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Kuranishi Family of Vector Bundles and Algebraic Description of the Moduli Space of Einstein-Hermitian Connections

By

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Introduction

There are two different ways of defining complex structures of the moduli space of irreducible Einstein-Hermitian connections (cf. [Don 1], [Don2], [U-Y]); i.e. a differential geometric way (cf. [I], [Ko], [L-O]) and an algebro-geometric way (cf. [Ma]). It has been unclear whether these two complex structures are isomorphic when they are non-reduced i.e. their structure sheaves have nilpotent elements. A main reason for this is that the deformation theory of Kuranishi type for vector bundles (e.g. [Ak]) has not been fully generalized so that we can hardly deal with non-reduced structures in differential geometric arguments.

Main purposes of this paper are to give a complete generalisation of the deformation theory of Kuranishi type for vector bundles and to prove that the above two complex structures are isomorphic to each other.

In §§1 and 2, we will give a generalisation of the local deformation theory of Kuranishi type for vector bundles. In §1, we will show the existence of semi-universal local family of holomorphic structures (Theorem 1) using Banach analytic space argument in [Dou]. By [Ko], Ch. VII or [L-O] together with the arguments of §1, the moduli space of simple holomorphic structures will be a (non-

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reduced) complex space. In §2, by a power series argument, we will show that the semi-universal local family of holomorphic structures induces that of holomorphic vector bundles (Theorem 2). In §3, we will prove that, over a projective algebraic manifold, the analytic moduli space of simple holomorphic structures (cf. [Ko], [L-O]) and the algebraic one of simple vector bundles (cf. [A-K]) are isomorphic to each other as not necessarily reduced complex spaces (Theorem 3). By Theorem 3 together with the theorems of S. Donaldson, K. Uhlenbeck and S. T. Yau, we will obtain the isomorphism between the two complex structures of the moduli space of irreducible Einstein-Hermitian connections. Finally in §4, we will prove along [F-S] that the forgetful map from the moduli space of irreducible Einstein-Hermitian connections into that of simple holomorphic structures is an open embedding as not necessarily reduced real analytic spaces (Proposition 4.2). This implies that the two complex structures considered above are natural.

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§1. Kuranishi Family of Holomorphic Structures

In this section, we will prove the existence of semi-universal family of holomorphic structures, using the Banach analytic space argument in [Dou]. The following notion of a Banach analytic space is due to A. Douady (cf. [Dou]).

Let E and F be Banach spaces over $\boldsymbol{C}$ (resp. $\boldsymbol{R}$) and $\theta : E \supset U \to F$ be a $\boldsymbol{C}$- (resp. $\boldsymbol{R}$-) analytic map from an open neighbourhood U of 0 in E into F with $\theta(0)=0$. $\boldsymbol{C}$- (resp. $\boldsymbol{R}$-) analytic functions h on an open neighbourhood in U with the form $h(u)=\langle\theta(u), f(u)\rangle$, where f is a $\boldsymbol{C}$- (resp. $\boldsymbol{R}$-) analytic map from the same neighbourhood

into a Banach space $L(F, \mathbb{C})$ (resp. $L(F, \mathbb{R})$), the space of bounded linear forms on F, forms an ideal of the sheaf $\mathcal{O}_U$. We consider a ringed space $(\theta^{-1}(0), \mathcal{O}_U/\mathcal{I}_\theta)$ a local model of a Banach analytic space.

Definition 1.1. A Banach analytic space is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to a local model of the above type.

Remark. If E is, in particular, finite dimensional, then a Banach analytic space is an analytic space in usual sense, because $\mathcal{O}_U$ is a sheaf of noetherian rings.

A ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to local models of the above type with $\theta=0$ is called a Banach analytic manifold. A ringed subspace $(Y, \mathcal{O}_Y)$ of a Banach analytic manifold X is called *a direct submanifold* if, for each point $y\in Y$, there exists a local chart φ: $U\to W\subset E$ of X with a direct sum decomposition $E=F_1+F_2$ by closed Banach subspaces such that $\varphi(Y\cap U)=W\cap(F_1+0)$ and $\mathcal{O}_{Y|Y\cap U}\simeq \mathcal{O}_W/\mathcal{I}_{p_2}$, where p_2: $E=F_1+F_2\to F_2$ is the projection operator. A direct submanifold $(Y, \mathcal{O}_Y)$ with local models as above induces a Banach manifold $(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ with local charts $(Y\cap U, \varphi_{|Y\cap U})$.

Proposition 1.1. *$(Y, \mathcal{O}_Y)\simeq(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ as ringed spaces.*

Proof. We infer from the following lemma that $\mathrm{Ker}(\mathcal{O}_E\to \mathcal{O}_{F_1})\simeq\mathcal{I}_{p_2}$ Q. E. D

Lemma 1.2. *Let E and G be Banach spaces with a direct sum decomposition $E=F_1+F_2$. If a local analytic mapping h from E into G vanishes identically on F_2, then there exists a local analytic mapping f from E into $L(F_2, G)$ such that $h(t, s)=\langle f(t, s), s\rangle$.*

Proof. Let $h=\sum_{n\geq 0}h_n$ be an expansion of h into a sum of polynomial maps and u_n be a symmetric n-form on $F_1\times F_2$ with values in G associated with h_n. Let a_1, a_2 be non-negative integers with

$a_1+a_2=n$ and u_{a_1,a_2} be a (a_1, a_2)-form induced from u_n. Then $h_n(x) = \sum_{a=0}^{n} {}_nC_a\ u_{a,n-a}(x, \ldots, x)$. Since $h(t, 0)=0$, we have $u_{n,0}=0$. If $a<n$, let $\bar{u}_{a,n-a}$ be a $(a, n-a-1)$-form with values in $L(F_2, G)$ induced from $u_{a,n-a}$. Because $||\bar{u}_{a,n-a}||=||u_{a,n-a}||$, $\sum_{n\geq 0}\sum_{a=0}^{n-1} {}_nC_a||\bar{u}_{a,n-a}||r^{n-1}$ is convergent for some $r>0$. Hence we have a local analytic map f: $F_1\times F_2\to L(F_2, G)$ by $f=\sum_{n\geq 0} f_n$, where $f_n(x)=\sum_{a=0}^{n-1} {}_nC_a\bar{u}_{a,n-a}(x, \ldots, x)$. It is clear that $h(t, s)=\langle f(t, s), s\rangle$ by the construction of f.

Q. E. D.

Throughout this paper except §3, by "*analytic*" we mean "*$\mathcal{C}$-analytic*".

Let E be a differentiable complex vector bundle over a compact complex manifold X.

Definition 1.2. A semi-connection on E is a $\mathcal{C}$-linear map $\bar{\partial}_{A\cdot}$: $A^{0,0}(E)\to A^{0,1}(E)$ satisfying the Leibnitz-rule $\bar{\partial}_{A\cdot}(f\circ s)=(\bar{\partial}f)\otimes s+f\cdot(\bar{\partial}_{A\cdot}s)$ for $f\in C^\infty(X)$ and $s\in A^{0,0}(E)$.

A semi-connection induces a first order differential operator $\bar{\partial}_{A\cdot}$: $A^{p,q}(E)\to A^{p,q+1}(E)$.

Definition 1.3. A semi-connection $\bar{\partial}_{A\cdot}$ is called a holomorphic structure if it satisfies $\bar{\partial}_{A\cdot}\circ\bar{\partial}_{A\cdot}=0$.

Definition 1.4. A semiconnection $\bar{\partial}_{A_1^\cdot}$ is isomorphic to $\bar{\partial}_{A_2^\cdot}$ if there exists a $g\in GL(E)$ such that $g\cdot\bar{\partial}_{\ 2}^{\ \cdot}=\bar{\partial}_{A_1^\cdot}\cdot g$ holds.

