^{(\alpha)}_{(\beta)}| \le C(d_1, |\alpha|+|\beta|)B^{|\alpha|+|\beta|} \times \langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|}/|p(x,\xi)|. \tag{3.2}$$

Using the identity $\{p(x,\xi)\partial(1/p(x,\xi))\}^{(\alpha)}_{(\beta)} = -\{\partial p(x,\xi)/p(x,\xi)\}^{(\alpha)}_{(\beta)}$, (3.2) can be proved by induction on $|\alpha|+|\beta|$. **Q. E. D.**

Lemma 3.3. *Assume that $p(x,\xi)$ satisfies the condition $(H; C, A, d_0, d_1, B, N_0)$ and that $\delta\le\rho$. Put $q(x,\xi)=\sigma(p(x,D)\circ(1/p)(x,D))(x,\xi)-1$ and $\tilde{q}(x,\xi)=\sigma((1/p)(x,D)\circ p(x,D))(x,\xi)-1$. Then*

$$|q^{(\alpha)}_{(\beta)}(x,\xi)| \le \{C(d_1, N_0, B^2h^{\delta-\rho})B^2h^{\delta-\rho}B^{|\alpha|+|\beta|} + C(C, A, 1/d_0, d_1, B, N_0)/h\}\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|} \tag{3.3}$$

for $|\alpha|, |\beta|\le N_0'$ and $N_0'\le N_0-r-2L(N_0')-1$, where $r=[(m-m'+n+1)/(1-\delta)]$ and $L(N_0')=[(\rho N_0'+\delta(r+1)+|m|-m'+n+1)/(2-2\delta)]+1$, and $\tilde{q}(x,\xi)$ also satisfies the estimates (3.3) for $|\alpha|\le N_0-2[n/2]-\tilde{r}-3$ and $|\beta|\le N_0-\tilde{r}$, where $\tilde{r}=[(m-m'+1)/\rho]$.

Proof. Let $\phi\in\mathscr{D}^{(\kappa')}$ $(1<\kappa'<\kappa)$ be a function such that $\phi(\xi)=1$ for $|\xi|\le 1/4$ and $\phi(\xi)=0$ for $|\xi|\ge 1/2$, and write

$$q(x,\xi)=q_1(x,\xi)+\int_0^1 (r+1)(1-\theta)^r\{q_2(x,\xi,\theta)+q_3(x,\xi,\theta)\}d\theta,$$

where $q_1(x,\xi)=\sum_{1\le|\alpha|\le r}p^{(\alpha)}(x,\xi)(1/p(x,\xi))_{(\alpha)}/\alpha!$ and $q_2(x,\xi,\theta) =\sum_{|\alpha|=r+1}\alpha!^{-1}Os-\int e^{-iy\cdot\eta}p^{(\alpha)}(x,\xi+\eta)(1/p(x+\theta y,\xi))_{(\alpha)}\phi(\eta/\langle\xi\rangle_h)\,dy\,d\!\!{}^-\eta$. Then we have

$$|q^{(\alpha)}_{1(\beta)}(x,\xi)| \le C(d_1, N_0, B^2h^{\delta-\rho})B^2h^{\delta-\rho}B^{|\alpha|+|\beta|}\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|} \quad \text{for } |\alpha|, |\beta|\le N_0-r, \tag{3.4}$$

$$|q^{(\alpha)}_{2(\beta)}(x,\xi,\theta)| \le C(C, A, 1/d_0, d_1, B, N_0)\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|-1} \quad \text{for } |\alpha|\le N_0 \text{ and } |\beta|\le N_0-r-1. \tag{3.5}$$

A simple calculation yields

$$\begin{aligned}|q^{(\alpha)}_{3(\beta)}(x,\xi,\theta)| &\le \int\langle\eta\rangle^{-2L}\sum_{|\gamma|=r+1}\sum_{\alpha^1+\alpha^2=\alpha,\beta^1+\beta^2=\beta}\alpha!\beta! \\ &\quad\times(\gamma!\alpha^1!\alpha^2!\beta^1!\beta^2!)^{-1}|\partial_\xi^{\alpha^1}D_x^{\beta^1}\langle D_\eta\rangle^{2M}\{p^{(\gamma)}(x,\xi+\eta)(1- \\ &\quad\phi(\eta/\langle\xi\rangle_h))\}||\partial_\xi^{\alpha^2}D_x^{\gamma+\beta^2}\langle D_y\rangle^{2L}(\langle y\rangle^{-2M}/p(x+\theta y,\xi))|\,dy\,d\!\!{}^-\eta \\ &\le C(C, A, 1/d_0, d_1, B, N_0)\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|-1}\end{aligned} \tag{3.6}$$

for $|\alpha|, |\beta|\le N_0'$ and $N_0'\le N_0-r-2L-1$, where $L=L(N_0')$ and $M=[n/2]+1$. (3.4)-(3.6) prove (3.3). Write

$$\tilde{q}(x,\xi)=\tilde{q}_1(x,\xi)+\int_0^1(\tilde{r}+1)(1-\theta)^{\tilde{r}}\tilde{q}_2(x,\xi,\theta)d\theta,$$

where $\tilde{q}_1(x,\xi)=\sum_{1\le|\alpha|\le\tilde{r}}(1/p(x,\xi))^{(\alpha)}p_{(\alpha)}(x,\xi)/\alpha!$. Then, it is obvious that $\tilde{q}_1(x,\xi)$ satisfies the estimates (3.4) for $|\alpha|, |\beta|\le N_0-\tilde{r}$. Moreover, we have

$$\begin{aligned}|\tilde{q}^{(\alpha)}_{2(\beta)}(x,\xi)|&\le\sum_{|\gamma|=\tilde{r}+1}\gamma!^{-1}\int\langle\eta\rangle^{-2L}|\partial_\xi^\alpha D_x^\beta\langle D_\eta\rangle^{2M}\langle D_y\rangle^{2L}\\&\quad\times\{\langle y\rangle^{-2M}(1/p(x,\xi+\eta))^{(\gamma)}p_{(\gamma)}(x+\theta y,\xi)\}|dy\bar{d}\eta\\&\le C(C,A,1/d_0,d_1,B,N_0)\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|-1}\end{aligned}$$

for $|\alpha|\le N_0-2M-\tilde{r}-1$ and $|\beta|\le N_0$, where $\tilde{L}=[\{(\delta+\rho)N_0+(1-\rho)n+|m'|\}/2]+1$, which proves the lemma. **Q.E.D.**

Proposition 3.4. *Let N_0 be a sufficiently large positive integer, and assume that $p(x,\xi)$ satisfies the condition $(H;C,A,d_0,d_1,B,N_0)$ and that $\delta\le\rho$. Then there are positive constants $\hat{\varepsilon}_0$, $h_p(1/d_0,d_1,B)$, c_{d_1} and $B(d_1)$ and an operator Q such that Q maps continuously $L^2_{\kappa,\varepsilon}$ to $H^{m'}_{\kappa,\varepsilon}$ and $H^{m-m'}_{\kappa,\varepsilon}$ to $H^m_{\kappa,\varepsilon}$ and satisfies $Qp(x,D)=I$ on $H^m_{\kappa,\varepsilon}$ (or on $L^2_{\kappa,\varepsilon}$) and $p(x,D)Q=I$ on $H^{m-m'}_{\kappa,\varepsilon}$ (or on $L^2_{\kappa,\varepsilon}$), if $|\varepsilon|<\varepsilon_0\equiv\hat{\varepsilon}_0A^{-1/\kappa}$ and $h\ge h_p(1/d_0,d_1,B)$, and if $|\varepsilon|B\le c_{d_1}$ when $\delta=1-1/\kappa$ or $\rho=1/\kappa$, and if $B\le B(d_1)$ when $\rho=\delta$. Here I denotes the identity operator, and $h_p(\cdots)$ is a constant depending on A, C, $\cdots$. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be conic sets in $T^*\boldsymbol{R}^n\backslash 0$ such that the distance between $\mathscr{C}_1\cap\{|\xi|=1\}$ and $\mathscr{C}_2\cap\{|\xi|=1\}$ is not less than $d_2>0$. If $\chi_j\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$, supp $\chi_j(x,\xi)\subset\mathscr{C}_j$ and $|\chi^{(\alpha)}_{j(\beta)}(x,\xi)|\le C_d d^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa\langle\xi\rangle^{-|\alpha|}$ for any $d>0$ $(j=1,2)$, then there are positive constants $\hat{a}_{d_2}$, $h_{p,d_2}(1/d_0,d_1,B)$, c_{d_1,d_2} and $B'(d_1)$ such that $\chi_2(x,D)Q\chi_1(x,D)$ maps continuously $L^2_{\kappa,-a}$ to $L^2_{\kappa,a}$ if $0\le a\le\hat{a}_{d_2}A^{-1/\kappa}$ and $h\ge h_{p,d_2}(1/d_0,d_1,B)$, and if $aB\le c_{d_1,d_2}$ when $\delta=1-1/\kappa$ or $\rho=1/\kappa$, and if $B\le B'(d_1)$ when $\rho=\delta$.*

Proof. From Lemma 3.1 with $a\Lambda(x,\xi)$ replaced by $\varepsilon\langle\xi\rangle_h^{1/\kappa}$ it follows that there are $\hat{\varepsilon}_0>0$ and symbols $p_\varepsilon(x,\xi)$ and $r_\varepsilon(x,\xi)$ for $|\varepsilon|\le\varepsilon_0\equiv\hat{\varepsilon}_0A^{-1/\kappa}$ such that

$$\exp[\varepsilon\langle D\rangle_h^{1/\kappa}]p(x,D)\exp[-\varepsilon\langle D\rangle_h^{1/\kappa}]=p_\varepsilon(x,D)+r_\varepsilon(x,D),$$
$$|r^{(\alpha)}_{\varepsilon(\beta)}(x,\xi)|\le C_A(|\alpha|,|\beta|,C)\langle\xi\rangle_h^{m-|\alpha|}\exp[-\varepsilon_0\langle\xi\rangle_h^{1/\kappa}]$$

if $|\varepsilon|\le\varepsilon_0$, and $p_\varepsilon(x,\xi)$ satisfies the condition $(H;C_A(C),2^{2\kappa}A,d_0/2,C(d_1),B,N_0-r)$, if ε and h satisfy the following conditions;

(3.7) $|\varepsilon|\le\varepsilon_0$ *and* $h\ge h_p(1/d_0,d_1,B)$, *and* $|\varepsilon|B\le c_{d_1}$ *when* $\delta=1-1/\kappa$ *or* $\rho=1/\kappa$, where $r=[(m-m'+1)/(1-1/\kappa)]$ and $C_A(|\alpha|, |\beta|, C)$, $C_A(C)$, $C(d_1)$, $h_p(1/d_0, d_1, B)$ and c_{d_1} are positive constants. We set

$$q_\varepsilon(x, \xi)=\sigma((p_\varepsilon(x, D)+r_\varepsilon(x, D))(1/p_\varepsilon)(x, D))(x, \xi)-1.$$

Applying Lemmas 3.2 and 3.3, we can see that $q_\varepsilon(x, \xi)$ satisfies the same estimates as (3.3) if $C(C, A, 1/d_0, d_1, B, N_0)$ is replaced by $C_p(1/d_0, d_1, B, N_0)$ and ε and h satisfy (3.7) and if $|\alpha|, |\beta| \leq \hat{N}_{m-m'}$ and $N_0 \geq \hat{N}_{m-m'}+r+l_1+l_2$, where $l_1=[(m-m'+n+1)/(1-\delta)]$, $l_2=2[(\rho\hat{N}_{m-m'}+\delta(l_1+1)+|m|-m'+n+1)/(2-2\delta)]+3$ and $\hat{N}_{m-m'}$ is the constant in Lemma 2.11. In fact, $\sigma(r_\varepsilon(x, D)(1/p_\varepsilon)(x, D))(x, \xi)=Os-\int e^{-iy\cdot\eta} r_\varepsilon(x, \xi+\eta)\, p_\varepsilon(x+y, \xi)^{-1}\, dy đ\eta$ can be estimated similarly. Therefore, it follows from Lemma 2.11 that there is the inverse $(1+q_\varepsilon(x, D))^{-1}$ of $(1+q_\varepsilon(x, D))$ such that $(1+q_\varepsilon)^{-1}(1+q_\varepsilon)=(1+q_\varepsilon)(1+q_\varepsilon)^{-1}=I$ on $H^{m-m'}$ if ε and h satisfy (3.7), and if $B \leq B(d_1)$ when $\rho=\delta$, where $h_p(1/d_0, d_1, B)$, c_{d_1} and $B(d_1)$ are suitable positive constants. Put $Q_\varepsilon=\exp[-\varepsilon\langle D\rangle_h^{1/\kappa}](1/p_\varepsilon)(x, D)(1+q_\varepsilon(x, D))^{-1}\exp[\varepsilon\langle D\rangle_h^{1/\kappa}]$. Then Q_ε maps continuously $H_{\kappa,\varepsilon}^{m-m'}$ to $H_{\kappa,\varepsilon}^{m}$ and satisfies $p(x, D)Q_\varepsilon=I$ on $H_{\kappa,\varepsilon}^{m-m'}$ if $|\varepsilon| \leq \varepsilon_0$. Here we have assumed that $N_0 \geq \hat{N}_m+r$, and applied Lemma 2.11 to $(1/p_\varepsilon)(x, D)$. Put $\tilde{q}_\varepsilon(x, \xi)=\sigma((1/p_\varepsilon)(x, D)(p_\varepsilon(x, D)+r_\varepsilon(x, D)))(x, \xi)-1$. Similarly $(1+\tilde{q}_\varepsilon(x, D))$ has the inverse $(1+\tilde{q}_\varepsilon(x, D))^{-1}$ on $H^{m'}$ if ε and h satisfy (3.7), and if $B \leq B(d_1)$ when $\rho=\delta$, modifying the constants. If we set $\tilde{Q}_\varepsilon=\exp[-\varepsilon\langle D\rangle_h^{1/\kappa}](1+\tilde{q}_\varepsilon(x, D))^{-1}(1/p_\varepsilon)(x, D)\exp[\varepsilon\langle D\rangle_h^{1/\kappa}]$, then $\tilde{Q}_\varepsilon$ maps continuously $L^2_{\kappa,\varepsilon}$ to $H_{\kappa,\varepsilon}^{m'}$ and satisfies $\tilde{Q}_\varepsilon p(x, D)=I$ on $H_{\kappa,\varepsilon}^{m}$ if $|\varepsilon| \leq \varepsilon_0$. Here we have assumed that $N_0 \geq \hat{N}_{m'}+r+l_3+2[n/2]+3$, where $l_3=[(m-m'+1)/\rho]$. It is easy to see that $Q_\varepsilon=Q_{\varepsilon'}$ on $H_{\kappa,\varepsilon}^{m-m'}$ if $\varepsilon \geq \varepsilon'$ and $\tilde{Q}_\varepsilon=Q_\varepsilon$ on $H_{\kappa,\varepsilon}^{m-m'}$, which proves the first part of the proposition. Choose a symbol $\Lambda(x, \xi)$ satisfying

$$|\Lambda_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{d_2} A_{d_2}^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\langle\xi\rangle_h^{1/\kappa-|\alpha|}, \tag{3.8}$$

$$\inf_{L>0} \sup_{|\xi| \geq L} |\Lambda(x, \xi)|\langle\xi\rangle_h^{-1/\kappa}<2, \tag{3.9}$$

$$\inf_{L>0} \sup_{(x,\xi)\in\mathscr{C}_1, |\xi| \geq L} \Lambda(x, \xi)\langle\xi\rangle_h^{-1/\kappa}<-1, \tag{3.10}$$

$$\sup_{L>0} \inf_{(x,\xi)\in\mathscr{C}_2, |\xi| \geq L} \Lambda(x, \xi)\langle\xi\rangle_h^{-1/\kappa}>1. \tag{3.11}$$

For example, let $\varphi(x, \xi)$ be a function in $C^1(T^*\boldsymbol{R}^n)$ such that the first order derivatives of φ are bounded, and $|\varphi(x, \xi)|<5/3$, $\varphi(x, \xi)=-4/3$ in $\mathscr{C}_1 \cap \{|\xi| \geq 1/2\}$ and $\varphi(x, \xi)=4/3$ in $\mathscr{C}_2 \cap \{|\xi| \geq 1/2\}$. Put

