Publ. RIMS, Kyoto Univ.
25 (1989), 263-277

Some Classes of Functions with Exponential Decay in the Unit Ball of $\boldsymbol{C}^n$

By

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§ 1. Introduction

One of the characterizing properties of the functions of bounded mean oscillation (BMO) is that their distribution functions have an exponential decay effect, i. e. the famous John-Nirenberg's theorem[1]. In 1980, Baernstein[2] proved that the distribution functions of the non-tangential maximal functions decrease exponentially for a bounded subset of the Nevanlinna class in the unit disk, and as corollaries, he obtained an analytic form of John-Nirenberg's theorem with a weaker integrability assumption and pointed out that in the analytic category BMO is equivalent to BMO of logarithmic type. Long Ruilin and Yang Le[3] obtained similar results to Baernstein's theorem in the n-dimensional real and complex ball by showing that $\mathrm{BMO}_{(\log)^k} = \mathrm{BMO}$ for spaces of homogeneous type.

In this paper, we try to generalize a series of the famous Baernstein's results for the unit disk to the unit ball with respect to different topological structures applying Rudin's function theory in the unit ball of $\boldsymbol{C}^n$[4]. In order to lead to the discussion, we define a class of point sets in the ball in §2, where we point out that there is a useful geometric property of the intersections of this class of sets and the admissible domains $D_\alpha(\zeta)$ defined by Korányi[5]. The key part of this paper will be found in §3, where the decay characterizations will be studied for a bounded subset of a function space larger than the Nevanlinna class, which shall be referred to as H_φ class in the present article. Hence the maximal function in the admissible domain

Communicated by K. Saito, July 8, 1988.
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plays an important role in the proof. As the corollaries of the main results in §3, in §4, for the BMOA functions in the ball, we will prove the John-Nirenberg theorem with respect to the harmonic measure as well as the results derived thereof.

The main results had been reported in [6].

§2. A Class of Point Sets in the Unit Ball

Let B be the unit ball $\{z\in C^n: \langle z, z\rangle=\sum_{j=1}^{n} z_j\bar{z}_j<1\}$ and let S be its boundary, i. e. the unit sphere. By Rudin's expression, the group $\mathscr{M}$ of the Möbius transformations of the unit ball is written as

$$\mathscr{M}=\left\{\psi=\varphi_a U:\ \varphi_a=\frac{a-P_a z-s_a Q_a z}{1-\langle z, a\rangle}\right\},$$

where $a\in B$, and U are unitary transformations,

$$P_a z=\begin{cases}\dfrac{\langle z, a\rangle}{\langle a, a\rangle}\ a, & a\neq 0,\\ 0, & a=0,\end{cases}\quad Q_a=I-P_a,\quad s_a=(1-|a|^2)^{1/2}$$

Obviously $\psi(0)=a$. Since, φ_a maps the unit sphere S into itself, $\mathscr{M}$ is also the transformation group of S into itself. Note that $\psi\in\mathscr{M}\Leftrightarrow$ $\exists a$, $U\in\mathscr{U}$ (unitary group), such that $\psi(z)=\varphi_a U(z)$.

Now we define a class of point sets in the unit ball.

Definition. For $\beta>0$ and $a\in B(a\neq 0)$, $\zeta\in S$, define

$$R_{\beta,a}(\zeta)=\{z\in\bar{B}:\ |1-\langle z, a\rangle|\leq\beta|1-\langle a,\zeta\rangle|\}.$$

The point sets $R_{\beta,a}(\zeta)$ possess the following properties:

1) When $\beta<\dfrac{1-|a|}{1+|a|}$, $R_{\beta,a}(\zeta)$ is the empty set.

As a matter of fact, $\forall z\in B$, $\zeta\in S$, from Schwarz inequality,

$$|1-\langle z, a\rangle|\geq 1-|z|\circ|a|\geq 1-|a|,$$

and

$$\beta|1-\langle a, \zeta\rangle|\leq\beta(1+|\langle a, \zeta\rangle|)\leq\beta(1+|a|).$$

Thus, when $1-|a|>\beta(1+|a|)$, $|1-\langle z, a\rangle|>\beta|1-\langle a,\zeta\rangle|$.

2) When $\beta\geq\dfrac{1+|a|}{1-|a|}$, for every fixed $\zeta\in S$, $R_{\beta,a}(\zeta)$ fills the closed ball $\bar{B}$.

3) When $\beta \geq 1+|a|$, $a \in R_{\beta,a}(\zeta)$ for any $\zeta \in S$.

The reason of properties 2) and 3) is the same as 1).

4) For any unitary transformation, $U(R_{\beta,a}(\zeta)) = R_{\beta,Ua}(U\zeta)$.

