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Qu^*-Algebras and Twisted Product

By

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Abstract

Fundamental properties of Qu^*-algebras of operators are shown, and the class of CQ^*-algebras is introduced. They are the natural generalization of C^*-algebras to the case of unbounded operators. The CQ^*-algebra $(\mathscr{S}_2, \mathscr{S}^+)$ of distributions with the twisted product is defined, and some of their Qu^*-subalgebras are described.

§ 1. Introduction

The noncommutativity of the multiplication of observables is the fundamental fact in quantum theory. This leads to the realization of the observables as (in general) unbounded operators in a Hilbert space. One can assume that the observables form a *-algebra. But already the fundamental procedure of Weyl quantization of classical observables leads to unbounded operators which cannot be multiplied in any cases.

Let $\mathscr{S}=\mathscr{S}(R^d)$ resp. $\mathscr{S}'=\mathscr{S}'(R^d)$ be the Schwartz spaces of test functions resp. tempered distributions with their strong topologies t resp. t'. We put $\mathscr{S}_2=\mathscr{S}(R^{2d})$ and $\mathscr{S}'_2=\mathscr{S}'(R^{2d})$. For every $f\in\mathscr{S}'_2$ we denote by $\tilde{f}$ the Fourier transform $\tilde{f}(q,p)=(2\pi)^{-2d}\int e^{-i(qu+pv)}f(u,v)\,du\,dv$. qu, pv are the Euclidean scalar products in R^d. Let $Q=(Q_1,\dots,Q_d)$, $P=(P_1,\dots,P_d)$ be the position and momentum operators $Q_j\phi=q_j\phi$, $p_j\phi=\frac{1}{i}\partial_{q_j}\phi$ defined on $\phi(q)\in\mathscr{S}\subset\mathscr{H}=L_2(R^d)$. $W(q,p)=e^{i(qQ+pP)}$ is a unitary operator on $\mathscr{H}$. They Weyl quantization or Weyl correspondence [42] of a classical distribution $f\in\mathscr{S}'_2$ is the operator

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$$\hat{f} \equiv W(f) = \int W(q, p) \tilde{f}(q, p) \ dq \ dp. \tag{1.1}$$

$W(f)$ is a well-defined operator of $\mathscr{L}(\mathscr{S}, \mathscr{S}')$, i.e. a continuous linear map of $\mathscr{S}$ into $\mathscr{S}'$, and the Weyl correspondence $f \to W(f)$ is an isomorphism between $\mathscr{S}'_2$ and $\mathscr{L}(\mathscr{S}, \mathscr{S}')$ (see §4). f is called the symbol of the Operator $\hat{f} \equiv W(f)$.

$\mathscr{L}(\mathscr{S}, \mathscr{S}')$ is not an algebra, and therefore the product $W(f)$ $W(g) = W(h)$ is not always defined. It is defined e.g. if $W(f)$, $W(g)$ leave $\mathscr{S}$ invariant. Then $h = f \circ g$ is called the twisted product of f, g, and it can be calculated by the formulae

$$(f \circ g)(q, p) = \pi^{-2d} \int f(q+q_1, p+p_1) g(q+q_2, p+p_2) e^{2i(q_1 p_2 - q_2 p_1)} \prod_i dq_i dp_i \tag{1.2}$$

if the integral is well-defined in a certain sense. Thus the twisted product is only defined for partial pairs f, $g \in \mathscr{S}'_2$ and therefore $\mathscr{S}'_2$ has the structure of quasi *-algebra which is isomorphic to the quasi *-algebra $\mathscr{L}(\mathscr{S}, \mathscr{S}')$. This we had pointed out in [26, 27].

We repeat the fundamental facts in §4 and discuss the problem of extending the twisted multiplication.

In §§2, 3 we collect some basic properties of quasi *-algebras of operators and define the class of CQ^*-algebras, which are the natural generalization of C^*-algebras to the case of unbounded operators.

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§ 2. Quasi-Uniform Topologies

The rigged Hilbert space $\mathscr{S} \subset L_2 \subset \mathscr{S}'$ is a special case of such spaces generated by *a* Hilbert scale $\mathscr{H}^s = \mathscr{D}(T^s)$, $-\infty < s < +\infty$, where $T \geq I$ is a unbounded selfadjoint operator in a Hilbert space $\mathscr{H} = \mathscr{H}^0$. The scalar product in $\mathscr{H}^s = \mathscr{D}(T^s)$ is $\langle \phi, \psi \rangle_s = \langle T^s \phi, T^s \psi \rangle$, where $\langle \ , \ \rangle$ is the scalar product in $\mathscr{H}$. Let us put $\mathscr{D}[t] = \mathscr{D}^\infty(T) = \lim_s \text{proj } \mathscr{H}^s$ and $\mathscr{D}'[t'] = \mathscr{D}^{-\infty}(T^{-\infty}) = \lim_s \text{ind } \mathscr{H}^s$. t resp. t' are the projective resp. inductive limits of the topologies of $\mathscr{H}^s$. Thus we get a rigged Hilbert space

$$\mathscr{D}[t]\subset\mathscr{H}\subset\mathscr{D}'[t']. \tag{2.1}$$

For $\phi\in\mathscr{H}^s$ and $\psi\in\mathscr{H}^{-s}$, $s\in[-\infty, \infty]$ the scalar products $\langle\phi, \psi\rangle=\overline{\langle\psi, \phi\rangle}$ are well-defined and the elements $F\in\mathscr{D}'$ define by $\langle F, \phi\rangle$ all linear continuous functionals on $\mathscr{D}'$. Therefore, $\mathscr{D}'$ is the dual space of $\mathscr{D}$ (equipped with the dual linear structure) and t' is the strong topology of the dual pair $(\mathscr{D}', \mathscr{D})$. The sesquilinear form $\langle F, \phi\rangle$ $F\in\mathscr{D}'$, $\phi\in\mathscr{D}$ is antilinear in the first factor. Without loss of generality we can suppose T to have only integer eigenvalues t_i in the spectrum, $1\leq t_1<t_2<t_3<\dots$, $t_i\to\infty$. Let $\mathscr{H}=\sum_i\oplus\mathscr{H}_i$ be the corresponding decomposition of the Hilbert space, $T\phi=\sum_i t_i\phi_i$ for $\phi=\sum\phi_i\in\mathscr{D}$, then we get [22]

$$\mathscr{D}=\{\phi:\ \sum t_i^{2k}||\phi_i||^2<\infty,\ k=0, 1, 2\dots\} \tag{2.2}$$

and the topology t is defined by the seminorms, $k=0, 1, 2, \dots$,

$$||\phi||_k=||T^k\phi||=(\sum t_i^{2k}||\phi_i||^2)^{1/2}. \tag{2.3}$$

Let Γ_T be the set of all decreasing sequences (a_i) of positive numbers, $a_1\geq a_2\geq\dots>0$, that $\sum a_i^2t_i^{2k}<\infty$ for every $k=0, 1, 2, \dots$. The elements $F=\{\psi_1, \psi_2, \dots\}\in\mathscr{D}'$, $\psi_i\in\mathscr{H}_i$, are determined by the conditions

$$||F||_{(a_i)}=\sum ||\psi_i||a_i<\infty \tag{2.4}$$

for all $(a_i)\in\Gamma_T$. The seminorms $||\cdot||_{(a_i)}$, $(a_i)=\Gamma_T$, define the topology t' on $\mathscr{D}'$.