$\varphi_j(x,\xi)=E_j*\varphi(x,\xi)$ and $\Lambda(x,\xi)=\varphi_j(x,\xi/\langle\xi\rangle_h)\langle\xi\rangle_h^{1/\kappa}$, where $E_j(x,\xi)=j^n(4\pi)^{-n}\exp[-j(|x|^2+|\xi|^2)/4]$. If j is sufficiently large, then $\Lambda(x,\xi)$ satisfies (3.8)-(3.11). Let symbols $p_\Lambda^a(x,\xi)$ and $r_\Lambda^a(x,\xi)$ be as defined in Lemma 3.1 for $0\le a\le\hat{a}_{d_2}A^{-1/\kappa}$, where $\hat{a}_{d_2}$ is a positive constant. By Lemma 3.1 $r_\Lambda^a(x,D)$ maps continuously $L^2_{\kappa,\varepsilon}$ to $L^2_{\kappa,2\varepsilon_1}$ for $|\varepsilon|\le\varepsilon_1\equiv\hat{\varepsilon}_{d_2}A^{-1/\kappa}$, where $\hat{\varepsilon}_{d_2}>0$. Applying the same argument as in the first part of the proof to p_Λ^a instead of p, we can show that there is an operator Q_Λ^a which maps continuously $L^2_{\kappa,\varepsilon}$ to $H^{m'}_{\kappa,\varepsilon}$ and satisfies $Q_\Lambda^a p_\Lambda^a=I$ on $H^m_{\kappa,\varepsilon}$, if $|\varepsilon|<\varepsilon_0/4$ and $h\ge h_{p,d_2}(a,1/d_0,d_1,B)$, and if $|\varepsilon|B\le c'_{d_1}$ and $aC_{d_2}A_{d_2}B\le c'_{d_1}$ when $\delta=1-1/\kappa$ or $\rho=1/\kappa$, and if $B\le B'(d_1)$ when $\rho=\delta$, where $h_{p,d_2}(a,1/d_0,d_1,B)$, c'_{d_1} and $B'(d_1)$ are positive constants. Here we have assumed that $N_0\ge\hat{N}_{m-m'}+2r+l_1+l_2$, $N_0\ge\hat{N}_m+2r$ and $N_0\ge\hat{N}_{m'}+2r+l_3+2[n/2]+3$. Lemma 2.14 and the same argument as in the first part of the proof show that ${}^R(e^{-a\Lambda})(x,D)(e^{a\Lambda})(x,D)$ can be written as ${}^R(e^{-a\Lambda})(x,D)(e^{a\Lambda})(x,D)=1+\hat{q}_\Lambda^a(x,D)+\hat{r}_\Lambda^a(x,D)$ and that $(1+\hat{q}_\Lambda^a(x,D)+\hat{r}_\Lambda^a(x,D))$ has the inverse on $L^2_{\kappa,\varepsilon}$ if $|\varepsilon|\le\varepsilon_0$ and $h\ge h_{d_2}(a,\varepsilon_0)$. In fact, using an oscillatory integral, we can estimate a symbol $\sigma(\exp[\varepsilon\langle D\rangle_h^{1/\kappa}]\hat{r}_\Lambda^a(x,D)\exp[-\varepsilon\langle D\rangle_h^{1/\kappa}])(x,\xi)$. Let $0\le a\le\min(\hat{a}_{d_2}A^{-1/\kappa},\varepsilon_0/12,\varepsilon_1/3)$, and put $u=Q\chi_1(x,D)f\ (\in H^m_{\kappa,-a})$ for $f\in H^{m-m'}_{\kappa,-a}$. By Lemma 2.10, Proposition 2.12 and (3.10), we have $(e^{a\Lambda})(x,D)\chi_1(x,D)f\in H^{m-m'}$. Similarly, Propositions 2.8 and 2.12 and (3.11) imply that $\chi_2(x,D)^R(e^{-a\Lambda})(x,D)$ maps continuously $H^s_{\kappa,\varepsilon}$ to $H^s_{\kappa,\varepsilon+a}$ if $|\varepsilon|\le\varepsilon_0$. Thus we have

$$\begin{aligned}\chi_2Q\chi_1f&=\chi_2{}^R(e^{-a\Lambda})Q_\Lambda^a p_\Lambda^a e^{a\Lambda}(1+\hat{q}_\Lambda^a+\hat{r}_\Lambda^a)^{-1}u\\&=\chi_2{}^R(e^{-a\Lambda})Q_\Lambda^a(e^{a\Lambda}p\ {}^R(e^{-a\Lambda})-r_\Lambda^a)e^{a\Lambda}(1+\hat{q}_\Lambda^a+\hat{r}_\Lambda^a)^{-1}u\\&=\chi_2{}^R(e^{-a\Lambda})Q_\Lambda^a e^{a\Lambda}\chi_1f-\chi_2{}^R(e^{-a\Lambda})Q_\Lambda^a r_\Lambda^a e^{a\Lambda}(1+\hat{q}_\Lambda^a+\hat{r}_\Lambda^a)^{-1}u\in H^{m'}_{\kappa,a},\end{aligned}$$

which completes the proof. **Q. E. D.**

Let $(x^0,\xi^0)\in T^*\boldsymbol{R}^n\backslash 0$ and $|\xi^0|=1$, and let $\mathscr{C}$ be a convex conic neighborhood of (x^0,ξ^0). Choose a neighborhood U of x^0 and a conic neighborhood Γ of ξ^0 so that $U\times\Gamma\subset\mathscr{C}$. Moreover, let $\mathscr{C}_1$ be a conic neighborhood of (x^0,ξ^0) such that $\mathscr{C}_1\Subset U\times\Gamma$. Choose $\psi_1(x)\in\mathscr{D}^{(\kappa')}$ and $\psi_2(\xi)\in\mathscr{E}^{(\kappa')}$ for a fixed $\kappa'<\kappa$ so that $\psi_2(\xi)$ is positively homogeneous of degree 0 for $|\xi|\ge 1$, $0\le\psi_1(x)\psi_2(\xi)\le 1$, supp $\psi_1(x)\psi_2(\xi)\cap\{|\xi|=1\}\Subset U\times\Gamma$ and $\psi_1(x)\psi_2(\xi)=1$ on $\mathscr{C}_1\cap\{|\xi|\ge 1\}$. Let

$\sigma(\xi)\in\mathscr{E}^{(\kappa')}$ be a function such that $\sigma(\xi)=0$ for $|\xi|\leq 1$ and $\sigma(\xi)=1$ for $|\xi|\geq 2$, and write $\sigma_h(\xi)=\sigma(\xi/h)$ for $h\geq 1$. We set

$$X(x)=(1-\psi_1(x))x^0+\psi_1(x)x,$$
$$\Xi_h(\xi)=\{h(1-\sigma_h(\xi))+\sigma_h(\xi)(1-\psi_2(\xi))|\xi|\}\xi^0+\sigma_h(\xi)\psi_2(\xi)\xi,$$
$$\tilde{p}_h(x,\xi)=p(X(x),\ \Xi_h(\xi)).$$

Then it is obvious that $\tilde{p}_h(x,\xi)=p(x,\xi)$ if $(x,\xi)\in\mathscr{C}_1$ and $|\xi|\geq 2h$.

Lemma 3.5. *Assume that* $h\geq 1$ *and that*

(3.12) $\quad |\lambda\xi^0+\xi|\geq(\lambda+|\xi|)/2$ *for* $\lambda>0$ *and* $(x,\xi)\in\mathscr{C}$.

Then we have $(X(x),\ \Xi_h(\xi))\in\mathscr{C}$, $|\Xi_h(\xi)|\geq h/2$ *and*

$$(2\sqrt{5})^{-1}\langle\xi\rangle_h\leq|\Xi_h(\xi)|\leq\langle\Xi_h(\xi)\rangle\leq\sqrt{2}\langle\xi\rangle_h$$

Lemma 3.6. *Assume that* (3.12) *is satisfied. If a symbol* $p(x,\xi)$ *satisfies*

$$|p_{(\beta)}^{(\alpha)}(x,\xi)|\leq CA^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^{\kappa}\langle\xi\rangle^{m-|\alpha|}\quad \textit{for } (x,\xi)\in\mathscr{C} \textit{ with } |\xi|\geq h_0\ (\geq 1),$$

then

$$|\tilde{p}_{h(\beta)}^{(\alpha)}(x,\xi)|\leq C(C,1/A_1)A_1^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^{\kappa}\langle\xi\rangle_h^{m-|\alpha|}$$

for $h\geq 2h_0$ *and* $(x,\xi)\in T^*\boldsymbol{R}^n$, *where* $A_1\equiv C_{\sigma,\psi_1,\psi_2}A$ *and* $C_{\sigma,\psi_1,\psi_2}>0$.

Proof. It is sufficient to verify that

$$\begin{aligned}(3.13)\quad |\partial_\xi^\alpha D_x^\beta p_{(\nu)}^{(\gamma)}(X(x),\ \Xi_h(\xi))|&\leq(2\sqrt{5})^{|m|}CA_1^{|\alpha|+|\beta|}(2\sqrt{5}A)^{|\gamma|+|\nu|}\\ &\times(|\alpha|+|\beta|+|\gamma|+|\nu|)!^{\kappa}\langle\xi\rangle_h^{m-|\alpha|-|\gamma|}\textstyle\sum_{k=0}^{|\alpha|+|\beta|}b^k/k!^{\kappa-1}\end{aligned}$$

for $h\geq 2h_0$, where $b\equiv b(1/A_1)$. (3.13) can be proved by induction on $|\alpha|+|\beta|$ if $A_1\geq C_{\sigma,\psi_1,\psi_2}A$. **Q.E.D.**

Lemma 3.7. *Assume that* (3.12) *is satisfied. If a symbol* $p(x,\xi)$ *satisfies*

(3.14) $\quad |p(x,\xi)|\geq d_0\langle\xi\rangle^{m'}$,

(3.15) $\quad |p_{(\beta)}^{(\alpha)}(x,\xi)/p(x,\xi)|\leq d_1B^{|\alpha|+|\beta|}\langle\xi\rangle^{\delta|\beta|-\rho|\alpha|}$

for $(x,\xi)\in\mathscr{C}$ *with* $|\xi|\geq h_0(\geq 1)$ *and* $|\alpha|,|\beta|\leq N_0$, *then*

(3.16) $\quad |\tilde{p}_h(x,\xi)|\geq(2\sqrt{5})^{-|m'|}d_0\langle\xi\rangle_h^{m'}$

for $(x, \xi) \in T^*\mathbf{R}^n$ *and* $h \geq 2h_0$,

(3.17) $\quad |\tilde{p}_{h(\beta)}^{(\alpha)}(x, \xi)/\tilde{p}_h(x, \xi)| \leq d_1 B_1^{|\alpha|+|\beta|}\langle\xi\rangle_h^{\delta|\beta|-\rho|\alpha|}$

for $(x, \xi) \in T^*\mathbf{R}^n$, $|\alpha|$, $|\beta| \leq N_0$ *and* $h \geq h(1/B_1, h_0, N_0)$,

where $B_1 = C_{\sigma,\phi_1,\phi_2} B$, $C_{\sigma,\phi_1,\phi_2} > 0$ *and* $h(1/B_1, h_0, N_0) > 0$.

Proof. (3.16) is obvious. We can prove by induction on $|\alpha| + |\beta|$ that

$$|\partial_\xi^\alpha D_x^\beta p_{(\nu)}^{(\gamma)}(X(x), \Xi_h(\xi))/\tilde{p}_h(x, \xi)|$$
$$\leq d_1 B_1^{|\alpha|+|\beta|}(2\sqrt{5}B)^{|\gamma|+|\nu|}\langle\xi\rangle_h^{\delta(|\beta|+|\nu|)-\rho(|\alpha|+|\gamma|)}$$

if $h \geq h(1/B_1, h_0, N_0)$, $|\alpha| + |\gamma| \leq N_0$ and $|\beta| + |\nu| \leq N_0$, using $\rho < 1$ and $\delta > 0$. This proves (3.17). Q.E.D.

Proposition 3.8. *Let* $(x^0, \xi^0) \in T^*\mathbf{R}^n \backslash 0$, *and let* $\mathscr{C}$ *be a conic neighborhood of* (x^0, ξ^0). *Assume that* $0 < \delta \leq 1 - 1/\kappa$, $1/\kappa \leq \rho < 1$ *and* $\delta < \rho$, *and that* $p(x, \xi) \in S_*^m$ *satisfies* (3.14) *and* (3.15) *for* $(x, \xi) \in \mathscr{C}$ *with* $|\xi| \geq h_0 (\geq 1)$ *and* $|\alpha|$, $|\beta| \leq N_0$, *and* $p(x, D)$ *is properly supported, where* N_0 *is a sufficiently large positive integer,* m, $m' \in \mathbf{R}$ *and* h_0, d_0, d_1 *and* B *are positive constants. Moreover assume that* $\delta < 1 - 1/\kappa$ *and* $\rho > 1/\kappa$ *if* $* = (\kappa)$. *Then there is an operator* Q, *which maps continuously* $\mathscr{D}^{*\prime}$ *to* $\mathscr{D}^{*\prime}$, *such that*

$$(x^0, \xi^0) \notin WF_*(pQf - f) \cup WF_*(Qpf - f) \quad \textit{for } f \in \mathscr{D}^{*\prime},$$
$$(x^0, \xi^0) \notin WF_*(Qf) \quad \textit{if } (x^0, \xi^0) \notin WF_*(f) \textit{ and } f \in \mathscr{D}^{*\prime}.$$

Remark. (i) Taniguchi [25] essentially proved the proposition by his method of multi-products of pseudo-differential operators, and constructed Q as a pseudo-differential operator. (ii) The proposition implies that $p(x, D)$ has a microlocal parametrix at (x^0, ξ^0) modulo $\mathscr{E}^*$ and, therefore, $p(x, D)$ is hypoelliptic at (x^0, ξ^0) (in $\mathring{\mathscr{C}}$) with respect to $\mathscr{E}^*$. (iii) When $\rho = 1$ or $\delta = 0$, the proposition is valid, modifying ρ and δ.

Proof. We may assume that for any $A > 0$ there is $C \equiv C_A > 0$ (resp. there are $A > 0$ and $C > 0$) such that

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq CA^{|\alpha|+|\beta|}(|\alpha| + |\beta|)!^\kappa \langle\xi\rangle^{m-|\alpha|} \quad \textit{for } (x, \xi) \in T^*\mathbf{R}^n$$

if $* = (\kappa)$ (resp. if $* = \{\kappa\}$), and that $|\xi^0| = 1$ and (3.12) is satisfied.

