On the p-Regularity of Stiefel Manifolds

Dedicated to Professor Shôrô Araki for his 60th birthday

By

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Introduction

A topological space which has the homotopy type of a finite CW-complex is called p-regular if its localization at a prime p is homotopy equivalent to a product of a certain number of spheres localized at p. We note that, if X is p-regular, the rational cohomology of X determines the number and dimensions of spheres which appear in the decomposition into the product. J.-P. Serre gave, in one of his celebrated papers ([19]), a necessary and sufficient condition for the p-regularity of classical Lie groups. For example, $U(n)$ (resp. $Sp(n)$) is p-regular if and only if $n \leq p$ (resp. $2n \leq p$). The purpose of this paper is to generalize the result of Serre to complex and quaternionic Stiefel manifolds. The p-regularity of Stiefel manifolds was also studied by Y. Hemmi ([6]) from the H-space theoretical point of view. He showed that, for an odd prime p, there are infinitely many complex and quaternionic Stiefel manifolds which are p-regular (see [6] Theorem 3.3 for details). Our result is somewhat stronger than his in the sense that, for an odd prime p, we determine whether a complex (quaternionic) Stiefel manifold is p-regular or not except for finitely many undecided cases, and we also deal with the case $p = 2$. In particular, we settle the p-regularity problem on complex Stiefel manifolds for $p = 2$ and 3, and there are still two (resp. six) undecided cases for $p = 5$ (resp. $p = 7$) (see Example 4.10).

Following James ([8]), we denote by $V_{n,k}$, $W_{n,k}$ and $X_{n,k}$ the real, complex and quaternionic Stiefel manifolds $O(n)/O(n - k)$, $U(n)/U(n - k)$ and $Sp(n)/Sp(n - k)$ respectively. For an odd prime p, by examining the action of the Steenrod operation P^1 on $H^*(W_{n+k,k}; \mathbf{F}_p)$ and $H^*(X_{n+k,k}; \mathbf{F}_p)$, we easily derive a necessary condition that the p-regularity of $W_{n+k,k}$ (resp. $X_{n+k,k}$) implies $k \leq p - 1$, or $k = p$ and $p|n$ (resp. $k \leq (p - 1)/2$, or $k = (p + 1)/2$ and $p|2n + 1$).

The main results of this paper are as follows.

Theorem 2.17. $W_{n+2,2}$ is not 2-regular unless $n = 0$ or 2, and $X_{n+2,2}$ is not 2-regular for any $n \geq 0$.

We define sets of integers $C_{n+k,k}(\mathbb{F})$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$) and $\Pi(n; p)$ (p: odd prime) by $C_{n+k,k}(\mathbb{F}) = \{i \in \mathbb{Z} \mid i = \sum_{j=1}^{k} \varepsilon_j(dn + dj - 1), \varepsilon_j = 0 \text{ or } 1\}$, where $d = 2$ if $\mathbb{F} = \mathbb{C}$ and $d = 4$ if $\mathbb{F} = \mathbb{H}$, $\Pi(n; p) = \{i \in \mathbb{Z} \mid i = 2pn + 2j(p - 1) \text{ for } j \geq 1, \text{ or } i = 2(r_n + 1)(p - 1) + 2pn - 1 \text{ if } r_n > 0\}$, where r_n is defined by $-n - 1 = pq_n + r_n$, $0 \leq r_n < p$. Note that $\Pi(n; p) = \{i \in \mathbb{Z} \mid i = 2pn + 2j(p - 1) \text{ for } j \geq 1\}$ in the case $r_n = 0$.

Theorem 4.7. i) *For* $k \leq p - 1$, $W_{n+k,k}$ *is* *p-regular* *if* $\bigcup_{j=0}^{k-1}(C_{n+k,k-j}(\mathbb{C}) \cap \Pi(n + j; p))$ *is empty. In particular, it is p-regular if* $n \geq (k^2 - 2p + 3)/(2p - 2k)$. *For* $k = p$, $W_{mp+p,p}$ *is p-regular if* $m \geq (p - 1)/2$.
ii) *For* $k \leq (p - 1)/2$, $X_{n+k,k}$ *is p-regular if* $\bigcup_{j=0}^{k-1}(C_{n+k,k-j}(\mathbb{H}) \cap \Pi(2n + 2j + 1; p))$ *is empty. In particular, it is p-regular if* $n \geq (2k^2 + k - 4p + 3)/(4p - 4k)$. *For* $k = p$, $X_{n+(p+1)/2,(p+1)/2}$ *is p-regular if* $p \mid 2n + 1$.

If follows from the above theorem that, for each odd prime p, almost all (= except for finitely many undecided cases) $W_{n+k,k}$ and $X_{n+k,k}$ are p-regular if the necessary condition we mentioned above is satisfied. And as a corollary of Theorem 2.17, we show that the attaching maps of the top cells of $W_{n+2,2}$ and $X_{n+2,2}$ produce two families of infinite number of elements of the 2-components of the unstable homotopy groups of spheres, and that the loop space $\Omega W_{n+2,2}$ (resp. $\Omega X_{n+2,2}$) has an exotic loop space structure if n is even and greater than two (resp. $n \equiv 6$ modulo 8). We also discuss the non-2-regularity of real Stiefel manifolds in the appendix and prove an analogous theorem.

We prove the non-2-regularity of $W_{n+2,2}$ and $X_{n+2,2}$ by showing that the classifying maps of spherical fibrations $S^{2n+1} \to W_{n+2,2} \to S^{2n+3}$ and $S^{4n+3} \to X_{n+2,2} \to S^{4n+7}$ are non-trivial. For this end, we examine the J-homomorphism in the unstable range. On the other hand, we prove the p-regularity for odd prime p under the condition of the above theorem by showing that spherical fibrations $S^{2n+2j+1} \to W_{n+k,k-j} \to W_{n+k,k-j-1}$ and $S^{4n+4j+3} \to X_{n+k,k-j} \to X_{n+k,k-j-1}$ are fiber homotopy equivalent to product fibrations for $j = 0, 1,..., k - 1$. To show this, we demonstrate the existence of left homotopy inverses of the canonical inclusions $S^{2n+2j+1} \to W_{n+k,k-j}$ and $S^{4n+4j+3} \to X_{n+k,k-j}$ by using obstruction theory.

In Section 1, we give a necessary and sufficient condition for the splitting of a quasiprojective space into a wedge of spheres at a prime p. In fact, we prove the following three conditions are equivalent; a) $k \leq p - 1$, or $k = p$ and $p \mid n$. b) $Q_{n+k,k}(\mathbb{C}) = \Sigma(\mathbb{CP}^{n+k-1}/\mathbb{CP}^{n-1})$ is homotopy equivalent to $\bigvee_{i=1}^{k} S^{2n+2i-1}$ at prime p. c) $\Omega W_{n+k,k}$ is homotopy equivalent to $\Omega(\prod_{i=1}^{k} S^{2n+2i-1})$ at prime p. A similar result is also obtained for the quaternionic case. In Section 2, we discuss the non-2-regularity of $W_{n+2,2}$ and $X_{n+2,2}$. We reduce the problem on the unstable J-homomorphism, which is examined by applying theorems of Toda ([21], Theorem 11.7) and Mahowald-Milgram ([10] Theorem A, [14] Theorem 8.5, 8.21). Section 3 is a preparation for Section 4. We compute some p-

localized homotopy groups of the homotopy fiber of the natural inclusion S^{2n+1} $\rightarrow QS^{2n+1}$. We apply the Adams-Novikov spectral sequence based on the BP-homology theory to the computation, and determine the necessary and sufficient condition for the vanishing of "obstruction groups" $\pi_i(QS^{2n+1}, S^{2n+1})$ for sufficiently many i's. In Section 4, we first show the existence of maps $f:(W_{n+k,k}, S^{2n+1}) \rightarrow (QS^{2n+1}, S^{2n+1})$ and $g:(X_{n+k,k}, S^{4n+3}) \rightarrow (QS^{4n+3}, S^{4n+3})$ such that the restrictions to S^{2n+1} and S^{4n+3} are maps of degree 1 if n and k satisfy the necessary condition. And by applying obstruction theory, we see that these maps can be deformed to left homotopy inverses of the canonical inclusions. We prove in the appendix that the real Stiefel manifold $V_{n+2,2}$ is not 2-regular unless $n = 0$, 2 or 6.