This is obvious from the defining equation of $R_{\beta,a}(\zeta)$. Hence, without loss of generality, take $\zeta = e_1 = (1, 0, \dots, 0)$, and especially put $a = re_1 (0 < r < 1)$, then

$$R_{\beta,re_1}(e_1) = \{z \in \bar{B}:\ |1 - rz_1| \leq \beta(1-r)\}.$$

Again denote $z_1 = \rho e^{i\theta}$. For instance, if we take $r = \frac{2}{3}$, $\beta = 3$, then

$$R_{3,(2/3)e_1}(e_1) = \left\{z \in \bar{B}:\ \rho \leq 3 \cos\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}.$$

It is the point set in the closed unit ball where those first variables z_1 belong to the closed disk $\rho \leq 3 \cos\theta$. This special case provided some geometrically intuitional information for point sets $R_{\beta,a}(\zeta)$.

In the following, the relationship between $R_{\beta,a}(\zeta)$ and the admissible domain is discussed. According to the Rudin's[4] expression

$$D_\alpha(\zeta) = \left\{z \in B:\ |1 - \langle z, \zeta\rangle| < \frac{\alpha}{2}(1 - |z|^2),\ \zeta \in S,\ \alpha > 1\right\}.$$

In the discussion of function theory in the unit ball, its situation is similar to Stolz region $\Gamma_\alpha(\theta)$ in the unit disk. From the defining equation of $D_\alpha(\zeta)$, it is easy to know that for radial point $a = r_1\zeta (0 < r_1 < 1)$ of ζ, $a \in D_\alpha(\zeta)$, when $r_1 > \frac{2}{\alpha} - 1$, therefore, radial points $z = r\zeta (r_1 \leq r < 1)$ belong to $D_\alpha(\zeta)$. Combining property 3) of point set $R_{\beta,a}(\zeta)$, then, when $\alpha > \frac{2}{1+r_1}$, $\beta \geqq 1 + r_1$, $z = r\zeta (r_1 \leqq r < 1)$ belongs to $D_\alpha(\zeta) \cap R_{\beta,a}(\zeta)$. Generally, it is always possible to set point sets $D_\alpha(\zeta) \cap R_{\beta,a}(\zeta)$ to be nonempty when the values of α and β are suitably chosen. Thus we have

Proposition. *If $\psi \in \mathcal{M}$, choose α and β so that $\alpha\beta > 1$ and $D_\alpha(\zeta) \cap R_{\beta,a}(\zeta)$ is nonempty, then*

$$D_\alpha(\zeta) \cap R_{\beta,a}(\zeta) \subset \psi(D_{\alpha\beta}(\xi)),$$

where $\psi(\xi) = \zeta$.

Proof. It is pointed out in theorem 2.2.5 of [4] that if $\psi \in \mathrm{Aut}$

(B), and $a=\phi^{-1}(0)$, then there exists a unique U, such that $\phi=U\varphi_a$; and

$$1-\langle\phi(z),\ \phi(\zeta)\rangle=\frac{(1-\langle a,a\rangle)(1-\langle z,\zeta\rangle)}{(1-\langle z,a\rangle)(1-\langle a,\zeta\rangle)},\ \text{for all } z,\ \zeta\in\bar{B}.$$

Since, if $\zeta\in S$, $z\in D_\alpha(\zeta)\cap R_{\beta,a}(\zeta)$, then

$$|1-\langle\phi(z),\ \phi(\zeta)\rangle|<\frac{|1-\langle a,a\rangle|\cdot(\alpha/2)(1-|z|^2)}{|1-\langle z,a\rangle|\cdot(1/\beta)\,|1-\langle z,a\rangle|}$$
$$=\frac{\alpha\beta}{2}\cdot\frac{|1-\langle a,a\rangle|\cdot|1-\langle z,z\rangle|}{|1-\langle z,a\rangle|^2}=\frac{\alpha\beta}{2}(1-|\phi(z)|^2).$$

But then $\alpha\beta>1$, thus $\phi(z)\in D_{\alpha\beta}(\phi(\zeta))$. Then it is deduced that

$$\phi(D_\alpha(\zeta)\cap R_{\beta,a}(\zeta))\subset D_{\alpha\beta}(\phi(\zeta)).$$

Notice that φ_a is the $1-1$ mapping on $\bar{B}$, then

$$D_\alpha(\zeta)\cap R_{\beta,a}(\zeta)\subset\phi^{-1}(D_{\alpha\beta}(\xi)),\quad \xi=\phi(\zeta).$$

However, since $\varphi_a^{-1}=\varphi_a$, then $\phi^{-1}=(U\varphi_a)^{-1}=\varphi_a U^{-1}\in\mathscr{M}$, denoting ϕ as ϕ^{-1}, then $\phi(\xi)=\zeta$. Thus

$$D_\alpha(\zeta)\cap R_{\beta,a}(\zeta)\subset\phi(D_{\alpha\beta}(\xi)).$$

Remark. This result means that under Möbius transformation, the admissible domain for a point ξ on the unit sphere includes the intersection of the admissible domain $D_\alpha(\zeta)$ and $R_{\beta,a}(\zeta)$ for its image point ζ. This is an interesting fact for the complex geometry of $\boldsymbol{C}^n$. We are to use it below.