By $\bar{\mathcal{D}}(E)$ we denote the set of all semi-connections on E. The following proposition is well known (cf. [Ko] Ch. VII).

Proposition 1.3. (1) *For any $\bar{\partial}_{A\cdot}\in\bar{\mathcal{D}}(E)$, $\bar{\mathcal{D}}(E)=\bar{\partial}_{A\cdot}+A^{0,1}(End\ E)$.*

(2) *Let $\bar{\partial}_{A\cdot}$ be a holomorphic structure. Then $(\bar{\partial}_{A\cdot}+\alpha)\circ(\bar{\partial}_{A\cdot}+\alpha)=0$ if and only if $P_{A\cdot}(\alpha)=\bar{D}_{A\cdot}\alpha+\alpha\wedge\alpha=0$, where $\bar{D}_{A\cdot}\alpha=[\bar{\partial}_{A\cdot}, \alpha]$.*

(3) *Let $g\in GL(E)$. Then $(\bar{\partial}_{A\cdot}+\alpha_2)\cdot g=g\cdot(\bar{\partial}_{A\cdot}+\alpha_1)$ if and only if $g\cdot\alpha_1-\alpha_2\cdot g-\bar{D}_{A\cdot}g=0$.*

Let E be a holomorphic vector bundle over a compact complex manifold X and $\bar{\partial}$: $A^{0,0}(E)\to A^{0,1}(E)$ be the canonical holomorphic structure.

We fix an integer $k>\dim_R X+1$ and denote the Sobolev completion of order k by a subscript k.

Let P be an analytic map $A^{0,1}(End\ E)_k\to A^{0,2}(End\ E)_{k-1}$ defined by $P(\alpha)=\bar{D}\alpha+\alpha\wedge\alpha$ where $\bar{D}\alpha=[\bar{\partial},\alpha]$.

Definition 1.5. By a local family of deformations of $\bar{\partial}$, we mean a pair $(\omega,\ (T,o))$ of a germ of an analytic space (T,o) and an analytic map ω from (T,o) into a germ of a Banach analytic space $(P^{-1}(0),\ 0)$, which is of class C^∞ on $X\times D$ where D is an ambiant space of T.

Definition 1.6. Two local families $(\omega,\ (T,o))$ and $(\omega',\ (T,o))$ are equivalent if there exists an analytic map g: $(T,o)\to(GL(E)_{k+1}, id_E)$ such that it is of class C^∞ and the map $t\to g(t)\omega(t)-\omega'(t)g(t)-\bar{D}g(t)$ is a 0-map from (T,o) into $A^{0,1}(End\ E)_k$.

For a local family $(\omega,\ (T,o))$ and a holomorphic map σ: $(S,o)\to(T,o)$, we denote by $(\sigma^*\omega,\ (S,o))$ the family given by $\omega\circ\sigma$: $(S,o)\to(P^{-1}(0),0)$.

Definition 1.7. A local family $(\omega,\ (T,o))$ is complete at o if for any local family $(\omega',\ (S,o))$ there exists a holomorphic map σ: $(S,o)\to(T,o)$ such that $(\sigma^*\omega,\ (S,o))$ is equivalent to $(\omega',\ (S,o))$.

Let $(\omega,\ (T,o))$ be a local family of deformations of $\bar{\partial}$. For $v\in T_oT$ we have $\bar{D}(d\omega(v))=0$ because $\langle\bar{D}(d\omega(v)),f\rangle=v(\langle P(\omega),f\rangle)=0$ for any $f\in L(A^{0,2}(End\ E)_{k-1},\ \boldsymbol{C})$. Thus we define *the infinitesimal deformation map* ρ: $T_oT\to H^{0,1}(End\ E)$ by $\rho(v)=[d\omega(v)]$.

Theorem 1. *There exists a local family* $(\alpha,\ (T,o))$ *such that*
(1) *the infinitesimal deformation map* ρ: $T_oT\to H^{0,1}(End\ E)$ *is bijective, and*
(2) *it is complete at any point of* T.

Proof. Let $Q=\{\alpha\in A^{0,1}(End\ E)_k\mid\ ||\alpha||_k<\varepsilon,\ \bar{D}^*\alpha=0,\ \bar{D}^*(\bar{D}\alpha+\alpha\wedge\alpha)$

$=0\}$. Then $Q\subset A^{0,1}(End\,E)$ because α satisfies an elliptic partial differential equation $\square\alpha+\bar{D}^*(\alpha\wedge\alpha)=0$.

Lemma 1.4. *Q is a direct submanifold of a neighbourhood of 0 in $A^{0,1}(End\ E)_k$ for sufficiently small $\varepsilon>0$, whose local chart is $(Q, p_{1|Q})$, where by p_1 we denote the projection onto the harmonic space $H^{0,1}(End\,E)$. In particular, Q is finite dimensional.*

Proof. Let $q: A^{0,1}(End\,E)_k\to H^{0,1}(End\,E)+\bar{D}^*A^{0,1}(End\ E)_k+\bar{D}^*A^{0,2}(End\,E)_{k-1}$ be an analytic map defined by $q(\alpha)=(p_1\alpha, \bar{D}^*\alpha, \bar{D}^*(\bar{D}\alpha+\alpha\wedge\alpha))$. Since $dq_{(0)}(v)=(p_1v, \bar{D}^*v, \bar{D}^*\bar{D}v)$, $dq_{(0)}$ is a topological linear isomorphism. Indeed, $dq_{(0)}^{-1}(u_0, u_1, u_2)=u_0+G\bar{D}u_1+Gu_2$. Hence, by the inverse mapping theorem in Banach analytic manifolds, q is a local analytic isomorphism at 0. Since $Q=q^{-1}(H^{0,1}(End\ E)\times 0\times 0)$ locally around 0, Q is a direct submanifold for a sufficiently small $\varepsilon>0$ and p_1: $Q\to H^{0,1}(End\,E)$ gives its local chart. Q. E. D.

Remark. On Q, p_1 coincides with the Kuranishi map $K(\alpha)=\alpha-\bar{D}^*G(\alpha\wedge\alpha)$.

Let $P: A^{0,1}(End\,E)_k\to A^{0,2}(End\,E)_{k-1}$ be a local analytic map defined by the integrability condition $P(\alpha)=\bar{D}\alpha+\alpha\wedge\alpha$. Then we have a local family $(\alpha, (T, o))$ of deformations of $\bar{\partial}$ parametrized by a germ of an analytic space $T=P_{|Q}^{-1}(0)$.

Remark. T is isomorphic to a Banach analytic space $\{\alpha\in A^{0,1}(End\ E)_k \mid \|\alpha\|_k<\varepsilon,\ \bar{D}^*\alpha=0,\ \bar{D}\alpha+\alpha\wedge\alpha=0\}$.

Since Q is finite dimensional, $\mathcal{I}_P$ is finitely generated. We will find a canonical generator of $\mathcal{I}_P$.

Let $F=\{(\alpha,\theta)\in Q\times A^{0,2}(End\ E)_{k-1} \mid \|\alpha\|_k<\varepsilon,\ \|\theta\|_{k-1}<\varepsilon,\ \bar{D}^*\theta=0,\ \bar{D}^*(\bar{D}\theta-2\theta\wedge\alpha)=0\}$.