By Lemmas 3.5–3.7, Proposition 3.4 can be applicable and there is an operator $\tilde{Q}_h$, which maps continuously $L^2_{\kappa,\varepsilon}$ to $H^{m'}_{\kappa,\varepsilon}$ and $H^{m-m'}_{\kappa,\varepsilon}$ to $H^m_{\kappa,\varepsilon}$ and satisfies $\tilde{Q}_h\tilde{p}_h=I$ on $H^m_{\kappa,\varepsilon}$ and $\tilde{p}_h\tilde{Q}_h=I$ on $H^{m-m'}_{\kappa,\varepsilon}$ *if* ε and h satisfy the following conditions;

(3.18) $$|\varepsilon|<\varepsilon_1\equiv\hat{\varepsilon}_{U\times\Gamma,\mathscr{C}_1}A^{-1/\kappa} \text{ and } h\geq h_{p,B,U\times\Gamma,\mathscr{C}_1}(1/d_0, d_1, h_0), \text{ and}$$
$$|\varepsilon|B\leq c_{d_1,U\times\Gamma,\mathscr{C}_1} \text{ when } \delta=1-1/\kappa \text{ or } \rho=1/\kappa.$$

Fix $h>0$ so that h satisfies (3.18). If $*=(\kappa)$, then A can tend to zero. So, for any $\varepsilon\in\boldsymbol{R}$ we can define

$$\tilde{Q}f=\tilde{Q}_{h'}f-\tilde{Q}_h(\tilde{p}_h-\tilde{p}_{h'})\tilde{Q}_{h'}f \quad \text{for } f\in H^{m-m'}_{\kappa,\varepsilon},$$

where h' $(\geq h)$ is sufficiently large according to $|\varepsilon|$, when $*=(\kappa)$. In fact, $\tilde{p}_h\tilde{Q}f=f$ and $\tilde{Q}\tilde{p}_hf=f+(\tilde{Q}_{h'}-\tilde{Q}_h)(\tilde{p}_h-\tilde{p}_{h'})f-\tilde{Q}_h(\tilde{p}_h-\tilde{p}_{h'})\tilde{Q}_{h'}(\tilde{p}_h-\tilde{p}_{h'})f=f$ for $f\in H^{m-m'}_{\kappa,\varepsilon}$, since $\tilde{p}_h(x,\xi)=\tilde{p}_{h'}(x,\xi)$ for $|\xi|\geq 2h'$. This implies that $\tilde{Q}$ does not depend on h'. When $*=\{\kappa\}$, we define $\tilde{Q}=\tilde{Q}_h$. Then we have

(3.19) $$\tilde{Q}f-\tilde{Q}_{h'}f=\tilde{Q}f-\tilde{Q}_{h'}\tilde{p}_h\tilde{Q}f=\tilde{Q}_{h'}(\tilde{p}_{h'}-\tilde{p}_h)\tilde{Q}f\in H^{m'}_{\kappa,\varepsilon_2}$$

if $f\in H^{m-m'}_{\kappa,\varepsilon}$, $0<\varepsilon_2<\varepsilon_1$ and $|\varepsilon|<\varepsilon_1$, and if h' $(\geq h)$ is sufficiently large according to ε_1 (or A^{-1}) when $*=(\kappa)$, and if ε and h' $(\geq h)$ satisfy (3.18) when $*=\{\kappa\}$, modifying $\hat{\varepsilon}_{U\times\Gamma,\mathscr{C}_1}$. Let $\varphi(x)\in\mathscr{D}^{(\kappa)}$ and $\chi(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ be functions such that $\varphi(x)=1$ in a neighborhood of U, $\chi(x,\xi)$ is positively homogeneous of degree 0 in ξ for $|\xi|\geq 1$, $0\leq\chi(x,\xi)\leq 1$, supp $\chi(x,\xi)\cap\{|\xi|=1\}\Subset\mathscr{C}_1$ and $\chi(x,\xi)=1$ if (x,ξ) belongs to a conic neighborhood of (x^0,ξ^0) and $|\xi|\geq 1$. We write $\chi\Subset\mathscr{C}_1$ for (x^0,ξ^0) if $\chi(x,\xi)$ has the above properties. Let $\chi_1(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ satisfy $\chi_1\Subset\chi$ for (x^0,ξ^0), i.e., $\chi_1\Subset\{(x,\lambda\xi);\chi(x,\xi)=1, |\xi|\geq 1 \text{ and } \lambda>0\}$ for (x^0,ξ^0). We set $Qf=\tilde{Q}\chi(x,D)\varphi(x)f$. Then it follows from (3.19) that Q maps continuously $\mathscr{D}^{*\prime}$ to $\mathscr{D}^{*\prime}$ and $\mathscr{E}^*$ to $\mathscr{E}^*$. In fact, if $*=(\kappa)$, then A can tend to zero and ε_1 can tend to $+\infty$. So, for any $\varepsilon_1>0$ $\tilde{Q}_{h'}$ maps continuously $L^2_{\kappa,\varepsilon}$ to $H^{m'}_{\kappa,\varepsilon}$ if $*=(\kappa)$, $|\varepsilon|<\varepsilon_1$ and h' is sufficiently large. We have also

$$\chi_1\varphi pQf-\chi_1\varphi f=\chi_1\varphi(\chi\varphi-1)f+\chi_1\varphi(p-\tilde{p}_h)Qf\in\mathscr{E}^*$$

for $f\in\mathscr{D}^{*\prime}$. Taking h' sufficiently large according to χ_1 and χ, it follows from Proposition 3.4 (pseudo-locality of $\tilde{Q}_{h'}$) and (3.19) that

$$\chi_1Qpf-\chi_1f=\chi_1\tilde{Q}_{h'}(\chi\varphi p-\tilde{p}_h)f+\chi_1(\tilde{Q}-\tilde{Q}_{h'})(\chi\varphi p-\tilde{p}_h)f\in L^2_{\kappa,\varepsilon_1}$$

if $f\in L^2_{\kappa,\varepsilon}$ and $|\varepsilon|<\varepsilon_1$, and if $|\varepsilon|$ is sufficiently small according to B, d_1, χ_1 and χ when $\delta=1-1/\kappa$ or $\rho=1/\kappa$, modifying $\hat{\varepsilon}_{U\times\Gamma,\mathscr{C}_1}$. This implies that $\chi_1 Qpf-\chi_1 f\in\mathscr{E}^*$ for $f\in\mathscr{D}^{*\prime}$, since $\delta<1-1/\kappa$ and $\rho>1/\kappa$ when $*=(\kappa)$. Assume that $f\in\mathscr{D}^{*\prime}$ and $(x^0,\xi^0)\notin WF_*(f)$. We may assume that $f\in\mathscr{E}^{*\prime}$. Then there is $\chi_2(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ such that $\chi_2\Subset\chi_1$ for (x^0,ξ^0) and $\chi_2(x,D)f\in\mathscr{E}^*$. In fact, by definition there is $\tilde{\chi}(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ such that $\tilde{\chi}\Subset\chi_1$ for (x^0,ξ^0) and ${}^R\tilde{\chi}(x,D)f\in\mathscr{E}^*$. If $\chi_2\Subset\tilde{\chi}$ for (x^0,ξ^0), Lemma 2.15 implies that $\chi_2(x,D)\,{}^R\tilde{\chi}(x,D)f-\chi_2(x,D)f\in\mathscr{E}^*$. Let $\chi_3(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ satisfy $\chi_3\Subset\chi_2$ for (x^0,ξ^0). Taking h' sufficiently large according to χ_2 and χ_3, Lemma 2.15 and Proposition 3.4 give

$$\chi_3(x,D)\tilde{Q}_{h'}\chi(x,D)\varphi(x)(1-\chi_2(x,D))f\in L^2_{\kappa,\varepsilon_1(h')},$$

where $\varepsilon_1(h')>0$ and $\varepsilon_1(h')\to\infty$ as $h'\to\infty$ when $*=(\kappa)$. In fact, we can write

$$\chi\varphi(1-\chi_2)f=b_1\varphi(1-\chi_2)f+b_2\varphi(1-\chi_2)f,$$
$$b_1(x,\xi)=\chi(x,\xi)(1-\chi_4(x,\xi)),\ b_2(x,\xi)=\chi(x,\xi)\chi_4(x,\xi),$$

where $\chi_4(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ satisfies $\chi_3\Subset\chi_4\Subset\chi_2$ for (x^0,ξ^0). On the other hand, we have $\chi_3(x,D)Q\chi_2(x,D)f\in\mathscr{E}^*$. Therefore, by (3.19) we have $\chi_3(x,D)Qf\in\mathscr{E}^*$. Corollary 2 of Lemma 2.15 implies that $(x^0,\xi^0)\notin WF_*(Qf)$. **Q. E. D.**

§ 4. The Microlocal Cauchy Problem

Modifying $p(x,D)$ and using pseudo-differential operators of infinite order, we shall reduce the problem in Gevrey classes to the problem in the Sobolev spaces and construct the inverses of the reduced operators in this section. Then we can construct microlocal parametrices of the microlocal Cauchy problem in Gevrey classes and prove microlocal well-posedness (see Theorem 4.11 below). Theorem 1.4 easily follows from Theorem 4.11 (see §5). In this section we assume that $p(x,\xi)$ satisfies the conditions (A-1) and (A-2) with κ_1 replaced by κ (>1). Let $z^0=(x^0,\xi^0)\in T^*\boldsymbol{R}^n\backslash 0$, $|\xi^0|=1$ and $\vartheta\in T_{z^0}(T^*\boldsymbol{R}^n)$, and assume that $p_m(x,\xi)$ is microhyperbolic with respect to ϑ at z^0.

Lemma 4.1. ([30], [33]). *Let $M\Subset\Gamma(p_{mz^0},\vartheta)$. Then there is a* *neighborhood* $\mathscr{U}$ *of* z^0 *in* $T^*\boldsymbol{R}^n\backslash 0$ *such that* p_m *is microhyperbolic with respect to* $\tilde{\vartheta}\in M$ *at* $z\in\mathscr{U}$, *and* $M\subset\Gamma(p_{mz},\vartheta)$ *for* $z\in\mathscr{U}$.

Define for $v\in T_{z^0}(T^*\boldsymbol{R}^n)$

$$p_m(x,\xi;v,t)=\textstyle\sum_{j=0}^{l}(-itv)^j p_m(x,\xi)/j!,$$

where $l=\mu(z^0)$ and v is regarded as a vector field. By definition there are a neighborhood $\mathscr{U}$ of z^0 in $T^*\boldsymbol{R}^n\backslash 0$ and positive constants c and t_0 such that

$$|p_m(x,\xi;\vartheta,t)|\geq ct^l \quad \textit{for}\ (x,\xi)\in\mathscr{U}\ \textit{and}\ 0\leq t\leq t_0. \tag{4.1}$$

Lemma 4.2. *Let* M *be a compact subset of* $\Gamma(p_{mz^0},\vartheta)$. *Modifying* $\mathscr{U}$, c *and* t_0, *we have*

$$|p_m(x,\xi;v,t)|\geq ct^l \tag{4.2}$$

$$|\partial_t^j p_{m(\beta)}^{(\alpha)}(x,\xi;v,t)/p_m(x,\xi;v,t)|\leq C(\alpha,\beta)t^{-|\alpha|-|\beta|-j} \tag{4.3}$$

if $(x,\xi)\in\mathscr{U}$, $v\in M$ *and* $0<t\leq t_0$.

Remark. Without applying the Malgrange preparation theorem, we can also prove the lemma if only $p_m(x,\xi)\in C^{l+\delta}(T^*\boldsymbol{R}^n)$ and $0<\delta$ (<1).

Proof. Let $\chi(x,\xi)\in C_0^\infty(T^*\boldsymbol{R}^n)$ satisfy supp $\chi\subset\{|x|+|\xi|\leq 2h\}$ and $\chi(x,\xi)=1$ for $|x|+|\xi|\leq h$, where $h>0$, and define an almost analytic extension of $p_m(x,\xi)$ by

$$\begin{aligned}p_m(x+iy,\xi+i\eta)&=\textstyle\sum_{\alpha,\beta}(\alpha!\beta!)^{-1}(i\eta)^\alpha(-y)^\beta p_{m(\beta)}^{(\alpha)}(x,\xi)\\&\quad\times\chi(b_{|\alpha|+|\beta|}y,b_{|\alpha|+|\beta|}\eta)\ \textit{for}\ (x,\xi)\in\bar{\mathscr{U}}\ \textit{and}\ (y,\eta)\in\boldsymbol{R}^n\times\boldsymbol{R}^n,\end{aligned}$$

where $b_0=1$ and $\{b_k\}\subset\boldsymbol{R}$ is a rapidly increasing sequence. Then we have

$$|\partial_t^j\partial_\xi^\alpha D_x^\beta\{p_m((x,\xi)-itv)-p_m(x,\xi;v,t)\}|\leq c_0|t|^{l+1-j} \tag{4.4}$$

if $t\in\boldsymbol{R}$, $|t|\leq t_0$, $(x,\xi)\in\mathscr{U}$, $j\leq l$ and $|\alpha|+|\beta|\leq l$. From Lemma 2.6 in [33] it follows that

$$|p_m((x,\xi)-itv)|\geq c_1t^l \quad \textit{for}\ (x,\xi)\in\mathscr{U},\ v\in M\ \textit{and}\ 0\leq t\leq t_0,$$

modifying $\mathscr{U}$ and t_0 if necessary, which proves (4.2). Applying the Malgrange preparation theorem, there are a neighborhood $\mathscr{U}_1$ of z^0, $\delta>0$, $e(z,v,t)\in C^\infty(\bar{\mathscr{U}}_1\times M\times[-\delta,\delta])$ and $a_j(z,v)\in C^\infty(\bar{\mathscr{U}}_1\times M)$ $(1\leq j\leq l)$ such that $a_j(z^0,v)=0$ for $v\in M$ and $1\leq j\leq l$, $p_m(z+tv)=e(z,v,t)$

$\times g(z, v, t)$ and $e(z, v, t) \neq 0$ for $(z, v, t) \in \bar{\mathscr{U}}_1 \times M \times [-\delta, \delta]$, and $g(z, v, t) \equiv t^l + a_1(z, v) t^{l-1} + \cdots + a_l(z, v) \neq 0$ for $(z, v, t) \in \bar{\mathscr{U}}_1 \times M \times \boldsymbol{C}$ and Im $t<0$. In fact, by Mather's proof of the Malgrange preparation theorem we can obtain the above assertion without dividing M (see [30]). Applying Theorem 2 in [32] to $g(z, v, t)$, we have

$$|\partial_t^j g_{(\beta)}^{(\alpha)}(x, \xi; v, t)/g(x, \xi; v, t)| \leq C'(\alpha, \beta) |\mathrm{Im}\ t|^{-|\alpha|-|\beta|-j}$$

if $(x, \xi) \in \mathscr{U}_1$, $v \in M$, $t \in \boldsymbol{C}$ and $-1 \leq \mathrm{Im}\ t < 0$. On the other hand, (4.4) yields

$$|\partial_t^j \{p_m(x, \xi; v, t) - e(x, \xi, v, -it) g(x, \xi, v, -it))_{(\beta)}^{(\alpha)}| \leq C_0' |t|^{l+1-j}$$

for $t \in \boldsymbol{R}$, $|t| \leq t_0$, $(x, \xi) \in \mathscr{U}$, $j \leq l$ and $|\alpha| + |\beta| \leq l$, modifying $\mathscr{U}$ if necessary, where $e(x, \xi, v, t+i\tau)$ is an almost analytic extension of $e(x, \xi, v, t)$ in t. This proves (4.3) for $j \leq l$ and $|\alpha| + |\beta| \leq l$. It is obvious that (4.3) is valid for $j + |\alpha| + |\beta| \geq l$. **Q. E. D.**

Corollary. *Let M be a compact subset of $\Gamma(p_{mz^0}, \vartheta)$. Then there are a conic neighborhood $\mathscr{C}$ of z^0 and positive constants c and t_0 such that*

$$|p_m(x, \xi; v(\xi), t|\xi|)| \geq ct^l |\xi|^m,$$
$$|p_{m(\beta)}^{(\alpha)}(x, \xi)/p_m(x, \xi; v(\xi), t|\xi|)| \leq C(\alpha, \beta) t^{-|\alpha|-|\beta|} |\xi|^{-|\alpha|}$$

if $(x, \xi) \in \mathscr{C}$, $v \in M$ and $0 < t \leq t_0$, where $v(\xi) = (v_x/|\xi|, v_\xi)$ for $v = (v_x, v_\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$.

Proof. It is easy to see that

$$p_{m(\beta)}^{(\alpha)}(x, \xi/|\xi|) = \textstyle\sum_{j=0}^l (-t)^j \partial_t^j p_{m(\beta)}^{(\alpha)}(x, \xi/|\xi|; v, t)/j!.$$

On the other hand,

$$p_m(x, \xi/|\xi|; v, t) = p_m(x, \xi; v(\xi), t|\xi|) |\xi|^{-m}.$$

Therefore, we have

$$\begin{aligned} &|p_{m(\beta)}^{(\alpha)}(x, \xi)/p_m(x, \xi; v(\xi), t|\xi|)| \\ &\leq |\xi|^{-|\alpha|} \textstyle\sum_{j=0}^l t^j |\partial_t^j p_{m(\beta)}^{(\alpha)}(x, \xi/|\xi|; v, t) p_m(x, \xi/|\xi|; v, t)|/j! \\ &\leq C'(\alpha, \beta) t^{-|\alpha|-|\beta|} |\xi|^{-|\alpha|} \end{aligned}$$

if $(x, \xi/|\xi|) \in \mathscr{U}$, $v \in M$ and $0 < t \leq t_0$. This completes the proof.