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Notations and Terminology

When we use the term "at prime p" in a statement, this means that the statement is valid if we localize spaces and groups in the statement. We denote the mod p homology (resp. cohomology) of X by $H_*(X)$ (resp. $H^*(X)$) when the prime p is clear from the context. Otherwise, we denote by $H_*(X; R)$ (resp. $H^*(X; R)$) the homology (resp. cohomology) of X with coefficients in R. We list some notations which are frequently used in this paper.

$V_{n,k}, W_{n,k}$ and $X_{n,k}$: the real, complex and quaternionic Stiefel manifolds
$\qquad\qquad O(n)/O(n-k), U(n)/U(n-k)$ and $Sp(n)/Sp(n-k)$ respectively.
$R\{x_1, x_2, ...\}$: the free R-module generated by $x_1, x_2, ...$.
$\mathbf{F}_p[x_1, x_2, ...]$: the polynomial algebra over $\mathbf{F}_p$ generated by $x_1, x_2, ...$.
$E(x_1, x_2, ...)$: the exterior algebra over $\mathbf{F}_p$ generated by $x_1, x_2, ...$.
$M_{(p)}$: the localization at prime p of an abelian group M.
$\binom{n}{k}$: the binomial coefficient $n(n-1).....(n-k+1)/k!$.
$[x]$: the Gauss symbol of $x \in \mathbf{R}$, that is, the largest integer that does not exceed x.
$k|n$ $(k, n \in \mathbf{Z})$; k divides n.
$Q_{n,k}(\mathbf{F})$: the quasiprojective space over field $\mathbf{F}$ ($\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$).
$\mathbf{P}_n^m$: the stunted real projective space $\mathbf{RP}^m/\mathbf{RP}^{n-1}$.
$F(n)$: the homotopy fiber of the natural inclusion $S^n \rightarrow QS^n$.
$\pi_r^S(X)$: the stable homotopy group of a space (spectrum) X.

§1. Splitting of Quasiprojective Spaces

Let $Q_{n,k}(\mathbb{F})$ be the quasiprojective space over a field $\mathbb{F}(\mathbb{F} = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H})$ defined in [7](see also [8], [20]). We identify $Q_{n,k}(\mathbb{R})$, $Q_{n,k}(\mathbb{C})$ and $Q_{n,k}(\mathbb{H})$ with subcomplexes of $V_{n,k}$, $W_{n,k}$ and $X_{n,k}$ respectively, then the pair of CW-complexes $(W_{n+k,k}, Q_{n+k,k}(\mathbb{C}))$ (resp. $(X_{n+k,k}, Q_{n+k,k}(\mathbb{H}))$) is $(4n + 3)$-connected (resp. $(8n + 9)$-connected), and the pair $(V_{n+k,k}, Q_{n+k,k}(\mathbb{R}))$ is $2n$-connected at prime 2. Thus quasiprojective spaces are approximations of Stiefel manifolds in the stable range. The mod p cohomology of $Q_{n+k,k}(\mathbb{F})(\mathbb{F} = \mathbb{C} \text{ or } \mathbb{H})$ and the action of the Steenrod operations are given by the following.

Lemma 1.1. i) $\tilde{H}^*(Q_{n+k,k}(\mathbb{C})) = \mathbb{F}_p\{x_{2n+1}, x_{2n+3}, \ldots, x_{2n+2i-1}, \ldots, x_{2n+2k-1}\}$, $\deg x_i = i$. $P^s x_{2n+2i-1} = \binom{n+i-1}{s} x_{2n+2i+2s(p-1)-1}$ (*We put* $P^s = Sq^{2s}$ *if* $p = 2$).

ii) $\tilde{H}^*(Q_{n+k,k}(\mathbb{H})) = \mathbb{F}_p\{x_{4n+3}, x_{4n+7}, \ldots, x_{4n+4i-1}, \ldots, x_{4n+4k-1}\}$, $\deg x_i = i$. $P^s x_{4n+4i-1} = \binom{2n+2i-1}{s} x_{4n+4i+2s(p-1)-1}$ (*If* $p = 2$, *we put* $P^s = Sq^{2s}$ *as above, and* $x_i = 0$ *if* $i \not\equiv 3$ *modulo* 4).

Proof. Since $Q_{n+k,k}(\mathbb{C})$ is a suspension of a stunted complex projective space $\mathbb{C}P^{n+k-1}/\mathbb{C}P^{n-1}$, i) follows from the action of P^s on $H^*(\mathbb{C}P^\infty)$ ([20], Chap. I, VI). For ii), there is a map $c': Q_{n+k,k}(\mathbb{H}) \to Q_{2n+2k,2k}(\mathbb{C})$ induced by the canonical inclusion $Sp(n) \to U(2n)$ and $c'^*: \tilde{H}^*(Q_{2n+2k,2k}(\mathbb{C})) \to \tilde{H}^*(Q_{n+k,k}(\mathbb{H}))$ maps $x_{4n+4i-1}$ to $x_{4n+4i-1}$, and $x_{4n+4i-3}$ to zero. Hence ii) follows from i).

In particular, we have $P^1 x_{2n+2i-1} = (n+i-1)x_{2n+2(i+p-1)-1}$ in $\tilde{H}^*(Q_{n+k,k}(\mathbb{C}))$ and $P^1 x_{4n+4i-1} = (2n+2i-1)x_{4n+4i+2(p-1)-1}$ for $p > 2$, $Sq^4 x_{4n+4i-1} = (n+i-1)x_{4n+4i+3}$ for $p = 2$ in $\tilde{H}_*(Q_{n+k,k}(\mathbb{H}))$. Therefore

Corollary 1.2. *The action of the Steenrod operations on* $\tilde{H}^*(Q_{n+k,k}(\mathbb{F}))$ *is trivial in the following cases and non-trivial otherwise;*
For $\mathbb{F} = \mathbb{C}$, $k \leq p - 1$, *or* $k = p$ *and* $p|n$.
For $\mathbb{F} = \mathbb{H}$, $k \leq (p-1)/2$, *or* $k = (p+1)/2$ *and* $p|2n+1$ *if* $p > 2$. *In the case* $p = 2$, $k = 1$, *or* $k = 2$ *and* $2|n$.

We consider the case $p = k = 2$ first. Let $\Delta_C: \pi_{2n+3}(S^{2n+3}) \to \pi_{2n+2}(S^{2n+1})$ and $\Delta_H: \pi_{4n+7}(S^{4n+7}) \to \pi_{4n+6}(S^{4n+3})$ be the connecting homomorphisms of the long exact sequence associated with the spherical fibrations $S^{2n+1} \to W_{n+2,2} \to S^{2n+3}$, $S^{4n+3} \to X_{n+2,2} \to S^{4n+7}$ respectively. We denote by $\iota_n, \eta_n, \nu', \nu_n$ the generators of $\pi_n(S^n)$, $\pi_{n+1}(S^n)$, $\pi_6(S^3)$ and $\pi_{n+3}(S^n)$ $(n \geq 4)$ respectively. Then $Q_{n+2,2}(\mathbb{C})$ and $Q_{n+2,2}(\mathbb{H})$ are cofibers of maps $\Delta_C \iota_{2n+3}: S^{2n+2} \to S^{2n+1}$ and $\Delta_H \iota_{2n+7}: S^{4n+6} \to S^{4n+3}$. Since Sq^2 determines whether $\Delta_C \iota_{2n+3}$ is trivial or not, we have

Theorem 1.3. $Q_{n+2,2}(\mathbf{C})$ *is homotopy equivalent to* $S^{2n+1} \vee S^{2n+3}$ *at prime 2 if and only if n is even. And if n is even, $Q_{n+2,2}(\mathbf{C})$ is (globally) homotopy equivalent to* $S^{2n+1} \vee S^{2n+3}$.

