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Orbital Stability of the Periodic Solutions of Autonomous Systems with Impulse Effect

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Abstract

In the present paper the orbital asymptotic stability of the periodic solutions of autonomous systems with impulse effect is investigated. An analogue of the theorem of Andronov-Vitt is proved.

§ 1. Introduction

In the recent years still more works have been published dedicated to systems with impulse effect [1]-[9].

These systems describe evolution processes which in certain moments change rapidly their state. In the mathematical simulation of such processes it is convenient to neglect the duration of this rapid change and to assume that the system changes its state by jumps.

Processes with such a character are studied in numerous fields of science and techniques such as control theory, impulse techniques, populational dynamics, mass service, control of the reserves, etc.

Systems with impulse effect are defined by a system of ordinary differential equations $\dot{x} = f(t, x)$ and conditions which determine the moments and the magnitude of the impulse effect. The determination of the moments of impulse effect can be realized in various ways.

For instance, the moments of impulse effect for the system

$$\begin{aligned} &\frac{dx}{dt} = f(t, x), \quad t \neq \tau_k(x), \\ &\Delta x|_{t=\tau_k(x)} = I_k(x) \end{aligned} \tag{a}$$

occur when the mapping point $(t, x(t))$ of the extended phase space meets some of the hypersurfaces σ_k defined by the equations $t = \tau_k(x)$, $k = 1, 2, \ldots$.

The moments of impulse effect of the system

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$$\frac{dx}{dt} = f(t, x), \; t \neq \tau_k,$$
$$\Delta x|_{t=\tau_k} = I_k(x), \; k = 1, 2, \ldots \tag{b}$$

are $t = \tau_k$, $k = 1, 2, \ldots$ and are fixed.

The moments of impulse effect of the system

$$\frac{dx}{dt} = g(x), \; x \bar{\in} M,$$
$$\Delta x|_{x \in M} = I(x) \tag{c}$$

occur when the point $x(t)$ of the phase space X meets the set $M \subset X$.

The solutions $x(t)$ of systems (a, b, c) are piecewise continuous functions. At the moment $t = \tau_k$ of impulse effect the solution $x(t)$ has a discontinuity of first type and we assume that the following equalities hold

$$x(\tau_k - 0) = x(\tau_k), \; x(\tau_k + 0) = x(\tau_k) + \Delta x(\tau_k).$$

Between two consecutive moments of impulse effect ($t \in (\tau_k, \tau_{k+1}]$) the solution $x(t)$ of system (a) coincides with the solution $\xi(t)$ of the initial value problem

$$\frac{d\xi}{dt} = f(t, \xi), \; \xi(\tau_k) = x(\tau_k + 0).$$

The solutions of systems (b) and (c) for $t \in (\tau_k, \tau_{k+1}]$ are defined analogously.

We shall note that the moments of impulse effect of the different solutions of systems (a) and (c) are different. That is why in the investigation of such systems there are some additional difficulties which do not occur when we study system (b).

Up to the present moment systems of type (a) and (b) have been profoundly studied [1]-[9] while systems of type (c) have been almost not studied.

Note that in the case when $f(t, x)$ does not depend on t (i.e. $f(t, x) = g(x)$), systems (a) and (b) do not possess the property of autonomy. System (c) has this property. On the other hand, in practice the most frequently one meets autonomous systems with impulse effect. That is why the investigation of systems of type (c) represents an undoubted interest.

In the present paper the orbital asymptotical stability of a periodic solution of system (c) is investigated and an analogue of the theorem of Andronov-Vitt is proved [10], [11].

§2. Preliminary Notes

Consider the autonomous system with impulse effect

$$\frac{dx}{dt} = g(x), \quad x \bar{\in} M,$$
$$\Delta x|_{x \in M} = I(x), \tag{1}$$

where $t \in \mathbf{R}$; g, I: $\Omega \to \mathbf{R}^n$; Ω is a domain contained in the n-dimensional Euclidean space $\mathbf{R}^n$ with elements $x = \operatorname{col}(x_1, \ldots, x_n)$, scalar product $(x, y) = x_1 y_1 + \cdots + x_n y_n$ and norm $|x| = (x, x)^{1/2}$; M is an $(n-1)$-dimensional manifold contained in Ω.

Further on we shall use the following notations: $|A| = \sup\limits_{|x|=1} |Ax|$—*norm of the* $(n \times n)$-matrix A; $\operatorname{diag}(A_1, A_2)$— a block-diagonal matrix with blocks A_1 and A_2; E_m—the unit $(m \times m)$-matrix; 0_m—the zero $(m \times m)$-matrix; $B_\varepsilon(x_0) = \{x \in \mathbf{R}^n \colon |x - x_0| < \varepsilon\}$—$\varepsilon$-neighbourhood of the point $x_0 \in \mathbf{R}^n$; $\bar{G}$—the closure of the set $G \subset \mathbf{R}^n$; $\rho(x, L) = \inf\limits_{y \in L} |x - y|$—the distance from the point $x \in \mathbf{R}^n$ to the set $L \subset \mathbf{R}^n$; $\dfrac{\partial f}{\partial x} = \left(\dfrac{\partial f_i}{\partial x_j}\right)_1^n$—the Jacobian matrix of the function f: $\mathbf{R}^n \to \mathbf{R}^n$; $[a; b]$—the interval $[a, b]$ if $a \le b$ or the interval $[b, a]$ if $b < a$; $x(t; t_0, x_0)$—the solution of system (1) satisfying the initial condition $x(t_0 + 0; t_0, x_0) = x_0$ and $J^+(t_0, x_0)$—the maximal interval of the form (t_0, ω) in which the solution $x(t; t_0, x_0)$ is continuable to the right.

Let $\varphi(t)$ $(t \in \mathbf{R}_+ = [0, \infty))$ be a solution of system (1) with moments of impulse effect $\{\tau_k\}$:

$$0 < \tau_1 < \tau_2 < \cdots, \quad \lim_{k \to \infty} \tau_k = \infty \qquad \text{and}$$
$$L = \{x \in \mathbf{R}^n \colon x = \varphi(t), \ t \in \mathbf{R}_+\}.$$

Definition 1. The solution $\varphi(t)$ of system (1) is called:

1.1. orbitally stable if

$(\forall \varepsilon > 0)(\forall \eta > 0)(\forall t_0 \in \mathbf{R}_+, |t_0 - \tau_k| > \eta)\ (\exists \delta > 0)$

$(\forall x_0 \in \Omega, \ \rho(x_0, L) < \delta, \ x_0 \bar{\in} \bar{B}_\eta(\varphi(\tau_k)) \cup \bar{B}_\eta(\varphi(\tau_k + 0)))(\forall t \in J^+(t_0, x_0))$

$\rho(x(t; t_0, x_0), L) < \varepsilon$;

1.2. orbitally attractive if

$(\forall \eta > 0)(\forall t_0 \in R_+, |t_0 - \tau_k| > \eta)\ (\exists \lambda > 0)$

$(\forall x_0 \in \Omega, \ \rho(x_0, L) < \lambda, \ x_0 \bar{\in} \bar{B}_\eta(\varphi(\tau_k)) \cup \bar{B}_\eta(\varphi(\tau_k + 0)))(\forall \varepsilon > 0)\ (\exists \sigma > 0)$

$t_0 + \sigma \in J^+(t_0, x_0)(\forall t \ge t_0 + \sigma, \ t \in J^+(t_0, x_0))$

$\rho(x(t; t_0, x_0), L) < \varepsilon$;

1.3. orbitally asymptotically stable if it is orbitally stable and orbitally attractive.

Definition 2. We shall say that the solution $\varphi(t)$ of system (1) has the property asymptotic phase if

$$(\forall \eta > 0)(\forall t_0 \in \mathbb{R}_+,\ |t_0 - \tau_k| > \eta)(\exists \lambda > 0)$$

$$(\forall x_0 \in \Omega,\ |x_0 - \varphi(t_0)| < \lambda)(\exists c \in \mathbb{R})(\forall \varepsilon > 0)(\exists \sigma > |c|)$$

$$t_0 + \sigma \in J^+(t_0, x_0)(\forall t \geq t_0 + \sigma,\ t \in J^+(t_0, x_0),\ |t - \tau_k| > \eta)$$

$$|x(t + c;\ t_0, x_0) - \varphi(t)| < \varepsilon.$$

Remark 1. Let the functions $g(x)$ and $I(x)$ be differentiable, the manifold M be smooth and the normal vectors n_k to M at the points $\varphi(\tau_k)$ be such that $(n_k, g(\varphi(\tau_k))) \neq 0$, $k = 1, 2, \ldots$. Then a straightforward verification shows that the function $\xi(t) = \dfrac{d\varphi}{dt}(t)$ satisfies the following linear system with impulse effect at fixed moments of time

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial g}{\partial x}(\varphi(t))u,\ t \neq \tau_k, \\ \Delta u|_{t=\tau_k} &= N_k u,\ k = 1, 2, \ldots, \end{aligned} \tag{2}$$

where

$$N_k u = \frac{\partial I}{\partial x}(\varphi(\tau_k))u + \left[g(\varphi(\tau_k + 0)) - g(\varphi(\tau_k)) - \frac{\partial I}{\partial x}(\varphi(\tau_k))g(\varphi(\tau_k)) \right] \frac{(n_k, u)}{(n_k, g(\varphi(\tau_k)))}.$$

System (2) is called system in variations (of system (1) with respect to the solution $\varphi(t)$).