Lemma 1.5. *F is a finite dimensional direct submanifold of a neighbourhood of $(0,0)$ in $Q\times A^{0,2}(End\ E)_{k-1}$ for sufficiently small $\varepsilon>0$, whose local chart is $(F, (p_1\times p_2)_{|F})$ where p_i is the projection onto the harmonic space $H^{0,i}(End\ E)$ for $i=1, 2$.*

Proof. Let $f\colon Q\times A^{0,2}(End\,E)_{k-1}\to Q\times A^{0,2}(End\,E)_{k-3}$ be an analytic map given by $f(\alpha,\theta)=(\alpha, p_2\theta+\bar{D}\bar{D}^*\theta+\bar{D}^*(\bar{D}\theta-2\theta\wedge\alpha))$. Since $df_{(0,0)}(v_1,v_2)=(v_1, p_2v_2+\bar{D}\bar{D}^*v_2+\bar{D}^*\bar{D}v_2)$, $df_{(0,0)}$ is a topologicall linear isomorphism. Indeed $df^{-1}_{(0,0)}(u_1,u_2)=(u_1, p_2u_2+Gu_2)$. Hence, by the inverse mapping theorem in Banach manifolds, f is a local isomorphism at $(0,0)$. Since $F=f^{-1}(Q\times H^{0,2}(End\,E))$ locally around $(0,0)$, F is a direct submanifold of a neighbourhood of $(0,0)$ in $Q\times A^{0,2}(End\,E)_{k-1}$. Since $f_{|F}$ induces a local isomorphism from F onto a neighbourhood of $(0,0)$ in $Q\times H^{0,2}(End\,E)_{k-1}$, $(p_1\times p_2)_{|F}\colon F\to H^{0,1}(End\,E)\times H^{0,2}(End\,E)$ is a local chart of F. Q. E. D.

Let θ' be the expression of an analytic map $\theta=pr_{2|F}\colon F\to A^{0,2}(End\,E)_{k-1}$ with respect to the local chart $(F, (p_1\times p_2)_{|F})$, where pr_2 is the projection onto the second factor.

Lemma 1.6. $\theta'(t,0)=0$.

Proof. Let $(\alpha,\theta)\in F$ and $p_2\theta=0$. Then $\theta=G\bar{D}^*\bar{D}\theta=2G\bar{D}^*(\theta\wedge\alpha)$. Since $||\theta||_{k-1}\leq c_1||\theta\wedge\alpha||_{k-2}\leq c_2||\theta||_{k-1}||\alpha||_k$ for some constants c_1 and c_2, we have $\theta=0$ if $||\alpha||_k<1/c_2$. Q. E. D.

Lemma 1.7. $\mathscr{I}_P=\mathscr{I}_{p_2\circ P}$.

Proof. It is clear that $\mathscr{I}_{p_2\circ P}\subset\mathscr{I}_P$.

Let λ be a local analytic map from Q into F defined by $\lambda(\alpha)=(\alpha, \bar{D}\alpha+\alpha\wedge\alpha)$. If λ' is the expression of λ with respect to the local charts $(Q, p_{1|Q})$ and $(F, (p_1\times p_2)_{|F})$, then $\lambda'(t)=(t, p_2\circ P(\alpha(t)))$. Since $P(\alpha)=\theta\circ\lambda(\alpha)$, we infer from Lemma 1.2 that the expression P' of P with respect to the chart $(Q, p_{1|F})$ has the form $P'(t)=\theta'(t, p_2\circ P(\alpha(t)))=\langle\xi'(t, p_2\circ P(\alpha(t))), p_2\circ P(\alpha(t))\rangle$. Therefore we have $\mathscr{I}_P\subset\mathscr{I}_{p_2\circ P}$. Q. E. D.

By Lemma 1.7, $h_1,\ldots,h_r$ is a system of generators of $\mathscr{I}_P$ if $p_2\circ P(\alpha(t))=h_1(t)e_1+\ldots+h_r(t)e_r$ where $e_1,\ldots,e_r$ is a base of $H^{0,2}(End\,E)$.

Since $p_2\circ P(\alpha(t))$ has no linear terms, we have $T_oT\simeq H^{0,1}(End\,E)$.

Thus we proved (1) of Theorem 1.

Proposition 1.8. *The family* $(\alpha, (T, o))$ *is complete at any point of* T *sufficiently close to* o.

Proof. We may assume that T is so small that the map $\bar{D}^*A^{0,1}(End\ E)_{k+2} \ni v \to \bar{D}^*\bar{D}_{\alpha(t_1)}v \in \bar{D}^*A^{0,1}(End\ E)_k$ is topologicall isomorphism for any $t_1 \in T$, where $\bar{D}_\alpha v = \bar{D}v + [\alpha, v]$.

Let $(\omega', (S, o))$ be any local family of deformations of $\bar{\partial}_{\alpha(t_1)}$, i.e. ω' is a local analytic map $(S, o) \to (P^{-1}(0), \alpha(t_1))$, where S is an analytic subspace of $D \subset \mathbb{C}^m$.

Let F: $D \times \bar{D}^*A^{0,1}(End\ E)_{k+2} \to \bar{D}^*A^{0,1}(End\ E)_k$ be a local analytic map given by $F(s, g) = \bar{D}^*(e(g)^{-1} \circ \omega'(s) \circ e(g) + e(g)^{-1} \circ \bar{D}(e(g)))$ where we denote $id_E + g$ by $e(g)$. Since $dF_{g,(0,0)}(v) = \bar{D}^*\bar{D}_{\alpha(t_1)}v$, $dF_{g,(0,0)}$ is a topological linear isomorphism between Banach spaces. Hence, by the implicit mapping theorem in Banach manifolds, we have an analytic map $g : D \to \bar{D}^*A^{0,1}(End\ E)_{k+2}$ such that $g(0) = 0$ and $\bar{D}^*(e(g)^{-1} \circ \omega'(s) \circ e(g) + e(g)^{-1} \circ \bar{D}(e(g))) = 0$. $g(s)$ is of class C^∞ on $X \times D$ because g satisfies an elliptic partial differential equation $\frac{\partial^2 g}{\partial s \partial \bar{s}} + \bar{D}^*\bar{D}g + \bar{D}^*R(\omega'(s), g) + \bar{D}^*\omega'(s) = 0$, where $R(u\omega', ug) = u^2R'(\omega', g, u)$ for a real parameter u, with $R'(\omega', g, u)$ depending differentiably on ω', g and u for small u. If we set $\omega(s) = e(g(s))^{-1} \circ \omega'(s) \circ e(g(s)) + e(g(s))^{-1} \circ \bar{D}(e(g(s)))$, then $P(\omega(s)) = e(g(s))^{-1} \circ P(\omega'(s)) \circ e(g(s))$. Hence we have a local family $(\omega, (S, o))$ equivalent to $(\omega', (S, o))$. Because $\bar{D}^*\omega(s) = 0$, ω is a local analytic map from $(D, 0)$ into $(Q, \alpha(t_1))$ which maps (S, o) into $(T, \alpha(t_1))$. Thus we have a holomorphic map $\sigma : (S, o) \to (T, \alpha(t_1))$ such that $(\sigma^*\alpha, (S, o)) \sim (\omega', (S, o))$.

Q. E. D.

This completes the proof of Theorem 1.

§2. Equivalence of Deformations of Holomorphic Structures and Those of Holomorphic Vector Bundles

In this section, we will show that the family obtained in Theorem 1

induces the semi-universal family of vector bundles (cf. [F-K]). In order to compare local families of holomorphic structures and those of holomorphic vector bundles, we discuss at first another description of local families of holomorphic structures.

Let (T, o) be a germ of an analytic subspace of a neighbourhood of the origin in $\boldsymbol{C}^m$ and $\mathscr{I}_{T,o}$ be its ideal. Because analytic maps from $(\boldsymbol{C}^m, 0)$ into a Banach space F correspond bijectively to convergent power series in $(t_1, \ldots, t_m)$ with coefficients in F, we have the following description of Definitions 1.2 and 1.3. Where $(t_1, \ldots, t_m)$ denotes a coordinate function of $\boldsymbol{C}^m$ and we abbreviate it as simply (t) in the rest of this section.