In the quaternionic case, the condition of Corollary 1.2 does not necessarily imply the splitting of $Q_{n+2,2}(\mathbf{H})$ at prime 2, because Sq^4 cannot detect all elements of $\pi_{4n+6}(S^{4n+3})$. We need the following result of James.

Theorem 1.4 ([7], [8]). $\Delta_H l_{4n+7} = \begin{cases} v' & \text{for } n = 0 \\ (n+2)v_{4n+3} & \text{for } n \geq 1 \end{cases}$

Since v_{4n+3} has order 24, the above immediately implies

Theorem 1.5. $Q_{n+2,2}(\mathbf{H})$ *is homotopy equivalent to* $S^{4n+3} \vee S^{4n+7}$ *at prime 2 if and only if $n \equiv 6$ modulo 8, and it is homotopy equivalent to $S^{4n+3} \vee S^{4n+7}$ at prime 3 if and only if $n \equiv 1$ modulo 3. Moreover, if $n \equiv 22$ modulo 24, $Q_{n+2,2}(\mathbf{H})$ is globally homotopy equivalent to* $S^{4n+3} \vee S^{4n+7}$.

And also Theorem 1.4 implies

Theorem 1.6. $X_{n+2,2}$ *is not 2-regular if $n \not\equiv 6$ modulo 8.*

Proof. In fact, if $X_{n+2,2}$ is 2-regular, $\pi_{4n+6}(X_{n+2,2})_{(2)} \cong \pi_{4n+6}(S^{4n+3} \times S^{4n+7})_{(2)}$

$\cong \mathbf{Z}/8$. On the other hand, $\pi_{4n+6}(X_{n+2,2})_{(2)} \cong (\pi_{4n+6}(S^{4n+3})/\text{Im } \Delta_H)_{(2)} \cong \mathbf{Z}/(8, n+2)$

which is not isomorphic to $\mathbf{Z}/8$ if $n \not\equiv 6$ modulo 8.

We therefore consider the following conditions (C), (H) on p, k, n instead of those of Corollary 1.2.

(C) $k \leq p - 1$, *or* $k = p$ *and* $p|n$

(H) $k \leq (p-1)/2$, *or* $k = (p+1)/2$ *and* $p|2n + 1$ *if* $p > 2$,

   $k = 2$ *and* $n \equiv 6$ *modulo 8 if* $p = 2$.

Lemma 1.7. *Let p be an odd prime and let X be a space of finite type. Suppose that X is $2n$-connected at prime p and $H_{2n+2i}(X; \mathbf{F}_p) = 0$ for $i < p - 1$, then $\pi_{2n+2i}(X)_{(p)} = 0$ for $i < p - 1$.*

Proof. We may assme that $\pi_{2n+j}(X)_{(p)} \neq 0$ for some $j < 2(p-1)$, otherwise there is nothing to prove. Then, by the assumption, $\min\{j | \pi_{2n+j}(X)_{(p)} \neq 0\}$ is odd, and we define an integer $i(X)$ by $2i(X) + 1 = \min\{j | \pi_{2n+j}(X)_{(p)} \neq 0\}$. We prove the assertion by induction on $l = p - 2 - i(X)$. First we note that $l \geq 0$ by the above assumption and that the assertion holds if $l = 0$. Considering the universal coefficient theorem, it follows from the assumption that $H_{2n+2i+1}(X; \mathbf{Z}_{(p)})$ is torsion free for $i < p - 2$. Hence so is the lowest non-trivial homotopy

group $\pi_{2n+2i(X)+1}(X)_{(p)}$. Applying the killing method to X, we have a fibering $X' \to X \to K(\mathbb{Z}^r, 2n + 2i(X) + 1)$ (r = the rank of $\pi_{2n+2i(X)+1}(X)_{(p)}$) such that f_*: $\pi_{2n+2i(X)+1}(X)_{(p)} \to \pi_{2n+2i(X)+1}(K(\mathbb{Z}^r, 2n + 2i(X) + 1))_{(p)}$ is an isomorphism. Then, the standard argument on the Serre spectral sequence associated with this fibering shows that X' is $(2n + 2i(X) + 2)$-connected at prime p and $H_{2n+2j}(X'; \mathbb{F}_p) = 0$ for $j < p - 1$. Thus $i(X') > i(X)$ and $p - 2 - i(X') < l$, and the inductive assuption applies to X'.

Let $f_{n,i}: S^{2n+2i-2} \to Q_{n+i-1,i-1}(\mathbb{C})$ and $g_{n,i}: S^{4n+4i-2} \to Q_{n+i-1,i-1}(\mathbb{H})$ be the attaching maps of $(2n + 2i - 1)$-cell and $(4n + 4i - 1)$-cell. Note that quasi-projective spaces $Q_{n+i-1,i-1}(\mathbb{C})$ and $Q_{n+i-1,i-1}(\mathbb{H})$ satisfy the assumptions of the preceding lemma. It follows that $f_{n,i} \simeq 0$ if $i < p$ and $g_{n,i} \simeq 0$ if $i < (p + 1)/2$, when we localize spaces at prime p.

Theorem 1.8. *Let p be an odd prime. $Q_{n+k,k}(\mathbb{C})$ is homotopy equivalent to $\bigvee_{i=1}^{k} S^{2n+2i-1}$ at prime p if $k < p$, or $k = p$ and $p|n$. $Q_{n+k,k}(\mathbb{H})$ is homotopy equivalent to $\bigvee_{i=1}^{k} S^{4n+4i-1}$ at prime p if $k < (p + 1)/2$, or $k = (p + 1)/2$ and $p|2n + 1$.*

Proof. In the case $\mathbb{F} = \mathbb{C}$, $k < p$ or $\mathbb{F} = \mathbb{H}$, $k < (p + 1)/2$, the assertion follows by induction of k since attaching maps are all null homotopic at prime p. We consider the case $\mathbb{F} = \mathbb{C}$, $k = p$ and $p|n$. If $n = 0$, $Q_{p,p}(\mathbb{C}) = \Sigma(\mathbb{CP}^{p-1}{}_+)$ which is homotopy equivalent to $S^1 \vee \Sigma\mathbb{CP}^{p-1}$. Since $\Sigma\mathbb{CP}^{i-1}$ is 2-connected and satisfies the assumptions of Lemma 1.7, it follows that each attaching map $S^{2i} \to \Sigma\mathbb{CP}^{i-1}$ is null homotopic at prime p if $i < p$. Hence $\Sigma\mathbb{CP}^{p-1}$ is homotopy equivalent to $\bigvee_{i=2}^{p} S^{2i-1}$ at prime p. If $n > 0$, then $n \geqq p$ since $p|n$. Hence homotopy group $\pi_{2n+2p-2}(Q_{n+p-1,p-1}(\mathbb{C}))$ is in the stable range. By the homotopy equivalence $Q_{n+p-1,p-1}(\mathbb{C}) \simeq \bigvee_{i=1}^{p-1} S^{2n+2i-1}$ at prime p, we have $\pi_{2n+2p-2}(Q_{n+p-1,p-1}(\mathbb{C}))_{(p)} \cong \sum_{i=1}^{p-1} \pi_{2n+2p-2}(S^{2n+2i-1})_{(p)} = \pi_{2n+2p-2}$ $(S^{2n+1})_{(p)}(\cong \mathbb{Z}/p)$, which implies that attaching map $f_{n,p}$ can be factorized as $S^{2n+2p-2} \xrightarrow{f'} S^{2n+1} = Q_{n+1,1}(\mathbb{C}) \to Q_{n+p-1,p-1}(\mathbb{C})$ at prime p. Let $C_{f'}$ be the cofiber of f', then we have the following homotopy commutative diagram whose horizontal rows are cofiberings

$$
\begin{array}{ccccc}
S^{2n+2p-2} & \xrightarrow{\;f'\;} & Q_{n+1,1}(\mathbb{C}) & \longrightarrow & C_{f'} \\
\| & & \downarrow & & \downarrow{\phi} \\
S^{2n+2p-2} & \xrightarrow{\;f_{n,p}\;} & Q_{n+p-1,p-1}(\mathbb{C}) & \longrightarrow & Q_{n+p,p}(\mathbb{C})
\end{array}
$$

$\phi^*: H^s(Q_{n+p,p}(\mathbb{C}); \mathbb{F}_p) \to H^s(C_{f'}; \mathbb{F}_p)$ is an isomorphism for $s = 2n + 1$, $2n + 2p - 1$. Thus P^1 acts on $H^*(C_{f'}; \mathbb{F}_p)$ trivially by Corollary 1.2. It follows $f' \simeq 0$, hence $f_{n,p} \simeq 0$ at prime p. This proves the assertion in the case $\mathbb{F} = \mathbb{C}$, $k = p$. Noting that the assumption $p|2n + 1$ implies that

$\pi_{4n+2p}(Q_{n+(p-1)/2,(p-1)/2}(\mathbf{H}))$ is in the stable range, we can prove the assertion similarly in the case $\mathbf{F} = \mathbf{H}$, $k = (p+1)/2$ and $p\,|\,2n+1$.