We shall say that conditions (A) hold if the following conditions are satisfied:

A1. System (1) has a T-periodic solution $p(t)$ with moments of impulse effect τ_k ($\tau_k < \tau_{k+1}$, $k = 0, \pm 1, \pm 2, \ldots$) and the positive integer q be such that

$$\tau_0 < 0 < \tau_1 < \cdots < \tau_q < T < \tau_{q+1},$$

$$\tau_{k+q} = \tau_k + T, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

A2. $\dfrac{dp}{dt}(t) \neq 0$ for $t \neq \tau_k$,

A3. There exists a constant $H > 0$ such that:

A3.1. The function $g\colon \Omega \to \mathbb{R}^n$ is differentiable in the set $D = \bigcup_{k=1}^{q} D_k(H) \subset \Omega$ and is continuous in $\bar{D}$ where

$$D_k(H) = \{x \in \mathbf{R}^n\colon\ |x - y| < H,\ y \in \Gamma_k\},$$

$$\Gamma_k = \{x \in \mathbf{R}^n\colon\ x = p(t),\ t \in (\tau_{k-1}, \tau_k]\} \cup \{p(\tau_{k-1} + 0)\}.$$

A3.2. The function $I\colon \Omega \to \mathbf{R}^n$ is differentiable in the set $\bigcup_{k=1}^{q} B_H(p(\tau_k)) \subset \Omega$.

A3.3. The set $M \cap B_H(p(\tau_k))$, $k = 1, \ldots, q$ coincides with the set of solutions of the equation $\phi(x) = 0$ where the function $\phi\colon B_H(p(\tau_k)) \to \mathbf{R}$ is differentiable in $B_H(p(\tau_k))$.

A4. The following relations hold

$$\phi(p(\tau_k)) = 0,\ k = 1, \ldots, q,$$

$$\frac{\partial \phi}{\partial x}(p(\tau_k))g(p(\tau_k)) \neq 0,\ k = 1, \ldots, q. \tag{3}$$

A5. For any $h \in (0, H)$ there exists $\gamma > 0$ such that $\rho(x, M) \geq \gamma$ for $k = 1, \ldots, q$ and $x \in \Gamma_k \setminus B_h(p(\tau_k))$.

A6. $\rho_1 = 1$ and $|\rho_k| < 1$, $k = 2, \ldots, n$ where ρ_k, $k = 1, \ldots, n$ are the multiplicators of the T-periodic linear system with fixed moments of impulse effect

$$\begin{aligned} \frac{dy}{dt} &= \frac{\partial g}{\partial x}(p(t))y,\ t \neq \tau_k \\ \Delta y|_{t=\tau_k} &= N_k y,\ k = 0,\ \pm 1,\ \pm 2, \ldots \end{aligned} \tag{4}$$

and

$$N_k y = \frac{\partial I}{\partial x}(p(\tau_k))y + \left[g(p(\tau_k + 0)) - g(p(\tau_k)) - \frac{\partial I}{\partial x}(p(\tau_k))g(p(\tau_k)) \right] \frac{\frac{\partial \phi}{\partial x}(p(\tau_k))y}{\frac{\partial \phi}{\partial x}(p(\tau_k))g(p(\tau_k))}. \tag{5}$$

Remark 2. Let conditions A1-A4 hold. Then, by Remark 1, the function $y = \frac{dp}{dt}$ is a T-periodic solution of system (4). In view of condition A2 we conclude that the linear T-periodic system (4) has a multiplicator equal to one.

System (4) coincides with the system in variations of system (1) with respect to the solution $p(t)$.

Remark 3. Condition (3) means that the trajectories of system (1) are not tangent to the manifold M in some neighbourhood of the point $p(\tau_k)$.

§3. Main Results

Before we go to the proof of the main theorem we shall prove some auxiliary assertions.

Denote by $z(t\,;\, t_0,\, z_0)$ the solution of the initial value problem

$$\frac{dz}{dt} = g(z),\ z(t_0) = z_0 \in \mathbf{R}^n.$$

Lemma 1. *Let conditions* A1-A4 *hold.*

Then there exists a number $h \in (0, H)$ *such that for any* $k = 1, \dots, q$ *there exists a unique differentiable function* $T_k\colon B_h(p(\tau_k)) \to \mathbf{R},\ w \to T_k(w)$ *so that*

$$T_k(p(\tau_k)) = \tau_k,$$

$$\phi(z(T_k(w);\, \tau_k,\, w)) \equiv 0,$$

$$\frac{\partial \phi}{\partial x}(z(T_k(w);\, \tau_k,\, w))\left[g(z(T_k(w);\, \tau_k,\, w))\frac{\partial T_k}{\partial w}(w) + \frac{\partial z}{\partial w}(T_k(w);\, \tau_k,\, w) \right] \equiv 0$$

for $w \in B_h(p(\tau_k))$.

Proof. Lemma 1 follows immediately from the implicit function theorem applied to the function

$$\varphi_k\colon (\tau_k - H,\ \tau_k + H) \times B_H(p(\tau_k)) \to \mathbf{R},\ (t,\, w) \to \varphi_k(t,\, w)$$
$$= \phi(z(t;\, \tau_k,\, w)) \text{ and the system } \varphi_k(t,\, w) = 0.$$

Corollary 1. There exists a constant $\tau > 0$ such that

$$|T_k(w) - T_k(u)| \leq \tau |w - u| \text{ for } k = 1, \dots, q \text{ and } w,\ u \in B_h(p(\tau_k)).$$

Let $n > 0$ and $m > 0$ be integers and $D \subset R$. Denote by $PC(D, R^{n\times m})$ the space of $(n \times m)$-matrix-valued functions which are defined for $t \in D$, continuous in $t \in D$, $t \neq \tau_k$, at the points τ_k they have discontinuities of first type and are left continuous.

The following two lemmas are related to the system

$$\begin{aligned} &\frac{dy}{dt} = P(t)y,\ t \neq \tau_k, \\ &\Delta y|_{t=\tau_k} = P_k y,\ k = 0,\ \pm 1,\ \pm 2, \dots, \end{aligned} \tag{6}$$

where $P(t)$ and P_k $(t \in R,\ k = 0,\ \pm 1,\ \pm 2, \dots)$ are $(n \times n)$-matrices.

Introduce the following conditions (C):

C1. $P(\cdot) \in PC(R, R^{n\times n})$ and $P(t + T) = P(t)$ for $t \in \mathbf{R}$.

C2. There exists an integer $q > 0$ such that

$\tau_{k+q} = \tau_k + T,\ P_{k+q} = P_k,\ k = 0,\ \pm 1,\ \pm 2, \ldots$

C3. $\rho_1 = 1,\ |\rho_k| < 1,\ k = 2, \ldots, n,$

where ρ_k, $k = 1, \ldots, n$ are the multiplicators of system (6).