Q. E. D.

Now assume that $1 < \kappa \leq \kappa_0 \equiv \min\{2, \mu(z^0)/(\mu(z^0) - 1)\}$ if $* = (\kappa)$ and $1 < \kappa < \kappa_0$ if $* = \{\kappa\}$. Let $\varphi(x, \xi) \in C^2(T^*\boldsymbol{R}^n \backslash 0)$ be a real-valued positively homogeneous function of degree 0 in ξ such that $\varphi(z^0)=0$ and $-H_\varphi(z^0) \equiv -\sum_{j=1}^n \{(\partial\varphi/\partial\xi_j)(z^0)(\partial/\partial x_j)-(\partial\varphi/\partial x_j)(z^0)(\partial/\partial\xi_j)\} \in \Gamma(p_{mz^0}, \vartheta)$. Choose a compact subset M of $\Gamma(p_{mz^0}, \vartheta)$ so that $\vartheta \in M$ and $-H_\varphi(z^0) \in M$. Then there is a neighborhood $\mathscr{U}$ of z^0 such that $M \Subset \Gamma(p_{mz}, \vartheta)$ for $z \in \mathscr{U}$. For given $f \in \mathscr{D}^{*\prime}$ with $WF_*(f) \cap \{\varphi(x, \xi) < 0\} \cap \mathscr{U} = \emptyset$, we shall consider the microlocal Cauchy problem at z^0

$$\text{(MCP)} \qquad \begin{cases} p(x, D)u=f, \\ WF_*(u) \cap \{\varphi(x, \xi)<0\} \cap \mathscr{U} = \emptyset, \end{cases}$$

where $u \in \mathscr{D}^{*\prime}$.

Lemma 4.3. *Let M_1 be a compact convex subset of $\Gamma(p_{mz^0}, \vartheta)$ such that $M \Subset M_1 \Subset \Gamma(p_{mz}, \vartheta)$ for $z \in \mathscr{U}$. Then there are symbols $\Lambda_h(x, \xi)$ $(h \geq 1)$, a convex conic neighborhood $\mathscr{C}$ of z^0, and positive constants ε_1, C_0, A_0, c_1 and c_2 such that*

$$(4.5) \qquad |\Lambda_h(x, \xi)| \leq \langle\xi\rangle_h^{1/\kappa} \quad \text{for } (x, \xi) \in T^*\boldsymbol{R}^n,$$

$$(4.6) \qquad |\Lambda_{h(\beta)}^{(\alpha)}(x, \xi)| \leq C_0 A_0^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\langle\xi\rangle_h^{1/\kappa-|\alpha|} \quad \text{for } (x, \xi) \in T^*\boldsymbol{R}^n,$$

$$(4.7) \qquad |\Lambda_h(x^0, \lambda\xi^0)| \leq c_1 h^4 \lambda^{1/\kappa-4} \quad \text{for } \lambda \geq h,$$

$$(4.8) \qquad \Lambda_h(x, \xi) \leq -(\varphi(x, \xi)+|x-x^0|^2+|\xi/|\xi|-\xi^0|^2)\langle\xi\rangle_h^{1/\kappa}+c_2 h^{1/\kappa} \quad \text{for } (x, \xi) \in \mathscr{C} \text{ and } |\xi| \geq h,$$

$$(4.9) \qquad \begin{aligned} &\langle\xi\rangle_h^{-1/\kappa}\textstyle\sum_{j=1}^n\{|\xi|(\partial\Lambda_h/\partial\xi_j)(x, \xi)(\partial/\partial x_j) \\ &-(\partial\Lambda_h/\partial x_j)(x, \xi)(\partial/\partial\xi_j)\}+\textstyle\sum_{j=1}^n\{y_j(\partial/\partial x_j)+\eta_j(\partial/\partial\xi_j)\} \in M_1 \\ &\text{for } (x, \xi) \in \mathscr{C}, \ |\xi| \geq h \text{ and } |y|^2+|\eta|^2 \leq \varepsilon_1^2. \end{aligned}$$

Proof. Put

$$\begin{aligned} \varphi_1(x, \xi) &= (x-x^0)\cdot\nabla_x\varphi(z^0)+\xi\cdot\nabla_\xi\varphi(z^0) \\ &\quad +B_\varphi(|x-x^0|^2+|\xi-\xi^0|^2), \end{aligned}$$

$$(4.10) \qquad \varphi_2(x, \xi) = \varphi_1(x, \xi)(1+\varphi_1(x, \xi)^2)^{-1/2},$$

and choose $B_\varphi \in \boldsymbol{R}$ so that $\varphi(x, \xi) \leq \varphi_1(x, \xi)-2(|x-x^0|^2+|\xi-\xi^0|^2)$ for $|x-x^0|^2+|\xi-\xi^0|^2 \leq 1$ and $|\xi|=1$. If we set $\Lambda_h(x, \xi) = -\varphi_2(x, \xi/\langle\xi\rangle_{\theta h})\langle\xi\rangle_h^{1/\kappa}$ and θ $(0<\theta\leq 1)$ is chosen appropriately, we can show that $\Lambda_h(x, \xi)$ satisfies (4.5)-(4.9). It is obvious that $\Lambda_h(x, \xi)$ satisfies (4.5). Noting that $\varphi_1(x, \eta)$ is a polynomial of (x, η), there are $L>0$ and $(1\gg)$ $c>0$ such that Re $\varphi_1(x, \eta)^2 \geq 0$ if (x, η) $\in \boldsymbol{C}^n \times$

$\boldsymbol{C}^n$. $|\mathrm{Re}\ x-x^0|\geq L$, $|\mathrm{Im}\ x|\leq c|\mathrm{Re}\ x-x^0|$ and $|\eta|\leq 3$. Then Cauchy's estimates yield

$$|\varphi_{2(\beta)}^{(\alpha)}(x,\eta)|\leq C_\varphi A_\varphi^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!$$

for $(x,\eta)\in T^*\boldsymbol{R}^n$ with $1/2\leq|\eta|\leq 2$. From Lemma 2.2 (4.6) easily follows, where A_0 depends on θ. (4.7) is trivial. It is easy to see that

$$\begin{aligned}&\varphi_2(x,\xi/\langle\xi\rangle_{\theta h})-\varphi(x,\xi)\\&\geq\varphi_1(x,\xi/|\xi|)-\varphi(x,\xi)-|\varphi_1(x,\xi/|\xi|)-\varphi_1(x,\xi/\langle\xi\rangle_{\theta h})|\\&\quad-|\varphi_1(x,\xi/\langle\xi\rangle_{\theta h})-\varphi_2(x,\xi/\langle\xi\rangle_{\theta h})|,\\&|\varphi_1(x,\xi/|\xi|)-\varphi_1(x,\xi/\langle\xi\rangle_{\theta h})|\leq h^2|\nabla_\xi\varphi(z^0)|\langle\xi\rangle_{\theta h}^{-1}|\xi|^{-1}/2\\&\quad+2h^2|B_\varphi|\langle\xi\rangle_{\theta h}^{-1}|\xi|^{-1}\quad \textit{for}\ \xi\neq 0,\end{aligned}$$

and that there are $\delta_\varphi>0$ and $\theta_\varphi>0$ such that $\theta_\varphi\leq 1$ and

$$\begin{aligned}&|\varphi_1(x,\xi/\langle\xi\rangle_{\theta h})-\varphi_2(x,\xi/\langle\xi\rangle_{\theta h})|\leq|\varphi_1(x,\xi/\langle\xi\rangle_{\theta h})|^3/2\\&\leq|x-x^0|^2+|\xi/|\xi|-\xi^0|^2+c_2'h^2\langle\xi\rangle_h^{-2}\end{aligned}$$

if $0<\theta\leq\theta_\varphi$, $|x-x^0|^2+|\xi/|\xi|-\xi^0|^2\leq\delta_\varphi$ and $|\xi|\geq h$, where $c_2'>0$. Here we have used the inequality that $|\xi/\langle\xi\rangle_{\theta h}-\xi/|\xi||\leq\theta^2h^2\langle\xi\rangle_{\theta h}^{-1}|\xi|^{-1}/2$. This proves (4.8), taking $\theta\leq\theta_\varphi$. A simple calculation gives

$$\begin{aligned}&\varphi_{2(\beta)}^{(\alpha)}(x,\xi)=\varphi_{1(\beta)}^{(\alpha)}(x,\xi)(1+\varphi_1(x,\xi)^2)^{-3/2}\quad \textit{for}\ |\alpha|+|\beta|=1,\\&\varphi_{1(\beta)}^{(\alpha)}(x,\xi)=\varphi_{(\beta)}^{(\alpha)}(z^0)+2B_\varphi(|\beta|(x-x^0)^\beta/i+|\alpha|(\xi-\xi^0)^\alpha)\\&\qquad\qquad \textit{for}\ |\alpha|+|\beta|=1,\\&\Lambda_h^{(e_j)}(x,\xi)=-\textstyle\sum_{k=1}^n\varphi_2^{(e_k)}(x,\xi/\langle\xi\rangle_{\theta h})(\delta_{jk}\langle\xi\rangle_{\theta h}^{-1}-\xi_j\xi_k\langle\xi\rangle_{\theta h}^{-3})\langle\xi\rangle_h^{1/\kappa}\\&\qquad -\varphi_2(x,\xi/\langle\xi\rangle_{\theta h})\xi_j\langle\xi\rangle_h^{1/\kappa-2}/\kappa,\\&\Lambda_{h(e_j)}(x,\xi)=-\varphi_{2(e_j)}(x,\xi/\langle\xi\rangle_{\theta h})\langle\xi\rangle_h^{1/\kappa},\end{aligned}$$

where $e_j=(0,\cdots,\overset{\subset j}{1},\cdots,0)\in(\boldsymbol{N}\cup\{0\})^n$. Moreover, with $C_\varphi>0$, we have

$$\begin{aligned}&|\varphi_1(x,\xi/\langle\xi\rangle_{\theta h})|\leq C_\varphi(|x-x^0|^2+|\xi/\langle\xi\rangle_{\theta h}-\xi^0|^2)^{1/2}\\&\qquad\qquad \textit{for}\ |x-x^0|^2+|\xi/\langle\xi\rangle_{\theta h}-\xi^0|^2\leq 1,\\&|\textstyle\sum_{k=1}^n\varphi_2^{(e_k)}(x,\xi/\langle\xi\rangle_{\theta h})\xi_k\langle\xi\rangle_{\theta h}^{-1}|\leq(|\nabla_\xi\varphi(z^0)|+2|B_\varphi|)\\&\qquad\qquad\times|\xi/\langle\xi\rangle_{\theta h}-\xi^0|,\end{aligned}$$

since $\xi^0\cdot\nabla_\xi\varphi(z^0)=0$. Therefore, we have

$$\begin{aligned}&|\langle\xi\rangle_h^{-1/\kappa}|\xi|\Lambda_h^{(e_j)}(x,\xi)+\varphi^{(e_j)}(z^0)|+|\langle\xi\rangle_h^{-1/\kappa}\Lambda_{h(e_j)}(x,\xi)+\varphi_{(e_j)}(z^0)|\\&\leq C_\varphi'\{(|x-x^0|^2+|\xi/|\xi|-\xi^0|^2)^{1/2}+\theta^2h^2\langle\xi\rangle_h^{-2}\},\end{aligned}$$

if $\theta\leq 1/2$, $|x-x^0|^2+|\xi/|\xi|-\xi^0|^2\leq 1/2$ and $|\xi|\geq h$. Taking $\mathscr{C}$ and θ sufficiently small, $\Lambda_h(x, \xi)$ satisfies (4.9). **Q. E. D.**

Lemma 4.4. *Modifying a conic neighborhood $\mathscr{C}$ of z^0, there are symbols $W_h(x, \xi)$ $(h\geq 1)$ and positive constants C_0' and A_0' such that*

$$W_h(x, \xi)\leq 2\langle\xi\rangle_h^{1/\kappa},$$
$$|W_{h(\beta)}^{(\alpha)}(x, \xi)|\leq C_0'A_0'^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!\langle\xi\rangle_h^{1/\kappa-|\alpha|},$$
$$\langle\xi\rangle_h^{-2/\kappa}\{|\xi|^2|\nabla_\xi W_h(x, \xi)|^2+|\nabla_x W_h(x, \xi)|^2\}\leq\varepsilon_1^2$$
$$\text{for } (x, \xi)\in\mathscr{C} \text{ with } |\xi|\geq h,$$
$$\varepsilon_1\langle\xi\rangle_h^{1/\kappa}/2+|\Lambda_h(x, \xi)|<W_h(x, \xi)+c_1h^{1/\kappa} \quad \text{for } |\xi|\geq h,$$

where $\varepsilon_1(<2)$ and c_1 are the constants in Lemma 4.3.

Proof. There is a conic neighborhood $\tilde{\mathscr{C}}$ of z^0 such that $\mathscr{C}\Subset\tilde{\mathscr{C}}$ and

$$|\Lambda_h(x, \xi)|\leq\varepsilon_1\langle\xi\rangle_h^{1/\kappa}/8+c_1h^{1/\kappa} \quad \text{for } (x, \xi)\in\tilde{\mathscr{C}} \text{ with } |\xi|\geq h,$$

modifying $\mathscr{C}$. Choose $w(x, \xi)\in C^2(T^*\boldsymbol{R}^n)$ so that the first and second order derivatives of w are bounded, and $3\varepsilon_1/4\leq w(x, \xi)\leq 2$, $w(x, \xi)=3\varepsilon_1/4$ for $(x, \xi)\in\mathscr{C}$ with $|\xi|\geq 1/2$ and $w(x, \xi)=2$ for $(x, \xi)\notin\tilde{\mathscr{C}}$ with $|\xi|\geq 1/2$. Put $w_j(x, \xi)=E_j*w(x, \xi)$ and $W_h(x, \xi)=w_j(x, \xi/\langle\xi\rangle_h)\times\langle\xi\rangle_h^{1/\kappa}$, where $E_j(x, \xi)=j^n(4\pi)^{-n}\exp[-j(|x|^2+|\xi|^2)/4]$. If j is sufficiently large, then $W_h(x, \xi)$ satisfies the conditions in the lemma. **Q. E. D.**

We may assume that for any $A>0$ there is $C\equiv C_A>0$ (resp. there are $A>0$ and $C>0$) such that

$$|p_{(\beta)}^{(\alpha)}(x, \xi)|\leq CA^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa\langle\xi\rangle^{m-|\alpha|}$$

if $*=(\kappa)$ (resp. if $*=\{\kappa\}$). From (4.1) it follows that

$$\text{Re}\{(\bar{c}_0/|c_0|)p_m(x, \xi;\vartheta, t)\}\geq|c_0|t^l/2 \tag{4.11}$$

for $(x, \xi)\in\mathscr{U}$ and $t_0/3\leq t\leq t_0$, where $c_0=p_{mz^0}(-i\vartheta)$ $(=p_m(x^0, \xi^0;\vartheta, t)\times t^{-l})$, modifying $\mathscr{U}$ if necessary. Let $\mathscr{C}$ and $\mathscr{C}_j(j=1, 2)$ be conic neighborhoods of z^0 such that $\mathscr{C}\cap\{|\xi|=1\}\Subset\mathscr{U}$, $\mathscr{C}_1\Subset\mathscr{C}_2\Subset\mathscr{C}$ and Lemmas 4.3 and 4.4 hold for $\mathscr{C}$. Let $\chi_j(x, \xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n\backslash 0)$ $(j=1, 2)$ be positively homogeneous functions of degree 0 such that $0\leq\chi_j(x, \xi)$

≤ 1, $\chi_j(x,\xi)=1$ on $\mathscr{C}_j$, supp $\chi_1 \subset \mathscr{C}_2$ and supp $\chi_2 \subset \mathscr{C}$, and let $\sigma(\xi) \in \mathscr{E}^{(\kappa)}$ be a function such that $0 \leq \sigma(\xi) \leq 1$, $\sigma(\xi)=1$ for $|\xi| \geq 1$ and $\sigma(\xi)=0$ for $|\xi| \leq 1/2$. We set for $h \geq 1$

$$\begin{aligned}\tilde{p}_{m,h}(x,\xi) &= \sigma_h(\xi)\chi_2(x,\xi)p_m(x,\xi;\vartheta(\xi), t_0\{(1-\sigma_{2h}(\xi))h \\ &\quad + (1-\chi_1(x,\xi))|\xi|\}/3) + c_0(t_0/3)^l|\xi|^m\sigma_h(\xi)(1-\chi_2(x,\xi))/2 \\ &\quad + c_0(t_0/3)^l h^m(1-\sigma_h(\xi)), \\ \tilde{p}_h(x,\xi) &= \tilde{p}_{m,h}(x,\xi) + \sigma_h(\xi)(p(x,\xi)-p_m(x,\xi)),\end{aligned}$$

where $\sigma_h(\xi)=\sigma(\xi/h)$, $\vartheta=(\vartheta_x,\vartheta_\xi)$ and $\vartheta(\xi)=(\vartheta_x/|\xi|,\vartheta_\xi)$. Note that $\tilde{p}_{m,h}(x,\xi)=p_m(x,\xi)$ and $\tilde{p}_h(x,\xi)=p(x,\xi)$ for $(x,\xi)\in\mathscr{C}_1$ with $|\xi|\geq 2h$, and that $\tilde{p}_h(x,\xi)=\tilde{p}_{h'}(x,\xi)$ if $h'\geq h$ and $|\xi|\geq 2h'$.