Remark 1.9. If the condition (C) or (H) is not satisfied, it follows from Corollary 1.2 and Theorem 1.4 that neither $Q_{n+k,k}(\mathbf{C})$ nor $Q_{n+k,k}(\mathbf{H})$ splits into a wedge of spheres at prime p, even stably.

After localizing at prime p (if necessary), we have maps ϕ_C:
$\vee_{i=1}^{k} S^{2n+2i-1} \xrightarrow{\simeq} Q_{n+k,k}(\mathbf{C}) \to W_{n+k,k}$ and ϕ_H: $\vee_{i=1}^{k} S^{4n+4i-1} \xrightarrow{\simeq} Q_{n+k,k}(\mathbf{H})$
$\to X_{n+k,k}$ for n, k satisfying (C), (H) in each case. Using the loop space structures of $\Omega W_{n+k,k}$ and $\Omega X_{n+k,k}$, we can extend the compositions $\vee_{i=1}^{k} \Omega S^{2n+2i-1}$
$\to \Omega(\vee_{i=1}^{k} S^{2n+2i-1}) \xrightarrow{\Omega\phi_C} \Omega W_{n+k,k}$ and $\vee_{i=1}^{k} \Omega S^{4n+4i-1} \to \Omega(\vee_{i=1}^{k} S^{4n+4i-1})$
$\xrightarrow{\Omega\phi_H} \Omega X_{n+k,k}$ to maps $\bar{\phi}_C : \prod_{i=1}^{k} \Omega S^{2n+2i-1} \to \Omega W_{n+k,k}$ and $\bar{\phi}_H$:
$\prod_{i=1}^{k} \Omega S^{4n+4i-1} \to \Omega X_{n+k,k}$. Since the Pontrijagin rings $H_*(\Omega W_{n+k,k})$ and $H_*(\Omega X_{n+k,k})$ are commutative, $\bar{\phi}_C$ and $\bar{\phi}_H$ induce algebra homomorphisms between homologies. By the construction of $\bar{\phi}_C$ and $\bar{\phi}_H$ $\bar{\phi}_{C*}$: $\otimes_{i=1}^{k} H_*(\Omega S^{2n+2i-1}) \to H_*(\Omega W_{2n+k,k})$ and $\bar{\phi}_{H*}$: $\otimes_{i=1}^{k} H_*(\Omega S^{4n+4i-1})$ $\to H_*(\Omega X_{n+k,k})$ map each polynomial generator of $H_*(\Omega S^{2n+2i-1})$ and $H_*(\Omega S^{4n+4i-1})$ to a polynomial generator of $H_*(\Omega W_{n+k,k})$ and $H_*(\Omega X_{n+k,k})$ respectively. Thus $\bar{\phi}_{C*}$ and $\bar{\phi}_{H*}$ are isomorphisms, and hence they are homotopy equivalences.

Proposition 1.10. i) $\Omega W_{n+k,k}$ *is homotopy equivalent to* $\prod_{i=1}^{k} \Omega S^{2n+2i-1}$ *at prime p if and only if the condition* (C) *is satisfied. And* $\Omega W_{n+2,2}$ *is globally homotopy equivalent to* $\Omega S^{2n+1} \times \Omega S^{2n+3}$ *if* $2\,|\,n$.
ii) $\Omega X_{n+k,k}$ *is homotopy equivalent to* $\prod_{i=1}^{k} \Omega S^{4n+4i-1}$ *at prime p if and only if the condition* (H) *is satisfied. And* $\Omega X_{n+2,2}$ *is globally homotopy equivalent to* $\Omega S^{4n+3} \times \Omega S^{4n+7}$ *if* $n \equiv 22$ *modulo* 24.

Proof. We already showed the homotopy equivalence if (C) or (H) is satisfied. The converse is easily proved by examining the action of the Steenrod operations on $H_*(\Omega W_{n+k,k})$ and $H_*(\Omega X_{n+k,k})$ and by comparing the homotopy group $\pi_{4n+5}(\Omega X_{n+2,2})_{(2)}$ with $\pi_{4n+5}(\Omega S^{4n+3} \times \Omega S^{4n+7})_{(2)}$ (See the proof of Theorem 1.6).

Corollary 1.11. $\pi_*(W_{n+k,k})_{(p)}$ *is isomorphic to* $\pi_*(\prod_{i=1}^{k} S^{2n+2i-1})_{(p)}$ *if* (C) *is satisfied and* $\pi_*(X_{n+k,k})_{(p)}$ *is isomorphic to* $\pi_*(\prod_{i=1}^{k} S^{4n+4i-1})_{(p)}$ *if* (H) *is satisfied.*

Thus the homotopy groups cannot distinguish $W_{n+k,k}$ from $\prod_{i=1}^{k} S^{2n+2i-1}$, nor $X_{n+k,k}$ from $\prod_{i=1}^{k} S^{4n+4i-1}$ for n, k satisfying (C) and (H) respectively, if we localize these spaces at prime p.

§2. The Metastable J-Homomorphism and the Non-2-Regularity

We first note that Corollary 1.2 implies,

Proposition 2.1. *Neither* $W_{n+k,k}$ *nor* $X_{n+k,k}$ *can be 2-regular if* $k \geqq 3$, *or* $k = 2$ *and* $2 \nmid n$.

Let $f_n\colon S^{2n+3} \to BU(n+1)$ and $g_n\colon S^{4n+7} \to BSp(n+1)$ be the classifying maps of sphere bundles $S^{2n+1} \to W_{n+2,2} \to S^{2n+3}$ and $S^{4n+3} \to X_{n+2,2} \to S^{4n+7}$ respectively. We begin with the following exceptional cases.

Proposition 2.2. $W_{2,2} = U(2)$ *is homeomorphic to* $S^1 \times S^3$ *and* $W_{4,2}$ *is homeomorphic to* $S^5 \times S^7$.

Proof. The first part is obvious. For the second part, consider the composite $S^7 \xrightarrow{f_2} BU(3) \to BSO(6)$. There is an isomorphism of Lie groups $Spin(6) \cong SU(4)$. Hence $\pi_7(BSO(6)) \cong \pi_7(BSpin(6)) \cong \pi_7(BSU(4)) \cong \pi_6(SU(4)) = 0$. Thus the above composition is null homotopic and $S^5 \to W_{4,2} \to S^7$ is a trivial bundle with structure group SO(6).

We therefore concentrate on $W_{n+2,2}$ for even $n \geqq 4$, and on $X_{n+2,2}$ for $n \equiv 6$ modulo 8 (Recall Theorem 1.6). For the rest of this section, we localize all spaces and groups at prime 2.

Theorem 2.3. ([5]) i) $\pi_{2n+3}(BU(n+1)) \cong \pi_{2n+2}(U(n+1)) \cong \mathbb{Z}_{(2)}/(n+1)!$, *and* f_n *generates* $\pi_{2n+3}(BU(n+1))$.
ii) $\pi_{4n+7}(BSp(n+1)) \cong \pi_{4n+6}(Sp(n+1)) \cong \mathbb{Z}_{(2)}/2(2n+3)!$ *for   even   n,*
$\cong \mathbb{Z}_{(2)}/(2n+3)!$ *for odd n, and* g_n *generates* $\pi_{4n+7}(BSp(n+1))$.