Lemma 2. *Let conditions* (C) *hold.*

Then system (6) *has a fundamental matrix of the form*

$$Y(t) = \psi(t)\ \mathrm{diag}(E_1,\ e^{Ct}), \tag{7}$$

where the matrix $\psi(\cdot) \in PC(\mathbf{R}, \mathbf{R}^{n \times n})$ *is non-singular and* $2T$*-periodic and* C *is a real* $(n-1) \times (n-1)$*-matrix with eigenvalues which have negative real parts*:

$$\mathrm{Re}\lambda_j(C) < 0,\ j = 1, \ldots, n-1. \tag{8}$$

Proof. Let $\tilde{Y}(t)$ be the fundamental matrix of system (6) for which $\tilde{Y}(0) = E_n$. From the T-periodicity of system (6) it follows that

$$\tilde{Y}(t+T) = \tilde{Y}(t)\tilde{Y}(T) \text{ for } t \in \mathbf{R}. \tag{9}$$

From condition C3 it follows that the monodromy matrix $\tilde{Y}(T)$ has eigenvalues

$$\rho_1 = 1 \text{ and } |\rho_k| < 1, \quad k = 2, \ldots, n.$$

Hence there exists a non-singular matrix S such that

$$S^{-1}\tilde{Y}(T)S = \mathrm{diag}(E_1,\ B_1) = B, \tag{10}$$

where B_1 is a real $(n-1) \times (n-1)$-matrix with eigenvalues which are in modulus less than one:

$$|\lambda_j(B_1)| < 1, \quad j = 1, \ldots, n-1. \tag{11}$$

Set

$$C = \frac{1}{T} \ln B_1, \tag{12}$$

$$A = \frac{1}{T} \ln B = \mathrm{diag}(0_1,\ C),$$

$$Y(t) = \tilde{Y}(t)S. \tag{13}$$

Since from (13), (9) and (10) it follows that

$$Y(t+T) = \tilde{Y}(t+T)S = \tilde{Y}(t)\tilde{Y}(T)S = \tilde{Y}(t)SS^{-1}\tilde{Y}(T)S = Y(t)B$$

then by the representation of Floquet the fundamental matrix $Y(t)$ has the form

$$Y(t) = \psi(t)e^{At} = \psi(t)\mathrm{diag}(E_1,\ e^{Ct}),$$

where the matrix $\psi(\cdot) \in PC(\mathbf{R}, \mathbf{R}^{n \times n})$ is non-singular and $2T$-periodic.

From (11) and (12) there follows (8) which proves Lemma 2.

Lemma 3. *Let conditions* (C) *hold, the fundamental matrix* $Y(t)$ *of system* (6) *have the form* (7) *and*

$$G(t,\ s) = \begin{cases} Y(t)\operatorname{diag}(0_1,\ E_{n-1})Y^{-1}(s), & t > s \\ -\ Y(t)\operatorname{diag}(E_1,\ 0_{n-1})Y^{-1}(s), & t \le s. \end{cases} \tag{14}$$

Then:

1) *The matrix* $G(t, s)$ *satisfies the relations*

$$G(t,\ t-0) - G(t,\ t+0) = E_n, \quad t \in \mathbf{R}, \tag{15}$$

$$\frac{\partial G}{\partial t}(t,\ s) = P(t)G(t,\ s), \qquad t \neq \tau_k, \tag{16}$$

$$G(\tau_k + 0,\ s) = (E_n + P_k)G(\tau_k,\ s),\ \tau_k \neq s, \tag{17}$$

$$G(\tau_k + 0,\ \tau_k) = (E_n + P_k)G(\tau_k,\ \tau_k) + E_n,\ k = 0,\ \pm 1,\ \pm 2, \ldots, \tag{18}$$

$$|G(t,\ s)| \le \begin{cases} Ke^{-2\alpha(t-s)}, & t > s \\ K, & t \le s, \end{cases} \tag{19}$$

where $\alpha > 0$ *and* $K \ge 1$ *are constants.*

2) *If* $b_k \in \mathbf{R}^n$, $f \in PC(\mathbf{R}_+,\ \mathbf{R}^{n \times 1})$, $a \in \mathbf{R}^n$, $e_1 = \operatorname{col}(1,\ 0, \ldots,\ 0)$ *and*

$$\int_0^\infty |f(t)|dt + \sum_{k=1}^\infty |b_k| < \infty,$$

$$(a,\ e_1) = 0 \tag{20}$$

then the function

$$y(t,\ a) = Y(t)a + \int_0^\infty G(t,\ s)f(s)ds + \sum_{k=1}^\infty G(t,\ \tau_k)b_k \quad (t \in \mathbf{R}_+) \tag{21}$$

is a solution of the system

$$\begin{aligned} \frac{dy}{dt} &= P(t)y + f(t), \quad t \neq \tau_k, \\ \Delta y|_{t=\tau_k} &= P_k y + b_k, \qquad k = 1,\ 2, \ldots \end{aligned} \tag{22}$$

and

$$\lim_{t \to \infty} y(t,\ a) = 0. \tag{23}$$

3) *If, moreover,*

$$|f(t)| \le re^{-\alpha t},\ |b_k| \le re^{-\alpha \tau_k},\ t \in \mathbf{R}_+,\ \tau_k \in \mathbf{R}_+, \tag{24}$$

then the function

$$u(t) = \int_0^\infty G(t, s)f(s)ds + \sum_{k=1}^{\infty} G(t, \tau_k)b_k$$

satisfies the estimate

$$|u(t)| \leq Qre^{-\alpha t}, \ t \in \mathbf{R}_+, \tag{25}$$

where the constant $Q > 0$ *is independent of* r, $f(t)$ *and* b_k.

Proof. Equalities (15)–(18) follows immediately from formula (14).

From representation (7) and formula (14) it follows that

$$G(t, s) = \begin{cases} \psi(t)\mathrm{diag}(0_1, e^{C(t-s)})\psi_{(s)}^{-1}, \ t > s \\ -\psi(t)\mathrm{diag}(E_1, 0_{n-1})\psi^{-1}(s), \ t \leq s. \end{cases} \tag{26}$$

By Lemma 2 there exists a number $\alpha > 0$ such that

$$\mathrm{Re}\lambda_j(C) < -2\alpha < 0, \ j = 1, \ldots, n-1 \tag{27}$$

and the matrices $\psi(t)$ and $\psi^{-1}(t)$ are bounded on $\mathbf{R}$ (since they are periodic and belong to $PC(\mathbf{R}, \mathbf{R}^{n\times n})$). Then (26) and (27) imply estimate (19).

2) Since $|G(t, s)| \leq K$ for $t, s \geq 0$, then

$$\int_0^\infty |G(t, s)f(s)|ds + \sum_{k=1}^{\infty} |G(t, \tau_k)b_k| \leq K\int_0^\infty |f(s)|ds + K\sum_{k=1}^{\infty} |b_k| < \infty,$$

hence the improper integral and the series in (21) are absolutely convergent.

Write (21) in the form

$$y(t, a) = Y(t)a + \int_0^t G(t, s)f(s)ds + \int_t^\infty G(t, s)f(s)ds$$
$$+ \sum_{k=1}^{\infty} G(t, \tau_k)b_k.$$

After a differentiation with respect to $t \neq \tau_k$, in view of (15) and (16), we obtain

$$\frac{dy}{dt}(t, a) = \frac{dY}{dt}(t)a + [G(t, t-0) - G(t, t+0)]f(t)$$
$$+ \int_0^\infty \frac{\partial G}{\partial t}(t, s)f(s)ds + \sum_{k=1}^{\infty} \frac{\partial G}{\partial t}(t, \tau_k)b_k$$
$$= P(t)y(t, a) + f(t).$$

The differentiation is possible since the improper integral and the sum obtained as a result of the formal differentiation are convergent uniformly with respect to t belonging to any finite subinterval of $\mathbf{R}_+$.

Applying (17) and (18), we find that

$$y(\tau_i + 0,\ a) = Y(\tau_i + 0,\ a) + \int_0^\infty G(\tau_i + 0,\ s)f(s)ds$$
$$+ \sum_{\substack{k \neq 1 \\ k \neq i}}^{\infty} G(\tau_i + 0,\ \tau_k)b_k + G(\tau_i + 0,\ \tau_i)b_i$$
$$= (E_n + P_i)y(\tau_i,\ a) + b_i.$$

Hence $y(t,\ a)$ is a solution of (22).

In view of the structure of $Y(t)$ and of (20), we obtain that for $t \in \mathbb{R}_+$

$$|Y(t)a| \leq K|a|e^{-2\alpha t}. \tag{28}$$

Moreover,

$$0 \leq \left| \int_0^\infty G(t,\ s)f(s)ds + \sum_{k=1}^{\infty} G(t,\ \tau_k)b_k \right|$$
$$\leq \int_0^{t/2} |G(t,\ s)||f(s)|ds + \int_{t/2}^\infty |G(t,\ s)||f(s)|ds$$
$$+ \sum_{0<\tau_k<t/2} |G(t,\ \tau_k)||b_k| + \sum_{\tau_k \geq t/2} |G(t,\ \tau_k)||b_k|$$
$$\leq \int_0^{t/2} Ke^{-2\alpha(t-s)}|f(s)|ds + \int_{t/2}^\infty K|f(s)|ds$$
$$+ \sum_{0<\tau_k<t/2} Ke^{-2\alpha(t-\tau_k)}|b_k| + \sum_{\tau_k \geq t/2} K|b_k|$$
$$\leq Ke^{-\alpha t}\int_0^\infty |f(s)|ds + K\int_{t/2}^\infty |f(s)|ds$$
$$+ Ke^{-\alpha t}\sum_{k=1}^{\infty} |b_k| + K\sum_{\tau_k \geq t/2} |b_k| \to 0 \quad \text{as } t \to \infty. \tag{29}$$

Hence from (21), (28) and (29) it follows that relation (23) is fulfilled.