Proposition 2.1. *Let $\omega(t)$ be in $A^{0,1}(End\, E)_k\{t_1, \ldots, t_m\}$ and ω be an analytic map from $(\boldsymbol{C}^m, 0)$ into $A^{0,1}(End\, E)_k$ corresponding to $\omega(t)$. Then $(\omega, (T, o))$ is a local family of deformations of $\bar{\partial}$ if and only if $\omega(t)$ satisfies the following conditions.*

(1) $\omega(0) = 0$,

(2) *$\omega(t)$ is of class C^∞ on $X \times D$ for some neighbourhood D of 0 in $\boldsymbol{C}^m$,*

(3) $P(\omega(t)) \in \mathscr{I}_{T,o} A^{0,2}(End\, E)_{k-1}\{t_1, \ldots, t_m\}$.

Proof. If $\omega(t)$ satisfies (1)$\sim$(3), then $\langle P(\omega(t)), f(t)\rangle \in \mathscr{I}_{T,o}$ for all $f(t) \in L(A^{0,2}(End\, E)_{k-1}, \boldsymbol{C})\{t_1, \ldots, t_m\}$. Hence ω induces an analytic map from (T, o) into $(P^{-1}(0), 0)$.

Conversely, let $(\omega, (T, o))$ be a local family of deformations of $\bar{\partial}$. We will show $P(\omega(t)) \in \mathscr{I}_{T,o}\, A^{0,2}(End\, E)_{k-1}\{t_1, \ldots, t_m\}$ by applying the Grauert division theorem (cf. [Gra] or [F-K]). Let $D = \{ord(f) \mid f \in \mathscr{I}_{T,o}\}$ and Λ be the reducing system of D. Then we have a system of generators $\{f_\lambda\}_{\lambda \in \Lambda}$ of $\mathscr{I}_{T,o}$ with the form $f_\lambda = t^\lambda + h_\lambda$ satisfying $ord(h_\lambda) > \lambda$. By the Grauert division theorem, there exists a unique $r = \sum_{\nu \notin D} r_\nu t^\nu$ in $A^{0,2}(End\, E)_{k-1}\{t_1, \ldots, t_m\}$ such that $P(\omega(t)) - r(t) \in \mathscr{I}_{T,o}\, A^{0,2}(End\, E)_{k-1}\{t_1, \ldots, t_m\}$. Since $\langle P(\omega(t)), f(t)\rangle \in \mathscr{I}_{T,o}$ for all $f(t) \in L(A^{0,2}(End\, E)_{k-1}, \boldsymbol{C})\{t_1, \ldots, t_m\}$, we have $r(t) = 0$.

Q. E. D.

Proposition 2.2. *Let $\omega(t)$, $\omega'(t)$ be in $A^{0,1}(End\, E)_k\{t_1, \ldots, t_m\}$ and satisfy the conditions in Proposition 2.1, and let $(\omega, (T, o))$, $(\omega', (T, o))$ be corresponding local families of deformations of $\bar{\partial}$. Then $(\omega, (T, o))$ is*

equivalent to $(\omega', (T, o))$ *if and only if there exists a* $g(t)$ *in* $GL(E)_{k+1}\{t_1, \ldots, t_m\}$ *satisfying the following conditions.*

(1) $g(0) = id_E$,

(2) g *is of class* C^∞ *on* $X \times D$ *for some neighbourhood* D *of* 0 *in* $\boldsymbol{C}^m$,

(3) $g(t)\omega(t) - \omega'(t)g(t) - \bar{D}g(t) \in \mathscr{I}_{T,o}A^{0,1}(End\, E)_k\{t_1, \ldots, t_m\}$.

This is proved by the same argument of the proof of Proposition 2.1.

Hence, in the rest of this section, we call $\omega(t) \in A^{0,1}(End\, E)_k\{t_1, \ldots, t_m\}$ satisfying (1)$\sim$(3) in Proposition 2.1 a local family of deformations of $\bar{\partial}$ and their equivalence is defined by the existence of $g(t) \in GL(E)_{k+1}\{t_1, \ldots, t_m\}$ satisfying (1)$\sim$(3) in Proposition 2.2.

Next we consider two local deformation functors given by $\mathscr{F}_H((T, o)) = \{\omega(t) \in A^{0,1}(End\, E)_k\{t_1, \ldots, t_m\}$ satisfying (1)$\sim$(3) in Proposition 2.1$\}/\sim$, where the equivalence is defined as Proposition 2.2, and $\mathscr{F}_V((T, o)) = \{\mathscr{E} \to X \times (T, o) \mid$ a vector bundle with i: $E \simeq \mathscr{E}_{|X\times o}\}/\sim$, where $\mathscr{E} \sim \mathscr{E}'$ if there exists a bundle isomorphism χ: $\mathscr{E} \to \mathscr{E}'$ with $\chi \cdot i = i'$.

Remark. When E is a simple vector bundle, the following local deformation functors are equivalent to $\mathscr{F}_H$ and $\mathscr{F}_V$ respectively: $\mathscr{F}'_H((T, o)) = \{\omega(t) \in A^{0,1}(End\, E)_k\{t_1, \ldots, t_m\} \mid \bar{\partial} + \omega(0) \sim \bar{\partial}$ and $\omega(t)$ satisfies (2) and (3) in Proposition 2.1$\}/\sim$, where $\omega(t) \sim \omega'(t)$ if there exists $g(t)$ satisfying (2) and (3) in Proposition 2.2. $\mathscr{F}'_V((T, o)) = \{\mathscr{E} \to X \times (T, o) \mid$ a vector bundle$\}/\sim$, where $\mathscr{E} \sim \mathscr{E}'$ if there exists a bundle isomorphism $\mathscr{E} \simeq \mathscr{E}'$.

Theorem 2. $\mathscr{F}_H$ *and* $\mathscr{F}_V$ *are isomorphic to each other.*

Proof. Both families correspond to each other via differentiable trivializations of families of holomorphic vector bundles, i.e. a differentiable isomorphism s: $E \times T \to \mathscr{E}$ with $s_{|E\times o} = i$.

Let $\mathscr{E} \to X \times (T, o)$ be a vector bundle and $\{e_{\alpha\beta}(t)\}$ be a system of its transition matrices with respect to a Stein covering $X = \cup_\alpha V_\alpha$. We may assume that $\{e_{\alpha\beta}(o)\}$ gives a system of transition matrices of E. A differentiable trivialization of $\mathscr{E}$ is represented by $\{s_\alpha(t)\}$ with

the following properties;

(2.1) $s_\alpha(t) \in A^{0,0}(\overline{V}_\alpha, GL(r, \boldsymbol{C}))_{k+1}\{t_1, \ldots, t_m\}$ and is of class C^∞ on $\overline{V}_\alpha \times D$ for some neighbourhood D of 0 in $\boldsymbol{C}^m$,

(2.2) $s_\alpha(t) e_{\alpha\beta}(t) - e_{\alpha\beta}(0) s_\beta(t) \in \mathscr{I}_{T,o}\, A^{0,0}(\overline{V}_\alpha \cap \overline{V}_\beta, GL(r, \boldsymbol{C}))_{k+1}\{t_1, \ldots, t_m\}$,

where $r = rank\ E$.

A local family $\omega(t)$ of deformations of $\bar{\partial}$ corresponding to the local family of vector bundles $\mathscr{E} \to X \times (T, o)$ is characterized by

(2.3) $\bar{\partial} s_\alpha(t) + \omega(t) s_\alpha(t) \in \mathscr{I}_{T,o}\, A^{0,1}(\overline{V}_\alpha, GL(r, \boldsymbol{C})_k\{t_1, \ldots, t_m\}$.