Let $\mathscr{F}$ be the set of all positive, monotone and continuous functions $f(x)$ on R^1_+, which are decreasing faster than any inverse power, i.e. $\sup_{x\geq 0} x^kf(x)<\infty$ for all $k=0, 1, 2, \dots$. Now we can characterize the bounded sets of $\mathscr{D}[t]$ [8, 22, 23]:

Lemma 2.1.

i) *For* $(a_n)\in\Gamma_T$ *we put* $\mathscr{M}_{(a_n)}=\{\phi=\sum a_n\phi_n:\ \phi_n\in\mathscr{H}_n,\ ||\phi_n||\leq 1\}$ *and* $\mathscr{M}_f=\{f(T)\phi;\ \phi\in\mathscr{H}, ||\phi||\leq 1\}$ *for* $f\in F$. *The two systems* $\{\mathscr{M}_{(a_n)};\ (a_n)\in\Gamma_T\}$ *and* $\{\mathscr{M}_f;\ f\in\mathscr{F}\}$ *of bounded sets coincide and form a fundamental system of bounded sets in* $\mathscr{D}[t]$.

ii) *The sets* $\{T^k\phi:\ ||\phi||\leq 1\}$, $k=0, 1, 2, \dots$, *are the unit spheres in* $\mathscr{H}_{-k}$ *and therefore they form a total system of bonded sets in* $\mathscr{D}'[t']$.

Now we recall some fundamental facts about the unbounded operators on a rigged Hilbert space [21, 22] which lead to the concept of quasi *-algebras.

Let $\mathscr{D}$ be a unitary space (incomplete Hilbert space) with the scalar product $\langle .,. \rangle$, $\mathscr{H}$ its completion. By $\mathscr{L}^+(\mathscr{D})$ we denote the set of all endomorphisms $A \in \text{End}\,\mathscr{D}$ for which an $A^+ \in \text{End}\,\mathscr{D}$ exists with $\langle \psi, A\phi \rangle = \langle A^+\psi, \phi \rangle$ for all $\phi, \psi \in \mathscr{D}$. $\mathscr{L}^+(\mathscr{D})$ is a *-algebra with the usual algebraic operation with operators and the involution $A \to A^+$. If $\mathscr{D} = \mathscr{H}$, then $\mathscr{L}^+(\mathscr{D}) = \mathscr{B}(\mathscr{H})$ the C^*-algebra of all bounded operators on $\mathscr{H}$. We call a *-subalgebra $\mathscr{A}$ of $\mathscr{L}^+(\mathscr{D})$ containing the identity *Op*-algebra* [21].

On $\mathscr{D}$ we define a locally convex topology t by the following system of seminorms

$$t:\ ||\phi||_A = ||A\phi||,\quad A \in \mathscr{L}^+(\mathscr{D}). \tag{2.5}$$

A domain in $\mathscr{H}$ is called a *closed domain*, if $\mathscr{D}[t]$ is a complete space. Then $\mathscr{D} = \bigcap_{A \in \mathscr{L}^+(\mathscr{D})} \mathscr{D}(\bar{A})$, where $\mathscr{D}(\bar{A})$ is the domain of the closure $\bar{A}$ of the operator A.

The dual space of $\mathscr{D}[t]$ we denote by $\mathscr{D}'[t']$, where t' is the strong topology on $\mathscr{D}'$. The Hilbert space $\mathscr{H}$ is canonical imbedded into $\mathscr{D}'[t']$. Hence, any dense domain $\mathscr{D} \subset \mathscr{H}$ defines in a canonical way a rigged Hilbert space

$$\mathscr{D}[t] \to \mathscr{H} \to \mathscr{D}'[t']$$

where the scalar product $\langle F, \phi \rangle$ is defined for $\phi \in \mathscr{D}$, $F \in \mathscr{D}'$. In what follows we regard only such $\mathscr{D}$ for which $\mathscr{D}[t]$ is a reflexive space. Let $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ be the linear space of all continuous maps of $\mathscr{D}[t]$ into $\mathscr{D}'[t']$. Further we write $\mathscr{L}(\mathscr{D}) = \mathscr{L}(\mathscr{D}, \mathscr{D})$ and $\mathscr{L}(\mathscr{D}') = \mathscr{L}(\mathscr{D}', \mathscr{D}')$ which are algebras with respect to the usual operations with maps. Then we get ([22, Lemma 2.1]):

Lemma 2.2. *Let $\mathscr{D}[t]$ be a reflexive space. Then*

i) *if $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$, so the adjoint operator $A^+ \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ is uniquely defined by $\langle A\phi, \psi \rangle = \overline{\langle A^+\psi, \phi \rangle}$. $A \to A^+$ is an involution on $\mathscr{L}(\mathscr{D}, \mathscr{D}')$.*

ii) *$\mathscr{L}(\mathscr{D})$, $\mathscr{L}(\mathscr{D}') \subset \mathscr{L}(\mathscr{D}, \mathscr{D}')$ and $\mathscr{L}(\mathscr{D})^+ = \mathscr{L}(\mathscr{D}')$*

iii) $\mathscr{L}^+(\mathscr{D})$ *is a subspace of* $\mathscr{L}(\mathscr{D})$ *and it is* $\mathscr{L}^+(\mathscr{D})=\mathscr{L}(\mathscr{D})\cap \mathscr{L}(\mathscr{D}')$.

If E, F are two locally convex spaces, then the topology τ of *uniformly bounded convergence* on $\mathscr{L}(E, F)$ is defined by all seminorms $q_{\alpha,\mathscr{M}}(A)=\sup_{\phi\in\mathscr{M}} p_\alpha(A\phi)$ where p_α runs over the seminorms defining the topology of F and $\mathscr{M}$ runs over all bounded sets in E.

The topologies of uniformly bounded convergences on the spaces $\mathscr{L}(\mathscr{D}, \mathscr{D}')$, $\mathscr{L}(\mathscr{D})$ and $\mathscr{L}(\mathscr{D}')$ we denote by $\tau_{\mathscr{D}}$, $\tau^{\mathscr{D}}$ and $\tau^{\mathscr{D}'}$. Let us describe the seminorms determining these topologies more explicitly [8, 22].

$$\begin{aligned}
&\tau_{\mathscr{D}}: \ \|A\|_{\mathscr{M}}=\sup_{\phi,\psi\in\mathscr{M}} |\langle A\psi, \phi\rangle|, \ \mathscr{M} \text{ bounded in } \mathscr{D}[t]\\
&\tau^{\mathscr{D}}: \ \|A\|^{\mathscr{M},B}=\sup_{\phi\in\mathscr{M}} \|BA\phi\|, \ B\in\mathscr{L}^+(\mathscr{D}), \ \mathscr{M} \text{ bounded in } \mathscr{D}[t]\\
&\tau^{\mathscr{D}'}: \ \|A\|^{\mathscr{N}',\mathscr{M}}=\sup_{\substack{\phi\in\mathscr{M}\\ \psi\in\mathscr{N}'}} |\langle A\psi, \phi\rangle|, \ \mathscr{M} \text{ bounded in } \mathscr{D}[t]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathscr{N}' \text{ bounded in } \mathscr{D}'[t'].
\end{aligned} \tag{2.6}$$

This definition of the topologies makes sense also for non-reflexive $\mathscr{D}[t]$.

Lemma 2.3 [8, 22]. *Let* $\mathscr{D}[t]$ *be reflexive. Then*

i) *the topology* $\tau^{\mathscr{D}'}$ *is given by the seminorms* $\|A\|_+^{\mathscr{M},B}=\|A^+\|^{\mathscr{M},B}$ *where* B *runs over all operators of* $\mathscr{L}^+(\mathscr{D})$ *and* $\mathscr{M}$ *over all bounded sets of* $\mathscr{D}[t]$.

ii) $\mathscr{L}(\mathscr{D})[\tau^{\mathscr{D}}]$, $\mathscr{L}(\mathscr{D}')[\tau^{\mathscr{D}'}]$ *are topological algebras of operators.*

iii) $A\to A^+$ *is one-to-one between* $\mathscr{L}(\mathscr{D})[\tau^{\mathscr{D}}]$ *and* $\mathscr{L}(\mathscr{D}')[\tau^{\mathscr{D}'}]$.

iv) $\mathscr{L}^+(\mathscr{D})[\tau_{\mathscr{D}}]$ *is a locally convex *-algebra.*

Let us still introduce $\tau_*^{\mathscr{D}}=\max(\tau^{\mathscr{D}}, \tau^{\mathscr{D}'})$ on $\mathscr{L}^+(\mathscr{D})$. Then $\mathscr{L}^+(\mathscr{D})$ becomes a locally convex *-algebra with respect to the topology $\tau_*^{\mathscr{D}}$. The relations between the different linear spaces of operators and their topologies are expressed by the following scheme.