Lemma 4.5. *There are positive constants $C'\equiv C'_{\chi_1,\chi_2,\sigma,A}(C)$ and d_0 and $h_0\geq 1$ such that*

(4.12) $\quad |\tilde{p}^{(\alpha)}_{h(\beta)}(x,\xi)| \leq C'(2^2A)^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa \langle\xi\rangle_h^{m-|\alpha|}$,

(4.13) $\quad |\tilde{p}_h(x,\xi)| \geq d_0\langle\xi\rangle_h^m$ *if* $(x,\xi)\notin\mathscr{C}_2$ *or* $|\xi|\leq h$ *and if* $h\geq h_0$.

Proof. Since $\langle\xi\rangle\leq\langle\xi\rangle_h\leq\sqrt{5}\langle\xi\rangle$ for $h\geq 1$ and $|\xi|\geq h/2$, (4.12) easily follows. If $|\xi|\leq h/2$, then $\tilde{p}_{m,h}(x,\xi)=c_0(t_0/3)^l h^m$. If $h/2\leq |\xi|\leq h$, then $t_0/3\leq t_0\{h/|\xi|+(1-\chi_1(x,\xi))\}/3\leq t_0$. Therefore, (4.11) gives

$$\begin{aligned}\mathrm{Re}\{(\bar{c}_0/|c_0|)\tilde{p}_{m,h}(x,\xi)\} &\geq |c_0|(t_0/3)^l\{(h+(1-\chi_1(x,\xi))|\xi|) \\ &\quad\times|\xi|^{m-l}\sigma_h(\xi)\chi_2(x,\xi)/2+|\xi|^m\sigma_h(\xi)(1-\chi_2(x,\xi))/2 \\ &\quad+h^m(1-\sigma_h(\xi))\}\geq 5^{-m_+/2}|c_0|(t_0/3)^l\langle\xi\rangle_h^m/2 \quad \text{if } h/2\leq|\xi|\leq h,\end{aligned}$$

where $m_+=\max(m,0)$. Since $t_0/3\leq t_0\{(1-\sigma_{2h}(\xi))h/|\xi|+1\}/3\leq t_0$ for $|\xi|\geq h$, it follows from (4.11) that

$$\begin{aligned}\mathrm{Re}\ &\{(\bar{c}_0/|c_0|)\tilde{p}_{m,h}(x,\xi)\}\geq |c_0|(t_0/3)^l \\ &\times\{((1-\sigma_{2h}(\xi))h+|\xi|)^l|\xi|^{m-l}\chi_2(x,\xi)+|\xi|^m(1-\chi_2(x,\xi))\}/2 \\ &\geq 2^{-1-m_+/2}|c_0|(t_0/3)^l\langle\xi\rangle_h^m \quad \textit{if } |\xi|\geq h \textit{ and } (x,\xi)\notin\mathscr{C}_2.\end{aligned}$$

So there is $d_0>0$ such that

$$|\tilde{p}_{m,h}(x,\xi)|\geq 2d_0\langle\xi\rangle_h^m \quad \textit{if } (x,\xi)\notin\mathscr{C}_2 \textit{ or } |\xi|\leq h.$$

This proves (4.13). **Q.E.D.**

Applying Proposition 2.13, we can write

$$\exp[a\Lambda_h^b](x, D)\tilde{p}_h(x, D)^R\exp[-a\Lambda_h^b](x, D) = \tilde{p}_h^{a,b}(x, D) + r_h^{a,b}(x, D)$$

if $0 \le a \le \hat{a}_0 A^{-1/\kappa}$ and $-1 \le b \le 1$, where $\Lambda_h^b(x, \xi) = \Lambda_h(x, \xi) + bW_h(x, \xi)$, $\hat{a}_0 > 0$, $\hat{\varepsilon} > 0$ and

$$|\hat{p}_{h(\beta)}^{a,b(\alpha)}(x, \xi)| \le C_A(C)(2^{2+2\kappa}A)^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa \times \langle\xi\rangle_h^{m-(1-1/\kappa)(l+1)-|\alpha|},$$

$$\hat{p}_h^{a,b}(x, \xi) = \tilde{p}_h^{a,b}(x, \xi) - \sum_{|\alpha|+|\beta|\le l}(\alpha!\beta!)^{-1} \times \{\tilde{p}_{h(\beta)}(x, \xi)\omega^\beta(a\Lambda_h^b; x, \xi)\omega_\alpha(-a\Lambda_h^b; x, \xi)\}^{(\alpha)},$$

$$|r_{h(\beta)}^{a,b(\alpha)}(x, \xi)| \le C_A(|\alpha|, C)(2^{2+4\kappa}A)^{|\beta|}|\beta|!^\kappa \exp[-\hat{\varepsilon}A^{-1/\kappa}\langle\xi\rangle_h^{1/\kappa}].$$

Lemma 4.6. *For a fixed N_0 there are positive constants a_p, $h_p(a)$, $C_A(C)$, C_p, c and c_1 such that $\tilde{p}_h^{a,b}(x, \xi)$ satisfies the condition $(H; C_A(C), c_1A, ca^l, C_p, a^{-1}, N_0)$ with $m' = m-(1-1/k)l$, $\delta = 1-1/\kappa$ and $\rho = 1/\kappa$ defined in* §3 *if $0 < a \le \hat{a}_0 A^{-1/\kappa}$, $-1 \le b \le 1$ and $h \ge h_p(a)$, and if $a \ge a_p$ when $\kappa = \kappa_0$. Here a_p, $h_p(a)$ and C_p do not depend on the choice of A when $*1 = (\kappa_1)$.*

Proof. When $(x, \xi) \notin \mathscr{C}_2$ or $|\xi| \le h$, by Lemma 4.5 we can apply Lemma 3.1 with $\delta = (1-1/\kappa)/2$ and $\rho = \delta + 1/\kappa$. So it is sufficient to estimate $\tilde{p}_h^{a,b}(x, \xi)$ for $(x, \xi) \in \mathscr{C}_2$ with $|\xi| \ge h$. Let $(x, \xi) \in \mathscr{C}_2$ and $|\xi| \ge h$. Then we have

$$\tilde{p}_h^{a,b}(x, \xi) = p_{m,h}^{a,b}(x, \xi) + s_{h,1}^{a,b}(x, \xi) + s_{h,2}^{a,b}(x, \xi),$$

$$\tilde{p}_{m,h}(x, \xi) = p_m(x, \xi; \vartheta(\xi), d_1(x, \xi)),$$

where

$$p_{m,h}^{a,b}(x, \xi) = \sum_{|\alpha|+|\beta|\le l}(\alpha!\beta!)^{-1}\tilde{p}_{m,h(\beta)}^{(\alpha)}(x, \xi) \times (ia\nabla_x\Lambda_h^b(x, \xi))^\alpha(a\nabla_\xi\Lambda_h^b(x, \xi))^\beta,$$

$$s_{h,1}^{a,b}(x, \xi) = \sum_{|\alpha|+|\beta|\le l}(\alpha!\beta!)^{-1} \times \{\tilde{p}_{m,h(\beta)}(x, \xi)\omega^\beta(a\Lambda_h^b; x, \xi)\omega_\alpha(-a\Lambda_h^b; x, \xi)\}^{(\alpha)} - p_{m,h}^{a,b}(x, \xi),$$

$$s_{h,2}^{a,b}(x, \xi) = \hat{p}_h^{a,b}(x, \xi) + \sum_{|\alpha|+|\beta|\le l}(\alpha!\beta!)^{-1}\{(p_{(\beta)}(x, \xi) - p_{m(\beta)}(x, \xi))\omega^\beta(a\Lambda_h^b; x, \xi)\omega_\alpha(-a\Lambda_h^b; x, \xi)\}^{(\alpha)},$$

$$d_1(x, \xi) = t_0\{(1-\sigma_{2h}(\xi))h + (1-\chi_1(x, \xi))|\xi|\}/3.$$

Put

$$\tilde{p}_{m,h}^{a,b}(x, \xi) = p_m(x - X(x, \xi), \xi - \Xi(x, \xi); v_x(x, \xi), v_\xi(x, \xi), d(x, \xi)),$$

$$X(x, \xi) = a\{\Lambda_h^b(x, \xi), d_1(x, \xi)/|\xi|\}\vartheta_x,$$

$$\Xi(x,\xi)=a\{\Lambda_h^b(x,\xi),\ d_1(x,\xi)\}\vartheta_\xi,$$
$$v_x(x,\xi)=d(x,\xi)^{-1}\{d_1(x,\xi)\vartheta_x/|\xi|+a\nabla_\xi\Lambda_h^b(x,\xi)\},$$
$$v_\xi(x,\xi)=d(x,\xi)^{-1}\{d_1(x,\xi)\vartheta_\xi-a\nabla_x\Lambda_h^b(x,\xi)\},$$
$$d(x,\xi)=d_1(x,\xi)+a\langle\xi\rangle_h^{1/\kappa},$$

where $\{f,g\}=\sum_{j=1}^n\{(\partial f/\partial\xi_j)(\partial g/\partial x_j)-(\partial f/\partial x_j)(\partial g/\partial\xi_j)\}$. Note that there is $h(a)>0$ such that $(x-X(x,\xi),\ \xi-\Xi(x,\xi))\in\mathscr{C}$ if $h\geq h(a)$ and $-1\leq b\leq 1$, $((x,\xi)\in\mathscr{C}_2$ and $|\xi|\geq h)$. From Lemmas 4.3 and 4.4 it follows that $(|\xi|v_x(x,\xi),\ v_\xi(x,\xi))\in M_1$. Therefore, by Corollary of Lemma 4.2 we have

$$\begin{aligned}(4.14)\qquad &|\tilde{p}_{m,h}^{a,b}(x,\xi)|\geq cd(x,\xi)^l|\xi|^{m-l},\\ &|p_{m(\beta)}^{(\alpha)}(x-X(x,\xi),\ \xi-\Xi(x,\xi))/\tilde{p}_{m,h}^{a,b}(x,\xi)|\\ &\leq C(\alpha,\beta)d(x,\xi)^{-|\alpha|-|\beta|}|\xi|^{|\beta|}\end{aligned}$$

if $h\geq h(a)$, modifying $h(a)$. A simple calculation gives

$$\begin{aligned}&|p_{m(\beta)}^{(\alpha)}(x,\xi)/\tilde{p}_{m,h}^{a,b}(x,\xi)|\leq C''d(x,\xi)^{-|\alpha|-|\beta|}\langle\xi\rangle_h^{|\beta|}\\ &\leq C''a^{-|\alpha|-|\beta|}\langle\xi\rangle_h^{(1-1/\kappa)|\beta|-|\alpha|/\kappa}\quad \text{if } h\geq h_p(a) \text{ and } |\alpha|+|\beta|\leq l,\end{aligned}$$

where $h_p(a)>0$. It is easy to see that

$$\begin{aligned}(4.15)\qquad &\textstyle\sum_{|\tilde\alpha|+|\tilde\beta|\leq l}(\tilde\alpha!\tilde\beta!)^{-1}\{p_{m(\beta)}^{(\alpha)}(x,\xi)\,(d_1(x,\xi)\vartheta_x/|\xi|)^\beta\\ &\quad\times(-id_1(x,\xi)\vartheta_\xi)^\alpha\}_{(\tilde\beta)}^{(\tilde\alpha)}(ia\nabla_x\Lambda_h^b(x,\xi))^{\tilde\alpha}(a\nabla_\xi\Lambda_h^b(x,\xi))^{\tilde\beta}\\ &=\textstyle\sum_{|\alpha^0|+|\beta^0|\leq l-N}(\alpha^0!\beta^0!)^{-1}p_{m(\beta+\beta^0)}^{(\alpha+\alpha^0)}(x,\xi)\\ &\quad\times(ia\nabla_x\Lambda_h^b(x,\xi))^{\alpha^0}(a\nabla_\xi\Lambda_h^b(x,\xi))^{\beta^0}(-id_1(x,\xi)\vartheta_\xi-\Xi(x,\xi))^\alpha\\ &\quad\times(d_1(x,\xi)\vartheta_x/|\xi|-iX(x,\xi))^\beta+\Sigma_1,\end{aligned}$$

$$\begin{aligned}\Sigma_1=&\textstyle(\sum_{|\alpha^0|+\cdots+|\alpha^N|+|\beta^0|+\cdots+|\beta^N|\leq l}\\ &\textstyle-\sum_{|\alpha^0|+|\beta^0|\leq l-N,\,|\alpha^j|+|\beta^j|\leq 1,\,1\leq j\leq N})\,(\alpha^0!\beta^0!)^{-1}p_{m(\beta+\beta^0)}^{(\alpha+\alpha^0)}(x,\xi)\\ &\times(ia\nabla_x\Lambda_h^b(x,\xi))^{\alpha^0}(a\nabla_\xi\Lambda_h^b(x,\xi))^{\beta^0}\textstyle\prod_{j=1}^{|\alpha|}(\alpha^j!\beta^j!)^{-1}d_{1(\beta^j)}^{(\alpha^j)}(x,\xi)\\ &\times(ia\nabla_x\Lambda_h^b(x,\xi))^{\alpha^j}(a\nabla_\xi\Lambda_h^b(x,\xi))^{\beta^j}\textstyle\prod_{j=|\alpha|+1}^N(\alpha^j!\beta^j!)^{-1}\\ &\times(d_1(x,\xi)/|\xi|)_{(\beta^j)}^{(\alpha^j)}(ia\nabla_x\Lambda_h^b(x,\xi))^{\alpha^j}(a\nabla_\xi\Lambda_h^b(x,\xi))^{\beta^j}(-i\vartheta_\xi)^\alpha\vartheta_x^\beta,\end{aligned}$$

where $N=|\alpha|+|\beta|\leqq l$. Then we have

$$\begin{aligned}&|\Sigma_1/\tilde{p}_{m,h}^{a,b}(x,\xi)|\\ &\leq C_p(\textstyle\sum_{|\alpha^0|+|\beta^0|\leq l-N,\,1\leq|\alpha^1|+\cdots+|\beta^N|\leq N+1}a^{|\alpha^0|+\cdots+|\beta^N|}\\ &\quad\times d(x,\xi)^{-N-|\alpha^0|-|\beta^0|}d_1(x,\xi)^{N-(|\alpha^1|+\cdots+|\beta^N|-1)}\\ &\quad\times\langle\xi\rangle_h^{(|\alpha^0|+\cdots+|\alpha^N|)/\kappa+(|\beta^0|+\cdots+|\beta^N|)/\kappa-1}\\ &\quad+\textstyle\sum_{|\alpha^0|+|\beta^0|\leq l-N,\,|\alpha^1|+\cdots+|\beta^N|>N+1}a^{|\alpha^0|+\cdots+|\beta^N|}\\ &\quad\times d(x,\xi)^{-N-|\alpha^0|-|\beta^0|}\langle\xi\rangle_h^{|\beta|+|\beta^0|+(|\alpha^0|+\cdots+|\alpha^N|)/\kappa}\end{aligned}$$

$$\times \langle\xi\rangle_h^{-(1-1/\kappa)(|\beta^0|+\cdots+|\beta^N|)+|\alpha|-|\alpha^1|-\cdots-|\alpha^N|}$$
$$+\sum_{l-N<|\alpha^0|+|\beta^0|\le l,\,|\alpha^1|+\cdots+|\beta^N|\le N-1} a^{|\alpha^0|+\cdots+|\beta^N|}$$
$$\times d(x,\xi)^{-l} d_1(x_1,\xi)^{N-(|\alpha^1|+\cdots+|\beta^N|)}$$
$$\times \langle\xi\rangle_h^{l-N-(1-1/\kappa)(|\alpha^0|+|\beta^0|)+(|\alpha^1|+\cdots+|\beta^N|)/\kappa}\}$$
$$\le aC_p'\langle\xi\rangle_h^{1/\kappa-1} \quad if \ h\ge h_p(a),$$

modifying $h_p(a)$. Therefore, we have

$$p_{m,h}^{a,b}(x,\xi)=\sum_{|\alpha^1|+|\alpha^2|+|\beta^1|+|\beta^2|\le l}(\alpha^1!\alpha^2!\beta^1!\beta^2!)^{-1}$$
$$\times p_{m(\beta^1+\beta^2)}^{(\alpha^1+\alpha^2)}(x,\xi)(-\Xi(x,\xi))^{\alpha^1}(-iX(x,\xi))^{\beta^1}(-id(x,\xi)v_\xi(x,\xi))^{\alpha^2}$$
$$\times(d(x,\xi)v_x(x,\xi))^{\beta^2}+\Sigma_2=\tilde{p}_{m,h}^{a,b}(x,\xi)+\Sigma_2+\Sigma_3,$$

where $|\Sigma_2/\tilde{p}_{m,h}^{a,b}(x,\xi)|\le aC_p''\langle\xi\rangle_h^{1/\kappa-1}$ if $h\ge h_p(a)$ and

$$\Sigma_3=\sum_{|\alpha|+|\beta|\le l}(\alpha!\beta!)^{-1}\{p_{m(\beta)}^{(\alpha)}(x-X(x,\xi),\ \xi-\Xi(x,\xi))$$
$$-\sum_{|\tilde\alpha|+|\tilde\beta|\le l-|\alpha|-|\beta|}(\tilde\alpha!\tilde\beta!)^{-1}p_{m(\beta+\tilde\beta)}^{(\alpha+\tilde\alpha)}(x,\xi)(-\Xi(x,\xi))^{\tilde\alpha}$$
$$\times(-iX(x,\xi))^{\tilde\beta}\}(-id(x,\xi)v_\xi(x,\xi))^{\alpha}(d(x,\xi)v_x(x,\xi))^{\beta}.$$