Proposition 2.3. (1) *Let $\mathscr{E}$ be a vector bundle over $X \times (T, o)$. Then there exists a differentiable trivialization which induces a local family of deformations of $\bar{\partial}$.*

(2) *Conversely, let $\omega(t)$ be a local family of deformations of $\bar{\partial}$. Then there exists a vector bundle over $X \times (T, o)$ and a differentiable trivialization which induces $\omega(t)$.*

(3) *Let $\omega(t)$ (resp. $\omega'(t)$) be a local family of deformations of $\bar{\partial}$ induced from a vector bundle $\mathscr{E}$ (resp. $\mathscr{E}'$) over $X \times (T, o)$ via a differentiable trivialization s: $\mathscr{E} \simeq E \times T$ (resp. s': $\mathscr{E}' \simeq E \times T$). Then $\mathscr{E} \sim \mathscr{E}'$ if and only if $\omega(t) \sim \omega'(t)$.*

Proof. (1) If we set $s_\alpha(t) = \sum_\gamma e_{\alpha\gamma}(0) \rho_\gamma e_{\gamma\alpha}(t)$, then $s_\alpha(t)$ satisfies (2.1), where $\{\rho_\gamma\}$ is a partition of unity subordinate to the covering $\{V_\gamma\}$. Since $s_\alpha(t) e_{\alpha\beta}(t) - e_{\alpha\beta}(0) s_\beta(t) = \sum_\gamma e_{\alpha\gamma}(0) \rho_\gamma (e_{\gamma\alpha}(t) e_{\alpha\beta}(t) - e_{\gamma\beta}(t))$ and $e_{\gamma\alpha}(t) e_{\alpha\beta}(t) - e_{\gamma\beta}(t) \in \mathscr{I}_{T,o}\, \Gamma(\overline{V}_\alpha \cap \overline{V}_\beta, \mathcal{O}_X)\{t_1, \ldots, t_m\}$, we infer that $s_\alpha(t)$ satisfies (2.2).

Let $\omega(t) = \{\sum_\gamma \rho_\gamma e_{\alpha\gamma}(0) (\bar{\partial} s_\gamma(t)) (s_\gamma(t))^{-1} e_{\gamma\alpha}(0)\}_\alpha$. Then $\omega(t)$ satisfies (1) and (2) in Proposition 2.1. We infer from (2.2) that $\omega(t)$ satisfies (2.3). We will show that $\omega(t)$ satisfies (3) in Proposition 2.1. Since $\omega(t)_{|V_\alpha} - (\bar{\partial} s_\alpha(t)) (s_\alpha(t))^{-1} \in \mathscr{I}_{T,o}\, A^{0,1}(\overline{V}_\alpha, End\ E))_k\{t_1, \ldots, t_m\}$, we have $P(\omega(t))_{|V_\alpha} \in \mathscr{I}_{T,o}\, A^{0,2}(\overline{V}_\alpha, End\ E)_{k-1}\{t_1, \ldots, t_m\}$. We infer from this that $P(\omega(t)) \in \mathscr{I}_{T,o} A^{0,2}(End\ E)_{k-1}\{t_1, \ldots, t_m\}$.

(2) (Due to H. Flenner.) It is enough to prove the assertion for a semi-universal family $(\alpha(t), (T, o))$ obtained in Theorem 1.

Let $(\mathscr{E}, (S, o))$ be a semi-universal family of vector bundles with $\mathscr{E}_{|X \times o} \simeq E$ (cf. [F-K]). By (1), we have an inducing morphism σ:

$(S, o) \to (T, o)$. It is enough to show that σ induces isomorphisms $(S_n, o) \simeq (T_n, o)$ between every infinitesimal neighbourhoods. This will be achieved by showing the equivalence of functors over Artinian bases.

Clearly $\mathcal{F}_V((T, o)) \to \mathcal{F}_H((T, o))$ is injective, it is enough to show the surjectivity. We will show it by induction on n. If $n=1$ then it is clear by the Dolbeault isomorphism. Let (T, o) be a small extension of an infinitesimal neighbourhood (T', o) such that $\mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_{T'}) \simeq \boldsymbol{C}$. Let $\mathcal{E}'$ be a vector bundle over $X \times (T', o)$. Then we have a vector bundle $\mathcal{E}$ over $X \times (T, o)$ with $\mathcal{E}_{|X \times (T', o)} \simeq \mathcal{E}'$ by the following exact diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{E}_0 & \longrightarrow & A^{0,0}(E) & \longrightarrow & A^{0,1}(E) & \longrightarrow & A^{0,2}(E) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & A^{0,0}(E) \otimes_{\boldsymbol{C}} \mathcal{O}_T & \longrightarrow & A^{0,1}(E) \otimes_{\boldsymbol{C}} \mathcal{O}_T & \longrightarrow & A^{0,2}(E) \otimes_{\boldsymbol{C}} \mathcal{O}_T \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & A^{0,0}(E) \otimes_{\boldsymbol{C}} \mathcal{O}_{T'} & \longrightarrow & A^{0,1}(E) \otimes_{\boldsymbol{C}} \mathcal{O}_{T'} & \longrightarrow & A^{0,2}(E) \otimes_{\boldsymbol{C}} \mathcal{O}_{T'} \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & 0 & & 0 & & 0
\end{array}
$$

where the horizontal arrows are induced from $\alpha(t) \in \mathcal{F}_H((T, o))$.

(3) Suppose $\chi = \{\chi_\alpha(t)\}$ represents a bundle isomorphism $\mathcal{E} \simeq \mathcal{E}'$ i.e. it satisfies

$$(2.4) \qquad \chi_\alpha(t) e_{\alpha\beta}(t) - e'_{\alpha\beta}(t) \chi_\beta(t) \in \mathcal{I}_{T,o} \Gamma(\overline{V}_\alpha \cap \overline{V}_\beta, \mathcal{O}) \{t_1, \ldots, t_m\},$$

where $\{e'_{\alpha\beta}(t)\}$ is a system of transition matrices of $\mathcal{E}'$ with respect to the covering $X = \cup_\alpha V_\alpha$. We may assume that $e_{\alpha\beta}(0) = e'_{\alpha\beta}(0)$. If we set $g_\alpha(t) = s'_\alpha(t) \chi_\alpha(t) (s_\alpha(t))^{-1}$ then $g_\alpha(t) \in A^{0,0}(\overline{V}_\alpha, GL(r, \boldsymbol{C}))_{k+1}\{t_1, \ldots, t_m\}$ and we infer from (2.4) that

$$(2.5) \qquad e_{\alpha\beta}(0) g_\beta(t) - g_\alpha(t) e_{\alpha\beta}(0) \in \mathcal{I}_{T,o} A^{0,0}(\overline{V}_\alpha \cap \overline{V}_\beta, GL(r, \boldsymbol{C}))_{k+1}\{t_1, \ldots, t_m\},$$

$$(2.6) \qquad \bar{\partial} g_\alpha(t) - \omega'(t) g_\alpha(t) + g_\alpha(t) \omega(t) \in \mathcal{I}_{T,o} A^{0,1}(\overline{V}_\alpha, GL(r, \boldsymbol{C}))_k\{t_1, \ldots, t_m\}.$$

If we set $g(t) = \sum_\gamma \rho_\gamma e_{\alpha\gamma}(0) g_\gamma(t) e_{\gamma\alpha}(0)$, then $g(t)$ satisfies (1)$\sim$(3) of Proposition 2.2 because of (2.5) and (2.6).