$$\begin{array}{ccccc}
 & \nearrow & \mathscr{L}(\mathscr{D})[\tau^{\mathscr{D}}] & \searrow & \\
\mathscr{L}^+(\mathscr{D})[\tau_*^{\mathscr{D}}] & & & & \mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau_{\mathscr{D}}] \\
 & \searrow & \mathscr{L}(\mathscr{D}')[\tau^{\mathscr{D}'}] & \nearrow &
\end{array} \tag{2.7}$$

where $\longrightarrow$ denotes a continuous injection. If $\mathscr{D}=\mathscr{H}$ then all four spaces coincide with $\mathscr{B}(\mathscr{H})$ and all topologies with the operator norm topology. Among the four topologies $\tau_{\mathscr{D}}$ plays an exceptional role [21, 22]. Therefore, we call it the *uniform topology* on $\mathscr{L}^+(\mathscr{D})$. The other topologies are called *quasi-uniform topologies.*

Now we go back to rigged Hilbert spaces (2. 1), associated to $\mathscr{D}[t]=\mathscr{D}^\infty(T)$. Then the quasi-uniform topologies (2. 6) are defined by the following systems of seminorms [23] :

$$\begin{aligned} \tau_{\mathscr{D}} &: \ \|A\|_f=\|f(T)Af(T)\| \\ \tau^{\mathscr{D}} &: \ \|A\|^{f,k}=\|T^kAf(T)\| \\ \tau^{\mathscr{D}\prime} &: \ \|A\|_+^{f,k}=\|f(T)AT^k\| \\ \tau_*^{\mathscr{D}} &: \ \|A\|^{f,k}=\max\{\|T^kAf(T)\|,\ \|f(T)AT^k\|\}, \end{aligned} \tag{2.8}$$

where f runs over F, $k=0,1,2,\ldots$, and the norm on the right-hand side is the usual operator norm.

Another explicitly given system of seminorms for the quasi-uniform topologies we get by using the decomposition $\mathscr{H}=\sum\oplus\mathscr{H}_i$ in eigen-spaces $\mathscr{H}_i$ of T(see (2. 2)). Let P_i be the projection of $\mathscr{H}$ to $\mathscr{H}_i$. Then every operator $A\in\mathscr{L}(\mathscr{D},\mathscr{D}')$ gives a matrix $A=(A_{ij})$, i. e. $A=\sum_{i,j} A_{ij}$ with

$$A_{ij}=P_iAP_j,\quad P_i\mathscr{H}=\mathscr{H}_i\subset\mathscr{D} \tag{2.9}$$

and the quasi-uniform topologies are defined by the following seminorms [22], where (a_i) runs over all $(a_i)\in\Gamma_T$ and $k=0,1,2,\ldots$

$$\begin{aligned} \tau_{\mathscr{D}} &: \ \|A\|_{(a_i)}=\sum_{i,j}\|A_{ij}\|a_ia_j \\ \tau^{\mathscr{D}} &: \ \|A\|^{(a_i),k}=\sum_{i,j}\|A_{ij}\|t_i^ka_j \\ \tau^{\mathscr{D}\prime} &: \ \|A\|_+^{(a_i),k}=\sum_{i,j}\|A_{ij}\|a_it_j^k \\ \tau_*^{\mathscr{D}} &: \ \|A\|^{(a_i),k}=\sum_{i,j}\|A_{ij}\|(t_i^ka_j+a_it_j^k). \end{aligned} \tag{2.10}$$

The linear spaces (resp. algebras) of operators (2. 7) are formed exactly by all operator-matrices $A=(A_{ij})$, A_{ij}: $\mathscr{H}_j\to\mathscr{H}_i$, for which the corresponding seminorms in (2. 10) are finite. Furthermore, one can see immediately from (2. 10) that with respect to each of the four topologies every A can be approximated by finite matrices $A_N=(A_{ij})$, $A_{ij}=0$ for $i, j\geq N$. Therefore we have ([22, Lemma 2. 6]) :

Lemma 2. 4.

i) *$\mathscr{L}^+(\mathscr{D})$ is dense in the three other locally convex spaces of* *operators* $\mathscr{L}(\mathscr{D})[\tau^{\mathscr{D}}]$, $\mathscr{L}(\mathscr{D}')[\tau^{\mathscr{D}'}]$, *and* $\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau_{\mathscr{D}}]$.

ii) *All these four locally convex spaces of operators are complete.*

All the above results about $(\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau_{\mathscr{D}}], \mathscr{L}^+(\mathscr{D})[\tau_*^{\mathscr{D}}])$, especially the fundamental Lemmas 2.1 and 2.4, can be generalized to a wide class of rigged Hilbert spaces $\mathscr{D}\subset\mathscr{H}\subset\mathscr{D}'$ [19, 33]. But there exist also remarkable counterexamples, where $\mathscr{D}[t]$ is not separable and $\mathscr{L}^+(\mathscr{D})$ not dense in $\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau_{\mathscr{D}}]$ [18].

We conclude this Section with a third characterization of the topologies on the operators. They are defined by the following systems of seminorms, where (a_i) runs over all Γ_T, $k=0, 1, 2, \ldots$

$$\begin{aligned}
\tau_{\mathscr{D}} &: q_{(a_i)}(A) = \sup_{i,j} \|A_{ij}\| a_i a_j \\
\tau^{\mathscr{D}} &: q^{(a_i),k}(A) = \sup_{i,j} \|A_{ij}\| t_i^k a_j \\
\tau^{\mathscr{D}'} &: q_+^{(a_i),k}(A) = \sup_{i,j} \|A_{ij}\| a_i t_j^k \\
\tau_*^{\mathscr{D}} &: q_*^{(a_i),k}(A) = \sup_{i,j} \|A_{ij}\| (t_i^k a_j + a_i t_j^k).
\end{aligned} \tag{2.11}$$

Since we supposed t_i, $i=1, 2, \ldots$, to be integers, we have $\sum t_i^{-2}=\kappa<\infty$. Therefore, we get e.g. for the seminorms of the topology $\tau_{\mathscr{D}}$ in (2.10) and (2.11) the estimation $q_{(a_i)}(A)\le\|A\|_{(a_i)}\le q_{(t_i^2 a_i)}(A)\cdot\kappa^2$. In the same way one can prove the equivalence of the corresponding seminorms in (2.10) and (2.11).

§ 3. Qu^*-Algebras

One of the fundamental ingredients of the C^*- and W^*-theories is the relation $\mathscr{B}(\mathscr{H})=\mathfrak{S}_1'$, i.e. the space $\mathfrak{S}_1$ of all nuclear operators is the predual of the C^*-algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators on $\mathscr{H}$. This property can be generalized to $\mathscr{L}(\mathscr{D}, \mathscr{D}')$. For that first we describe the set $\mathfrak{S}_1(\mathscr{D})$ of the nuclear operators associated to a rigged Hilbert space [28]:

$$\mathfrak{S}_1(\mathscr{D}) = \{\rho : \rho\in\mathscr{L}^+(\mathscr{D}),\ A\rho B\in\mathfrak{S}_1 \text{ for } A, B\in\mathscr{L}^+(\mathscr{D})\}. \tag{3.1}$$

Let $\|\rho\|_1=tr(\rho^+\rho)^{1/2}$ be the trace-norm on the nuclear operators, then by β^* we denote a locally convex topology on $\mathfrak{S}_1(\mathscr{D})$ defined by all seminorms $\|A\rho B\|_1$, $A, B\in\mathscr{L}^+(\mathscr{D})$. In the case $\mathscr{D}=\mathscr{D}^\infty(T)$ every operator $A\in\mathscr{L}^+(\mathscr{D})$ can be estimated by a power of T, and therefore the topology β^* is defined by the following denumerable system of seminorms (see also [25]):

$$\beta^*: \ \|\rho\|_{(k)} = \|T^k \rho T^k\|_1, \ k=0, 1, \dots . \tag{3.2}$$

In correspondence with (2.10-2.11) the β^*-topology can also be given by the seminorms:

$$\beta^*: \quad \begin{aligned} p_k(\rho) &= \sum_{i,j} \|\rho_{ij}\|_1 t_i^k t_j^k \\ q_k(\rho) &= \sup_{i,j} \|\rho_{ij}\|_1 t_i^k t_j^k . \end{aligned} \tag{3.3}$$

This follows from the estimations

$$\begin{aligned} &\|T^k \rho T^k\|_1 \le \sum_{i,j} \|P_i T^k \rho T^k P_j\|_1 = p_k(\rho) \\ &p_k(\rho) = \sum_{i,j} \|\rho_{ij}\|_1 t_i^k t_j^k \le q_{k+2}(\rho)\, \kappa^2 \\ &q_k(\rho) \le \|T^k \rho T^k\|_1 = \|\rho\|_{(k)} \end{aligned} \tag{3.4}$$

where $\kappa = \sum t_i^{-2}$.