§ 3. Baernstein Theorem for the H_φ Class of Functions

Let E be a measurable set on the sphere S, for $a\in B$, we define[6]

$$\mu_a(E)=\int_E d\mu_a(\zeta)=\int_E\frac{(1-|a|^2)^n}{|1-\langle a,\zeta\rangle|^{2n}}d\sigma(\zeta)$$

to be the harmonic measure of E at a with respect to B. The integrand function at the right-handed side of the above equation is the Poisson integral kernel $P(a,\zeta)$ in B. For any $E\subset S$, the normalized Lebesgue measure $\sigma(E)=\mu_0(E)$.

As in [4], let φ: $[-\infty,\infty)\to[0,\infty)$ be a nondecreasing convex function, not identically 0, the class of functions in B

$$H_\varphi(B)=\begin{cases} f \text{ is holomorphic in } B \\ I_1(f)=\sup\limits_{0<r<1}\int_S \varphi(\log|f_r|)\,d\sigma<\infty,\ f_r=f(r\zeta). \end{cases}$$

If $\varphi(x)=x^+=\max(0,x)$, then $H_\varphi(B)=N(B)$. Here $N(B)$ denotes the Nevanlinna class

$$N(B)=\begin{cases} f \text{ is holomorphic in } B \\ T(f)=\sup\limits_{0<r<1}\int_S \log^+|f_r|\,d\sigma<\infty. \end{cases}$$

If $\varphi(x)=e^{px}$, then $H_\varphi(B)=H^p(B)$, $0<p<\infty$. When $p_1>p_2$, we have $H^{p_1}\subset H^{p_2}\subset N$.

For the admissible domain $D_\alpha(\zeta)$ and a continuous function F in B, the maximal function in the admissible domain is defined[4] by

$$(M_\alpha F)(\zeta)=\sup\{|F(z)|:\ z\in D_\alpha(\zeta)\}.$$

Obviously, if there is a K-limit $F^*(\zeta)=\lim\limits_{z\in D_\alpha(\zeta),z\to\zeta}F(z)$ of function F for a point ζ, then $(M_\alpha F)(\zeta)\geq|F^*(\zeta)|$.

Lemma 1. *If $f\in H_\varphi$, then for every t satisfying $\varphi(\log t)>0$, we have the following inequality*

$$\mu_0(\{\zeta\in S:\ (M_\alpha f)(\zeta)>t\})<\frac{C(\alpha)}{\varphi(\log t)}I_1(f),$$

where $C(\alpha)$ is a constant depending on α.

Proof. If $f\in H_\varphi$, then $I_1(f)=\sup\limits_{0<r<1}\int_S\varphi(\log|f_r|)\,d\sigma<\infty$. Hence from theorem 5.6.2. (a) in [4], there is a positive measure ν on S, such that $u=P[\nu]$ is the least $\mathscr{M}$-harmonic majorant of $\varphi(\log|f|)$, and $\|\nu\|=I_1(f)$, here $\|\nu\|=|\nu|(S)$ is the total variation measure of ν on S.

From the definition of the least $\mathscr{M}$-harmonic majorant function, for every $z\in B$, there is $\varphi(\log|f|)\leq P[\nu]$. However, since the compound function $\varphi\circ\log$ is nondecreasing, so that for definite $D_\alpha(\zeta)$ we have $\varphi(\log(M_\alpha f)(\zeta))\leq(M_\alpha P[\nu])(\zeta)$. Therefore, if $(M_\alpha f)(\zeta)>t$, then

$$(M_\alpha P[\nu])(\zeta)\geq\varphi(\log t).$$

Hence

$$\{\zeta\in S:\ (M_\alpha f)(\zeta)>t\}\subset\{\zeta\in S:(M_\alpha P[\nu])(\zeta)\geq\varphi(\log t)\}.$$

Thus it is only necessary to prove

$$\mu_0(\{\zeta\in S:\ (M_\alpha P[\nu])(\zeta)\geq\varphi(\log t)\})<\frac{C(\alpha)}{\varphi(\log t)}I_1(f).$$

Now let us prove this inequality. According to Rudin [4, p68], define the maximal function of a complex measure on S by

$$(M\nu)(\zeta)=\sup_{\delta>0}\frac{|\nu|Q(\zeta,\delta)}{\sigma(Q_\delta)},$$

where $Q_\delta\cong Q(\zeta,\delta)=\{\eta\in S: |1-\langle\zeta,\eta\rangle|^{1/2}<\delta,\ \zeta,\ \eta\in S,\ \delta>0\}$ is the "ball" on S, ζ is the center of the ball and δ is the radius. Then one has the corresponding inequality of weak type (1, 1):

$$\mu_0(\{\zeta\in S: (M\nu)(\zeta)>\tau\})\leq A_3\tau^{-1}||\nu||,\ \text{for every }\tau>0.$$