Conversely, suppose that $g(t) \in GL(E)_{k+1}\{t_1, \ldots, t_m\}$ satisfies (1)$\sim$(3) in Proposition 2.2. If we set $\eta_\alpha(t) = (s'_\alpha(t))^{-1} g(t) s_\alpha(t)$ then $\eta_\alpha(t) \in A^{0,0}(\overline{V}_\alpha, GL(r, \boldsymbol{C}))_{k+1}\{t_1, \ldots, t_m\}$ and it satisfies

$$\eta_\alpha(t) e_{\alpha\beta}(t) - e'_{\alpha\beta}(t)\eta_\beta(t) \in \mathscr{I}_{T,o}\, A^{0,0}(\overline{V}_\alpha \cap \overline{V}_\beta, GL(r, \boldsymbol{C}))_{k+1}\{t_1, \ldots, t_m\}, \tag{2.7}$$

$$\bar{\partial}\eta_\alpha(t) \in \mathscr{I}_{T,o}\, A^{0,1}(\overline{V}_\alpha, GL(r, \boldsymbol{C}))_{k+1}\{t_1, \ldots, t_m\}. \tag{2.8}$$

If we take a smaller Stein covering $X = \cup_\alpha U_\alpha$ with $U_\alpha \Subset V_\alpha$ and set $\chi_\alpha(t) = (I - \bar{\partial}^* N_\alpha \bar{\partial})\eta_\alpha(t)$, then $\chi_\alpha(t) \in A^{0,0}(\overline{U}_\alpha, GL(r, \boldsymbol{C}))_k\{t_1, \ldots, t_m\}$ and we infer from (2.7) and (2.8) that

$$\chi_\alpha(t) e_{\alpha\beta}(t) - e'_{\alpha\beta}(t)\chi_\beta(t) \in \mathscr{I}_{T,0} A^{0,0}(\overline{U}_\alpha \cap \overline{U}_\beta, GL(r, \boldsymbol{C}))_k\{t_1, \ldots, t_m\}, \tag{2.9}$$

$$\bar{\partial}\chi_\alpha(t) = 0, \tag{2.10}$$

where N_α is the L^2-Neumann operator over $\overline{U}_\alpha$. This implies that $\chi_\alpha(t) \in \Gamma((U_\alpha \cap U_\beta) \times D, \mathcal{O}^{r^2})$ and $\chi_\alpha(t) e_{\alpha\beta}(t) - e'_{\alpha\beta}(t)\chi_\beta(t) \in \Gamma((U_\alpha \cap U_\beta) \times D, \mathscr{I}_{T,o} \otimes_{\mathcal{O}_X} \mathcal{O}_{X\times D})$ for a sufficiently small neighbourhood D of 0 in $\boldsymbol{C}^m$. Thus we have a bundle isomorphism $\chi = \{\chi_\alpha(t)\}$. Q. E. D.

This completes the proof of Theorem 2.

§ 3. Comparison of Three Kinds of Moduli Spaces of Simple Structures on a Vector Bundle over a Projective Algebraic Manifold

Let E be a differentiable complex vector bundle over a projective algebraic manifold X.

Definition 3.1. A semi-connection $\bar{\partial}_{A'}$ is simple if $\mathrm{Ker}\,(\bar{D}_{A'|A^{0,0}(End^0 E)}) = \{0\}$, where by $End^0 E$ we denote the subbundle of $End\ E$ consisting of endomorphisms with trace 0.

Remark. A holomorphic structure is simple if and only if its associated holomorphic bundle is simple i.e. every holomorphic endomorphism is constant.

Let $\mathscr{M}^S(E)$ be the moduli space of simple holomorphic structures

on E. By [Ko] or [L-O] together with the arguments in §1, we can see that $\mathscr{M}^S(E)$ is a finite dimensional (non-reduced) complex space. Let $\mathscr{M}^S_{an}$ and $\mathscr{M}^S_{alg}$ be the moduli spaces of holomorphic simple vector bundles and of algebraic simple vector bundles on X respectively. In this section, we will prove that they are isomorphic to each other as not necessarily reduced complex spaces.

Let F^S_{an}: $(An)\to(Sets)$ be a functor given by $F^S_{an}(T)=\{\mathscr{E}\to X\times T \mid$ a T-simple holomorphic vector bundle such that $\mathscr{E}_{|X\times t}$ is differentiably equivalent to E for any $t\in T\}/\sim$, where $\mathscr{E}\sim\mathscr{E}'$ if $\mathscr{E}\simeq p^*\mathscr{L}\otimes\mathscr{E}'$ for some invertible sheaf $\mathscr{L}$ on T, and F^S_{alg}: $(Sch/\boldsymbol{C})\to(Sets)$ be a functor given by $F^S_{alg}(T)=\{\mathscr{E}\to X\times T \mid$ a T-simple algebraic vector bundle$\}/\sim$, where $\mathscr{E}\sim\mathscr{E}'$ if $\mathscr{E}\simeq p^*\mathscr{L}\otimes\mathscr{E}'$ for some invertible sheaf $\mathscr{L}$ over T.

Existence Theorem. (1) *(cf. [N], [K-O] and [F-S]) The sheafified functor $(F^S_{an})^\#$ is representable by a locally Hausdorff complex space $\mathscr{M}^S_{an}(E)$.*

(2) *(cf. [A-K]) The sheafified functor $(F^S_{alg})^{\#(\acute{e}t)}$ under the étale topology is representable by an algebraic space $\mathscr{M}^S_{alg}$.*

Theorem 3. *$\mathscr{M}^S(E)\simeq\mathscr{M}^S_{an}(E)\simeq(\mathscr{M}^S_{alg})_{an}(E)$, where by $(\mathscr{M}^S_{alg})_{an}(E)$ we denote the open part of the underlying complex space of $\mathscr{M}^S_{alg}$ consisting of simple vector bundles differentiably equivalent to E.*

Proof. At first, we will show the isomorphism $\mathscr{M}^S(E)\simeq\mathscr{M}^S_{an}(E)$. By $\mathscr{S}(E)$, we denote the set of all simple holomorphic structures on E. As is shown in [L-O], $\mathscr{M}^S(E)=\mathscr{S}(E)/G$ where $G=GL(E)/\boldsymbol{C}^*$, is a Banach $\boldsymbol{C}$-analytic space whose local model is given by a slice of a principal G_{k+1}-bundle π_k: $\mathscr{S}(E)_k\to\mathscr{M}^S_k(E)=\mathscr{S}(E)_k/G_{k+1}$, $V_{A',\varepsilon}=\{\bar\partial_{A'}+\alpha\,|\,\alpha\in A^{0,1}(End\ E)_k, ||\alpha||_k<\varepsilon, \bar D^*_A\cdot\alpha=0, \bar D_A\cdot\alpha+\alpha\wedge\alpha=0\}$ for $\bar\partial_{A'}\in\mathscr{S}(E)$. Hence $\mathscr{M}^S(E)$ represents a local universal family of simple holomorphic structures on E at each point. By Theorem 2, we have an open covering $\mathscr{M}^S(E)=\cup_i S_i$ such that

(i) there exists a S_i-simple holomorphic vector bundle $\mathscr{E}_i$ over $X\times S_i$,

(ii) $\mathscr{E}_{i|S_i\cap S_j}\sim\mathscr{E}_{j|S_i\cap S_j}$,

(iii) $(\mathscr{E}_i, S_i)$ is universal at any point of S_i.

From this, we infer that $\mathcal{M}^S(E)$ represents $(F^S_{an})^\sharp$. Hence we have $\mathcal{M}^S(E) \simeq \mathcal{M}^S_{an}(E)$.

Next we will show the second isomorphism $\mathcal{M}^S_{an}(E) \simeq (\mathcal{M}^S_{alg})_{an}(E)$. By the definition of algebraic space, we have an étale covering $(U_i) \to \mathcal{M}^S_{alg}$ by affine schemes U_i with an algebraic vector bundle $\mathcal{E}_i \to X \times S_i$ such that

(i) $\mathcal{E}_i$ is U_i-simple,

(ii) $\sigma_i^* \mathcal{E}_i \sim \sigma_j^* \mathcal{E}_j$ where σ_i and σ_j are the projections of $U_i \times_F U_j$ onto each components respectively,

(iii) $(\mathcal{E}_i, U_i)$ is formally universal at any point of U_i (cf. [Ar], Théorème 5.2).

By [GAGA] and [S], (iii) means

(iii)′ $(\mathcal{E}_i, U_i)$ is universal for analytic families at any point of U_i.