3) Let $f(t)$ and b_k satisfy estimates (24),

$$\min_k (\tau_k - \tau_{k-1}) = \theta \qquad \text{and } \tau_n < t \leq \tau_{n+1}.$$

Then the estimates

$$|u(t)| \leq \int_0^t Ke^{-2\alpha(t-s)}re^{-\alpha s}ds + \int_t^\infty Kre^{-\alpha s}ds$$
$$+ \sum_{k=1}^{n} Ke^{-2\alpha(t-\tau_k)}re^{-\alpha\tau_k} + \sum_{k=n+1}^{\infty} Kre^{-\alpha\tau_k};$$

$$e^{-2\alpha t}\int_0^t e^{\alpha s}ds + \int_t^\infty e^{-\alpha s}ds < \frac{2}{\alpha}e^{-\alpha t};$$

$$\begin{aligned} e^{-2\alpha t}\sum_{k=1}^{n} e^{\alpha\tau_k} &\le e^{-2\alpha t}\left[\frac{1}{\theta}\sum_{k=1}^{n-1}(\tau_{k+1}-\tau_k)e^{\alpha\tau_k} + e^{\alpha\tau_n}\right] \\ &\le e^{-2\alpha t}\left[\frac{1}{\theta}\int_{\tau_1}^{\tau_n} e^{\alpha s}ds + e^{\alpha\tau_n}\right] \\ &\le e^{-2\alpha t}\left[\frac{1}{\alpha\theta}(e^{\alpha\tau_n} - e^{\alpha\tau_1}) + e^{\alpha\tau_n}\right] \\ &< e^{-2\alpha t}\left(\frac{1}{\alpha\theta}+1\right)e^{\alpha\tau_n} \le \frac{1+\alpha\theta}{\alpha\theta}e^{-\alpha t}; \end{aligned}$$

$$\begin{aligned} \sum_{k=n+1}^{\infty} e^{-\alpha\tau_k} &\le e^{-\alpha\tau_{n+1}} + \frac{1}{\theta}\sum_{k=n+2}^{\infty}(\tau_k - \tau_{k-1})e^{-\alpha\tau_k} \\ &\le e^{-\alpha\tau_{n+1}} + \frac{1}{\theta}\int_{\tau_{n+1}}^{\infty} e^{-\alpha s}ds \\ &= \frac{1+\alpha\theta}{\alpha\theta}e^{-\alpha\tau_{n+1}} \le \frac{1+\alpha\theta}{\alpha\theta}e^{-\alpha t} \end{aligned}$$

imply estimate (25) with $Q = \dfrac{2K}{\alpha\theta}(\alpha\theta + \theta + 1)$.

This completes the proof of Lemma 3.

Theorem 1. *Let conditions* (A) *hold.*

Then the T-periodic solution $p(t)$ *of system* (1) *is orbitally asymptotically stable and has the property asymptotic phase.*

Proof. Let in system (1) the change of the variables be realized

$$x = Sz + p(0),$$

where the non-singular matrix S is chosen so that

$$S^{-1}\frac{dp}{dt}(0) = e_1 = \mathrm{col}(1, 0, \ldots, 0).$$

As a result of this change we obtain a new system which has a T-periodic solution $\pi(t) = S^{-1}(p(t) - p(0))$. An immediate verification shows that the new system and the solution $\pi(t)$ satisfy conditions (A). Moreover, $\pi(0) = 0$, $\dfrac{d\pi}{dt}(0) = e_1$. That is why, without loss of generality, we assume that the solution $p(t)$ of system (1) satisfies the conditions

$$p(0) = 0, \quad \frac{dp}{dt}(0) = e_1. \tag{30}$$

For the system in variations (4) the conditions of Lemma 2 hold with $P(t) = \frac{\partial g}{\partial x}(p(t))$ and $P_k = N_k$. Hence system (4) has a fundamental matrix of the form

$$Y(t) = \psi(t)\operatorname{diag}(E_1, e^{Ct}), \tag{31}$$

where the matrix $\psi(\cdot) \in PC(\mathbf{R}, \mathbf{R}^{n\times n})$ is $2T$-periodic non-singular and $\operatorname{Re}\lambda_j(C) < 0$, $j = 1, \dots, n-1$.

From (31) we obtain that the first column of $Y(t)$ is a $2T$-periodic solution of system (4). But from conditions A2 and A6 it follows that this solution is proportional to the T-periodic solution $\frac{dp}{dt}(t)$ of system (4). Hence we can assume that

$$Y(t) = \left[\frac{dp}{dt}(t),\ Y_1(t)\right],$$

where $Y_1(t)$ is an $n \times (n-1)$-matrix. Moreover, (30) implies

$$Y(0) = [e_1,\ Y_1(0)]. \tag{32}$$

Let

$$G(t, s) = \begin{cases} Y(t)\operatorname{diag}(0_1, E_{n-1})Y^{-1}(s), & t > s \\ -Y(t)\operatorname{diag}(E_1, O_{n-1})Y^{-1}(s), & t \le s. \end{cases} \tag{33}$$

By Lemma 3 $G(t, s)$ satisfies estimate (19) and $Y(t)$ satisfies estimate (28) if $a \in \mathbf{R}^n$, $(a, e_1) = 0$.

Choose successively $h \in (0, H)$ by Lemma 1 and $\gamma > 0$ by condition A5.

First we shall prove that there exist constants $\eta_0 > 0$ and $B > 0$ such that for any $a \in D(\eta_0) = \{a \in R^n\colon (a, e_1) = 0, |a| < \eta_0\}$ system (1) has a solution $x(t) = x(t, a)$ which is defined for $t \in \mathbf{R}_+$, has points of discontinuity $t_k = t_k(a)$ and satisfies the estimates

$$|t_k - \tau_k| \le B|a|e^{-\alpha\tau_k} = \delta_k, \tag{34}$$

$$|x(t) - p(t)| \le B|a|e^{-\alpha t} \qquad \text{for } |t - \tau_k| > \delta_k. \tag{35}$$

For this purpose we construct the sequences

$$w_n(t) = w_n(t, a) \text{ and } t_k^n = t_k^n(a),\ n = 0, 1, 2, \dots$$

setting for $t \in R$ and $k = 1, 2, \dots$

$$W_0(t) = p(t),\ t_k^0 = \tau_k$$

after which we successively define

$$w_{n+1}(t) = p(t) + Y(t)a + \int_0^\infty G(t, s)f_n(s)ds + \sum_{k=1}^\infty G(t, \tau_k)b_k^n, \tag{36}$$

where

$$f_n(t) = F(w_n(t),\ t) = g(w_n(t)) - g(p(t)) - \frac{\partial g}{\partial x}(p(t))(w_n(t) - p(t)), \tag{37}$$

$$\begin{aligned} b_k^n = \beta_k(w_n(\tau_k)) &= I(z(t_k^n;\ \tau_k,\ w_n\tau_k))) - I(p(\tau_k)) \\ &+ \int_{t_k^n}^{\tau_k} [g(z(s;\ \tau_k,\ w_n(\tau_k + 0))) - g(z(s;\ \tau_k,\ w_n(\tau_k)))]ds \\ &- N_k(w_n(\tau_k) - p(\tau_k)); \end{aligned} \tag{38}$$

t_k^{n+1} we determine as the unique solution with respect to t of the system

$$\phi(z(t;\ \tau_k,\ w_{n+1}(\tau_k))) = 0 \tag{39}$$

i.e. $t_k^{n+1} = T_k(w_{n+1}(\tau_k))$.

Let $|a| < \dfrac{1}{2K}\min(h,\ \gamma)$. Then for $n = 0$ from (37), (38), (36) and (28) it follows that

$$|w_1(t) - p(t)| \le K|a|e^{-2\alpha t},\ t \ge 0. \tag{40}$$

Moreover, since $|w_1(\tau_k) - p(\tau_k)| \le K|a| < h$, then by Lemma 1 and Corollary 1 there exists a unique solution $t_k^1 = T_k(w_1(\tau_k))$ of the system $\phi(z(t;\ \tau_k,\ w_1(\tau_k))) = 0$ and the following estimate holds

$$\begin{aligned} |t_k^1 - t_k^0| = |t_k^1 - \tau_k| &= |T_k(w_1(\tau_k)) - T_k(p(\tau_k))| \\ &\le \tau|w_1(\tau_k) - p(\tau_k)| \le \tau K|a|e^{-2\alpha\tau_k}. \end{aligned} \tag{41}$$

Let $\mu \in (0,\ h)$. The analysis shows that the functions $F(w,\ t)$ and $\beta_k(w)$ satisfy the inequalities

$$|F(w,\ t) - F(u,\ t)| \le L(\mu)|w - u| \quad \text{for } |w - p(t)| \le \mu,\ |u - p(t)| \le \mu \tag{42}$$

and

$$|\beta_k(w) - \beta_k(u)| \le L(\mu)|w - u| \tag{43}$$

for $|w - p(\tau_k)| \le \mu$, $|u - p(\tau_k)| \le \mu$ where $\lim\limits_{\mu \to 0+} L(\mu) = 0$.