A result of Korányi [5] is expressed as

$$(M_\alpha P[\nu])(\zeta)\leq A(\alpha)(M\nu)(\zeta).$$

Combining this inequality and the above inequality of weak type (1, 1), suitably enlarging constant $A(\alpha)$, then

$$\begin{aligned}&\mu_0(\{\zeta\in S: (M_\alpha P[\nu])(\zeta)\geq\varphi(\log t)\})\\ &\leq\mu_0\left(\left\{\zeta\in S: (M\nu)(\zeta)>\frac{\varphi(\log t)}{A(\alpha)}\right\}\right)\\ &\leq\frac{A_3A(\alpha)}{\varphi(\log t)}||\nu||<\frac{C(\alpha)}{\varphi(\log t)}I_1(f).\end{aligned}$$

Q. E. D.

Similar to that of [2], we introduce the set of functions

$$\mathscr{M}(f)=\{g: g(z)=f\circ\psi(z)-f\circ\psi(0),\ \ \psi\in\mathscr{M}\}.$$

Then we have

Theorem 1. *Suppose that $f(z)$ is holomorphic in B, if $\mathscr{M}(f)$ is a bounded subset of H_φ, then for every $g(z)\in\mathscr{M}(f)$, we have*

$$\mu_0(\{\eta\in S: (M_\alpha g)(\eta)>t\})<Ke^{-\lambda t},$$

where K denotes the absolute constant, and $\lambda=Ce^{-\varphi^{-1}(C(\alpha)\sigma_\varphi(f))}$, $\sigma_\varphi(f)=$ $\sup\{I_1(g):\ g\in\mathscr{M}(f)\}$ for an increasing convex function.

Proof. Before beginning the proof, we ought to have some necessary preparations.

Let G be any compact set on S whose measure is non-zero, and employing finite many balls $\{Q\}$, with the centers $\zeta\in G$, cover G. Since $\{Q\}$ are only finitely many, suitably choosing the positions of the centers of those balls and the radii of those balls, it is always possible that $\sigma(J)\geqq\frac{1}{M}\sigma(Q)$ holds for every Q, where $J=G\cap Q$, M is a larger positive constant given in advance. Next using the covering lemma[4], we choose a disjoint subcollection $\Gamma=\{Q_i\}$ from $\{Q\}$, thus $\sigma(G)\leq A_3\sum_{\Gamma}\sigma(Q_i)$. Writing

$$J_i=G\cap Q_i\,.$$

Obviously J_i are nonempty and pairwise disjoint, and each J_i has the possibility to be composed of the countable many of path-connected components. Denote M_1 as the product A_3M, then

$$\sigma(G)\leq M_1\sum_{\Gamma}\sigma(J_i). \tag{1}$$

In addition, for every ball $Q_i(\zeta_i,\delta_i)$ in Γ, we might as well suppose $\sigma(Q_i)<1/4$ and take $a_i=r_i\zeta_i$, $r_i=1-\delta_i^2$.

When η belongs to certain Q_i in the above-mentioned text, as (2) in [7]

$$d\mu_a(\eta)\geq C_1\frac{d\sigma(\eta)}{\sigma(Q_i)},$$

where C_1 only depends on the dimension $n(>1)$. Thus for every $J_i\subset Q_i$, there is

$$\mu_0(J_i)\leq C_1^{-1}\mu_a(J_i)\,\mu_0(Q_i). \tag{2}$$

When $\mathscr{M}(f)$ is bounded in H_φ class, set $\sigma_\varphi(f)=\sup\{I_1(g):\ g\in\mathscr{M}(f)\}$. For constant M_1 in (1) and constant C_1 in (2), choose τ large enough so that

$$\frac{C(\alpha)\sigma_\varphi(f)}{\varphi(\log\tau)}\leq\min\left(\frac{C_1}{2M_1},\ \frac{1}{4}\right), \tag{3}$$

where $C(\alpha)$ is the constant in Lemma 1.

Now the proof of the theorem is carried out successively.

1° For the fixed $g(z)\in\mathscr{M}(f)$, define

$$E_k=\{\eta\in S:\ (M_\alpha g)(\eta)>k\tau\},\quad k=1,2,\ldots$$

Obviously $E_{k+1}\subset E_k$. Assume that E_{k+1} is nonempty, let G be a compact subset of the open set E_{k+1} whose measure is non-zero, same as the statement in the preparation in this proof, cover G by those balls $\{Q\}$ which are included in E_k with their centers $\zeta\in G$, and also write $\Gamma=\{Q_i\}$ to be the disjoint subcollection chosen by the covering lemma.

2° Now let us prove

$$\mu_0(E_{k+1})\leq(1/2)\mu_0(E_k),\quad k\geq 1. \tag{4}$$

Since $f(z)$ is holomorphic in B, and $\varphi_a:\bar{B}\to\bar{B}$ is the holomorphic mapping, thus $g(z)$ is holomorphic function in B. Hence in closed ball $\bar{B}_{r_0}=\{z\in B: |z|\leq r_0\}$ with its radius r_0 near to 1 (i. e. $1-r_0=o(\delta_i^2)$), for enough large τ in (3), we have $|g(z)|\leq k\tau$. Therefore when $\eta\in Q_i(\subset E_k)$,

$$\begin{aligned}(M_\alpha g)(\eta)&=\sup\{|g(z)|:z\in D_\alpha(\eta)\}\\&=\sup\{|g(z)|:z\in D_\alpha(\eta)\cap\bar{B}^c_{r_0}\}\end{aligned} \tag{5}$$

where $\bar{B}^c_{r_0}=\{z\in B: |z|>r_0\}$.