It follows from (4.14) that $|\Sigma_3/\tilde{p}_{m,h}^{a,b}(x,\xi)|\le aC_p'''\langle\xi\rangle_h^{1/\kappa-1}$ if $h\ge h_p(a)$. So we have

$$|p_{m,h}^{a,b}(x,\xi)-\tilde{p}_{m,h}^{a,b}(x,\xi)|/|\tilde{p}_{m,h}^{a,b}(x,\xi)|\le 1/6$$

if $h\ge h_p(a)$. The same argument as in (4.15) yields

$$|\tilde{p}_{m,h(\beta)}^{(\alpha)}(x,\xi)/\tilde{p}_{m,h}^{a,b}(x,\xi)|\le C_p a^{-|\alpha|-|\beta|}\langle\xi\rangle_h^{(1-1/\kappa)|\beta|-|\alpha|/\kappa}$$

if $h\ge h_p(a)$ and $|\alpha|+|\beta|\le l$. This implies that

$$|s_{h,1}^{a,b}(x,\xi)/\tilde{p}_{m,h}^{a,b}(x,\xi)|\le 1/6 \quad if\ h\ge h_p(a),$$

since

$$|\{\omega_\alpha(-a\Lambda_h^b;x,\xi)-(ia\nabla_x\Lambda_h^b(x,\xi))^\alpha\}_{(\tilde\beta)}^{(\tilde\alpha)}|\le C(\alpha,\tilde\alpha,\tilde\beta,a)$$
$$\times\langle\xi\rangle_h^{|\alpha|/\kappa-1/\kappa-|\tilde\alpha|},$$
$$|\{\omega^\beta(a\Lambda_h^b;x,\xi)-(a\nabla_\xi\Lambda_h^b(x,\xi))^\beta\}_{(\tilde\beta)}^{(\tilde\alpha)}|\le C(\beta,\tilde\alpha,\tilde\beta,a)$$
$$\times\langle\xi\rangle_h^{-(1-1/\kappa)|\beta|-1/\kappa-|\tilde\alpha|},$$
$$|\{\omega^\beta(a\Lambda_h^b;x,\xi)\omega_\alpha(-a\Lambda_h^b;x,\xi)\}_{(\tilde\beta)}^{(\tilde\alpha)}|\le C(\alpha,\beta,\tilde\alpha,\tilde\beta,a)$$
$$\times\langle\xi\rangle_h^{|\alpha|/\kappa-(1-1/\kappa)|\beta|-|\tilde\alpha|}.$$

It is obvious that

$$|s_{h,2}^{a,b}(x,\xi)/\tilde{p}_{m,h}^{a,b}(x,\xi)|\le 1/6 \quad if\ h\ge h_p(a)\ and\ \kappa<\kappa_0.$$

When $\kappa=\kappa_0$, we have also

$$|s_{h,2}^{a,b}(x,\xi)/\tilde{p}_{m,h}^{a,b}(x,\xi)|\le 1/6 \quad if\ h\ge h_p(a)\ and\ a\ge a_p,$$

where $a_p>0$. Thus we have

$$|\tilde{p}_h^{a,b}(x,\xi)|\geq|\tilde{p}_{m,h}^{a,b}(x,\xi)|/2\geq c'a^l\langle\xi\rangle_h^{m-(1-1/\kappa)l}$$

if $h\geq h_p(a)$, and if $a\geq a_p$ when $\kappa=\kappa_0$, where $c'>0$. Similarly, we have

$$|\tilde{p}_{h(\beta)}^{a,b(\alpha)}(x,\xi)/\tilde{p}_h^{a,b}(x,\xi)|\leq 2|\tilde{p}_{h(\beta)}^{a,b(\alpha)}(x,\xi)/\tilde{p}_{m,h}^{a,b}(x,\xi)|$$
$$\leq C_p(\alpha,\beta)a^{-|\alpha|-|\beta|}\langle\xi\rangle_h^{(1-1/\kappa)|\beta|-|\alpha|/\kappa}$$

if $h\geq h_p(a)$, and if $a\geq a_p$ when $\kappa=\kappa_0$. Here we have modified $h_p(a)$, if necessary. This proves the lemma. **Q. E. D.**

From Proposition 3.4 (or its proof) and Lemma 4.6 it follows that $\tilde{p}_h^{a,b}(x,D)+r_h^{a,b}(x,D)$ has the inverse $\tilde{Q}_h^{a,b}$, i.e., $\tilde{Q}_h^{a,b}$ maps continuously L^2 to $H^{m-(1-1/\kappa)l}(\subset H^m)$ and $H^{(1-1/\kappa)l}$ to H^m, and satisfies $\tilde{Q}_h^{a,b}(\tilde{p}_h^{a,b}(x,D)+r_h^{a,b}(x,D))=I$ on H^m and $(\tilde{p}_h^{a,b}(x,D)+r_h^{a,b}(x,D))\tilde{Q}_h^{a,b}=I$ on $H^{(1-1/\kappa)l}$ if a, b and h satisfy the following conditions;

(4.16) $0<a\leq\hat{a}_0A^{-1/\kappa}$, $-1\leq b\leq 1$ and $h\geq h_{p,a}$, and $a\geq a_p'$ when $\kappa=\kappa_0$,

where $h_{p,a}$ and a_p' are positive constants. By Lemma 2.14 and Proposition 3.4, for any $\varepsilon>0$ there is $h_a(\varepsilon)>0$ such that $1+q_h^{a,b}(x,D)\equiv{}^R\exp[-a\Lambda_h^b](x,D)\exp[a\Lambda_h^b](x,D)$ has the inverse $(1+q_h^{a,b}(x,D))^{-1}$ which maps continuously $L^2_{\kappa,\varepsilon'}$ to $L^2_{\kappa,\varepsilon'}$ if $|\varepsilon'|\leq\varepsilon$, $h\geq h_a(\varepsilon)$, $a\geq 0$ and $-1\leq b\leq 1$. Let us introduce the following spaces.

Definition 4.7. Let $\Lambda(x,\xi)$ satisfy (2.13). For $s\in\boldsymbol{R}$ we define $H^s_\Lambda=\{f\,;\,(e^\Lambda)(x,D)f\in H^s\}$, and write $L^2_\Lambda=H^0_\Lambda$, $H^s_{a,b}=H^s_{a\Lambda_h^b}$ and $L^2_{a,b}=H^0_{a,b}$.

Lemma 4.8. (i) *$f\in H^s$ if and only if ${}^R(e^{-\Lambda})(x,D)f\in H^s_\Lambda$.* (ii) *If $f\in H^s_{a\Lambda}$ and $k(x,\xi)$ satisfies*

$$|k_{(\beta)}^{(\alpha)}(x,\xi)|\leq C_1A_1^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa\langle\xi\rangle_h^{m-|\alpha|},$$

then $k(x,D)f\in H^{s-m}_{a\Lambda}$ for $|a|\leq c_\Lambda A_1^{-1/\kappa}$, where $c_\Lambda>0$.

Proof. By Lemma 2.14 we can write

$${}^R(e^{-a\Lambda})(x,D)(e^{a\Lambda})(x,D)=1+q_{h'}^a(x,D)+r_{h'}^a(x,D),$$
$$(e^{a\Lambda})(x,D){}^R(e^{-a\Lambda})(x,D)=1+\tilde{q}_{h'}^a(x,D)+\tilde{r}_{h'}^a(x,D),$$

where $h'\geq 1$ and

$$|q_{h'(\beta)}^{a(\alpha)}(x,\xi)|\leq C_{a,\Lambda,d}d^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa\langle\xi\rangle_h^{1/\kappa-1-|\alpha|}\quad \textit{for any } d>0,$$

$$q^a_{h'}(x, \xi)=0 \quad for \ |\xi|\leq h',$$
$$r^a_{h'}(x, D): L^2_{\kappa,\varepsilon} \to L^2_{\kappa,\varepsilon'} \quad for\ any\ \varepsilon,\ \varepsilon'\in \boldsymbol{R},$$

and $\tilde{q}^a_{h'}(x, \xi)$ and $\tilde{r}^a_{h'}(x, \xi)$ have the same properties as $q^a_{h'}(x, \xi)$ and $r^a_{h'}(x, \xi)$, respectively. From Proposition 3.4 it follows that $1+q^a_{h'}(x, D)$ and $1+\tilde{q}^a_{h'}(x, D)$ have the inverses $(1+q^a_{h'}(x, D))^{-1}$ and $(1+\tilde{q}^a_{h'}(x, D))^{-1}$, respectively, which map continuously $L^2_{\kappa,\varepsilon}$ to $L^2_{\kappa,\varepsilon}$ and $H^s_{\kappa,\varepsilon}$ to $H^s_{\kappa,\varepsilon}$, if h' is large enough and $|\varepsilon|\leq A_1^{-1/\kappa}$. Then the assertion (i) is obvious. Using (2.29), we have

$$\begin{aligned}(e^{a\Lambda})(x, D)k(x, D)f = e^{a\Lambda}k[{}^R(e^{-a\Lambda})(1+\tilde{q}^a_{h'})^{-1}e^{a\Lambda}\\ -(1+q^a_{h'})^{-1}\{{}^R(e^{-a\Lambda})\tilde{r}^a_{h'}-r^a_{h'}{}^R(e^{-a\Lambda})\}(1+\tilde{q}^a_{h'})^{-1}e^{a\Lambda}\\ -(1+q^a_{h'})^{-1}r^a_{h'}]f\end{aligned}$$

if $f\in H^s_{a\Lambda}$ and $|a|\leq c_\Lambda A_1^{-1/\kappa}$, where $c_\Lambda>0$. Here we have chosen h' sufficiently large according to A_1. Modifying c_Λ if necessary, Propositions 2.12 and 2.13 imply that $(e^{a\Lambda})(x, D)k(x, D)^R(e^{-a\Lambda})(x, D)$ maps continuously H^s to H^{s-m} for $|a|\leq c_\Lambda A_1^{-1/\kappa}$. This proves the assertion (ii). **Q.E.D.**

Lemma 4.9. *Let $\Lambda(x, \xi)$ and $\Lambda'(x, \xi)$ satisfy* (2.13), *where $h\geq 1$, and assume that a symbol $q(x, \xi)$ satisfies*

$$|q^{(\alpha)}_{(\beta)}(x, \xi)|\leq C_d d^{|\alpha|+|\beta|}|\alpha|!^\kappa|\beta|!^\kappa\langle\xi\rangle^{-|\alpha|} \ for\ any\ d>0.$$

If $\inf_{L>0}\sup_{(x,\xi)\in \mathrm{supp}\, q(x,\xi), |\xi|\geq L}(\Lambda(x, \xi)-\Lambda'(x, \xi))\langle\xi\rangle_h^{-1/\kappa}<\varepsilon$, then $q(x, D)(e^\Lambda)(x, D)\ {}^R(e^{-\Lambda'})(x, D)f$ and $(e^\Lambda)(x, D)q(x, D)\ {}^R(e^{-\Lambda'})(x, D)f$ belong to $L^2_{\kappa,a-\varepsilon}$ for $f\in L^2_{\kappa,a}$.

Proof. By Proposition 2.8 we can write $q(x, D)(e^\Lambda)(x, D)=\tilde{q}(x, D)+\tilde{r}(x, D)$, where $\tilde{q}(x, \xi)=\sum_{j=0}^\infty\sum_{|\alpha|=j}\alpha!^{-1}\varphi^R_j(\xi)q^{(\alpha)}(x, \xi)\times(e^{\Lambda(x,\xi)})_{(\alpha)}$ and $\tilde{r}(x, D): L^2_{\kappa,a}\to L^{2}_{\kappa,a'}$ continuously for any $a, a'\in\boldsymbol{R}$. It follows from Corollary of Lemma 2.9 that

$$\begin{aligned}|\tilde{q}^{(\alpha)}_{(\beta)}(x, \xi)|\leq C_{\Lambda,\rho,d}d^{|\alpha|+|\beta|}|\alpha|!^\kappa|\beta|!^\kappa\langle\xi\rangle^{-|\alpha|}\\ \times\exp[\rho\langle\xi\rangle^{1/\kappa}+\Lambda(x, \xi)] \quad for\ any\ d, \rho>0,\end{aligned}$$

and that $\mathrm{supp}\,\tilde{q}(x, \xi)\subset\mathrm{supp}\,q(x, \xi)$. Since $\Lambda(x, \xi)$ and $\Lambda'(x, \xi)$ satisfy (2.13), there are $c_1>0$ and $c_2>0$ such that $\inf_{L>0}\sup\{(\Lambda(x, \xi+\eta)-\Lambda'(x+y, \xi+\eta))\langle\xi\rangle^{-1/\kappa};\ (x, \xi+\eta)\in\mathrm{supp}\,q,\ |\xi|\geq L,\ |\eta|\leq c_1\langle\xi\rangle$ and $|y|\leq c_2\}<\varepsilon$. Applying Lemma 2.15 to $\tilde{q}(x, \xi)\ {}^R(e^{-\Lambda'})(x, D)$, we have $\tilde{q}(x, D)\ {}^R(e^{-\Lambda'})(x, D)f\in L^2_{\kappa,a-\varepsilon}$ for $f\in L^2_{\kappa,a}$, which proves that $q(x, D)$

$\times (e^{\Lambda})(x, D)\ {}^{R}(e^{-\Lambda'})(x, D)f \in L^2_{\kappa, a-\varepsilon}$ for $f \in L^2_{\kappa, a}$. Similarly, we can prove $(e^{\Lambda})(x, D) q(x, D)\ {}^{R}(e^{-\Lambda'})(x, D) f \in L^2_{\kappa, a-\varepsilon}$ for $f \in L^2_{\kappa, a}$. **Q. E. D**

Let us construct parametrices of (MCP). Define $Q_h^{a,b}$ by

$$Q_h^{a,b} f = {}^{R}\exp[-a\Lambda_h^b](x, D)\, \tilde{Q}_h^{a,b} \exp[a\Lambda_h^b](x, D) f.$$