Furthermore, in correspondence with Lemma 2.1 we have

Lemma 3.1. *For every* $f \in \mathscr{F}$ *we set*

$$\mathfrak{B}_f = \{ f(T) D f(T) ; \ D \in \mathfrak{S}_1, \ \|D\|_1 \le 1\} \tag{3.5}$$

is bounded in $\mathfrak{S}(\mathscr{D})$. *The system of bounded sets* $\{\mathfrak{B}_f ; f \in \mathscr{F}\}$ *is total in* $\mathfrak{S}_1(\mathscr{D})[\beta^*]$.

In fact, let $\mathfrak{B}$ be a bounded set in $\mathfrak{S}_1(\mathscr{D})[\beta^*]$ and $\alpha_{ij} = \sup_{\rho \in \rho_1(\mathscr{D})} \|\rho_{ij}\|_1$, then by (3.3) $\sup_{i,j} \alpha_{ij} t_i^k t_j^k < \infty$ for every $k=0, 1, 2, \dots$. Therefore, it exists a $f \in \mathscr{F}$ with $\alpha_{ij} \le f(t_i) f(t_j)$. Thus $\mathfrak{B} \subset \mathfrak{B}_f$. If $\rho \in \mathfrak{S}_1(\mathscr{D})$, $A \in \mathscr{L}^+(\mathscr{D})$, then

$$tr \rho A = \sum_{i,j,k} tr \rho_{ik} A_{kj} . \tag{3.6}$$

As a consequence of (3.3) and the characterization of the $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ by the finiteness of the first seminorm in (2.10), we see that the right-hand side of (3.6) is also defined for $\rho \in \mathfrak{S}_1(\mathscr{D})$ and $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$. We choose the notation $tr \rho A$ also for this general case, but ρA is in general not a nuclear operator (also not bounded).

Lemma 3.2.

i) $(\mathfrak{S}_1(\mathscr{D}),\ \mathscr{L}(\mathscr{D},\mathscr{D}'))$ *is a dual pair with respect to the binilinear form* $(\rho, A) = tr\rho A$ *defined by* (3.6).

ii) *The uniform topology* $\tau_{\mathscr{D}}$ *on* $\mathscr{L}(\mathscr{D},\mathscr{D}')$ *and the topology* β^* *on* $\mathfrak{S}_1(\mathscr{D})$ *are the strong topologies of the dual pair.*

iii) $\mathscr{L}(\mathscr{D},\mathscr{D}')[\tau_{\mathscr{D}}]$ *is the strong dual of* $\mathfrak{S}_1(\mathscr{D})[\beta^*] = \mathscr{D}\tilde{\otimes}_\pi\mathscr{D}$, *i.e.*

$$\mathscr{L}(\mathscr{D},\mathscr{D}')[\tau_{\mathscr{D}}] = \mathfrak{S}_1(\mathscr{D})[\beta^*]'. \tag{3.7}$$

This lemma (see [24, 26]) is essentially a consequence of the characterization of $\mathscr{L}(\mathscr{D},\mathscr{D}')[\tau_{\mathscr{D}}]$, $\mathfrak{S}_1(\mathscr{D})[\beta^*]$ by the seminorms (2.10), (3.3). The duality $\mathscr{L}(\mathscr{D},\mathscr{D}')[\tau_{\mathscr{D}}] = (\mathscr{D}\tilde{\otimes}_\pi\mathscr{D})'$ is a special case of the unsolved "Grothendieck-Problem", whether $\mathscr{L}(E, F') \equiv B(E\times F)[\tau_{bb}] = (E\tilde{\otimes}_\pi F)'$ (for arbitrary metric spaces E, F ([17, p. 1985]). For Frechet spaces $E = F = \mathscr{D}[t]$ which are closed domains (see (2.5)) the duality $\mathscr{L}(\mathscr{D},\mathscr{D}') = (\mathscr{D}\tilde{\otimes}_\pi\mathscr{D})'$ has been proved recently [20].

The duality $\mathscr{L}(\mathscr{D},\mathscr{D}')[\tau_{\mathscr{D}}] = \mathfrak{S}_1(\mathscr{D})[\beta^*]'$ is a consequence of (2.8) and Lemma 3.1, since

$$\|f(M)A\,f(M)\| = \sup_{\|D\|_1\le 1} |tr\ D\,f(M)Af(M)\,| = \sup_{\rho\in\mathfrak{B}_f} |tr\rho A\,|. \tag{3.8}$$

By Lemma 3.2, iii), $\mathscr{L}(\mathscr{D},\mathscr{D}')[\tau_{\mathscr{D}}]$ is the natural generalization of the W^*-algebra $\mathscr{B}(\mathscr{H})$ to unbounded operators. But $\mathscr{L}(\mathscr{D},\mathscr{D}')$ is not a *-algebra. It is a quasi *-algebra in the sense of the following definition [24]:

Definition 3.3. A locally convex quasi *-algebra, shortly Qu^*-algebra, $(\mathscr{A}[\xi], \mathscr{A}_0)$ is defined by the following conditions:

(1) $\mathscr{A}[\xi]$ is a locally convex space with a distinguished dense subspace $\mathscr{A}_0$.

(2) Partial multiplications $A\to AB$ and $A\to BA$ are defined on $\mathscr{A}$ for every $B\in\mathscr{A}_0$. They are continuous linear operators on $\mathscr{A}[\xi]$ and $\mathscr{A}$ is an $\mathscr{A}_0$-module with respect to these multiplications.

(3) A continuous involution $A\to A^+$ is defined on $\mathscr{A}[\xi]$, which leaves $\mathscr{A}_0$ invariant. $(AB)^+ = B^+A^+$, $(BA)^+ = A^+B^+$ for $B\in\mathscr{A}_0$, $A\in\mathscr{A}$.

Qu^*-algebras are special cases of the more general class of partial *-algebras [1, 2, 3], which have importance in mathematical physics. A simple consequence of the definition is the following

Lemma 3.4.

i) *Let* $(\mathscr{A}[\xi], \mathscr{A}_0)$ *be a* Qu^**-algebra and* $\widetilde{\mathscr{A}[\xi]}$ *the completion. Then the multiplication* $A, B \to AB, BA$ *can be extended by continuity for* $A \in \widetilde{\mathscr{A}[\xi]}$, $B \in \mathscr{A}_0$. $(\widetilde{\mathscr{A}[\xi]}, \mathscr{A}_0)$ *is a* Qu^**-algebra, the completion of* $(\mathscr{A}[\xi], \mathscr{A}_0)$.

ii) *The completion of a topological *-algebra* $\mathscr{A}[\xi]$ *is a* Qu^**-algebra* $(\widetilde{\mathscr{A}[\xi]}, \mathscr{A})$.

The completion functions leads beyond the category of topological *-algebras. The category of Qu^*-algebras is the smallest extension with completion. Since the completeness of the observable *-algebras in statistical physics is important for the existence of limits (e.g. thermodynamical limit), the fundamental results on general *-algebras in physics (see [4, 7, 14, 37, 38, 39, 40]) must be generalized to Qu^*-algebras.