Choose $\mu \in (0,\ \dfrac{h}{K})$ and $\eta_0 > 0$ so that

$$L(\mu)Q = L(\mu)\ \frac{2K}{\alpha\theta}\ (\alpha\theta + \theta + 1) < \frac{1}{2},\quad 2K\eta_0 < \mu. \tag{44}$$

Then by induction with respect to n we prove that the members of the sequences $w_n(t)$, t_k^n can be determined successively and the following estimates hold

$$|w_n(t) - w_{n-1}(t)| \le K|a|2^{1-n}e^{-\alpha t}, \tag{45}$$

$$|w_n(t) - p(t)| \le 2K|a|e^{-\alpha t}, \tag{46}$$

$$|t_k^n - t_k^{n-1}| \le \tau K|a|2^{1-n}e^{-\alpha\tau_k}, \tag{47}$$

$$|t_k^n - \tau_k| \le 2\tau K e^{-\alpha\tau_k} \tag{48}$$

for $n = 1, 2, \ldots, m$.

In fact, from (40) and (41) it follows that estimates (45)–(48) hold for $n = 1$. Let these estimates hold for $n = 1, \ldots, m$. Then $|w_k(t) - p(t)| < \mu$, $k = 1, \ldots, m$ and from (42), (43) and (45) we find

$$\begin{aligned}|f_m(t) - f_{m-1}(t)| &= |F(w_m(t), t) - F(w_{m-1}(t), t)| \\ &\le L(\mu)|w_m(t) - w_{m-1}(t)| \le L(\mu)K|a|2^{1-m}e^{-\alpha t}\end{aligned}$$

and

$$|b_k^m - b_k^{m-1}| \le L(\mu)K|a|2^{1-m}e^{-\alpha\tau_k}.$$

After this, by assertion 3 of Lemma 3, in view of (36) and (44), we obtain

$$\begin{aligned}|w_{m+1}(t) - w_m(t)| &= \left|\int_0^\infty G(t, s)[f_m(s) - f_{m-1}(s)]ds + \sum_{k=1}^\infty G(t, \tau_k)[b_k^m - b_k^{m-1}]\right| \\ &\le QL(\mu)K|a|2^{1-m}e^{-\alpha t} \le K|a|2^{-m}e^{-\alpha t}. \end{aligned}\tag{49}$$

From (49) it follows immediately that

$$\begin{aligned}|w_{m+1}(t) - w_m(t)| &\le \sum_{j=1}^{m+1} |w_j(t) - w_{j-1}(t)| \\ &\le K|a|(1 + 2^{-1} + 2^{-2} + \cdots)e^{-\alpha t} = 2K|a|e^{-\alpha t}.\end{aligned}$$

In particular, $|w_{m+1}(\tau_k) - p(\tau_k)| < 2K|a| < h$ and by Lemma 1 equation (39) has a unique solution $t_k^{m+1} = T_k(w_{m+1}(\tau_k))$ for which by corollary 1 we have

$$\begin{aligned}|t_k^{m+1} - t_k^m| &= |T_k(w_{m+1}(\tau_k)) - T_k(w_m(\tau_k))| \le \tau|w_{m+1}(\tau_k) - w_m(\tau_k)| \\ &\le \tau K|a|2^{-m}e^{-\alpha\tau_k}.\end{aligned}$$

Then

$$|t_k^{m+1} - \tau_k| \le \sum_{j=1}^{m+1} |t_k^j - t_k^{j-1}| \le 2\tau K|a|e^{-\alpha\tau_k}.$$

Thus estimates (45)–(48) hold for $n = m + 1$, hence for each $n = 1, 2, \ldots$.

From (45) and (47) it follows that the sequences $w_n(t)$ and t_k^n are convergent uniformly on $t \in \mathbb{R}$ and $k = 1, 2, \ldots$. Let $w(t) = w(t, a)$ and $t_k = t_k(a)$ be their limits. Then

$$|w(t) - p(t)| \leq 2K|a|e^{-\alpha t}, \tag{50}$$

$$|t_k - \tau_k| \leq 2\tau K|a|e^{-\alpha\tau_k} \tag{51}$$

and

$$w(t) = p(t) + Y(t)a + \int_0^\infty G(t, s)f(s, a)ds + \sum_{k=1}^\infty G(t, \tau_k)b_k(a), \tag{52}$$

where

$$f(t, a) = g(w(t)) - g(p(t)) - \frac{\partial g}{\partial x}(p(t))(w(t) - p(t)) \tag{53}$$

and

$$\begin{aligned} b_k(a) = {} & I(z(t_k;\ \tau_k,\ w(\tau_k))) - I(p(\tau_k)) \\ & + \int_{t_k}^{\tau_k} [g(z(s;\ \tau_k,\ w(\tau_k + 0))) - g(z(s;\ \tau_k,\ w(\tau_k)))]ds \\ & - N_k(w(\tau_k) - p(\tau_k)). \end{aligned} \tag{54}$$

In view of assertion 2 of Lemma 3 we obtain that $w(t)$ is a solution of the system with impulse effect in fixed moments of time

$$\begin{aligned} & \frac{dw}{dt} = g(w),\ t \neq \tau_k, \\ & \Delta w|_{t=\tau_k} = I(z(t_k;\ \tau_k,\ w(\tau_k))) \\ & \qquad + \int_{t_k}^{\tau_k} [g(z(s;\ \tau_k,\ w(\tau_k + 0))) - g(z(s;\ \tau_k,\ w(\tau_k)))]ds. \end{aligned} \tag{55}$$

Define the function $x(t) = x(t, a)$ by the formula

$$x(t) = \begin{cases} w(t) & \text{if } t \bar{\in} [\tau_k;\ t_k], \\ z(t;\ \tau_k,\ w(\tau_k + 0)) & \text{if } t_k < t \leq \tau_k, \\ z(t;\ \tau_k,\ w(\tau_k)) & \text{if } \tau_k \leq t < t_k, \end{cases} \tag{56}$$

$$x(t_k) = x(t_k - 0). \tag{57}$$

From (55)–(57) it follows that the function $x(t)$ is differentiable for $t \neq \tau_k$ and satisfies the system

$$\frac{dx}{dt} = g(x),\ t \neq \tau_k.$$

Moreover,

$$\Delta x(t_k) = \begin{cases} z(t_k;\ \tau_k,\ w(\tau_k + 0)) - w(t_k) & \text{if } t_k < \tau_k, \\ w(t_k) - z(t_k;\ \tau_k,\ w(\tau_k)) & \text{if } t_k > \tau_k, \\ \Delta w(\tau_k) & \text{if } t_k = \tau_k. \end{cases} \tag{58}$$

Let $t_k < \tau_k$. Then

$$\begin{aligned} \Delta x(t_k) &= z(t_k;\ \tau_k,\ w(\tau_k + 0)) - w(t_k) \\ &= \Delta w(\tau_k) + z(t_k;\ \tau_k,\ w(\tau_k + 0)) - w(\tau_k + 0) + w(\tau_k) - w(t_k) \\ &= I(z(t_k;\ \tau_k,\ w(\tau_k))) + \int_{t_k}^{\tau_k} [g(z(s;\ \tau_k,\ w(\tau_k + 0))) - g(z(s;\ \tau_k,\ w(\tau_k)))]\,ds \\ &\quad + \int_{\tau_k}^{t_k} g(z(s;\ \tau_k,\ w(\tau_k + 0)))\,ds - \int_{\tau_k}^{t_k} g(z(s;\ \tau_k,\ w(\tau_k)))\,ds \\ &= I(w(t_k)) = I(x(t_k)). \end{aligned}$$

By analogous calculations it is proved that in the cases $t_k > \tau_k$ and $t_k = \tau_k$ the following equality holds as well:

$$\Delta x(t_k) = I(x(t_k)).$$

Taking into account $z(t_k{:}\ \tau_k,\ w(\tau_k)) = x(t_k)$ and passing to the limit in the equality $\phi(z(t_k^n;\ \tau_k,\ w_n(\tau_k))) = 0$, we get $\phi(x(t_k)) = 0$, i.e. t_k are moments of impulse effect for $x(t)$. Moreover, in view of condition A5, from (50) and (56) it follows that $x(t)$ has no other moments of impulse effect.