Then $Q_h^{a,b}$ maps continuously $L^2_{a,b}$ to $H^{m-(1-1/\kappa)l}_{a,b}$ and $H^{(1-1/\kappa)l}_{a,b}$ to $H^m_{a,b}$ and satisfies $Q_h^{a,b} \tilde{p}_h(x, D) = I$ on $H^m_{a,b}$ and $\tilde{p}_h(x, D) Q_h^{a,b} = I$ on $H^{(1-1/\kappa)l}_{a,b}$ if a, b and h satisfy (4. 16). In fact, by Lemma 2. 14 and Proposition 3. 4 we may assume that $1 + \tilde{q}_h^{a,b}(x, D) \equiv \exp[a\Lambda_h^b](x, D)\ {}^{R}\exp[-a\Lambda_h^b](x, D)$ has the inverse $(1 + \tilde{q}_h^{a,b}(x, D))^{-1}$ which maps $H^s_{\kappa,\varepsilon}$ to $H^s_{\kappa,\varepsilon}$ if $|s| \leq |m| + (1 - 1/\kappa) l$, $|\varepsilon| \leq 4a$, $-1 \leq b \leq 1$ and $h \geq h_{p,a}$. We may also assume that $(1 + q_h^{a,b}(x, D))^{-1}$ maps continuously $L^2_{\kappa,\varepsilon}$ to $L^2_{\kappa,\varepsilon}$ if $|\varepsilon| \leq 4a$, $-1 \leq b \leq 1$ and $h \geq h_{p,a}$. From Corollary of Lemma 2. 9 and Proposition 2. 12 it follows taht ${}^{R}\exp[-a\Lambda_h^b](x, D) g \in L^2_{\kappa,-4a}$ for $g \in H^s$, where $s \in \boldsymbol{R}$, $a > 0$, $-1 \leq b \leq 1$ and $h \geq h_{p,a}$. So, by (2. 29) we have

$$f = {}^{R}\exp[-a\Lambda_h^b](x, D)(1 + \tilde{q}_h^{a,b}(x, D))^{-1} \exp[a\Lambda_h^b](x, D) f \tag{4. 17}$$

if $f \in H^s_{a,b}$, $s \in \boldsymbol{R}$, $a \geq 0$, $-1 \leq b \leq 1$ and $h \geq h_{p,a}$. This implies that

$$\begin{aligned} Q_h^{a,b} \tilde{p}_h(x, D) f &= {}^{R}\exp[-a\Lambda_h^b] \tilde{Q}_h^{a,b} \exp[a\Lambda_h^b] \tilde{p}_h\ {}^{R}\exp[-a\Lambda_h^b] g \\ &= {}^{R}\exp[-a\Lambda_h^b](1 + \tilde{q}_h^{a,b})^{-1} \exp[a\Lambda_h^b] f = f, \\ g &= (1 + \tilde{q}_h^{a,b})^{-1} \exp[a\Lambda_h^b] f \in H^m, \end{aligned}$$

if $f \in H^s_{a,b}$ and a, b and h satisfy (4. 16). Similarly, we can prove $\tilde{p}_h(x, D) Q_h^{a,b} f = f$ if $f \in H^{(1-1/\kappa)l}_{a,b}$ and a, b and h satisfy (4. 16). From Lemma 4. 9 and (4. 17) it follows that $H^s_{a,b} \subset H^s_{a',-1}$ for $s \in \boldsymbol{R}$, $0 \leq a \leq a'$, $-1 \leq b \leq 1$ and $h \geq h_{p,a}$. Therefore, we have $Q_h^{a,-1} f = Q_h^{a,b} f$ if $f \in H^{(1-1/\kappa)l}_{a,b}$ and a, b and h satisfy (4. 16). Fix h sufficiently large and define Q_h by

$$Q_h f = Q_{h'}^{a',-1} f - Q_h^{a,-1}(\tilde{p}_h(x, D) - \tilde{p}_{h'}(x, D)) Q_{h'}^{a',-1} f$$

for $f \in H^{(1-1/\kappa)l}_{a',-1}$, where a' and h' satisfy (4. 16). Then, by the same arguments as in the proof of Proposition 3. 8 Q_h does not depend on the choice of a, a' and h', and satisfies $\tilde{p}_h(x, D) Q_h f = Q_h \tilde{p}_h(x, D) f = f$ if $f \in L^2_{a',-1}$ and a' satisfies (4. 16). Moreover we have

$$Q_h f - Q_{h'}^{a',-1} f \in H^{m-(1-1/\kappa)l}_{a',1}$$

if $f \in H^{(1-1/\kappa)l}_{a',-1}$ and a' and h' satisfy (4. 16). Here we have modified $\hat{a}_0$ if necessary. So Q_h maps continuously $L^2_{a',b}$ to $H^{m-(1-1/\kappa)l}_{a',b}$ if a' and

b satisfy (4.16). To obtain this result we used Λ_h^b instead of Λ_h. Let $\chi(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ satisfy $\chi\Subset\mathscr{C}_1$ for z^0. Here we have used the notation in the proof of Proposition 3.8. Choose $\psi(x)\in\mathscr{D}^{(\kappa)}$ so that $\psi(x)=1$ in a neighborhood of $\{x\in\boldsymbol{R}^n;\ (x,\xi)\in\mathscr{C}_1$ for some $\xi\in\boldsymbol{R}^n\}$, and define Q by $Qf=Q_h\chi(x,D)\psi(x)f$. Let $\chi_1(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ satisfy $\chi_1\Subset\chi$ for z^0. Then we have

$$\chi_1(x,D)p(x,D)Qf-\chi_1(x,D)f=\chi_1(\chi\psi-1)f+\chi_1(p-\tilde{p}_h)Qf\in L^2_{\kappa,\varepsilon_0'}$$

if $f\in H^{(1-1/\kappa)l}_{a',-1}$ and a' satisfies (4.16), where $\varepsilon_0'=\hat{\varepsilon}_0'A^{-1/\kappa}$ and $\hat{\varepsilon}_0'>0$. Define $\Lambda_h'(x,\xi)$, replacing $\varphi_1(x,\xi)$ by $\varphi_1(x,\xi)-2(|x-x^0|^2+|\xi-\xi^0|^2)/3$ in the construction of $\Lambda_h(x,\xi)$ in the proof of Lemma 4.3. Then we may assume that $\Lambda_h'(x,\xi)$ satisfies

$$\begin{aligned}\Lambda_h(x,\xi)&\le\Lambda_h'(x,\xi)-(|x-x^0|^2+|\xi/|\xi|-\xi^0|^2)\langle\xi\rangle_h^{1/\kappa}/2+C_1(h)\\&\le-(\varphi(x,\xi)+|x-x^0|^2+|\xi/|\xi|-\xi^0|^2)\langle\xi\rangle_h^{1/\kappa}+C_2(h)\end{aligned}$$

for $(x,\xi)\in\mathscr{C}$ and $|\xi|\ge1$. Let us prove that Q is a left microlocal parametrix of p on $H^m_{a(\Lambda_h'+bw_h)}$. Let $f\in H^m_{a(\Lambda_h'+bw_h)}$, and let $\tilde{\chi}(x,\xi)\in\mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ be a positively homogeneous function of degree 0 in ξ for $|\xi|\ge1$ such that $0\le\tilde{\chi}(x,\xi)\le1$, supp $\chi\cap$supp $(1-\chi)\cap\{|\xi|=1\}$ $\Subset\{(x,\xi)\in T^*\boldsymbol{R}^n;\ \tilde{\chi}(x,\xi)=1\}$ and supp $\tilde{\chi}(x,\xi)$ is included in a small conic neighborhood of supp $\chi\cap$supp $(1-\chi)\cap\{|\xi|=1\}$. Assume that a and h satisfy (4.16). Then it follows from Proposition 2.8, Corollary 3 of Lemma 2.15 and its proof that

$$(1-\tilde{\chi}(x,D))\exp[a\Lambda_h^{b'}](x,D)[p,\chi\psi]f\in L^2$$

if $b, b'\in[-1,1]$, modifying $\hat{a}_0$. On the other hand, Lemma 4.9 and (4.17) yield

$$\begin{aligned}\tilde{\chi}(x,D)\exp[a\Lambda_h^{b'}](x,D)[p,\chi\psi]f&=\tilde{\chi}\exp[a\Lambda_h^{b'}]^R\exp[-a(\Lambda_h'+bW_h)]\\&\quad\times(1+\tilde{q}_h'^{a,b})^{-1}\exp[a(\Lambda_h'+bW_h)][p,\chi\psi]f\in L^2\end{aligned}$$

if $b'-b<2b_0\equiv\inf_{(x,\xi)\in\mathrm{supp}\,\tilde{\chi}}(|x-x^0|^2+|\xi/|\xi|-\xi^0|^2)/4$, where $1+\tilde{q}_h'^{a,b}(x,D)=\exp[a(\Lambda_h'+bW_h)](x,D)^R\exp[-a(\Lambda_h'+bW_h)](x,D)$. We may assume that $b_0\le2$. So we have $[p,\chi\psi]f\in L^2_{a,b+b_0}$ if $-1\le b\le1-b_0$. This gives $Q_h[p,\chi\psi]f\in H^{m-(1-1/\kappa)l}_{a,b_0/2}$ if $|b|\le b_0/2$. Modifying χ_1, we can assume that

$$\inf_{L>0}\ \sup_{(x,\xi)\in\mathrm{supp}\chi_1,|\xi|\ge L}|\Lambda_h(x,\xi)|\langle\xi\rangle^{-1/\kappa}<\varepsilon_1b_0/8.$$

Then, by Lemma 4.9 and (4.17) we have

$$\chi_1 Q_h [p, \chi\psi] f \in L^2_{\kappa, a\varepsilon_1 b_0/8} \text{ if } |b| \leq b_0/2.$$

This implies that

$$\chi_1(x, D) Q p(x, D) f - \chi_1(x, D) f = -\chi_1 Q_h [p, \chi\psi] f + \chi_1(\chi\psi - 1) f + \chi_1 Q_h (p - \tilde{p}_h) \chi\psi f \in L^2_{\kappa, a\hat{c}_0}$$

if $|b| \leq b_0/2$, where $\hat{c}_0 > 0$. Now assume that $f \in \mathscr{D}^{*\prime}$ and $WF_*(f) \cap \mathscr{C} \subset \{\varphi(x, \xi) \geq 0\}$. For a fixed b with $-b_0/2 \leq b < 0$ there is $a_f > 0$ such that

$$\chi(x, D) \psi(x) f \in H^{(1-1/\kappa)l}_{a,b} \cap H^m_{a(\Lambda'_h + bw_h)} \tag{4.18}$$

if $a \geq a_f$ when $* = (\kappa)$ and if $0 < a \leq a_f$ when $* = \{\kappa\}$. In fact, let $\chi_2(x, \xi) \in \mathscr{E}^{(\kappa)}(T^*\boldsymbol{R}^n)$ be a positively homogeneous function of degree 0 in ξ for $|\xi| \geq 1$ such that $0 \leq \chi_2(x, \xi) \leq 1$, $\chi_2(x, \xi) = 1$ if $|\xi| \geq 1$ and $\limsup_{\lambda \to +\infty} (\Lambda'_h(x, \lambda\xi) + bW_h(x, \lambda\xi)/4)(\lambda|\xi|)^{-1/\kappa} \leq 0$, and $\chi_2(x, \xi) = 0$ if $|\xi| \geq 1$ and $\liminf_{\lambda \to +\infty} (\Lambda'_h(x, \lambda\xi) + bW_h(x, \lambda\xi)/2)(\lambda|\xi|)^{-1/\kappa} \geq 0$. Since

$$\begin{aligned} &\liminf_{\lambda \to +\infty} \Lambda^b_h(x, \lambda\xi)(\lambda|\xi|)^{-1/\kappa} \\ &\leq \liminf_{\lambda \to +\infty} (\Lambda'_h(x, \lambda\xi) + bW_h(x, \lambda\xi))(\lambda|\xi|)^{-1/\kappa} \leq b\varepsilon_1/4 < 0 \end{aligned}$$

for $(x, \xi) \in \operatorname{supp} \chi_2 \cap \mathscr{C}$ with $|\xi| \geq 1$, applying Lemma 4.9 and the Paley-Wiener theorem for $\mathscr{E}^{*\prime}$ (see, e.g., [18]), there is $a_f > 0$ such that $\chi_2(x, D) \chi(x, D) \psi(x) f \in H^{(1-1/\kappa)l}_{a,b} \cap H^m_{a(\Lambda'_h + bw_h)}$ if $a \geq a_f$ when $* = (\kappa)$ and if $a > 0$ when $* = \{\kappa\}$. On the other hand, we have $(1 - \chi_2(x, D)) \chi(x, D) \psi(x) f \in \mathscr{E}^*$, since $\{\chi_2(x, \xi) \neq 1$ and $|\xi| = 1\} \cap \mathscr{C} \Subset \{\varphi(x, \xi) < 0\}$. This proves (4.18), taking $a_f(>0)$ small enough when $* = \{\kappa\}$. Noting that A can tend to zero when $* = (\kappa)$ and, therefore, a can tend to ∞, we have $\chi_1 p Q f - \chi_1 f \in \mathscr{E}^*$ and $\chi_1 Q p f - \chi_1 f \in \mathscr{E}^*$. In particular, if $f \in \mathscr{D}^{*\prime}$ and $WF_*(f) \cap \mathscr{C} = \emptyset$, then there is $a_f > 0$ such that $\chi(x, D) \psi(x) f \in L^2_{a,1}$ for any $a \geq 0$ when $* = (\kappa)$ and for $0 \leq a \leq a_f$ when $* = \{\kappa\}$. This implies that $Q f \in \mathscr{E}^*$ if $f \in \mathscr{D}^{*\prime}$ and $WF_*(f) \cap \mathscr{C} = \emptyset$. Therefore, we have just proved the following microlocal version of Holmgren's uniqueness theorem, which is necessary to prove that there is a conic neighborhood $\mathscr{C}_3$ of z^0 such that $WF_*(Qf) \cap \mathscr{C}_3 \subset \{\varphi(z) \geq 0\}$ for $f \in \mathscr{D}^{*\prime}$ with $WF_*(f) \subset \{\varphi(z) \geq 0\}$.

Proposition 4.10. *Assume that $p(x, \xi)$ satisfies the condition* (A-1) *and* (A-2) *with κ_1 replaced by κ (>1). Let $z^0 \in T^*\boldsymbol{R}^n \backslash 0$, and assume that $\varphi(z) \in C^2(T^*\boldsymbol{R}^n \backslash 0)$ is real-valued positively homogeneous of degree* 0 *in* ξ *and* $\varphi(z^0)=0$ *and that* $p_m(z)$ *is microhyperbolic with respect to* $-H_\varphi(z^0)$ *at* z^0. *Then* $z^0 \notin WF_*(u)$ *if* $u \in \mathscr{D}^{*\prime}$, $z^0 \notin WF_*(pu)$ *and* $WF_*(u) \cap \mathscr{C} \cap \{\varphi(z)<0\} = \emptyset$ *for a conic neighborhood* $\mathscr{C}$ *of* z^0. *Here* $*$ *denotes* (κ) *or* $\{\kappa\}$.