Let us call a dense domain $\mathscr{D}[t] \subset \mathscr{H}$ a *basic space*, if it is reflexive and Lemma 2.4 and Lemma 3.2 hold true.

In this paper, all basic spaces are of the form $\mathscr{D} = \mathscr{D}^\infty(T)$.

Theorem 3.5.

i) *Let* $\mathscr{D}$ *be a basic space. For the topologies of* (2.7) *we choose the abbreviation* $\tau \equiv \tau_{\mathscr{D}}$, $\tau_* = \tau_*^{\mathscr{D}}$. $(\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau], \mathscr{L}^+(\mathscr{D}))$ *is a* Qu^**-algebra.*

ii) *For* $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$ *the multiplications* $B \to AB, BA$ *are continuous linear maps from* $\mathscr{L}^+(\mathscr{D})[\tau_*]$ *to* $\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau]$.

Proof: i) By Lemma 2.4 we have only to show the continuity of the multiplication. But by (2.6) we get for $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$, $B \in \mathscr{L}^+(\mathscr{D})$

$$||BA||_{\mathscr{M}} \leq ||A||_{\mathscr{M} \cup B^+ \mathscr{M}}$$
$$||AB||_{\mathscr{M}} \leq ||A||_{B\mathscr{M} \cup \mathscr{M}}. \tag{3.9}$$

ii) Let $B \in \mathscr{L}^+(\mathscr{D})$ and $A \in \mathscr{L}(\mathscr{D}, \mathscr{D}')$, then $||f(M)BA\ f(M)|| =$

$\sup_{\phi\in\mathscr{N}}\|f(M)B\phi\|$, where $\mathscr{N}=\{Af(M)\phi\colon \phi\in\mathscr{H}, \|\phi\|\leq 1\}$. But by Lemma 2.1, ii) $\mathscr{M}$ is contained in a set const. $\{T^k\phi\colon \|\phi\|\leq 1\}$. Therefore $\|BA\|_f\leq$const. $\|f(M)BT^k\|=$const. $\|B\|_+^{f,k}$ (see (2.8)). The estimation for $\|AB\|_f$ is analogous.

Definition 3.6. $(\mathscr{B}[\xi], \mathscr{B}_0)$ is called a Qu^*-subalgebra of a Qu^*-algebra $(\mathscr{A}[\xi], \mathscr{A}_0)$, if $\mathscr{B}[\xi]$ is a topological subspace of $\mathscr{A}[\xi]$, $\mathscr{B}_0$ a *-subalgebra of $\mathscr{A}_0\cap\mathscr{B}$ dense in $\mathscr{B}[\xi]$ and if $\mathscr{B}$ is a $\mathscr{B}_0$-submodul of $\mathscr{B}_0$. $(\mathscr{B}[\xi], \mathscr{B}_0)$ is called a closed Qu^*-subalgebra of $(\mathscr{A}[\xi], \mathscr{A}_0)$ if $\mathscr{B}$ is a closed subspace of $\mathscr{A}[\xi]$ and $\mathscr{B}_0=\mathscr{A}_0\cap\mathscr{B}$.

The maximal Qu^*-algebra of operators $(\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau], \mathscr{L}^+(\mathscr{D}))$ (Theorem 3.5) on a rigged Hilbert space is the generalization of the C^*- and W^*-algebra $\mathscr{B}(\mathscr{H})$. Therefore, we propose the following definition

Definition 3.7. Let $\mathscr{D}$ be a basic space. A closed Qu^*-subalgebra $(\mathscr{A}[\tau], \mathscr{A}_0)$ of $(\mathscr{L}^+(\mathscr{D}, \mathscr{D})[\tau], \mathscr{L}^+(\mathscr{D}))$ is called a *CQ^*-algebra* of operators. It is called WQ^*-algebra of operators if $\mathscr{A}[\tau]$ is the strong dual of $\mathscr{A}_*=\mathfrak{S}_1(\mathscr{D})[\beta^*]/\mathscr{A}^0$, $\mathscr{A}^0=\{\rho\in\mathfrak{S}_1(\mathscr{D}) : tr\rho A=0$ for all $A\in\mathscr{A}\}$ is the polar of $\mathscr{A}$ in the dual pair $(\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau], \mathfrak{S}_1(\mathscr{D})[\beta^*])$. An arbitrary Qu^*-algebra $(\mathscr{A}[\xi], \mathscr{A}_0)$ we call CQ^*-algebra resp. WQ^*-algebra if it is isomorphic to a CQ^*-algebra resp. WQ^*-algebra of operators.

In the CQ^*-algebra $(\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau], \mathscr{L}^+(\mathscr{D}))$ the *-algebra $\mathscr{L}^+(\mathscr{D})$ is maximal in the sense of the following lemma. $\mathscr{D}$ is assumed to be a basic space.

Lemma 3.8.

i) *If for $A\in\mathscr{L}(\mathscr{D}, \mathscr{D}')$ the products AB, BA are in $\mathscr{L}^+(\mathscr{D})$ for every $B\in\mathscr{L}^+(\mathscr{D})$, then $A\in\mathscr{L}^+(\mathscr{D})$.*

ii) *If for $A\in\mathscr{L}(\mathscr{D}, \mathscr{D}')$ the products AB, BA are in $\mathfrak{S}_1(\mathscr{D})$ for all $B\in\mathfrak{S}_1(\mathscr{D})$, then $A\in\mathscr{L}^+(\mathscr{D})$. Consequently, $\mathscr{L}^+(\mathscr{D})=\{A\in\mathscr{L}(\mathscr{D}, \mathscr{D}')\colon AB, BA\in\mathfrak{S}_1(\mathscr{D})$ for all $B\in\mathfrak{S}_1(\mathscr{D})\}$.*

Proof: From the assumptions i) or ii) it follows $A: \mathscr{D}\to\mathscr{D}$ and also $A^+: \mathscr{D}\to\mathscr{D}$. By the closed graph theorem we have $A, A^+\in \mathscr{L}(\mathscr{D})$ and therefore $A\in\mathscr{L}^+(\mathscr{D})$ (Lemma 2.2, iii)).

In ii) we chose the fact that $\mathfrak{S}_1(\mathscr{D})$ is an ideal in $\mathscr{L}^+(\mathscr{D})$ ([28], see also [36, 37]). The maximality of $\mathscr{L}^+(\mathscr{D})$ in $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ is connected with completeness of $\mathscr{L}^+(\mathscr{D})[\tau_*]$, as we shall explain now.

Definition 3.9. Let $(\mathscr{A}[\xi], \mathscr{A}_0)$ be a Qu^*-algebra. By ξ_0 we denote the weakest locally convex topology on $\mathscr{A}_0$ such that for every bounded set $\mathfrak{M}\subset\mathscr{A}[\xi]$ the set of maps $\{B\to BA,\ B\to AB;\ A\in\mathfrak{M}\}$ from $\mathscr{A}_0[\xi_0]$ into $\mathscr{A}[\xi]$ is equicontinuous ([16, §15.13]).

Let $\mathscr{F}$ be a system of seminorms $p(\cdot)$ on $\mathscr{A}$ defining the topology ξ. We call $\mathscr{F}$ a $\mathscr{A}_0$-system of seminorms, if for every $p\in\mathscr{F}$ and $B\in\mathscr{A}_0$ also the seminorms $(Bp)(\cdot)$, $(pB)(\cdot)$, $p^+(\cdot)\in\mathscr{F}$

$$(Bp)(A)=p(BA),\quad (pB)(A)=p(AB),\quad p^+(A)=p(A^+). \tag{3.10}$$

Lemma 3.10.

i) *$\mathscr{A}[\xi_0]$ is a locally convex *-algebra*

ii) *If $\mathscr{F}$ is an $\mathscr{A}_0$-system of seminorms of $\mathscr{A}[\xi]$, then ξ_0 is defined by the following system of seminorms on $\mathscr{A}_0$*

$$p_{\mathfrak{M}}(B)=\sup_{A\in\mathfrak{M}} p(BA),\quad {}_{\mathfrak{M}}p(B)=\sup_{A\in\mathfrak{M}} p(AB) \tag{3.11}$$

where $p\in\mathscr{F}$ and $\mathfrak{M}$ runs over all bounded sets of $\mathscr{A}[\xi]$.