We also denote by f_n, g_n the generator of $\pi_{2n+2}(U(n+1))$, $\pi_{4n+6}(Sp(n+1))$ which correspond to f_n, g_n respectively by the natural isomorphisms. Consider the following commutative diagrams.

$$
(2.4)\quad
\begin{array}{ccccc}
\pi_{2n+3}(SO/SO(2n+1)) & \xrightarrow{\bar{j}_*} & \pi_{2n+3}(SO/SO(2n+2)) & \xleftarrow{\bar{r}_*} & \pi_{2n+3}(U/U(n+1)) \\
\downarrow{\partial} & & \downarrow{\partial} & & \downarrow{\Delta} \\
\pi_{2n+2}(SO(2n+1)) & \xrightarrow{j_*} & \pi_{2n+2}(SO(2n+2)) & \xleftarrow{r_*} & \pi_{2n+2}(U(n+1))
\end{array}
$$

$$
(2.5)\quad
\begin{array}{ccccc}
\pi_{4n+7}(SO/SO(4n+3)) & \xrightarrow{\bar{j}_*} & \pi_{4n+7}(SO/SO(4n+4)) & \xleftarrow{\bar{r}'_*} & \pi_{4n+7}(Sp/Sp(n+1)) \\
\downarrow{\partial} & & \downarrow{\partial} & & \downarrow{\Delta'} \\
\pi_{4n+6}(SO(4n+3)) & \xrightarrow{j_*} & \pi_{4n+6}(SO(4n+4)) & \xleftarrow{r'_*} & \pi_{4n+6}(Sp(n+1)),
\end{array}
$$

where ∂, Δ, Δ' are connecting homomorphisms, r, r', j are inclusions $U(n + 1) \hookrightarrow SO(2n + 2)$, $Sp(n + 1) \hookrightarrow SO(4n + 4)$, $SO(m) \hookrightarrow SO(m + 1)$, and $\bar{r}$, $\bar{r}'$, $\bar{j}$ are the maps induced by r, r', j respectively. Note that Δ, Δ' and both ∂'s in (2.5) are surjective by the Bott periodicity. We choose elements $\bar{f}_n \in \pi_{2n+3}(U/U(n + 1))$ and $\bar{g}_n \in \pi_{4n+7}(Sp/Sp(n + 1))$ such that $\Delta(\bar{f}_n) = f_n$, $\Delta'(\bar{g}_n) = \bar{g}_n$.

Let $k: \mathbf{P}_m^\infty \to SO/SO(m)$ be the canonical inclusion. Since the pair $(SO/SO(m), \mathbf{P}_m^\infty)$ is $2m$-connected, $k_*: \pi_i(\mathbf{P}_m^\infty) \to \pi_i(SO/SO(m))$ is an isomorphism for $i < 2m$.

Lemma 2.6. i) $k_*: \pi_{2n+3}(\mathbf{P}_{2n+i}^\infty) \to \pi_{2n+3}(SO/SO(2n + i))$ is an isomorphism for $i = 1, 2$. If n is even, we have $\pi_{2n+3}(\mathbf{P}_{2n+1}^\infty) \cong \mathbf{Z}/8$ and $\pi_{2n+3}(\mathbf{P}_{2n+2}^\infty) \cong \mathbf{Z}/4$. Moreover, the mod 2 Hurewicz homomorphism $h: \pi_{2n+3}(\mathbf{P}_{2n+i}^\infty) \to H_{2n+3}(\mathbf{P}_{2n+i}^\infty; \mathbf{F}_2)$ $(i = 1, 2)$ maps an element x to a non-zero element if and only if x generates $\pi_{2n+3}(\mathbf{P}_{2n+i}^\infty)$.

ii) $k_*: \pi_{4n+7}(\mathbf{P}_{4n+i}^\infty) \to \pi_{4n+7}(SO/SO(4n + i))$ is an isomorphism for $i = 3, 4$. If n is even, we have $\pi_{4n+7}(\mathbf{P}_{4n+3}^\infty) \cong \mathbf{Z}/16$ and $\pi_{4n+7}(\mathbf{P}_{4n+4}^\infty) \cong \mathbf{Z}/16 \oplus \mathbf{Z}/4$. Moreover, the mod 2 Hurewicz homomorphism $h: \pi_{4n+7}(\mathbf{P}_{4n+i}^\infty) \to H_{4n+7}(\mathbf{P}_{4n+i}^\infty; \mathbf{F}_2)(i = 3, 4)$ maps an element x to a non-zero element if and only if the order of x is 16.

Proof. Consider the Adams spectral sequence $E_2^{s,t} = \text{Ext}_{A_*}^{s,t}(\mathbf{F}_2, H_*(\mathbf{P}_m^\infty)) \Rightarrow \pi_{t-s}^S(\mathbf{P}_m^\infty)$, where $A_* = \mathbf{F}_2[\xi_1, \xi_2, \ldots, \xi_i, \ldots]$ is the dual of the mod 2 Steenrod algebra. The E_2-terms we need are given as follows (See [10] Chapter III, and [11]);

$$m = 2n + 1: E_2^{s,2n+3+s} = \mathbf{F}_2\{h_0^s x_{2n+3}\} \quad \text{for } s = 0, 1, 2,$$

$$E_2^{s,2n+3+s} = 0 \text{ for } s \geq 3$$

$$m = 2n + 2: E_2^{s,2n+3+s} = \mathbf{F}_2\{h_0^s x_{2n+3}\} \quad \text{for } s = 0, 1,$$

$$E_2^{s,2n+3+s} = 0 \text{ for } s \geq 2$$

(2.7) $$m = 4n + 3: E_2^{s,4n+7+s} = \mathbf{F}_2\{h_0^s x_{4n+7}\} \quad \text{for } s = 0, 1, 2, 3,$$

$$E_2^{s,4n+7+s} = 0 \text{ for } s \geq 4$$

$$m = 4n + 4: E_2^{s,4n+7+s} = \mathbf{F}_2\{h_0^s x_{4n+7}\} \quad \text{for } s = 0, 3,$$

$$E_2^{s,4n+7+s} = \mathbf{F}_2\{h_0^s x_{4n+7}, h_0^{s-1} h_2 x_{4n+4}\} \quad \text{for } s = 1, 2,$$

$$E_2^{s,4n+7+s} = 0 \text{ for } s \geq 4,$$

where $h_i \in \text{Ext}_{A_*}^{1,2^i}(\mathbf{F}_2, \mathbf{F}_2)$ is the element represented by $[\xi_1^{2^i}]$ in the cobar complex of $A_*([1])$. It is also easy to see that $x_{2n+3} \in E_2^{0,2n+3}$ and $x_{4n+7} \in E_2^{0,4n+3}$ are permanent cycles in each spectral sequence and that $h_1^s x_{2n+3}$ and $h_1^s x_{4n+7}$ are not zero in the E_∞-terms if they are not zero in the E_2-terms. Noting that $\pi_{2n+3}(\mathbf{P}_{2n-i}^\infty) \cong \pi_{2n+3}^S(\mathbf{P}_{2n+i}^\infty)(i = 1, 2)$ and $\pi_{4n+7}(\mathbf{P}_{4n+i}^\infty) \cong \pi_{4n+7}^S(\mathbf{P}_{4n+i}^\infty)(i = 3, 4)$, the result follows.