For the given α, when $z\in D_\alpha(\eta)\cap\bar{B}^c_{r_0}$, there is

$$|1-\langle z,\eta\rangle|<\frac{\alpha}{2}(1-|z|^2)<\alpha(1-|z|)<\alpha(1-r_0)<\delta_i^2,$$

if not, then suitably increase the value of r_0. However

$$|1-\langle a_i,\eta\rangle|\geq 1-|a_i|\cdot|\eta|=1-r_i=\delta_i^2,$$

hence

$$|1-\langle z,a_i\rangle|^{1/2}\leq|1-\langle z,\eta\rangle|^{1/2}+|1-\langle a_i,\eta\rangle|^{1/2}<2|1-\langle a_i,\eta\rangle|^{1/2}.$$

By the definition of set $R_{\beta,a}(\zeta)$, we know that $z\in R_{4,a_i}(\eta)$. Therefore, we deduce that $D_\alpha(\eta)\cap\bar{B}^c_{r_0}\subset R_{4,a_i}(\eta)$ which could be written as

$$D_\alpha(\eta)\cap\bar{B}^c_{r_0}\subset D_\alpha(\eta)\cap R_{4,a_i}(\eta).$$

Defining $\psi_{a_i}=\varphi_{a_i}U\in\mathscr{M}$ and $g_i(z)=g\circ\psi_{a_i}(z)-g\circ\psi_{a_i}(0)=g\circ\psi_{a_i}(z)-g(a_i)$, and applying the proposition in §2, then we have the following inclusion relation

$$D_\alpha(\eta)\cap\bar{B}^c_{r_0}\subset\psi_{a_i}(D_{4\alpha}(\xi)), \tag{6}$$

where $\psi_{a_i}(\xi)=\eta$.

For J_i is nonempty, then, when $\eta\in J_i(\subset E_{k+1})$, from Eq. (5) and (6)

$$(k+1)\tau<(M_\alpha g)(\eta)=\sup\{|g(z)|:z\in D_\alpha(\eta)\cap\bar{B}^c_{r_0}\}$$

$$\leq \sup\{|g(z)|: z\in\psi_{a_i}(D_{4\alpha}(\xi))\}$$
$$=\sup\{|g\circ\psi_{a_i}(z)|: z\in D_{4\alpha}(\xi)\}.$$

Since we have taken $a_i=r_i\zeta_i$, $r_i=1-\delta_i^2\ll r_0$, thus $a_i\in\bar{B}_{r_0}$, therefore $|g(a_i)|\leq k\tau$. Thus, there is

$$\begin{aligned}(M_{4\alpha}g_i)(\xi) &= \sup\{|g_i(z)| : z\in D_{4\alpha}(\xi)\}\\ &= \sup\{|g\circ\psi_{a_i}(z)-g(a_i)| : z\in D_{4\alpha}(\xi)\}\\ &\geq \sup\{|g\circ\psi_{a_i}(z)|-|g(a_i)| : z\in D_{4\alpha}(\xi)\}\\ &> (k+1)\tau-k\tau=\tau,\end{aligned}$$

noting that $\eta=\psi_{a_i}(\xi)$, this means

$$J_i\subset\psi_{a_i}(\{\xi\in S : (M_{4\alpha}g_i)(\xi)>\tau\}). \tag{7}$$

Since $g_i\in\mathscr{M}(f)$, applying Lemma 1 and (3), we obtain

$$\mu_0(\{\xi\in S : (M_{4\alpha}g_i)(\xi)>\tau\})\leq\frac{C_1}{2M_1}. \tag{8}$$

Denote $I=\{\xi\in S : (M_{4\alpha}g_i)(\xi)>\tau\}$, combining (7) and (8), and applying Eq. (8. 11) in [8], then

$$\mu_{a_i}(J_i)\leq\mu_{a_i}(\psi_{a_i}(I))=\mu_{\psi_{a_i}}(0)(\psi_{a_i}(I))=\mu_0(I)\leq\frac{C_1}{2M_1}. \tag{9}$$

Together with (2) and (9), we deduce

$$\mu_0(J_i)\leq C_1^{-1}\cdot\frac{C_1}{2M_1}\mu_0(Q_i)=\frac{1}{2M_1}\mu_0(Q_i). \tag{10}$$

Now let us discuss as above for every J_i, and permute the order of all the path-connected components of J_i, it does not matter to write it as J_i. Applying (10), then

$$\mu_0(G)\leq M_1\sum_{\Gamma}\mu_0(J_i)\leq(1/2)\sum_{\Gamma}\mu_0(Q_i)\leq(1/2)\mu_0(E_k).$$

Now (4) follows by taking the supremum over all compact subsets $G\subset E_{k+1}$.