Hence the function $x(t)$ is a solution of system (1) with moments of impulse effect t_k. By (56), (50) and (51) the solution $x(t)$ and the moments t_k satisfy (34) and (35).

We shall find a relation between the initial values $x^0 = \operatorname{col}(x_1^0, \ldots, x_n^0) = x(0, a)$ of the solutions $x(t, a)$ and the parameter $a \in D(\eta_0)$. Put $t = 0$ into (56) and, in view of (52) and (30), obtain

$$x^0 = Y(0)a + \int_0^\infty G(0,\ s)f(s,\ a)\,ds + \sum_{k=1}^\infty G(0,\ \tau_k)b_k(a), \tag{59}$$

where $f(t,\ a)$ and $b_k(a)$ are given by formulae (53) and (54).

From equality (32) it follows that

$$Y(0)\ \operatorname{diag}(E_1,\ 0_{n-1}) = \operatorname{diag}(E_1,\ 0_{n-1}). \tag{60}$$

Let c_{ij} and C_{ij} $(i, j = 1, \ldots, n)$ be respectively the elements of the matrix $Y(0)$ and their cofactors.

Taking into account (60) and (33), we conclude that equation (59) has the form

$$x_1^0 = \sum_{j=2}^n c_{1j}a_j + \xi(a_2, \ldots,\ a_n), \tag{61}$$

$$x_i^0 = \sum_{j=2}^{n} c_{ij} a_j \qquad (i = 2, \dots, n), \tag{62}$$

where

$$\xi(a_2, \dots, a_n) = \left[\int_0^\infty G(0, s) f(s, a) ds + \sum_{k=1}^{\infty} G(0, \tau_k) b_k(a) \right]_1$$

(Here $[x]_1$ denotes the first coordinate of the vector x).

Since the determinant Δ of system (62) is equal to

$$\Delta = C_{11} = \det Y(0) \neq 0$$

then system (62) is solvable with respect to $a_2, \dots, a_n$, i.e.

$$a_j = \sum_{k=2}^{n} d_{jk} x_k^o \qquad (j = 2, \dots, n), \tag{63}$$

where d_{jk} are constants.

Substitute the result obtained into (61) and obtain

$$x_1^0 = \sum_{k=2}^{n} h_k x_k^0 + \tilde{\xi}(x_2^0, \dots, x_n^0), \tag{64}$$

where h_k are constants and $\tilde{\xi}(x_2^0, \dots, x_n^0) = \xi(a_2, \dots, a_n)$.

In view of (53), (54) and estimates (50), (51) we conclude that the function $\xi(a_2, \dots, a_n)$ is differentiable in some neighbourhood of the point $0 \in D(\eta_0)$ and $\xi(a_2, \dots, a_n) = o(|a|)$ *as* $|a| \to 0$.

Then, in virtue of (63), the function $\tilde{\xi}(x_2^0, \dots, x_n^0)$ is differentiable in some neighbourhood $(x_2^0)^2 + \cdots + (x_n^0)^2 < r^2$ and

$$\tilde{\xi}(x_2^0, \dots, x_n^0) = O(|x|) \qquad as\ |x| \to 0. \tag{65}$$

This enables us to formulate the following assertions:

I. The graph of the function

$$x_1^0 = \sum_{k=2}^{n} h_k x_k^0 + \tilde{\xi}(x_2^0, \dots, x_n^0) \quad ((x_2^0)^2 + \cdots + (x_2^0)^2 < r^2)$$

defines in $\mathbf{R}^n$ a smooth hypersurface S and $0 \in S$.

II. The function

$$\varphi(x) = x_1 - \sum_{k=2}^{n} h_k x_k - \tilde{\xi}(x_2, \dots, x_n)$$

is differentiable in the neighbourhood $|x| < r$ and by (64), (65) and (30) we have

$$\varphi(0) = 0,$$

$$\frac{\partial \varphi}{\partial x}(0) = \mathrm{col}(1, \ -h_1, \dots, \ -h_n) \neq 0,$$

$$\frac{\partial\varphi}{\partial x}(0)g(p(0)) = \frac{\partial\varphi}{\partial x}(0)\frac{dp}{dt}(0) = 1 \neq 0.$$

III. For any $\rho \in (0, r)$ there exists $\rho_1 \in (0, \rho)$ such that if $w \in \mathbb{R}^n$, $|w| < \rho_1$, then the equation

$$\varphi(z(t;\ 0,\ w)) = 0$$

has a unique solution $\hat{t} = \hat{t}(w) \in [-r, r]$ and

$$|z(t;\ 0,\ w)| < \rho \qquad \text{for } t \in [0;\ \hat{t}].$$

Note that in the proof of Assertion III the arguments are as in the proof of Lemma 1.

Now let $\varepsilon > 0$ and $\eta > 0$ be given and $t_0 \in \mathbb{R}_+$, $|t_0 - \tau_k| > \eta$.

Choose successively the numbers:

$\eta_1 \in (0, \eta_0)$ so that $2\tau K \eta_1 < \eta_0$;

$\rho \in (0, \min(r, \varepsilon))$ so that if $(x_2^0)^2 + \cdots + (x_n^0)^2 < \rho^2$, then for the numbers a_j defined by (63), we should have

$$a_2^2 + \cdots + a_n^2 < \eta_1^2;$$

$\rho_1 \in (0, \rho)$ so that Assertion III should hold.

Let $iT - T < t_0 < iT$. Then $0 < T_0 = iT - t_0 < T$ and

$$p(t_0 + T_0) = p(iT) = p(0) = 0. \tag{66}$$

From Remark 3 and the continuous dependence of the solution of system (1) on the initial data it follows that there exists $\lambda > 0$ such that for any $x_0 \in \mathbb{R}^n$, $|x_0 - p(t_0)| < \lambda$ we have

$$\rho(x(t;\ t_0,\ x_0),\ L) < \varepsilon \qquad \text{for } t \in [t_0,\ t_0 + T_0] \tag{67}$$

and

$$|x(t;\ t_0,\ x_0) - p(t)| < \rho_1 < \varepsilon \qquad \text{for } t \in [t_0,\ t_0 + T_0],\ |t - \tau_k| > \eta. \tag{68}$$

In particular, for $w = x(t_0 + T_0;\ t_0,\ x_0)$, in view of (68) and (66), we have $|w| < \rho_1$. Then by Assertion III there exists a unique $\hat{t} \in [-r, r]$ such that

$$z(\hat{t};\ 0,\ w) \in S$$

and

$$|z(t;\ 0,\ w)| < \rho \qquad \text{for } t \in [0;\ \hat{t}]. \tag{69}$$

Since system (1) is autonomous, then

$$\begin{aligned} x_1 &= x(t_0 + T_0 + \hat{t};\ t_0,\ x_0) = x(t_0 + T_0 + \hat{t};\ t_0 + T_0,\ x(t_0 + T_0;\ t_0,\ x_0)) \\ &= x(t_0 + T_0 + \hat{t};\ t_0 + T_0,\ w) = x(\hat{t};\ 0,\ w) = z(\hat{t};\ 0,\ w). \end{aligned}$$

Hence $x_1 \in S$ and $|x_1| < \rho$ and to x_1 by (63) corresponds $a^* \in D(\eta_1)$. Then the solution $x(t;\, 0,\, x_1) = x(t,\, a^*)$ of system (1) and its moments of impulse effect t_k^* satisfy the estimates

$$|t_k^* - \tau_k| \leq \delta_k = 2\tau K|a^*|e^{-\alpha\tau_k} < 2\tau K\eta_1 < \eta,$$

$$|x(t;\, 0,\, x_1) - p(t)| < 2K|a^*|e^{-\alpha t} \qquad \text{for } t \geq 0,\ |t - \tau_k| > \delta_k.$$

Having taken into account $p(t) = p(t + t_0 + T_0)$,

$$x(t;\, 0,\, x_1) = x(t + t_0 + T_0 + \hat{t};\, t_0,\, x_0),\ \tau_k + t_0 + T_0 = \tau_{k+iq}$$

and setting $s = t + t_0 + T_0$, we obtain

$$|x(s + \hat{t};\, t_0,\, x_0) - p(s)| \leq 2K|a^*|e^{-\alpha(s - t_0 - T_0)} \tag{70}$$

for $s > t_0 + T_0$, $|s - \tau_{k+iq}| > 2\tau K|a^*|e^{-\alpha\tau_k}$.