Lemma 2.8. i) *In diagram* (2.4), $\bar{r}_*$ *and* $\bar{j}_*$ *are surjective. And* $\bar{r}_*(\bar{f}_n)$ *generates* $\pi_{2n+3}(SO/SO(2n+2))$.

ii) *In diagram* (2.5), $\bar{r}'_*(\bar{g}_n)$ *generates the images of* $\bar{j}_*$ *and* $\bar{r}'_*$ *if* $n \equiv 6$ *modulo* 8. *And* $\bar{j}_*$ *is a split monomorphism.*

Proof. We note that $\bar{r}_*: H_{2n+3}(U/U(n+1); \mathbb{F}_2) \to H_{2n+3}(SO/SO(2n+2); \mathbb{F}_2)$ and $\bar{r}_*: H_{4n+7}(Sp/Sp(n+1); \mathbb{F}_2) \to H_{4n+7}(SO/SO(4n+4); F_2)$ are isomorphisms, and that the mod 2 Hurewicz homomorphisms $h: \pi_{2n+3}(U/U(n+1)) \to H_{2n+3}(U/U(n+1); \mathbb{F}_2)$ and $h: \pi_{4n+7}(Sp/Sp(n+1)) \to H_{4n+7}(Sp/Sp(n+1); \mathbb{F}_2)$ map $\bar{f}_n$, $\bar{g}_n$ to each generator. It follows from the preceding lemma that $\bar{r}_*(\bar{f}_n)$ is a generator of $\pi_{2n+3}(SO/SO(2n+2))$ and that $\bar{r}_*(\bar{g}_n)$ is an element of order 16. r' induces a map from a sphere bundle $S^{4n+3} \to Sp/Sp(n) \to Sp/Sp(n+1)$ to a sphere bundle $S^{4n+3} \to SO/SO(4n+3) \to SO/SO(4n+4)$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\pi_{4n+7}(Sp/Sp(n+1)) & \xleftarrow{\cong} & \pi_{4n+7}(S^{4n+7}) \\
\downarrow{\scriptstyle \bar{r}_*} & \searrow{\scriptstyle \Delta''} & \downarrow{\scriptstyle \Delta_H} \\
\pi_{4n+7}(SO/SO(4n+4)) & \xrightarrow{\partial} & \pi_{4n+6}(S^{4n+3})
\end{array}
$$

By Theorem 1.4, $\Delta_H = 0$ if $n \equiv 6$ modulo 8, hence $\partial \circ \bar{r}_* = \Delta'' = 0$ and the image of $\bar{r}_*$ is contained in the image of $\bar{j}_*$. The preceding lemma implies that $\bar{j}_*$ in (2.4) is surjective and that $\bar{j}_*$ in (2.5) is a split monomorphism. This completes the proof.

We put $H(n) = \{f: S^{n-1} \to S^{n-1} | f:\text{homotopy equivalence}\}$ and $SH(n) = \{f: S^{n-1} \to S^{n-1} | \text{degree } f = 1\}$ and give them the compact-open topology. We denote by $J_{SH}: SO(n) \to SH(n)$ and $j': SH(n) \to H(n)$ the canonical inclusions, and put $J_H = j' \circ J_{SH}: SO(n) \to H(n)$. $H(n)$ and $SH(n)$ are topological monoids with respect to compositions of maps, and J_{SH}, j' and J_H are maps of topological monoids. Let $e: SH(n) \to S^{n-1}$ be the evaluation map at the base point. We denote by $\Omega_1^{n-1} S^{n-1}$ the component of $\Omega^{n-1} S^{n-1}$ which contains the identity map. Then $\Omega_1^{n-1} S^{n-1}$ is contained in $SH(n)$ and we have a fibration $\Omega_1^{n-1} S^{n-1} \to SH(n) \xrightarrow{e} S^{n-1}$. J_{SH} induces $J: SO(n-1) \to \Omega_1^{n-1} S^{n-1}$ and $J_*: \pi_i(SO(n-1)) \to \pi_i(\Omega_1^{n-1} S^{n-1}) \cong \pi_{i+n-1}(S^{n-1})$ coincides with the J-homomorphism ([23], [21] Chapter XI) up to sign. So, we rather denote J_* by J.

By the construction of Dold-Lashof, $H(n)$ and $SH(n)$ have classifying spaces $BH(n)$ and $BSH(n)$. The following is a special case of Theorem 7.5 of [3].

Theorem 2.9. *Let* $S^{n-1} \to E \to B$ *be an* $(n-1)$-*sphere bundle with structure group* $SO(n)$ *and let* $f: B \to BSO(n)$ *be the classifying map. Then* $S^{n-1} \to E \to B$ *is fiber homotopy equivalent to a trivial bundle if and only if the composition* $BJ_H \circ f: B \to BSO(n) \to BH(n)$ *is null homotopic.*

We also need the following fact.

Proposition 2.10 ([8], *Corollary* 4.5). *The sphere bundle* $S^{d(n+1)-1} \to O_{n+k,k}$ $\to O_{n+k,k-1}$ *is fiber homotopy equivalent to a trivial bundle if and only if* $O_{n+k,k}$ *is homotopy equivalent to* $S^{d(n+1)-1} \times O_{n+k,k-1}$, *where* $O_{n+k,k} = V_{n+k,k}$, $W_{n+k,k}$ *or* $X_{n+k,k}$ *and* $d = 1, 2$ *or* 4, *respectively.*

Now we reduce the problem to a problem on the J-homomorphism.

Lemma 2.11. i) *For even* n, $W_{n+2,2}$ *is not 2-regular if the image of* J: $\pi_{2n+2}(SO(2n+1)) \to \pi_{2n+2}(\Omega_1^{2n+1} S^{2n+1}))$ *contains an element of order four.*
ii) *For* $n \equiv 6$ *modulo* 8, $X_{n+2,2}$ *is not 2-regular if* J: $\pi_{4n+6}(SO(4n+3))$ $\to \pi_{4n+6}(\Omega_1^{4n+3} S^{4n+3})$ *is non-trivial.*

Proof. By the exactness of $\pi_{2n+3}(S^{2n+1}) \to \pi_{2n+2}(\Omega_1^{2n+1} S^{2n+1}) \xrightarrow{j''_*} \pi_{2n+2}$ $(SH(2n+2))$, the order of the kernel of j''_*: $\pi_{2n+2}(\Omega_1^{2n+1} S^{2n+1}) \to \pi_{2n+2}(SH(2n+2))$ is at most two. Hence the assumption of i) implies that J_{SH^*}: $\pi_{2n+2}(SO(2n+2)) \to \pi_{2n+2}(SH(2n+2))$ is non-trivial. Since f_n generates $\pi_{2n+2}(U(n+1))$ and r_*: $\pi_{2n+2}(U(n+1)) \to \pi_{2n+2}(SO(2n+2))$ is surjective by Lemma 2.8, it follows that $J_H \circ r \circ f_n$ is non-trivial in $\pi_{2n+2}(H(2n+2))$. Therefore $BJ_H \circ Br \circ f_n$ is non-trivial in $\pi_{2n+3}(BH(2n+2))$, and the result follows from Theorem 2.9 and Proposition 2.10. For the part ii), note that j''_*: $\pi_{4n+6}(\Omega_1^{4n+3} S^{4n+3}) \to \pi_{4n+6}(SH(4n+4))$ is injective since $\pi_{4n+7}(S^{4n+3}) = 0$. Then, the assumption of ii) implies the non-triviality of $J_{SH} \circ r' \circ g_n$. In fact, by the commutativity of (2.5) and Lemma 2.8, there exists a generator γ_n of $\pi_{4n+7}(SO/SO(4n+3))$ such that $\bar{r}'_*(\bar{g}_n) = \bar{j}_*(\gamma_n)$ and we have $J_{SH^*} \circ r'_*(g_n) = J_{SH^*} \circ \partial \circ \partial \circ \bar{r}'_*(\bar{g}_n) = J_{SH^*} \circ j_* \circ \partial(\gamma_n)$ $= j''_* \circ J \circ \partial(\gamma_n) \neq 0$ since ∂ is surjective and $j''_* \circ J \neq 0$. Therefore $BJ_H \circ Br' \circ g_n \neq 0$ in $\pi_{4n+7}(BH(4n+4))$.

In order to examine the J-homomorphisms, we use the relative J-homomorphisms constructed by Toda ([21]). We briefly review the construction.