3° Lemma 1 and (3) are to be used once more, then there is $\mu_0(E_1)<1/4<1/2$. Thus it can be deduced from (4) inductively that

$$\mu_0(E_k)\leq(1/2)\mu_0(E_{k-1})\leq\ldots\leq\frac{1}{2^{k-1}}\mu_0(E_1)\leq\frac{1}{2^k}.$$

For any $t>0$, when $t\geq\tau$, there always exists certain k, such that $k\tau\leq t<(k+1)\tau$, hence

$$\mu_0(\{\eta\in S:(M_\alpha g)(\eta)>t\})\leq\mu_0(E_k)$$

$$\leq 2^{-k}=2e^{-\left(\frac{1}{\tau}\log 2\right)\cdot(k+1)\tau}<2e^{-\left(\frac{1}{\tau}\log 2\right)t};$$

again when $0<t<\tau$, we have also

$$\mu_0(\{\eta\in S:(M_\alpha g)(\eta)>t\})\leq\mu_0(S)$$

$$=1=2e^{-\left(\frac{1}{\tau}\log 2\right)\tau}<2e^{-\left(\frac{1}{\tau}\log 2\right)t}.$$

Thus for any $t>0$, there is

$$\mu_0(\{\eta\in S:(M_\alpha g)(\eta)>t\})<Ke^{-\lambda t},$$

where $K=2$, for increasing convex φ, $\lambda=\frac{1}{\tau}\log 2=Ce^{-\varphi^{-1}(C(\alpha)\sigma_\varphi(f))}$. Now the theorem is proved.

Remark. For general nondecreasing convex functions, it does not seem that it is easy to yield the converse of Theorem 1, however, for some H_φ defined by φ, the converse is also true, just as the N class to be proved in the following.

Before the further discussion is going on, let us first introduce the elementary Lemma used in [2]:

Lemma 2. *Let h be a nonnegative measurable function on some measure space $(\Omega,\mathscr{F},\mu)$. The distribution function $\Lambda(t)$ of h is defined as*

$$\Lambda(t)=\mu(\{x\in\Omega:h(x)>t\}),\quad t>0.$$

Then the following two equalities hold

(i) $\int_\Omega h^p d\mu=-\int_0^\infty t^p d\Lambda(t)=p\int_0^\infty t^{p-1}\Lambda(t)\,dt,\ 0<p<\infty.$

(ii) $\int_\Omega\left(\log^+\frac{h}{\rho}\right)d\mu=-\int_\rho^\infty\left(\log\frac{t}{\rho}\right)d\Lambda(t)=\int_\rho^\infty t^{-1}\Lambda(t)\,dt,\ 0<\rho<\infty.$

Similarly, we could obtain the following lemma by the standard argument of real function theory, i. e. the simple functions approximate to an arbitrary measurable function.

Lemma 3. *h and its distribution function $\Lambda(t)$ are defined as above and $\mu(\Omega)=1$, then*

$$\int_{\Omega} e^{\rho h} d\mu = -\int_0^{\infty} e^{\rho t} dA(t) \leq 1 + \rho \int_0^{\infty} e^{\rho t} A(t)\, dt, \quad 0 < \rho < \infty.$$

As a deduction of Theorem 1 and Lemma 2, there is

Theorem 2. *Suppose that $f(z)$ is holomorphic in B, then the following are equivalent:*

(a) *$\mathscr{M}(f)$ is bounded in N.*

(b) *there exists an absolute constant K and a constant $\lambda = \lambda(\alpha, f)$ such that for every $g \in \mathscr{M}(f)$ and $t > 0$*

$$\mu_0(\{\eta \in S : (M_\alpha g)(\eta) > t\}) < Ke^{-\lambda t}.$$

Proof. (a)$\Rightarrow$(b) is the implication following directly from Theorem 1. In fact, if take $\varphi(x) = x^+ = \max(0, x)$, then $H_\varphi(B) = N(B)$. Thus by Theorem 1, it is clear that (a)$\Rightarrow$(b). At this moment, $\lambda = Ce^{-C(\alpha)\sigma(f)}$, where $\sigma(f) = \sup\{T(g) : g \in \mathscr{M}(f)\}$.

(b)$\Rightarrow$(a). Take $\Omega = S$ in Lemma 2(ii), $\rho = 1$, and set $h = M_\alpha g$, $A(t) = \mu_0(\{\eta \in S: (M_\alpha g)(\eta) > t\})$. If (b) holds, then

$$\int_S (\log^+ M_\alpha g)\, d\mu_0 = \int_1^{\infty} t^{-1} A(t)\, dt < \int_1^{\infty} t^{-1} K e^{-\lambda t} dt < \infty.$$

For the sake of definitivity, firstly, let $\alpha \geq 2$, when $0 < r < 1$ there is $r\eta \in D_\alpha(\eta)$. Thus for every $\eta \in S$, $|g_r(\eta)| \leq (M_\alpha g)(\eta)$. Therefore, for every $g \in \mathscr{M}(f)$,

$$\sup_{0<r<1} \int_S (\log^+ |g_r|)\, d\sigma \leq \int_S (\log^+ M_\alpha g)\, d\mu_0 < \infty,$$

that is, $\mathscr{M}(f)$ is bounded in N.