For $s \in [\tau_{k+iq} - \delta_k,\ \tau_{k+iq} + \delta_k]$ there holds either the estimate

$$|x(s + \hat{t};\, t_0, \quad x_0) - p(\tau_{k+iq} - \delta_k)| \leq 2K|a^*|((1 + \tau d)e^{2\alpha\eta} + \tau d e^{\alpha\eta})e^{-\alpha(s - t_0 - T_0)} \tag{71}$$

or the estimate

$$|x(s + \hat{t};\, t_0,\, t_0) - p(\tau_{k+iq} + \delta_k)| \leq 2K|a^*|(1 + \tau d + \tau d e^{\alpha\eta})e^{-\alpha(s - t_0 - T_0)}, \tag{72}$$

where $d = \sup_{x \in \bar{D}} |g(x)|$.

Then from (70) it follows that the solution $p(t)$ has the property asymptotic phase, from (70)–(72) it follows that $p(t)$ is orbitally attractive and (69)–(72) and (67) imply that $p(t)$ is orbitally stable, i.e. it is orbitally asymptotically stable.

Remark 4. Let the manifold M divide the domain Ω into q disjoint parts:

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_q \cup M,$$

$$\Omega_i \cap \Omega_j = \varnothing,\ \Omega_i \cap M = \varnothing,\ i,\ j = 1, \ldots,\ q,\ i \neq j$$

and

$$\Gamma_k \subset \Omega_k \cup M,\ k = 1, \ldots,\ q.$$

Let the function g_k: $\Omega \to \mathbf{R}^n$, $k = 1, \ldots,\ q$ be differentiable in $D_k(H)$ and continuous in $\bar{D}_k(H)$.

Then Theorem 1 holds if

$$g(x) = g_k(x) \qquad \text{for } x \in \Omega_k \cap \bar{D}_k(H)$$

or if condition A5 is replaced by the following condition A5′. For any $h \in (0,\, H)$ there exists $\gamma > 0$ such that:

$$\rho(x,\, M) \geq \gamma \text{ for } k = 1, \ldots,\ q \text{ and } x \in \Gamma_k \setminus (\bar{B}_h(p(\tau_k)) \cup \bar{B}_h(p(\tau_{k-1} + 0))).$$

$x + I(x) \in \Omega_k \cup M$ for $x \in M \cap B_h(p(\tau_{k-1}))$.

$\frac{\partial \phi}{\partial x}(p(\tau_{k-1}+0))g(p(\tau_{k-1}+0)) \neq 0,\ k = 1, \ldots, q.$

Consider the case $n = 2$. Assume that system (1) has a T-periodic solution $p(t)\left(\frac{dp}{dt} \neq 0\right)$ and in the interval $(0, T]$ the solution $p(t)$ has q moments of impulse effect. Let $Y(t)$ be the normalized for $t = 0$ fundamental matrix of the system in variations (4).

Then the multiplicators ρ_1, ρ_2 of system (4) satisfy the equation

$$\det(Y(T) - \rho E_2) = 0$$

or

$$\rho^2 - \rho \operatorname{Tr} Y(T) + \det Y(T) = 0. \tag{73}$$

Since system (4) has a non-trivial T-periodic solution $\frac{dp}{dt}$, then one of its multiplicators is $\rho_1 = 1$. From (73) and Viète's formulae it follows that

$$\rho_2 = \det Y(T),$$

or, in detail,

$$\rho_2 = \prod_{k=1}^{q} \det(E_2 + N_k) e^{\int_0^T \operatorname{Tr} \frac{\partial g}{\partial x}(p(t))dt} \tag{74}$$

Let system (1), in scalar notation, have the form

$$\left|\begin{array}{ll} \frac{dx}{dt} = P(x, y) & \\ \frac{dy}{dt} = Q(x, y) & \text{if } (x, y) \bar{\in} M, \end{array}\right.$$

$$\left|\begin{array}{ll} \Delta x = a(x, y) & \\ \Delta y = b(x, y) & \text{if } (x, y) \in M, \end{array}\right. \tag{75}$$

the set M is defined by the equation

$$\phi(x, y) = 0$$

and $p(t) = \operatorname{col}(\xi(t), \eta(t))$, $\left|\frac{d\xi}{dt}(t)\right| + \left|\frac{d\eta}{dt}(t)\right| \neq 0.$

Then system (4) assumes the form

$$\left|\begin{aligned} \frac{du}{dt} &= \frac{\partial P}{\partial x}(\xi(t),\ \eta(t))u + \frac{\partial P}{\partial y}(\xi(t))v,\ t \neq \tau_k, \\ \frac{dv}{dt} &= \frac{\partial Q}{\partial x}(\xi(t),\ \eta(t))u + \frac{\partial Q}{\partial y}(\xi(t),\ \eta(t),\ \eta(t))v,\ t \neq \tau_k, \end{aligned}\right. \tag{76}$$

$$\left|\begin{aligned} \Delta u|_{t=\tau_k} &= A_k u + B_k v, \quad k = 1,\ 2, \ldots, \\ \Delta v|_{t=\tau_k} &= C_k u + D_k v, \quad k = 1,\ 2, \ldots, \end{aligned}\right.$$

where the constants A_k, B_k, C_k and D_k have been calculated according to formula (5).

Then from (74) and (76) we obtain

$$\rho_2 = \prod_{k=1}^{q} \Delta_k e^{\int_0^T \left[\frac{\partial P}{\partial x}(\xi(t),\ \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t),\ \eta(t))\right] dt} \tag{77}$$

where $\Delta_k = \det(E_2 + N_k) = 1 + A_k + D_k + A_k D_k - B_k C_k$.

After detailed calculations we find that

$$\Delta_k = \frac{P_+\left(\frac{\partial b}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial b}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\right) + Q_+\left(\frac{\partial a}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right)}{P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y}}, \tag{78}$$

where P, Q, $\frac{\partial a}{\partial x}$, $\frac{\partial a}{\partial y}$, $\frac{\partial b}{\partial x}$, $\frac{\partial b}{\partial y}$, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$ have been calculated at the point $(\xi(\tau_k),\ \eta(\tau_k))$ and $P_+ = P(\xi(\tau_k + 0),\ \eta(\tau_k + 0))$,

$$Q_+ = Q(\xi(\tau_k + 0),\ \eta(\tau_k + 0)).$$

Thus for $n = 2$ the following corollary holds.

Corollary 2 (*Analogue of Poincaré's criterion*).

The solution $p(t) = \mathrm{col}(\xi(t),\ \eta(t))$ *of system* (75) *is orbitally asymptotically stable and has the property asymptotic phase if the multiplicator* ρ_2 *calculated by formula* (77) *satisfies the condition* $|\rho_2| < 1$.

Example 1. Consider the second order equation

$$\ddot{x} + 2h\dot{x} + \omega^2 x = 0 \quad (0 < h < \omega). \tag{79}$$

Let the solution $x(t)$ be subject to impulse effect at the moment $t = \tau_k$:

$$x(\tau_k + 0) = x(\tau_k),\ \dot{x}(\tau_k + 0) = \dot{x}(\tau_k) + b(x(\tau_k),\ \dot{x}(\tau_k)), \tag{80}$$

where τ_k is a moment of impulse effect if

$$x(\tau_k) = 0,\ \dot{x}(\tau_k) \geq 0. \tag{81}$$

The equation with impulse effect (79)–(81) represents the simplest mathematical model of a clock-work [12].

Set $\dot{x} = y$ and write down equations (79)–(81) in the form of a system:

$$\left|\begin{array}{ll} \dot{x} = y & \\ \dot{y} = -\omega^2 \times -2hy & \text{if } (x, y) \bar{\in} M \end{array}\right.$$

$$\left|\begin{array}{ll} \Delta x = 0 & \\ \Delta y = b(x, y) & \text{if } (x, y) \in M, \end{array}\right. \tag{82}$$

where M is defined by the conditions

$$\phi(x, y) \equiv x = 0, \; y \geq 0. \tag{83}$$

System (82)–(83) has a non-trivial T-periodic solution $x = \xi(t)$, $y = \eta(t)$ which coincides with the T-periodic continuation of the solution

$$x = \frac{y_0}{\beta} e^{-ht} \sin \beta t,$$

$$y = y_0 e^{-ht} (\cos \beta t - \frac{h}{\beta} \sin \beta t)$$

of (82) for $t \in (0, T]$, where y_0 is chosen so that

$$y_0 e^{-hT} + b(x(T), y(T)) = y_0$$

and $\quad T = 2\pi\beta^{-1}$, $\beta = \sqrt{\omega^2 - h^2}$.