For a pair (X, A) of topological spaces with base point $* \in A$, we denote by $\Omega(X, A)$ a topological space $\{l: [0, 1] \to X \mid l: \text{ continuous, } l(0) \in A, l(1) = *\}$ with compact-open topology. Let $e: \Omega(X, A) \to A$ be the evaluation map at 0. Then $\Omega(X, A)$ is the homotopy fiber of inclusion map $A \to X$ and $\Omega X \xrightarrow{\delta} \Omega(X, A) \xrightarrow{e} A \to X$ is a fiber sequence, where δ is the natural inclusion.

The natural inclusion $i: X \to \Omega\Sigma X$ is extended to a map $i': X \cup CA \to \Omega(\Sigma X, \Sigma A)$ defined by $i'([a, t])(s) = [a, st + 1 - t]$ on CA, where $X \cup CA = (X \amalg (A \times I))/\sim$, and $\sim$ is an equivalence relation generated by $a \sim (a, 1)$, $(a, 0) \sim *$ $\sim (*, t)$ for $a \in A$, $t \in [0, 1]$. We denote by $[a, t]$ the element of $X \cup CA$ (or ΣA) represented by $(a, t) \in A \times I$.

The map $J: SO(m) \to \Omega_1^m S^m$ induces the stable J-map $J^S: SO \to Q_1 S^0$, and let $J': \Sigma^m SO(m) \to S^m$ and $J_m^S: \Sigma^m SO \to QS^m$ be the adjoint maps of J and J^s. We put $F(m) = \Omega(QS^m, S^m)$, then J' and J_m^S induce $J^R: \Omega(\Sigma^m SO, \Sigma^m SO(m)) \to F(m)$. We define the relative J-map $J^{rel}: \Sigma^{m-1} \mathbb{P}_m^\infty \to F(m)$ to be the composition

$$\Sigma^{m-1} \mathbb{P}_m^\infty \xrightarrow{inv} \Sigma^{m-1}(\mathbb{P}^\infty \cup C\mathbb{P}^{m-1}) \xrightarrow{i'} \Omega(\Sigma^m \mathbb{P}^\infty, \Sigma^m \mathbb{P}^{m-1}) \xrightarrow{k'} \Omega(\Sigma^m SO, \Sigma^m SO(m))$$

$$\xrightarrow{J^R} F(m),$$

where *inv* is the $(m-1)$-fold suspension of the homotopy inverse of the collapsing map $\mathbb{P}^\infty \cup C\mathbb{P}^{m-1} \to \mathbb{P}_m^\infty (\mathbb{P}^n = \mathbb{R}\mathbb{P}^n)$, and k' is induced by the canonical inclusion $\mathbb{P}^\infty \hookrightarrow SO$.

Let $\rho: SO \to SO/SO(m)$ be the projection, then $(\Sigma^m \rho)_*: \pi_j(\Sigma^m SO, \Sigma^m SO(m)) \to \pi_j(\Sigma^m(SO/SO(m)))$ is an isomorphism for $j < 3m$. In fact, let Y be the homotopy fiber of $\Sigma^m \rho$, then there is a map $\zeta: \Sigma^m SO(m) \to Y$ such that the composition $\Sigma^m SO(m) \xrightarrow{\zeta} Y \to \Sigma^m SO$ is homotopic to inclusion $\Sigma^m SO(m) \hookrightarrow \Sigma^m SO$.

By an easy argument on the Serre spectral sequence associated with fibering $Y \to \Sigma^m SO \to \Sigma^m(SO/SO(m))$, we see that $\zeta_*: H_j(\Sigma^m SO(m)) \to H_j(Y)$ is an isomorphism for $j < 3m$ and an epimorphism for $j \leqq 3m$. Thus $\zeta_*: \pi_j(\Sigma^m SO(m)) \to \pi_j(Y)$ is an isomorphism for $j < 3m$ and the above assertion follows from the five-lemma.

We denote by $\bar{k}: \mathbb{P}^\infty \cup C\mathbb{P}^{m-1} \to SO/SO(m)$ the composition $\mathbb{P}^\infty \cup C\mathbb{P}^{m-1} \to \mathbb{P}_m^\infty \xrightarrow{k} SO/SO(m)$. For $j < 2m-1$, we have the following commutative diagram

$$\begin{array}{ccccc}
\pi_{j+1}(\mathbb{P}^\infty \cup C\mathbb{P}^{m-1}) & \xrightarrow{\bar{k}_*} & \pi_{j+1}(SO/SO(m)) & \xrightarrow{\rho_*^{-1}} & \pi_{j+1}(SO, SO(m)) \\
\downarrow{E^{m-1}} & & \downarrow{E^{m-1}} & & \downarrow \\
\pi_{j+1}(\Sigma^{m-1}(\mathbb{P}^\infty \cup C\mathbb{P}^{m-1})) \to \pi_{j+m}(\Sigma^{m-1}(SO/SO(m))) & \xrightarrow{E} & \pi_{j+m+1}(\Sigma^m(SO/SO(m))) & & \downarrow{E^m} \\
\downarrow{i'_*} & & & \searrow^{(\Sigma^m \rho)_*^{-1}} & \downarrow \\
\pi_{j+m}(\Omega(\Sigma^m \mathbb{P}^\infty, \Sigma^m \mathbb{P}^{m-1})) & \xrightarrow{\cong} & \pi_{j+m+1}(\Sigma^m \mathbb{P}^\infty, \Sigma^m \mathbb{P}^{m-1}) & \to & \pi_{j+m+1}(\Sigma^m SO, \Sigma^m SO(m)) \\
& \searrow^{k'_*} & & \nearrow^{\cong} & \\
& & \pi_{j+m}(\Omega(\Sigma^m SO, \Sigma^m SO(m))) & &
\end{array}$$

By the definition of the relative J-map and the commutativity of the above diagram, the following diagram commutes for $j < 2m-1$.

$$(2.12) \quad \begin{array}{ccccc} \pi_{j+1}(\mathbb{P}_m^\infty) & \xrightarrow{\ k_*\ } & \pi_{j+1}(SO/SO(m)) & \xrightarrow{\ \partial\ } & \pi_j(SO(m)) \\ \downarrow{\scriptstyle E^{m-1}} & & & & \downarrow{\scriptstyle J} \\ \pi_{j+m}(\Sigma^{m-1}\mathbb{P}_m^\infty) & \xrightarrow{\ J_*^{rel}\ } & \pi_{j+m}(F(m)) & \xrightarrow{\ e_*\ } & \pi_{j+m}(S^m) \cong \pi_j(\Omega_1^m S^m) \end{array}$$

Theorem 2.13 ([21], Theorem 11.7). $J_*^{rel}\colon \pi_j(\Sigma^{m-1}\mathbb{P}_m^\infty) \to \pi_j(F(m))$ *is an isomorphism for* $j < 4m - 3$.

In the case $m = 2n + 1$, $j = 2n + 2$ (n: even) or $m = 4n + 3$, $j = 4n + 6$ ($n \equiv 6$ modulo 8), k_*, E^{m-1} and J_*^{rel} are isomorphisms and ∂ is an epimorphism in the diagram (2.12). Consider the exact sequence associated with fiber sequence $QS^{m-1} \xrightarrow{\delta} F(m) \xrightarrow{e} S^m$. Then the following lemma is obvious since $\pi_{4n+3}(F(2n + 1)) \cong \mathbb{Z}/8$ if n is even by Lemma 2.8 and the above theorem. ($\pi_{8n+9}(F(4n + 3))$ is a cyclic group of order sixteen.)