When $1 < \alpha < 2$, take $r_1 = \dfrac{2}{\alpha} - 1$, clearly $0 < r_1 < 1$. Since the holomorphic function g_r is bounded in the closed ball $\bar{B}_{r_1}$, hence, when $r \leq r_1$, it is also bounded for $\int_S (\log^+ |g_r|)\, d\sigma$. While, when $r_1 < r < 1$, as $r\eta \in D_\alpha(\eta)$, therefore $|g_r(\eta)| \leq (M_\alpha g)(\eta)$. Similar to "$\alpha \geq 2$", $\int_S (\log^+ |g_r|)\, d\sigma$ is bounded. Summing up, for $\alpha > 1$, there exists the implication (b)$\Rightarrow$(a).

§4. The John-Nirenberg Theorem in the Sense of Harmonic Measure

For given $f\in L^1(\sigma)$, its Poisson extension to B is also denoted by f, i. e.

$$f(z)=\int_S P(z,\zeta)f(\zeta)d\sigma(\zeta)=\int_S f(\zeta)d\mu_z(\zeta).$$

In [7] we had defined a norm equivalent to the usual BMO norm

$$\|f\|_{**}=\sup_{a\in B}\int_S |f(\zeta)-f(a)|d\mu_a(\zeta).$$

If the Poisson extension of a BMO function is holomorphic in B, then we call this f to be the holomorphic function with bounded mean oscillation (BMOA).

Theorem 3. *$f\in BMOA\Leftrightarrow$ for every $a\in B$,*

$$\mu_a(\{\zeta\in S:\ |f(\zeta)-f(a)|>t\})<Ke^{-\lambda_1 t},$$

where K is an absolute constant, $\lambda_1=C/\|f\|_{**}$.

Proof. "$\Rightarrow$". First of all, notice that $(M_\alpha g)(\eta)\geqq|g(\eta)|$, a. e. on S. In fact, by theorem 5. 4. 8 and differentiation theorem (5. 3. 1) in [4], the K-limit of Poisson integral $f\circ\psi(z)$ of L^1 function $f\circ\psi(\eta)$ on S is still $f\circ\psi(\eta)$. Thus g^* is the K-limit of g,

$$\begin{aligned} g^*(\eta)&\triangleq(K\text{-lim } g)(\eta)\\ &=(K\text{-lim } f\circ\psi)(\eta)-f\circ\psi(0)\\ &=f\circ\psi(\eta)-f\circ\psi(0)=g(\eta),\ \text{a. e. on } S. \end{aligned}$$

Hence $(M_\alpha g)(\eta)\geq|g^*(\eta)|=|g(\eta)|$, a. e. on S.

If $f\in$BMOA, applying the corollary 3 in [7], then $\mathscr{M}(f)$ is bounded in H^1. And again, using theorem 1, and combining $(M_\alpha g)(\eta)\geq|g(\eta)|$, then

$$\begin{aligned} &\mu_a(\{\zeta\in S:|f(\zeta)-f(a)|>t\})=\mu_0(\{\eta\in S:|f\circ\psi(\eta)-f\circ\psi(0)|>t\})\\ &=\mu_0(\{\eta\in S:|g(\eta)|>t\})\leq\mu_0(\{\eta\in S:\ (M_\alpha g)(\eta)>t\})\\ &<Ke^{-\lambda t}, \end{aligned}$$

where $\lambda=C/C(\alpha)\sigma_1(f)$, $\sigma_1(f)=\sup\{\|g\|_{H^1}:\ g\in\mathscr{M}(f)\}=\|f\|_{**}$. Thus setting $\lambda_1=C\|f\|_{**}^{-1}\leq\lambda$, "$\Rightarrow$" is proved.

"$\Leftarrow$". Set $\Lambda(t)=\mu_a(\{\zeta\in S: |f(\zeta)-f(a)|>t\})$, by means of Lemma 2(i), then

$$\int_S |f(\zeta)-f(a)|d\mu_a(\zeta)=\int_0^\infty \Lambda(t)dt<\int_0^\infty Ke^{-\lambda_1 t}dt<\infty.$$

Thus

$$||f||_{**}=\sup_{a\in B}\int_S |f(\zeta)-f(a)|d\mu_a(\zeta)<\infty. \qquad \text{Q. E. D.}$$

Corollary. *For holomorphic function f in B, the following are equivalent*:

(a) $f\in BMOA$.

(b) $\mathscr{M}(f)$ *is bounded in* N.

(c) $\mathscr{M}(f)$ *is bounded in* H^p, $0<p<\infty$.