Proof: ii) is an immediate consequence of the definition of ξ_0. i) follows from ii) and the following relations for $B, C\in\mathscr{A}_0$, $p\in\mathscr{F}$, $\mathfrak{M}$ bounded in $\mathscr{A}[\xi]$:

$$p_{\mathfrak{M}}(BC)=p_{C_{\mathfrak{M}}}(B)=(Bp)_{\mathfrak{M}}(C)$$
$${}_{\mathfrak{M}}p(BC)={}_{\mathfrak{M}}(pC)(B)={}_{\mathfrak{M}}p(C)$$
$$p_{\mathfrak{M}}(A^+)={}_{\mathfrak{M}^+}p^+(A),\quad {}_{\mathfrak{M}}p(A^+)=p^+{}_{\mathfrak{M}^+}(A). \tag{3.12}$$

Lemma 3.11. *Let $\mathscr{D}$ be a basic space and $(\mathscr{L}(\mathscr{D}, \mathscr{D}')[\tau], \mathscr{L}^+(\mathscr{D}))$ the maximal Qu^*-algebra. Then $\tau_0=\tau_*$ on $L^+(\mathscr{D})$.*

Proof: The system $\|A\|_{\mathscr{M},\mathscr{N}}=\sup_{\phi\in\mathscr{M},\psi\in\mathscr{N}}|\langle A\phi, \psi\rangle|$, $\mathscr{M}$, $\mathscr{N}$ bounded in $\mathscr{D}[t]$, is a $\mathscr{L}^+(\mathscr{D})$-system of seminorms for the topology τ (see (2.6)). For example we have $\|A^+\|_{\mathscr{M},\mathscr{N}}=\|A\|_{\mathscr{N},\mathscr{M}}$. $\|BA\|_{\mathscr{M},\mathscr{N}}=\|A\|_{\mathscr{M},B^+\mathscr{N}}$, etc.

Let $\mathfrak{L}=\{A\in\mathscr{B}(\mathscr{H}), \|A\|\leqq 1\}$ the unit sphere of $\mathscr{B}(\mathscr{H})\subset\mathscr{L}(\mathscr{D},\mathscr{D}')$. $\mathfrak{L}$ is bounded in $\mathscr{L}(\mathscr{D},\mathscr{D}')[\tau]$. Since $\mathfrak{L}$ contains all projections $|\phi\rangle\langle\psi|$, $\|\phi\|=\|\psi\|=1$, there is a $\lambda>0$ for every bounded set $\mathscr{M}\subset\mathscr{D}[t]$ such that $\lambda\cdot\mathfrak{L}\cdot\mathscr{M}\supset\mathscr{K}=\{\phi\in\mathscr{H};\ \|\phi\|\leqq 1\}$. Now we put $\mathfrak{M}=\lambda T^+\cdot\mathfrak{L}$ for one $T\in\mathscr{L}^+(\mathscr{D})$. For the seminorm $p(\cdot)\equiv\|\cdot\|_{\mathscr{M}}\equiv\|\ \|_{\mathscr{M},\mathscr{M}}$ we estimate $p_{\mathfrak{M}}(\cdot)$ (3.11) from below and get

$$\begin{aligned} p_{\mathfrak{M}}(B) &=\sup_{A\in\mathfrak{M}}\|BA\|_{\mathscr{M}}=\sup_{A\in\mathfrak{M},\phi,\psi\in\mathscr{M}}|\langle BA\phi,\psi\rangle| \\ &=\sup_{C\in\mathfrak{L},\phi,\psi\in\mathscr{M}}\lambda|\langle C\phi, TB^+\psi\rangle| \\ &\geqq\sup_{\Omega\in\mathscr{K},\psi\in\mathscr{M}}|\langle\Omega, TB^+\psi\rangle|=\|B^+\|^{\mathscr{M},T}. \end{aligned} \tag{3.13}$$

Thus we have estimated a seminorm $\|B\|_+^{\mathscr{M},T}$ of τ_* (see Lemma 2.3) by a seminorm $p_{\mathfrak{M}}(B)$ of τ_0. On the same way every seminorm of τ_* can be estimated. Therefore, τ_0 is stronger than τ_*. But since $\mathscr{L}^+(\mathscr{D})[\tau_*]$ is a barrelled space, τ_0 cannot be stronger than τ_* by Theorem 3.5, ii). Therefore $\tau_*=\tau_0$, and the proof is complete.

Let $(\mathscr{A}[\xi],\mathscr{A}_0)$ be a Qu^*-algebra with complete $\mathscr{A}[\xi]$. Then the bilinear maps $A, B\to A\cdot B$, $B\cdot A$ from $\mathscr{A}_0\times\mathscr{A}$ to $\mathscr{A}$ can be extended to $\widetilde{\mathscr{A}_0[\xi_0]}\times\mathscr{A}$ by continuity. In that sense $\mathscr{L}^+(\mathscr{D})$ is maximal in $\mathscr{L}(\mathscr{D},\mathscr{D}')[\tau]$ by the last Lemma, since $\mathscr{L}^+(\mathscr{D})[\tau_*]$ is complete.

§4. Weyl Quantization and Twisted Product

The Weyl quantization $f\to W(f)$ leads to operators on the rigged Hilbert space $\mathscr{S}\subset L_2\subset\mathscr{S}'$. It is of the type (2.1) $\mathscr{S}=\mathscr{D}^\infty(T)$, $\mathscr{S}'=\mathscr{D}^{-\infty}(T)$ where for T we can take the operator $T=P^2+Q^2+1=P_1^2+\dots+P_d^2+Q_1^2+\dots+Q_d^2+1$ (number operator). Therefore, all definitions and statements of §§2, 3 are applicable. For the topologies of the Qu^*-algebra $(\mathscr{L}(\mathscr{S},\mathscr{S}')[\tau_{\mathscr{S}}], \mathscr{L}^+(\mathscr{S})[\tau_*^{\mathscr{S}}])$ we choose the abbreviations $\tau\equiv\tau_{\mathscr{S}}$, $\tau_*\equiv\tau_*^{\mathscr{S}}$ (see Theorem 3.5).

The differential operator T generates the Hilbert scale

$$\mathscr{H}_s=\mathscr{D}(T^s)=T^{-s}L_2(R^d),\ -\infty<s<\infty \tag{4.1}$$

of Sobolev spaces, $\mathscr{S}=\lim_S \operatorname{proj}\mathscr{H}_S$, $\mathscr{S}'=\lim_S \operatorname{ind}\mathscr{H}_S$. Since T^{-2d} is a nuclear operator, besides $\mathscr{L}(\mathscr{S}, \mathscr{S}') = \mathfrak{S}_1(\mathscr{S})'$ we have also $\mathfrak{S}_1(\mathscr{S}) = \mathscr{L}^+(\mathscr{S})[\tau]' = \mathscr{L}(\mathscr{S}, \mathscr{S}')[\tau]'$ [28] (see also [29]). $\mathscr{S}$ is a Montel space [16]. Therefore, the last duality relation is a special case of the more general results in [34].

Since $\mathscr{L}(\mathscr{S}, \mathscr{S}')[\tau]$ is a dual space with all good properties for an integration theory, the Weyl integral (1.1) $(W(f) = \int e^{i(qQ+pP)} \tilde{f}(q,p)\,dq\,dp$ is well-defined in $\mathscr{L}(\mathscr{S}, \mathscr{S}')$ for any $f \in \mathscr{S}'_2$ [26, 27].