In our case $q = 1$, $\tau_1 = T$, $x(T) = 0$, $x(T+0) = 0$,

$$y(T) = y_0 e^{-hT}, \; y(T+0) = y_0 e^{-hT} + b(x(T), y(T)) = y_0,$$

$$\frac{\partial a}{\partial x} = \frac{\partial a}{\partial y} = \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial x} = 1, \quad \frac{\partial P}{\partial x} = 0, \quad \frac{\partial Q}{\partial y} = -2h,$$

$$\int_0^T \left[\frac{\partial P}{\partial x}(\xi(t)), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t)) \right] dt = -2hT.$$

i) Let $b(x, y) = p > 0$.

Then $\dfrac{\partial b}{\partial x} = \dfrac{\partial b}{\partial y} = 0$,

$$\Delta_1 = \frac{P_+}{P} = \frac{y(T+0)}{y(T)} = \frac{y_0}{y_0 e^{-hT}} = e^{hT},$$

$$\rho_2 = e^{hT} e^{-2hT} = e^{-hT}.$$

ii) Let $b(x, y) = \sqrt{y^2 + \varepsilon} - y \quad (\varepsilon > 0)$.

Then

$$y_0e^{-hT}+\sqrt{y_0^2e^{-2hT}+\varepsilon}-y_0e^{-hT}=y_0,\ \frac{\partial b}{\partial x}=0,\ \frac{\partial b}{\partial y}=\frac{y}{\sqrt{y^2+\varepsilon}}-1$$

and

$$\Delta_1=\frac{P_+\left(\frac{\partial b}{\partial y}+1\right)}{P}=\frac{y(T+0)}{y(T)}\cdot\frac{y(T)}{\sqrt{y^2(T)+\varepsilon}}=\frac{Y_0}{\sqrt{y_0^2e^{-2hT}+\varepsilon}}=1.$$

Hence $\rho_2=e^{-2hT}$.

In the case (i) and (ii) $\rho_2\in(0,\ 1)$. Hence, by Corollary 2, the solution $x=\xi(t)$, $y=\eta(t)$ of the system (82) is orbitally asymptotically stable. It is to this solution that the normal work of the clock corresponds.

Example 2. The linear system with impulse effect

$$\frac{dx}{dt}=\begin{cases}\dfrac{kry}{krS-1}, & \text{if } |x|<\dfrac{V_s}{2}\\[2ex] -kry, & \text{if } |x|\geq\dfrac{V_s}{2}\end{cases}$$

$$\frac{dy}{dt}=\begin{cases}\dfrac{x}{krLC}+\dfrac{ry}{L(krS-1)}, & \text{if } |x|<\dfrac{V_s}{2}\\[2ex] \dfrac{x}{krLC}-\dfrac{ry}{L}, & \text{if } |x|\geq\dfrac{V_s}{2}\end{cases}\tag{84}$$

$$\Delta x=-2Skrx,\ \Delta y=-\frac{2Sr}{L}x,\ \text{if } |x|=\frac{V_s}{2},$$

models the work of the electronic scheme given on (Fig. 1) ([13], ch. V. § 17).

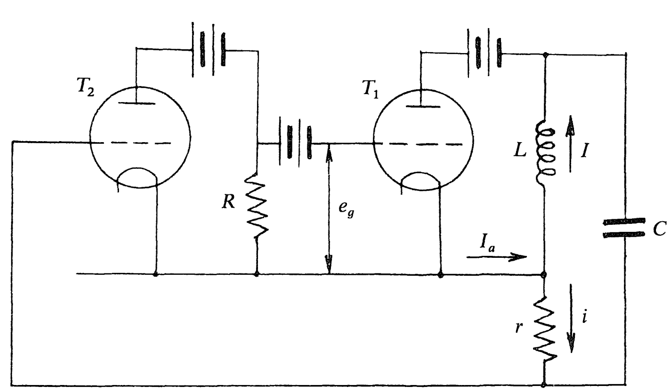

Fig.1

Here k is the amplification factor which depends on the parameters of the tube T_2 and on the anode resistance R, $x = e_g = kri$ is the lattice tension of the tube T_1; $y = \frac{dI}{dt}$; $S = \frac{I_s}{V_s}$ is the obliquity of the lattice characteristic of the tube T_1; I_s and V_s are respectively the current intensity and tension of satiation of the tube T_1; $krS - 1 > 0$.

In [13] it is proved that in the cases when L is small ($L \ll \frac{Cr^2}{4}$) or L is large ($L \gg \frac{Cr^2}{4}$) system (84) has a unique non-zero periodic solution $p(t)$ and the period T of this solution is estimated:

$$T \approx \frac{\pi}{\sqrt{\frac{1}{LC} + \frac{r^2}{4L^2}}} \qquad \left(\text{for } L \gg \frac{Cr^2}{4}\right)$$

$$T \approx 2\frac{L}{r}\ln(2Skr - 1) \qquad \left(\text{for } L \ll \frac{Cr^2}{4}\right)$$

The phase trajectory of $p(t)$ is given on Fig. 2 $\left(\text{for } L \gg \frac{Cr^2}{4}\right)$ and on Fig. 3 $\left(\text{for } L \ll \frac{Cr^2}{4}\right)$.

The motion of the mapping point $(x(t), y(t))$ is realized in the set defined by the inequality $|x| \geq \frac{V_s}{2}$. This motion is continuous from point A_2 to point A_3 and from point A_4 to point A_1 and by jumps from point A_1 into point A_2 and from point A_3 into point A_4 (Fig. 2, Fig. 3).

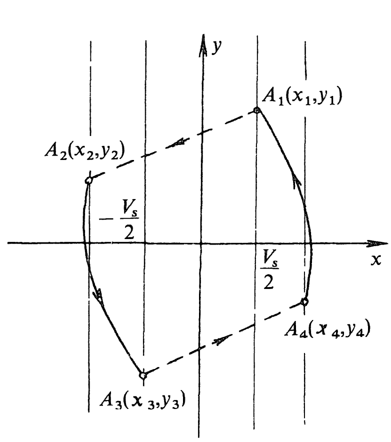

Fig.2

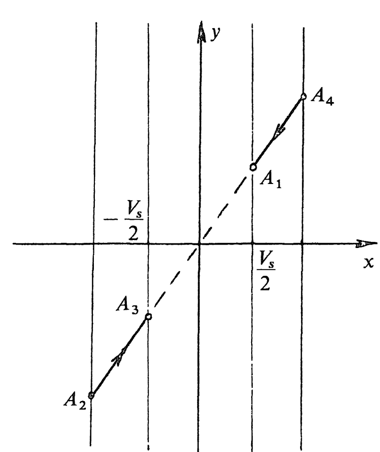

Fig.3

Using the above notations we obtain

$$P(x,\ y) = -\,kry,\ Q(x,\ y) = \frac{x}{krLC} - \frac{ry}{L},$$

$$a(x,\ y) = -\,2Skrx,\ b(x,\ y) = -\frac{2Sr}{L}x,\ \phi(x,\ y) = |x| - \frac{V_s}{2},$$

$$\frac{\partial P}{\partial x} = 0,\ \frac{\partial P}{\partial y} = -\,kr,\ \frac{\partial Q}{\partial x} = \frac{1}{krLC},\ \frac{\partial Q}{\partial y} = -\frac{r}{L},$$

$$\frac{\partial a}{\partial x} = -\,2Skr,\ \frac{\partial a}{\partial y} = 0,\ \frac{\partial b}{\partial x} = -\frac{2Sr}{L},\ \frac{\partial b}{\partial y} = 0,$$

$$\frac{\partial \phi}{\partial x}\left(\frac{V_s}{2},\ y\right) = 1,\ \frac{\partial \phi}{\partial x}\left(-\frac{V_s}{2},\ y\right) = -\,1,\ \frac{\partial \phi}{\partial y} = 0.$$

Then

$$\int_0^T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dt = -\frac{rT}{L},$$

$$\Delta_1 = \frac{P_+}{P} = \frac{P(x_2,\ y_2)}{P(x_1,\ y_1)} = \frac{y_1 - \frac{2Sr}{L}x_1}{y_1} = \frac{y_1 - \frac{r}{L}I_s}{y_1},$$

$$\Delta_2 = \frac{P_+}{P} = \frac{P(x_4,\ y_4)}{P(x_3,\ y_3)} = \frac{-\,y_1 - \frac{2Sr}{L}x_1}{y_1} = \Delta_1,$$

$$\rho_2 = \left(\frac{y_1 - \frac{r}{L}I_s}{y_1}\right)^2 e^{-\frac{rT}{2}}.$$

In the cases when $L \gg \frac{Cr^2}{4}$ or $L \ll \frac{Cr^2}{4}$, the condition $0 < \rho_2 < 1$ is satisfied. Then the periodic solution of system (84) is orbitally asymptotically stable.

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