(d) *There exists* $\rho=\rho_f>0$, *such that*

$$\sup_{g\in\mathscr{M}(f)}\ \sup_{0<r<1}\int_S e^{\rho|g_r|}d\sigma<\infty.$$

Proof. Since $H^p\subset N$, it is clear for the implication (c)$\Rightarrow$(b). Now we prove (b)$\Rightarrow$(c). Set $\Lambda(t)=\mu_0(\{\eta\in S: (M_\alpha g)(\eta)>t\})$. By means of Theorem 2 and Lemma 2(i), then for every $g\in\mathscr{M}(f)$, there is

$$\begin{aligned}\int_S [M_\alpha g]^p d\mu_0 &= p\int_0^\infty t^{p-1}\Lambda(t)dt\\ &<Kp\int_0^\infty t^{p-1}e^{-\lambda t}dt=\frac{K\Gamma(p+1)}{\lambda^p}<\infty.\end{aligned}$$

Similar to the treatment of proving (b)$\Rightarrow$(a) in Theorem 2, distinguishing the cases of $\alpha\geq 2$ and $1<\alpha<2$, we have both

$$\sup_{0<r<1}\int_S |g_r|^p\, d\sigma<C_2<\infty,$$

for every $g\in\mathscr{M}(f)$, i.e. $\mathscr{M}(f)$ is bounded in H^p, $0<p<\infty$.

The above assertion holds for every H^p or for some H^p. Especially, when $1\leq p<\infty$, since we have already supposed that the Poisson extension to B of an L^1 function g on S is also denoted by g. By theorem 3.3.4. (b) in [4], it follows

$$||g||_{H^p}^p\leq ||g||_{L^p}^p=\int_S |g(\eta)|^p d\sigma(\eta)$$

$$\leq\int_S[(M_a g)(\eta)]^p\,d\sigma<\frac{K\Gamma(p+1)}{\lambda^p}\leq\frac{K\Gamma(p+1)}{C^p}\|f\|_{**}^p.$$

Since $f\in\mathrm{BMOA}\Leftrightarrow\mathscr{M}(f)$ is bounded in H^1 (see [7]), thus (a)$\Leftrightarrow$(b) is the special case of (b)$\Leftrightarrow$(c) when $p=1$.

(d)$\Rightarrow$(a) is quite an obvious fact. Since $\rho|g_r|<e^{\rho|g_r|}$, thus

$$\|f\|_{**}=\sup_{g\in\mathscr{M}(f)}\ \sup_{0<r<1}\int_S|g_r|\,d\sigma<\rho^{-1}\sup_{g\in\mathscr{M}(f)}\ \sup_{0<r<1}\int_S e^{\rho|g_r|}d\sigma<\infty.$$

Now we would like to prove (a)$\Rightarrow$(d). Suppose $\Lambda(t)=\mu_a(\{\zeta\in S:|f(\zeta)-f(a)|>t\})$. Employing Theorem 3 and Lemma 3, then for every $a\in B$

$$\begin{aligned}\int_S e^{\rho|g(\eta)|}d\sigma(\eta)&=\int_S e^{\rho|f\circ\phi(\eta)-f\circ\phi(0)|}d\mu_0(\eta)\\&=\int_S e^{\rho|f(\zeta)-f(a)|}d\mu_a(\zeta)\leq 1+\rho\int_0^\infty e^{\rho t}\Lambda(t)\,dt\\&<1+K\rho\int_0^\infty e^{(\rho-\lambda_1)t}dt,\end{aligned}\tag{11}$$

it is only necessary to take $\rho<\lambda_1=C/\|f\|_{**}$, then this integral converges.

On the other hand, when $0<r<1$,

$$\begin{aligned}\int_S e^{\rho|g_r(\eta)|}d\sigma(\eta)&=\int_S\left(e^{\rho|\int_S P(r\eta,\zeta)g(\zeta)d\sigma(\zeta)|}\right)d\sigma(\eta)\\&\leq\int_S\left(e^{\rho\int_S|g(\zeta)|d\mu_{r\eta}(\zeta)}\right)d\sigma(\eta),\end{aligned}$$

as the measure $\mu_a(S)=1$, so by Jensen's convexity inequality and noting $\int_S P(r\eta,\zeta)d\sigma(\eta)=1$ (see[4]), then for $0<r<1$

$$\begin{aligned}&\int_S e^{\rho|g_r(\eta)|}d\sigma(\eta)\\&\leq\int_S\left(e^{\rho\int_S|g(\zeta)|d\mu_{r\eta}(\zeta)}\right)d\sigma(\eta)\leq\int_S\left(\int_S e^{\rho|g(\zeta)|}d\mu_{r\eta}(\zeta)\right)d\sigma(\eta)\\&=\int_S e^{\rho|g(\zeta)|}d\sigma(\zeta)\int_S P(r\eta,\zeta)\,d\sigma(\eta)\\&=\int_S e^{\rho|g(\zeta)|}d\sigma(\zeta).\end{aligned}$$

Combining (11) and taking $\rho=\rho_f\in(0,\ C/\|f\|_{**})$, then for every $g\in\mathscr{M}(f)$

$$\sup_{0<r<1}\int_S e^{\rho|g_r|}d\sigma<1+K\rho\int_0^\infty e^{(\rho-C\|f\|_{**}^{-1})t}\,dt<\infty.$$

(a)$\Rightarrow$(d) is proved.

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