Theorem 4.1 [13, 30, 32]. *The Weyl quantization $f \to W(f)$ is a linear continuous isomorphism between the locally convex spaces $\mathscr{S}'_2$ and $\mathscr{L}(\mathscr{S}, \mathscr{S}')[\tau]$ and also between $\mathscr{S}_2$ and $\mathfrak{S}_1(\mathscr{S})$, i.e.*

$$\begin{array}{lcccccc} \textit{symbol} & f: & \mathscr{S}'_2 & \supset & L_2 & \supset & \mathscr{S}_2 \\ & & \updownarrow & & \updownarrow & & \updownarrow \\ \textit{operator} & W(f): & \mathscr{L}(\mathscr{S}, \mathscr{S}') & \supset & \mathfrak{S}_2 & \supset & \mathfrak{S}_1(\mathscr{S}). \end{array} \tag{4.2}$$

From the classical Banach space only the symbols $f \in L_2$ are in correspondence to a well-known class of operators, namely to the Hilbert-Schmidt-operators $W(f) \in \mathfrak{S}_2$ [32], and for $f, g \in \mathfrak{S}_2$

$$tr\ W(f)\,W(g) = \frac{1}{(2\pi)^d} \int f(q,p) g(q,p)\,dq\,dp.$$

If $f \in \mathscr{S}_2$, then $W(f) \in \mathfrak{S}_1(\mathscr{S})$, but the symbols of all nuclear operators do not form a classical Banach space. In [6] it has been shown that there are nonsummable functions leading to nuclear operators but also bounded functions corresponding to unbounded operators (see also [9, 10, 15]).

If $f \in \mathscr{S}'_2$ and $g \in \mathscr{S}_2$, then the twisted products (1.2) $f \circ g$, $g \circ f$ are well-defined (in the sense of distribution) and elements of $\mathscr{S}'_2$. Furthermore, $f \circ g \in \mathscr{S}_2$ if $f, g \in \mathscr{S}_2$. All these multiplications are (separately) continuous in the corresponding topologies. More precisely, we have the following theorem [26, 27].

Theorem 4.2.

i) *$(\mathscr{S}'_2, \mathscr{S}_2)$ is a Qu*-algebra with respect to the twisted product (1.2) and the involution $f \to f^+ = \bar{f}$. It is also called the Qu*-algebra of symbols.*

ii) *The Weyl quantization $f \to W(f)$ is an isomorphism of the Qu*-*

algebra $(\mathscr{S}'_2, \mathscr{S}_2)$ *of symbols onto the* Qu^**-algebra* $(\mathscr{L}(\mathscr{S}, \mathscr{S}')[\tau], \mathfrak{S}_1(\mathscr{S})[\beta^*])$.

$(\mathscr{L}(\mathscr{S}, \mathscr{S}'), \mathfrak{S}_1(\mathscr{S}))$ is a Qu^*-algebra of operators but not yet a CQ^*-algebra since $\mathfrak{S}_1(\mathscr{S}) \subset \mathscr{L}^+(\mathscr{S})$. Its smallest CQ^*-extension on $\mathscr{S}$ is the CQ^*-algebra $(\mathscr{L}(\mathscr{S}, \mathscr{S}'), \mathscr{L}^+(\mathscr{S}))$, i. e. the maximal one on $\mathscr{S}$. Since $f \Leftrightarrow W(f)$ is an isomorphism we can define an extension $(\mathscr{S}'_2, \mathscr{S}^+)$ of the Qu^*-algebra $(\mathscr{S}'_2, \mathscr{S}_2)$ by

$$\mathscr{S}^+ \in f \leftrightarrow W(f) \in \mathscr{L}^+(\mathscr{S}). \tag{4.3}$$

In this way the twisted product $f \circ g$ of two elements f, $g \in \mathscr{S}^+$ is defined by $W(f \circ g) = W(f)W(g)$, where on the right-hand side we have the multiplication in the Op^*-algebra $\mathscr{L}^+(\mathscr{S})$. Thus the integral (1.2) is (formally) extended to a certain class of distributions, but we have not an explicit characterization of the symbols $f \in \mathscr{S}^+$. But an important *-subalgebra $S \subset \mathscr{S}^+$ is well-known from the theory of pseudodifferential operators [5, 12], studied in detail in [41], where they are called GLS-symbols (see [12]).

Definition 4.3. We use the abbreviation $x = (q, p) \in R^{2d}$, $x^2 = q^2 + p^2$, $\partial^\alpha f = \partial^{\alpha_1}_{q_1} \dots \partial^{\alpha_{2n}}_{p_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_{2d}$. A function $f \in C^\infty(R^{2d})$ is called GLS-symbol of order $\leqq m$, arbitrary real number, if for every $k \geqq \sigma$

$$p_{m,k}(f) = \sup_{x, |\alpha| \leq k} |\partial^\alpha f(x)| (1 + x^2)^{-(m-|\alpha|)/2} < \infty. \tag{4.4}$$

The space of symbols of order $\leq m$ we denote by S_m. It is a Frechet space with respect to the seminorms $p_{m,k}(\cdot)$, $k = 0, 1, 2, \dots$. Furthermore

$$S = \bigcup_m S_m \tag{4.5}$$

and we equip with the locally convex topology ξ_0 defined by $S = \lim_m \operatorname{ind} S_m$. The following facts are proved in [41, Theorems 2.3.1, 2.4.1 and Proposition 2.7.3].

Lemma 4.4.

i) *The Weyl quantization* $f \to W(f)$ *maps* S *into* $\mathscr{L}^+(\mathscr{S})$. *We denote* $\hat{S} = W(S)$ *and* $\hat{S}_m = W(S_m)$.

ii) $\hat{S}$ *is an Op*-algebra. More precisely,* $A\cdot B\in\hat{S}_{n+m}$ *for* $A\in\hat{S}_n$, $B\in\hat{S}_m$. *Therefore S is a *-algebra with respect to the twisted multiplication, and* $f\circ g\in S_{n+m}$ *for* $f\in S_n$, $g\in S_m$.

iii) *If* $A\in\hat{S}_m$ *and* $n\geq\frac{m}{2}$, *then A is a continuous map from* $\mathscr{H}_k$ *to* $\mathscr{H}_{k-n}$ *for all k. Therefore, A is an operator of order* $\leq\frac{m}{2}$ *for the Hilbert scale* $\{\mathscr{H}_S,\ -\infty\leq S\leq+\infty\}$ (4.1).

S is a subspace of the space $O_M=O_M(R^{2d})$ ([35, II, 5]) of multiplications for the distributions. O_M contains all functions $f\in C^\infty(R^{2d})$, such that for every α there exists a k with $\sup_x|\partial^\alpha f(x)|(1+x^2)^{-k}<\infty$.

Let us describe the structure of O_M in more detail: For two integers k, m we define

$$\|f\|_{m,k}=\sup_{x,|\alpha|\leq m}|\partial^\alpha f|(1+x^2)^{-k}<\infty. \tag{4.6}$$

Let O_k^m be the Banach space with the norm $\|\ \|_{m,k}$. From these spaces one gets O_M in the following way:

$$O_M=\bigcap_{m=1}^\infty O^m,\ O^m=\bigcup_{k=1}^\infty O_k^m. \tag{4.7}$$

We equip O_M with the natural locally convex topology ξ given by $O_M=\lim_m \operatorname{proj} O^m$, $O^m=\lim_k \operatorname{ind} O_k^m$.

Theorem 4.5.

i) $(O_M[\xi],S)$ *is a Qu*-algebra with respect to the twisted multiplication, i.e.* $f\circ g$, $g\circ f\in O_M$ *for* $f\in O_M$, $g\in S$ *and the multiplications are continuous.*

ii) *For every pair m, n of integers there exists an integer r such that*

$$O^r\circ S_n,\ \ S_n\circ O^r\subset O^m. \tag{4.8}$$

iii) *Let* $n, m, k\geq 0$ *be three given integers. Put* $r=m+2n+2d$ *and* $l=k+n$. *Then*

$$O_k^r\circ S_n,\ \ S_n\circ O_k^r\subset O_l^m. \tag{4.9}$$

Furthermore, the bilinear maps

$$O_k^r\times S_n\xrightarrow{f\circ g,\, g\circ f}O_l^m \tag{4.10}$$

$f\in O_k^r$, $g\in S_n$ *are continuous, i.e. there exists a seminorm* $P_{n,S}(\cdot)$ (4.4) *such that*

$$\|f\circ g\|_{m,l},\ \|g\circ f\|_{m,l}\leq c\|f\|_{r,k}P_{n,s}(g), \tag{4.11}$$

c is a constant. One can choose $s=m+2k+2d$.