Let $f \in \mathscr{D}^{*\prime}$ satisfy $WF_*(f) \cap \mathscr{C} \subset \{\varphi(z) \geq 0\}$. Then $Qf \in H_{a,b}^{(1-1/\kappa)l}$ for a fixed b with $-b_0/2 \leq b<0$, and $a \geq a_f$ when $*=(\kappa)$ and $0<a \leq a_f$ when $*=\{\kappa\}$, if a satisfies (4.16). Therefore, we have $WF_*(Qf) \cap \{\tilde{\varphi}_2(x,\xi)<3b\} = \emptyset$, since $\inf_{L>0} \sup \{\Lambda_h^b(x,\xi)|\xi|^{-1/\kappa}; \tilde{\varphi}_2(x,\xi)<3b$ and $|\xi| \geq L\} \geq -b$, where $\tilde{\varphi}_2(x,\xi)=\varphi_2(x,\xi/|\xi|)$ and $\varphi_2(z)$ is defined by (4.10). We may assume that $-H_\varphi(z)$ and $-H_{\tilde{\varphi}_2}(z)$ belong to $\Gamma(p_{mz},\vartheta)$ for $z \in \mathscr{C}_4 \equiv \{(x,\xi)\ ;\ |x-x^0|^2+|\xi/|\xi|-\xi^0|^2 \leq r_0^2\}$ and $WF_*(PQf-f) \cap \mathscr{C}_4 = \emptyset$, where $r_0>0$. Let $\zeta(x,\xi) \in C^2(T^*\boldsymbol{R}^n \backslash 0)$ be a real-valued positively homogeneous function in ξ such that $0 \leq \zeta(x,\xi) \leq 1$, and $\zeta(x,\xi)=1$ if $|x-x^0|^2+|\xi/|\xi|-\xi^0|^2 \leq r_1^2$ and $\zeta(x,\xi)=0$ if $|x-x^0|^2+|\xi/|\xi|-\xi^0|^2 \geq 4r_1^2$, where $0<r_1$ $(\leq r_0/2)$. Then we may assume that $|H_\zeta(x,\xi)|=O(r_1^{-1})$ for $|\xi|=1$. Since $|\varphi(x,\xi)-\tilde{\varphi}_2(x,\xi)|=O(|x-x^0|^2+|\xi/|\xi|-\xi^0|^2)$, we have $-H_{\varphi_\theta}(z) \in \Gamma(p_{mz},\vartheta)$ for $z \in \mathscr{C}_4$ and $\theta \in [0,1]$, if $|b|$ and r_1 are sufficiently small, where $\varphi_\theta(z)=\theta\{\zeta(z)\varphi(z)+(1-\zeta(z))(\tilde{\varphi}_2(z)-3b)\}+(1-\theta)(\tilde{\varphi}_2(z)-3b)$. Now assume that there is $\theta \in [0,1)$ such that $WF_*(Qf) \cap \{z \in \mathscr{C}_4; \varphi_\theta(z)=0\} \neq \emptyset$. We set $\theta_0=\inf\{\theta \in [0,1)\ ;\ WF_*(Qf) \cap \{z \in \mathscr{C}_4; \varphi_\theta(z)=0\} \neq \emptyset\}$. Then, $WF_*(pQf) \cap \{z \in \mathscr{C}_4; \varphi_{\theta_0}(z)=0\} = \emptyset$ and $WF_*(Qf) \cap \{z \in \mathscr{C}_4; \varphi_{\theta_0}(z)<0\} = \emptyset$. Therefore, Proposition 4.10 implies that $WF_*(Qf) \cap \{z \in \mathscr{C}_4; \varphi_{\theta_0}(z)=0\} = \emptyset$, which contradicts the definition of θ_0. This proves that $WF_*(Qf) \cap \mathscr{C}_3 \subset \{\varphi(z) \geq 0\}$, where $\mathscr{C}_3=\{(x,\xi)\ ;\ |x-x^0|^2+|\xi/|\xi|-\xi^0|^2 \leq r_1^2\}$. Thus we have the following

Theorem 4.11. *Let the hypothesis of Proposition* 4.10 *be satisfied, and let* $\mathscr{C}$ *be a conic neighborhood of* z^0. *Then there are a continuous operator* $Q: \mathscr{D}^{*\prime} \to \mathscr{D}^{*\prime}$ *and a conic neighborhood* $\mathscr{C}_1$ *of* z^0 *such that*

$$z^0 \notin WF_*(pQf-f) \cup WF_*(Qpf-f),$$
$$WF_*(Qf) \cap \mathscr{C}_1 \subset \{\varphi(z) \geq 0\},$$

if $f \in \mathscr{D}^{*\prime}$ *and* $WP_*(f) \cap \mathscr{C} \subset \{\varphi(z) \geq 0\}$. Moreover, $z^0 \notin WF_*(Qf)$ *if* $z^0 \notin WF_*(f)$.

Remark. (i) The theorem implies that (MCP) is microlocally well-posed in $\mathscr{D}^{*\prime}$ at z^0 modulo $\mathscr{E}^*$. (ii) By Theorems 1.5 and 4.11, (MCP) can be solved globally modulo $\mathscr{E}^*$ under reasonable assumptions.

§5. Proof of Theorem 1.4 and Some Remarks

Let us begin with some remarks on existence of time functions.

Proposition 5.1. *Let $z^0=(x^0,\xi^0)\in T^*\boldsymbol{R}^n\backslash 0$ and $\vartheta\in T_{z^0}(T^*\boldsymbol{R}^n)$, and assume that $p_m(x,\xi)$ is microhyperbolic with respect to ϑ at z^0. Then the following conditions are equivalent:* (i) *There are a conic neighborhood $\mathscr{C}$ of z^0 and a time function for p_m in $\mathscr{C}$.* (ii) *There is $\tilde{\vartheta}\in T_{z^0}(T^*\boldsymbol{R}^n)$ such that p_m is microhyperbolic with respect to $\tilde{\vartheta}$ at z^0 and $\sigma(r_0,\tilde{\vartheta})=0$, where $r_0=\sum_{j=1}^n \xi_j^0(\partial/\partial\xi_j)$.* (iii) *There is $\tilde{\vartheta}\in T_{z^0}(T^*\boldsymbol{R}^n)$ such that p_m is microhyperbolic with respect to $\tilde{\vartheta}$ at z^0 and $\pm r_0\notin\Gamma(p_{mz^0},\tilde{\vartheta})^\sigma$.*

Proof. Let $t(x,\xi)$ be a time function for p_m in $\mathscr{C}$. Then p_m is microhyperbolic with respect to $-H_t(z^0)$ at z^0 and $\sum_{j=1}^n(\partial t/\partial\xi_j)(x,\xi)\xi_j=0$, i.e., $\sigma(r_0,\ -H_t(z^0))=0$. This proves that the condition (i) implies the condition (ii). It is obvious that the condition (ii) implies the condition (iii). Assume that the condition (iii) holds. Then there are $\vartheta^j\in\Gamma(p_{mz^0},\tilde{\vartheta})$ $(j=1,2)$ such that $(-1)^j\sigma(\vartheta^j,r^0)>0$. Therefore, there is $\vartheta^0\in\Gamma(p_{mz^0},\tilde{\vartheta})$ such that $\sigma(\vartheta^0,r^0)=0$. Then $t(x,\xi)\equiv\sigma(\vartheta^0,\ (x-x^0,\ |\xi^0|\xi/|\xi|-\xi^0))$ is a time function for p_m in a conic neighborhood of z^0. **Q.E.D.**

We assume that the hypotheses of Theorem 1.4 be satisfied. We shall prove Theorem 1.4 by the same arguments as in [30], using Proposition 4.10 (and Theorem 4.11). If $p_m(z^0)\neq 0$, Proposition 3.8 implies that $z^0\notin WF_*(u)$ when $u\in\mathscr{D}^{*1\prime}$ and $z^0\notin WF_*(pu)$. So, in Theorem 1.4 $p_m(z)$ must vanish at z^0. If $\Gamma(p_{mz^0},\vartheta(z^0))^\sigma$ contains $r_0\equiv\sum_{j=1}^n\xi_j^0(\partial/\partial\xi_j)$ or $-r_0$, then Theorem 1.4 is trivial.

Proposition 5.2. *Let $z^0=(x^0,\xi^0)\in\Omega$ and $|\xi^0|=1$, and let M be a compact subset of $\Gamma(p_{mz^0},\vartheta(z^0))$. Assume that $p_m(z^0)=0$ and $\pm r_0\notin$*

$\Gamma(p_{mz^0}, \vartheta(z^0))^\sigma$ *and that* $\vartheta^0 \in \Gamma(p_{mz^0}, \vartheta(z^0))$ *and* $\sigma(\vartheta^0, r_0)=0$. *Then there is* $t_0>0$ *such that*

$$WF_*(u) \cap \{(x, \xi) \in z^0 - M^\sigma ; \sigma((x-x^0, \xi/|\xi|-\xi^0), \vartheta^0)=t\} \neq \emptyset \quad \text{for} \quad 0 \leq t \leq t_0$$

if $u \in \mathscr{D}^{*1'}$ *and* $z^0 \in WF_*(u) \setminus WF_*(pu)$.

Proof. We may assume that $u \in \mathscr{E}^{*1'}$ and that $\vartheta^0 \in \mathring{M}$, i.e., $\sigma(\vartheta^0, \delta z)>0$ for $\delta z \in M^\sigma \setminus \{0\}$. Let M_1 be a compact subset of $\Gamma(p_{mz^0}, \vartheta(z^0))$ such that $M \Subset M_1$. Then there are a neighborhood $\mathscr{U}$ of z^0 and $t_0>0$ such that $WF_*(pu) \cap \mathscr{U} = \emptyset$, p_m is microhyperbolic with respect to ϑ^0 at $z \in \mathscr{U}$, $M_1 \subset \Gamma(p_{mz}, \vartheta)$ for $z \in \mathscr{U}$, $t(x, \xi)$ is a time function for p_m in a conic neighborhood of $\mathscr{U}$ and $\{z \in z^0 - M^\sigma ; -t_0 \leq t(x, \xi) \leq 0\} \Subset \mathscr{U}$, where $t(x, \xi)=\sigma(\vartheta^0, (x-x^0, \xi/|\xi|-\xi^0))$. Now assume taht $WF_*(u) \cap \{z \in z^0 - M^\sigma ; t(x, \xi) = -t_1\} = \emptyset$ for some t_1 with $0<t_1 \leq t_0$. We can assume without loss of generality that $\xi^0=(0, \cdots, 0, 1)$. We denote by $S^*\boldsymbol{R}^n (\simeq \boldsymbol{R}^n \times S^{n-1})$ the cosphere bundle over $\boldsymbol{R}^n$ and we use inhomogeneous local coordinates (x, q). Let $\tau: T^*\boldsymbol{R}^n \setminus 0 \to S^*\boldsymbol{R}^n$ be the canonical map defined as $(x, \xi) \mapsto (x, -\xi_1/\xi_n, \cdots, -\xi_{n-1}/\xi_n)$ for $(x, \xi) \in T^*\boldsymbol{R}^n \setminus 0$ with $\xi_n \neq 0$. The map τ induces a map $d\tau_z : T_z(T^*\boldsymbol{R}^n \setminus 0) \ni (\delta x, \delta\xi) \mapsto (\delta x, \delta q) \in T_{\tau(z)}(S^*\boldsymbol{R}^n)$, where $\delta q_j = -\xi_n^{-1}(\delta\xi_j + q_j \delta\xi_n)$ $(1 \leq j \leq n-1)$, $z=(x, \xi)$ and $q_j = -\xi_j/\xi_n$ $(1 \leq j \leq n-1)$. Since $\pm r_0 \notin M^\sigma$ and M^σ is a closed proper convex cone, modifying $\mathscr{U}$ if necessary, there is a closed convex cone K with its vertex at the origin in R^{2n-1} $(\simeq T_{\tau(z)}(S^*\boldsymbol{R}^n))$ such that

$$d\tau_z(M^\sigma) \supset K \supset d\tau_z(M_1^\circ) \quad \text{for } z \in \mathscr{U},$$
$$\tau(z^0 - M^\sigma) \cap \tau(\mathscr{U}) \supset (\tau(z^0) - K) \cap \tau(\mathscr{U}).$$

Then there are $\varepsilon>0$ and $\hat{z} \in \mathring{K}$ such that

$$\tau(WF_*(u)) \cap K' \cap \{\tau(z) ; -t_1+\varepsilon \geq t(z) \geq -t_1\} = \emptyset,$$

where $K'=\tau(z^0)+\hat{z}-K$. Let $\psi(x, \xi) \in \mathscr{E}^{(\kappa_1)}(T^*\boldsymbol{R}^n)$ be a positively homogeneous functions of degree 0 in ξ for $|\xi| \geq 1/2$ such that $\psi(x, \xi)=1$ if $(x, \xi) \in \mathscr{C}$, $|\xi| \geq 1/2$ and $t(x, \xi) \geq -t_1+\varepsilon$, and $\psi(x, \xi)=0$ if $(x, \xi) \in \mathscr{C}$, $|\xi| \geq 1/2$ and $t(x, \xi) \leq -t_1$, where $\mathscr{C}$ is a conic neighborhood of $\mathscr{U}$. We set $v=\psi(x, D)u$ and $g=p(x, D)v$ $(=\psi pu + [p, \psi]u)$. Then,

$$WF_*(g) \cap \mathscr{U} = WF_*([p, \psi]u) \cap \mathscr{U} \subset \{z \in \mathscr{U} ; -t_1 + \varepsilon \geq t(z) \geq -t_1\},$$
$$\tau(WF_*(g)) \cap K' = \emptyset, \quad WF_{*1}(v) \cap \mathscr{U} \subset \{z \in \mathscr{U} ; t(z) \geq -t_1\}.$$

We may assume that the boundary $\partial(K' \cap T)$ of $K' \cap T$ in $S^*\boldsymbol{R}^n \cap T$ is smooth, where $T = \{\tau(z) ; t(z) = -t_1\}$. Let S be a C^2 hypersurface in $S^*\boldsymbol{R}^n$ such that $S \cap T = \partial(K' \cap T)$ and one of the normals $(\delta x, \delta q)$ at each point on $S \cap \tau(\mathscr{U})$ belongs to K^*, where $K^* = \{(\delta x, \delta q) ; \delta x \cdot \delta \tilde{x} + \delta q \cdot \delta \tilde{q} \geq 0$ for any $(\delta \tilde{x}, \delta \tilde{q}) \in K\}$. The family of hypersurfaces S with the above properties sweeps out the region $\mathring{K}' \cap \{\tau(z) ; t(z) \geq -t_1\}$ (see [12]). Assume that $\varphi \in C^2(T^*\boldsymbol{R}^n \backslash 0)$ is real-valued, positively homogeneous of degree 0 in ξ, $\tau^{-1}(S) \cap \mathscr{U} = \{z \in \mathscr{U} ; \varphi(z) = 0\}$, $d\tau_z(H_\varphi(z)) \in K^*$ on $\tau^{-1}(S) \cap \mathscr{U}$ and $WF_{*1}(v) \cap \{z \in \mathscr{U} ; \varphi(z) < 0\} = \emptyset$. We need the following

Lemma 5.3. (Lemma 3.1 in [30]). *For $z^1 = (x^1, \xi^1) \in \tau^{-1}(S) \cap \mathscr{U}$, we have $-H_\varphi(z^1) \in \Gamma(p_{mz^1}, \vartheta^0)$.*

From Lemma 5.3 and Theorem 4.11 it follows that $z^1 \notin WF_{*1}(Qg - v)$ for $z^1 \in \tau^{-1}(S \cap \mathring{K}')$, where Q is as defined in Theorem 4.11, replacing z^0 and κ by z^1 and κ_1, respectively. From the proof of Theorem 4.11 it follows that $Q : \mathscr{D}^{*\prime} \to \mathscr{D}^{*\prime}$ and Q satisfies the assertions in Theorem 4.11 with z^0 replaced by z^1. Therefore, we have $z^1 \notin WF_*(Qg)$ and $WF_*(v) \cap \tau^{-1}(S \cap \mathring{K}') = \emptyset$. The method of sweeping out in [12] shows that $z^0 \notin WF_*(v)$. This proves Proposition 5.2. **Q.E.D.**

From the same arguments as in the proof of Theorem 3.3 in [31], it follows that for every $z^0 \in \Omega$ there are neighborhood $\mathscr{U}(z^0)$ $(\subset \Omega)$ of z^0 and $t(z^0) > 0$ such that for any $z^1 \in \mathscr{U}(z^0)$ there is a Lipschitz continuous function $z(t)$ defined on $(-t(z^0), 0]$ with values in Ω satisfying $z(t) \in WF_*(u)$ for $t \in (-t(z^0), 0]$, $(d/dt)z(t) \in \Gamma(p_{mz(t)}, \vartheta(z(t)))^\sigma \cap \{\delta z ; |\delta z| = 1\}$ for *a.e.* $t \in (-t(z^0), 0]$ and $z(0) = z^1$ if $u \in \mathscr{D}^{*1\prime}$, $z^1 \in WF_*(u)$ and $WF_*(pu) \cap \Omega = \emptyset$. Therefore, by the same arguments as in the proof of extension theorem in theory of ordinary differential equations, we can prove Theorem 1.4.

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