Therefore $(\mathcal{M}^S_{alg})_{an}$ has an open covering $(\mathcal{M}^S_{alg})_{an} = \cup_i V_i$ such that

(i) there exists a V_i-simple holomorphic vector bundle $\mathcal{E}_i \to X \times V_i$,

(ii) $\mathcal{E}_{i|V_i \cap V_j} \sim \mathcal{E}_{j|V_i \cap V_j}$,

(iii)′ $(\mathcal{E}_i, V_i)$ is universal at any point of V_i.

Thus we proved that $(\mathcal{M}^S_{alg})_{an}(E)$ represents $(F^S_{an})^\sharp$. Hence we have $(\mathcal{M}^S_{alg})_{an}(E) \simeq \mathcal{M}^S_{an}(E)$. Q. E. D.

§ 4. Moduli Space of Irreducible Einstein-Hermitian Connections

Let (E, h) be a differentiable hermitian vector bundle over a compact Kähler manifold X with a Kähler form Φ.

Definition 4.1. An h-connection on E is a $\boldsymbol{C}$-linear map d_A: $A^0(E) \to A^1(E)$ satisfying

$$d_A(f \cdot s) = df \otimes s + f \cdot (d_A s) \quad \text{for } f \in C^\infty(X) \text{ and } s \in A^0(E).$$

and $d(h(s,t)) = h(d_A s, t) + h(s, d_A t)$ for $s, t \in A^0(E)$.

Every h-connection extends to a unique $\boldsymbol{C}$-linear map d_A: $A^p(E) \to A^{p+1}(E)$ and induces a unique $\boldsymbol{R}$-linear map D_A: $A^p(End(E,h)) \to A^{p+1}(End(E,h))$ where by $End(E,h)$ we denote the bundle of h-skew-hermitian endomorphisms of E.

Definition 4.2. An h-connection d_A is called h-Einstein if
(i) its curvature form F_A is of type $(1,1)$,
(ii) $\Lambda F_A = \lambda \cdot id_E$ for a $\lambda \in \boldsymbol{R}$,
where by Λ we denote the trace operator with respect to the Kähler form Φ.

For an h-Einstein connection d_A, we have the following elliptic complex (B^*) (cf. [Ko] Ch. VII) : Set $B^p = A^p(End(E,h))$, $B^{p,q} = A^{p,q}(End(E,h))$ and $B^2_+ = B^2 \cap (B^{2,0} + B^{0,2} + B^0\Phi)$. Then we have an elliptic complex,

$$(B^*) \qquad 0 \longrightarrow B^0 \xrightarrow{D_A} B^1 \xrightarrow{D_A^+} B^2_+ \xrightarrow{D_A^2} B^{0,3} \xrightarrow{\bar{D}_{A^\cdot}} \cdots$$

where $D_A^+ = p^+ \cdot D_A$, $D_A^2 = \bar{D}_{A^\cdot} \cdot p^{(0,2)}$, and by p^+ and $p^{(0,2)}$ we denote the projection operators $B^2 \to B^2_+$ and $B^2_+ \to B^{0,2}$ respectively.

Let $\mathscr{D}(E,h)$ be the set of all h-connections on E. Then the following proposition is well known (cf. [Ko] Ch. VII).

Proposition 4.1. (1) *For any* $d_A \in \mathscr{D}(E,h)$, $\mathscr{D}(E,h) = d_A + A^1(End(E,h))$.
(2) *Let* d_A *be an* h*-Einstein connection on* E. *Then* $d_A + \alpha$ *is* h*-Einstein if and only if* $D_A^+\alpha + p^+(\alpha \wedge \alpha) = 0$.

Definition 4.3. An h-Einstein connection d_A is irreducible if $\mathrm{Ker}\,(D_{A|A^0(End^0(E,h))}) = \{0\}$.

Remark. If d_A is an irreducible h-Einstein connection, then the semi-connection $\bar{\partial}_{A^\cdot}$ induced from d_A is simple.

Let $\hat{\mathscr{E}}(E,h)$ be the set of all irreducible h-Einstein connections on E.

Proposition 4.2. (cf. [F-S].) (1) *The moduli space* $\mathscr{M}_{HE} = \hat{\mathscr{E}}(E,h)/U(E,h)$ *of irreducible* h*-Einstein connections is a finite dimensional Hausdorff (non-reduced) real analytic space.*
(2) *The natural assignment* $d_A \to \bar{\partial}_{A^\cdot}$ *induces an injection* $\mathscr{M}_{HE} \to \mathscr{M}^S(E)$, *which is a local isomorphism of not necessarily reduced real analytic*

spaces.

Proof. We fix an integer $k>\dim_R X+1$.

(1) Because h-Einstein condition is preserved under $U(E,h)$-actions, we have an R-Banach analytic space $\mathcal{M}_{HE}$ by glueing together slices $U_{A,\varepsilon}=\{d_A+\alpha \mid \alpha\in A^1(End(E,h))_k, ||\alpha||_k<\varepsilon, D_A^+\alpha+p^+(\alpha\wedge\alpha)=0, D_A^*\alpha=0\}$ for $d_A\in\hat{\mathcal{E}}(E,h)$. By the same argument in §1, we can see that $\mathcal{M}_{HE}$ is a finite dimensional realanalytic space.

(2) Local models of $\mathcal{M}_{HE}$ and $\mathcal{M}^S(E)$ are given by slices $U_{A,\varepsilon}$ as above and $V_{A',\varepsilon'}=\{\bar\partial_{A'}+\alpha'' \mid \alpha''\in A^{0,1}(End\ E)_k, ||\alpha||_k<\varepsilon'', \bar D_{A'}\alpha''+\alpha''\wedge\alpha''=0, \bar D_A^*\alpha''=0\}$ respectively. We will show that the natural assignment $d_A+\alpha\to\bar\partial_{A'}+\alpha''$ gives a real analytic isomorphism $U_{A,\varepsilon}\to V_{A',\varepsilon'}$ via an intermediate slice

$$\tilde V_{A',\varepsilon'}=\{\bar\partial_{A'}+\alpha'' \mid \alpha''\in A^{0,1}(End\ E)_k,\ ||\alpha||_k<\varepsilon', \bar D_{A'}\alpha''+\alpha''\wedge\alpha''=0,$$
$$\bar D_A^*\alpha''-\frac{i}{2}\Lambda(\alpha'\wedge\alpha''+\alpha''\wedge\alpha')=0, \alpha'=-{}^t\bar\alpha''\}$$ introduced in [F-S].

At first, we note that the h-Einstein condition $D_A^+\alpha+p^+(\alpha\wedge\alpha)=0$ is divided into the following two conditions; $\Lambda(D_A\alpha+\alpha\wedge\alpha)=0$ and $\bar D_{A'}\alpha''+\alpha''\wedge\alpha''=0$, and will show the isomorphism $U_{A,\varepsilon}\simeq\tilde V_{A',\varepsilon'}$. Let

$$U_1=\{d_A+\alpha \mid \alpha\in A^1(End(E,h))_k, ||\alpha||_k<\varepsilon, \Lambda(D_A\alpha+\alpha\wedge\alpha)=0,$$
$$\bar D_{A'}^*(\bar D_{A'}\alpha''+\alpha''\wedge\alpha'')=0, D_A^*\alpha=0\}$$ and

$$\tilde V_1=\{\bar\partial_{A'}+\alpha'' \mid \alpha''\in A^{0,1}(End\ E)_k, ||\alpha''||_k<\varepsilon', \bar D_{A'}^*(\bar D_{A'}\alpha''+\alpha''\wedge\alpha'')=0,$$
$$\bar D_A^*\alpha''-\frac{i}{2}\Lambda(\alpha'\wedge\alpha''+\alpha''\wedge\alpha')=0\}.$$