Proof: We shall prove the estimation (4.11). ii) follows from iii), since the k in iii) can be chosen independently of r. Furthermore, the integers m, n in ii) can be arbitrary, and therefore i) is a consequence of ii).

Now we prove for $f, g\in S_2$, $r=m+2n+2d$, $l=k+n$,

$$\|f\circ g\|_{m,l}\leq c\|f\|_{r,k}P_{n,s}(g), \tag{4.12}$$

with a certain s. The second estimation of (4.11) can be proved in the same way. Since S_2 is dense in all spaces in (4.9), iii) is completely shown.

We use the following abbreviations: $\sigma(x_1, x_2)=p_2q_1-q_2p_1$,

$$\Delta_x=\sum_{i=1}^{d}(\partial_{q_i}^2+\partial_{p_i}^2),\ dx=\prod_i dq_i dp_i$$

$$(f\circ g)(x)=\frac{1}{\pi^{2d}}\int f(x+x_1)g(x+x_2)e^{i\sigma(x_1,x_2)}dx_1dx_2. \tag{4.13}$$

Now we have to estimate $\partial_x^{\gamma}(f\circ g)$ for $|\gamma|\leq m$. If we carry out the differentiation in (4.13), we get in the integral terms of the form $\partial^{\alpha}f(x+x_1)\partial^{\beta}g(x+x_2)$, $|\alpha|$, $|\beta|\leq m$. Now we use the relation

$$(1+x_1^2)^{-t}\left(1-\frac{1}{4}\Delta_{x_2}\right)^t e^{2i\sigma(x_1,x_2)}=e^{2i\sigma(x_1,x_2)}$$

$$(1+x_2^2)^{-t}\left(1-\frac{1}{4}\Delta_{x_1}\right)^t e^{2i\sigma(x_1,x_2)}=e^{2i\sigma(x_1,x_2)} \tag{4.14}$$

where t is an integer. Then we get

$$\begin{aligned}&\int\partial^{\alpha}f(x+x_1)\partial^{\beta}g(x+x_2)e^{2i\sigma(x_1,x_2)}dx_1dx_2\\&=\int\frac{\left(1-\frac{1}{4}\Delta_{x_1}\right)^{n+d}\partial^{\alpha}f(x+x_1)}{(1+x_1^2)^{k+d}}\cdot\left(1-\frac{1}{4}\Delta_{x_2}\right)^{k+d}\\&\qquad\frac{\partial^{\beta}g(x+x_2)}{(1+x_2^2)^{n+d}}e^{2i\sigma(x_1,x_2)}dx_1dx_2.\end{aligned} \tag{4.15}$$

We estimate the first factor by using $(1+(x+x_1)^2)^k\leq 2^k(1+x^2)^k(1+x_1^2)^k$ and get

$$\left|\frac{\left(1-\frac{1}{4}\Delta_{x_1}\right)^{n+d}\partial^\alpha f(x+x_1)}{(1+x_1^2)^{k+d}}\right| \leq \text{const. } \|f\|_{r,k}\frac{(1+x^2)^k}{(1+x_1^2)^d}. \tag{4.16}$$

The second factor we have to estimate by $P_{n,s}(g)$. We get

$$\left|\left(1-\frac{1}{4}\Delta_{x_2}\right)^{k+d}\frac{\partial^\beta g(x+x_2)}{(1+x_2^2)^{n+d}}\right| \leq \text{const. } \|g\|_{s,n}\frac{(1+x^2)^n}{(1+x_2^2)^d} \tag{4.17}$$

where $s=m+2k+2d$. Since the integral $\int(1+x_1^2)^{-d}(1+x_2^2)^{-d}dx_1dx_2$ is finite, we get from (4.15)-(4.17) the estimation

$$\|f\circ g\|_{m,n+k} \leq \text{const. } \|f\|_{m+2n+2d,k}\|g\|_{m+2k+2d,n}. \tag{4.18}$$

But since $\|g\|_{s,n}\leq p_{n,s}(g)$ we have proved $\|f\circ g\|_{m,l}\leq$const. $\|f\|_{r,k}p_{n,s}(g)$. Therefore, the proof of the theorem is complete.

Let us discuss the estimations above a little more. (4.12) means that the multiplication f, $g\to f\circ g$ is continuous from $O_k^{m+2n+2d}\times O_n^{m+2k+2d}$ into O_{n+k}^m, i.e.

$$O_k^{m+2n+2d}\circ O_n^{m+2k+2d}\subset O_{n+k}^m. \tag{4.19}$$

But from this last inclusion we cannot conclude $O_M\circ O_M\subset O_M$, as one could suppose. Let f, g be two elements of O_M, then for arbitrary m, n we can indeed choose k so large that $f\in O_k^{m+2n+2d}$, but then it is not clear that $g\in O_n^{m+2k+2d}$. If one takes n once more larger, then k has to be larger, and so on.

The indices in (4.19) mutually influence each other in such a way that one cannot conclude $O_M\circ O_M\subset O_M$. But this was stated in [31, Lemma 3.18, i)]. One cannot absolutely exclude such an extension of the twisted product that O_M becomes an algebra, but this is impossible in the sense of distributions and the proof in [31] is incorrect. This can be seen by the following counterexample.

Example 4.6. $f(q)=e^{iq^2}, g(p)=e^{ip^2}$, $(q,p)\in R^2$, are elements of O_M, but $(f\circ g)(q,p)=\sqrt{\pi/2}\,(1+i)e^{ip^2}\delta(p-q)\notin O_M$ (in the sense of distributions).

In fact, by using the relations $\int e^{-2ixt}\,dx\,dt=\pi$, $\int e^{ix^2}dx=\sqrt{\pi/2}(1+i)=c$, $\int f(n)e^{2int}dn=ce^{-t^2}$ we get

$$(f\circ g)(q,p)=\frac{1}{\pi}\int e^{i(q+q_1)^2}e^{i(p+p_2)^2}e^{2ip_2q_1}dq_1dp_2$$
$$=\frac{1}{\pi}\int e^{i(q+p_2+q_1)^2}e^{-p_2^2i}-e^{-2ip_2q}e^{i(p+p_2)^2}dq_1dp_2$$
$$=\frac{c}{\pi}e^{ip^2}\delta(p-q).$$

All topological spaces S_n, S, O_k^n, O^n, O_M of Theorem 4.5 contain S_2 as a dense subspace. Therefore, this spaces are admissible spaces in the sense of [27, Definition 3.4], i.e., the twisted product f, $g\to f\circ g$ as a bilinear map of (S_2, S_2) in S_2' can be extended by continuity to the following pairs of spaces

$$\begin{aligned}&(S_n, O_k^p),\ (O_k^p, S_n)\\&(S_n, O^p),\ (O^p, S_n)\\&(S, O_M),\ (O_M, S)\\&(O_k^{s+2n}, O_n^{s+2k}),\ \text{etc.}\end{aligned}\tag{4.20}$$

We conclude the paper with a remark. In [10] it has been introduced the set $\mathscr{M}\subset S_2'$ of such distributions f, for which the twisted products $f\circ g$, $g\circ f\in S_2$ for every $g\in S_2$. Then it was proved ([10, Proposition 7.8]) that $W(\mathscr{M})=\mathscr{L}(\mathscr{D})\cap\mathscr{L}(\mathscr{D}')=\mathscr{L}^+(\mathscr{D})$, i.e. $\mathscr{M}=S^+$ (see (4.3)). This statement is a consequence of Theorem 4.2, ii), and Lemma 3.8.

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