Lemma 4.3. (1) *U_1 is a finite dimensional real analytic direct submanifold of a neighbourhood of 0 in $A^1(End(E,h))_k$.*
(2) *$\tilde V_1$ is a finite dimensional real analytic direct submanifold of a neighbourhood of 0 in $A^{0,1}(End\ E)_k$.*

Proof. (1) Let f_1: $A^1(End(E,h))_k\to D_A^*A^1(End(E,h))_k+D_A^*A^1(End(E,h))_k+\bar D_A^*A^{0,2}(End\ E)_{k-1}+H_A^1$ be a real analytic map given by $f_1(\alpha)=(D_A^*\alpha, \Lambda(D_A\alpha+\alpha\wedge\alpha), \bar D_{A'}^*(\bar D_{A'}\alpha''+\alpha''\wedge\alpha''), \rho_A^1\alpha)$ where we denote by α'' the $(0,1)$-component of α and by ρ_A^1 the projection onto the harmonic space H_A^1 in B^1. Note that $\Lambda(D_A\alpha+\alpha\wedge\alpha)\in D_A^*A^1(End(E,h))_k$ because $\Lambda D_A\alpha\perp H_A^0$, $Tr(\alpha\wedge\alpha)=0$ and d_A is irreducible. Since $(df_1)_0$

$(v)=(D_A^*v,\ \Delta D_A v,\ \bar{D}_{A'}^*\bar{D}_{A'}v,\ \rho_A^1 v)$, $(df_1)_0$ is a topological isomorphism. We will show the surjectivity of $(df_1)_0$, its injectivity is clear because $D_A^*v=\Delta D_A v=\bar{D}_{A'}v''=0$ means $v\in H_A^1$. Take $(u_1,u_2,u_3'',u_0)\in D_A^*A^1(End(E,h))_k+D_A^*A^1(End(E,h))_k+\bar{D}_{A'}^*A^{0,2}(End\ E)_{k-1}+H_A^1$. Let $v_1=D_AG_Au_1+D_A^{+}{}^*G_A(u_2\Phi)$ where by G_A we denote the Green operator for the complex (B^*). Then $D_A^*v_1=u_1$ and $\Delta D_A^+v_1=u_2$ because $D_A^2(u_2\Phi)=0$ and $u_2\Phi\perp H_+^2$ where we denote by H_+^2 the harmonic space in B_+^2. Next, set $v_2''=G_{A'}(v_3''-\bar{D}_{A'}^*\bar{D}_{A'}v_1'')$ and $v_2=-{}^t\bar{v}_2''+v_2''$, where we denote by $G_{A'}$ the Green operator for the $End\ E$-valued Doubault complex. If we set $v=v_1+v_2+u_0$, then we have $(df_1)_0(v)=(u_1,u_2,u_3'',u_0)$. Hence, by the inverse mapping theorem in Banach manifolds, f_1 is a locally real analytic isomorphism. Since $U_1=f_1^{-1}(0\times 0\times 0\times H_A^1)$ locally around 0, U_1 is a finite dimensional real analytic direct submanifold of a neighbourhood of 0 in $A^1(End(E,h))_k$.

(2) Let $\tilde{f}_1\colon A^{0,1}(End\ E)_k\to\bar{D}_{A'}^*A^{0,1}(End\ E)_k+\bar{D}_{A'}^*A^{0,2}(End\ E)_{k-2}+H_{A'}^{0,1}$ be a real analytic map given by $\tilde{f}_1(\alpha'')=(\bar{D}_{A'}^*\alpha''-\frac{i}{2}\Delta(\alpha'\wedge\alpha''+\alpha''\wedge\alpha'),\ \bar{D}_{A'}^*(\bar{D}_{A'}\alpha''+\alpha''\wedge\alpha''),\ \rho_{A'}^{0,1}\alpha'')$ where we denote by $\rho_{A'}^{0,1}$ the projection onto the harmonic space $H_{A'}^{0,1}$ in $A^{0,1}(End\ E)$. Note that $\Delta(\alpha'\wedge\alpha''+\alpha''\wedge\alpha')\in\bar{D}_{A'}^*A^{0,1}(End\ E)_k$ because $Tr(\alpha'\wedge\alpha''+\alpha''\wedge\alpha')=0$ and $\bar{D}_{A'}$ is simple. Since $(d\tilde{f}_1)_0(v)=(\bar{D}_{A'}^*v'',\ \bar{D}_{A'}^*\bar{D}_{A'}v'',\ \rho_{A'}^{0,1}v'')$, $(d\tilde{f}_1)_0$ is topologically isomorphism. Indeed, $(d\tilde{f}_1)_0^{-1}(u_1'',u_2'',u_0'')=G_{A'}(\bar{D}_{A'}u_1''+u_2'')+u_0''$. Hence, by the inverse mapping theorem in Banach manifolds, $\tilde{f}_1$ is a locally real analytic isomorphism. Since $\tilde{V}_1=\tilde{f}_1^{-1}(0\times 0\times H_{A'}^{0,1})$ locally around 0, $\tilde{V}_1$ is a finite dimensional real analytic direct submanifold of a neighbourhood of 0 in $A^{0,1}(End\ E)_k$.

Q. E. D.

Because the assignment $d_A+\alpha\to\bar{\partial}_{A'}+\alpha''$ gives a real linear isomorphism $T_0U_1=H_A^1\simeq H^{0,1}(End\ E)=T_0\tilde{V}_1$ (cf. [Ko] Ch. VII, Theorem 2.21), by the inverse mapping theorem, we have a real analytic local isomorphism $U_1\simeq\tilde{V}_1$. Because both of $U_{A,\varepsilon}$ and $\tilde{V}_{A',\varepsilon'}$ are defined by the same equation $\bar{D}_{A'}\alpha''+\alpha''\wedge\alpha''=0$ in their ambients U_1 and $\tilde{V}_1$ respectively, we have $U_{A,\varepsilon}\simeq\tilde{V}_{A',\varepsilon'}$ as not necessarily reduced real analytic spaces.

Next, let $\tilde{V}_2=\{\bar{\partial}_{A'}+\alpha''\mid\ \alpha''\in A^{0,1}(End\ E)_k,\ \|\alpha''\|_k<\varepsilon',\ \bar{D}_{A'}^*\alpha''-\frac{i}{2}\Delta(\alpha'\wedge\alpha''+\alpha''\wedge\alpha')=0\}$ and $V_2=\{\bar{\partial}_{A'}+\alpha''\mid\ \alpha''\in A^{0,1}(End\ E)_k,\ \|\alpha''\|_k<\varepsilon'',$

$\bar{D}_A^* \cdot \alpha'' = 0\}$. Then they are $\boldsymbol{R}$-Banach analytic direct submanifolds of $A^{0,1}(End\ E)_k$ and transformed to each other by $GL(E)_{k+1}$-actions. Because $\tilde{V}_{A',\varepsilon'}$ and $V_{A',\varepsilon'}$ are $\boldsymbol{R}$-Banach analytic subspaces of $\tilde{V}_2$ and V_2 respectively defined by the same equation $\bar{D}_A \cdot \alpha'' + \alpha'' \wedge \alpha'' = 0$, which is preserved under $GL(E)_{k+1}$-actions, $\tilde{V}_{A',\varepsilon'}$ and $V_{A',\varepsilon'}$ are transformed to each other by $GL(E)_{k+1}$-actions. Thus $\tilde{V}_{A',\varepsilon'} \simeq V_{A',\varepsilon'}$ as not necessarily reduced real analytic spaces. Q. E. D.

Corollary. *$\mathcal{M}_{HE}$ is realized as an open Hausdorff (non-reduced) complex subspace of $\mathcal{M}^S(E)$.*

See [Ko], Ch. VII, Theorem (4.21) and Proposition (1.19) for openness and Hausdorffness respectively.

Note added in Proof. After submitting this paper, Professor D. Sundararaman informed me that he had independently proved Theorems 1, 2 and 3.

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