Lemma 2.14. i) *For even n, the image of $J\colon \pi_{2n-2}(SO(2n + 1)) \to \pi_{2n+2}(\Omega_1^{2n+1} S^{2n+1})$ contains an element of order four, if the order of the image of $\delta_*\colon \pi_{4n+3}(QS^{2n}) \to \pi_{4n+3}(F(2n + 1))$ is at most two.*
ii) *For $n \equiv 6$ modulo 8, $J\colon \pi_{4n+6}(SO(4n + 3)) \to \pi_{4n+6}(\Omega_1^{4n+3} S^{4n+3})$ is non-trivial if $\delta_*\colon \pi_{8n+9}(QS^{4n+2}) \to \pi_{8n+9}(F(4n + 3))$ is not surjective.*

Thus it suffices to examine $\delta_*\colon \pi_j(QS^{m-1}) \to \pi_j(F(m))$ in the above cases. We apply the following theorem which was first given by Mahowald ([10], Theorem A), and later it was reformulated by Milgram ([14], Part II) using a kind of unstable Adams spectral sequence. For a space X, we denote by $\{E_r^{s,t}(X), d_r\}$ a certain unstable Adams spectral sequence constructed in [14], Part II. It converges to $\pi_*(X) \otimes \mathbb{Z}_2^\wedge$ for "good" X ($X = \Sigma Y$, for example), and in the stable range, it coincides with the stable Adams spectral sequence. Thus $E_2^{s,t}(X) \cong \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_2, \tilde{H}_*(X))$ for $t - s < 2n - 1$ if X is $(n - 1)$-connected. In fact, there is a map of spectral sequences $E_r^{s,t}(X) \to E_r^{s,t+1}(\Sigma X)$ and let $\{E_r^{s,t}(\Sigma^\infty X), d_r\}$ be the spectral sequence defined by $E_r^{s,t}(\Sigma^\infty X) = \varinjlim_u E_r^{s,t+u}(\Sigma^u X)$, then $\{E_r^{s,t}(\Sigma^\infty X), d_r\}$ is the stable Adams spectral sequence (See [14], §6 for details). Note that $F(m)$ is $(2m - 2)$-connected and it is homotopy equivalent to $\Sigma^{m-1}\mathbb{P}^\infty$ through dimension $4m - 4$ by Theorem 2.13. Hence $\{E_r^{s,t}(F(m)), d_r\}$ converges to $\pi_{t-s}(F(m)) \otimes \mathbb{Z}_2^\wedge$ and $E_2^{s,t}(F(m)) \cong \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_2, \tilde{H}_*(F(m))) \cong \mathrm{Ext}_{A_*}^{s,t-m+1}(\mathbb{F}_2, \tilde{H}_*(\mathbb{P}_m^\infty))$ for $t - s < 4m - 3$.

Theorem 2.15 ([10] Theorem A, [14] Theorem 8.5 and Theorem 8.21).
There is a map of spectral sequences $\delta_r\colon E_r^{s,t}(\Sigma^\infty S^{m-1}) \to E_r^{s-1,t-1}(F(m))$ which converges to $\delta_*\colon \pi_{t-s}^S(S^{m-1}) \cong \pi_{t-s}(QS^{m-1}) \to \pi_{t-s}(F(m))$ for $t - s < 3m - 1$. In the E_2-term, δ_2 is identified with a homomorphism $\mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_2, \tilde{H}_*(S^{m-1}))$

$\to \mathrm{Ext}_{A_*}^{s-1,t-1}(\mathbb{F}_2, \tilde{H}_*(\Sigma^{m-1}\mathbb{P}_m^\infty))$ *for* $t - s < 3m - 1$.

Lemma 2.16 i) *If n is even and greater than 2, the order of the image of δ_*:* $\pi_{4n+3}(QS^{2n}) \to \pi_{4n+3}(F(2n+1))$ *is at most 2.*
ii) *If n is even and positive, δ_*: $\pi_{8n+9}(QS^{4n+2}) \to \pi_{8n+9}(F(4n+3))$ is not surjective.*

Proof. We denote by $F^{s,t}(\Sigma^\infty S^{m-1})$, $F^{s,t}(F(m))$ the filtrations on $\pi_*(QS^{m-1})$ and $\pi_*(F(m))$ associated with the corresponding spectral sequences in Theorem 2.15. Since $E_\infty^{s,s+j}(\Sigma^\infty S^{m-1}) = 0$ for $s = 0, 1, 2$ if j and m are odd and $j \neq m$, $m + 2$, $m + 6$ ([1] for $s = 1$, [11] for $s = 2$), it follows that $\pi_j(QS^{m-1}) = \pi_j^S(S^{m-1}) = F^{3,j+3}(\Sigma^\infty S^{m-1})$. Hence, by Theorem 2.15, we see that $\delta_*(\pi_{4n+3}(QS^{2n})) \subset F^{2,4n+5}(F(2n+1))$ if $n > 2$ and that $\delta_*(\pi_{8n+9}(QS^{4n+2})) \subset F^{2,8n+11}(F(4n+3))$ if $n > 0$. On the other hand, since $E_2^{s,t}(F(m)) \cong \mathrm{Ext}_{A_*}^{s,t-m+1}(\mathbb{F}_2, \tilde{H}_*(\mathbb{P}^\infty))$, it follows from (2.7) that $F^{2,4n+5}(F(2n+1)) \cong \mathbb{Z}/2$ and $F^{2,8n+11}(F(4n+3)) \cong \mathbb{Z}/4$ if n is even. This completes the proof.

Thus, by Lemmas 2.11, 2.14 and 2.16, we have proved

Theorem 2.17. $W_{n+2,2}$ *is not 2-regular unless $n = 0$ or 2, and $X_{n+2,2}$ is not 2-regular for any $n \geq 0$.*

Let ϕ: $S^{4n+3} \to Q_{n+2,2}(\mathbb{C})$ be the attaching map of the top cell of $W_{n+2,2}$. If n is even, we have a collapsing map c: $Q_{n+2,2}(\mathbb{C}) \cong S^{2n+1} \vee S^{2n+3} \to S^{2n+1}$ by Theorem 1.3. Proposition 2.10 and Theorem 2.17 imply that the composition $S^{4n+3} \xrightarrow{\phi} Q_{n+2,2}(\mathbb{C}) \to S^{2n+1}$ is not null homotopic if $n \geq 4$. Otherwise, we can extend c to $\bar{c}$: $W_{n+2,2} \to S^{2n+1}$ which is a left homotopy inverse of the canonical inclusion $S^{2n+1} \to W_{n+2,2}$. Then, fibration $S^{2n+1} \to W_{n+2,2} \to S^{2n+3}$ becomes retractible in the sense of James ([8]), it is fiber homotopy equivalent to the trivial bundle ([8], Corollary 4.3). This contradicts Theorem 2.17. Hence we have a family of non-trivial elements ϕ_j in the 2-component of $\pi_{8j+3}(S^{4j+1})$ ($j \geq 2$). Similarly, the attaching maps of the top cell of $X_{n+2,2}$ define a family of non-trivial elements ψ_j in the 2-component of $\pi_{64j+57}(S^{32j+27})(j \geq 0)$.

Corollary 2.18. *The attaching maps of the top cells of $W_{n+2,2}$ and $X_{n+2,2}$ define non-trivial elements ϕ_j $(j \geq 2)$ and ψ_j $(j \geq 0)$ in the 2-components of $\pi_{8j+3}(S^{4j+1})$ and $\pi_{64j+57}(S^{32j+27})$. Moreover, ϕ_j is in the kernel of E^3: $\pi_{8j+3}(S^{4j+1}) \to \pi_{8j+6}(S^{4j+4}) \cong \pi_{4j+2}^S(S^0)$ and has order at most eight, ψ_j is in the kernel of E^5: $\pi_{64j+57}(S^{32j+27}) \to \pi_{64j+62}(S^{32j+32}) \cong \pi_{32j+30}^S(S^0)$ and has order at most sixteen.*

Proof. The assertion that $\phi_j \in \mathrm{Ker}\,E^3$ and $\psi_j \in \mathrm{Ker}\,E^5$ is shown in Appendix (See the proof of Corollary A.10.). By the exactness of $\pi_*(F(m))$ $\xrightarrow{e_*} \pi_*(S^m) \xrightarrow{i_*} \pi_*(QS^m)$, $\phi_j \in \mathrm{Ker}\,E^3$ and $\psi_j \in \mathrm{Ker}\,E^5$ imply that $\phi_j \in \mathrm{Im}\,[e_*:$

$\pi_{8j+3}(F(4j+1)) \to \pi_{8j+3}(S^{4j+1})]$ and $\psi_j \in \mathrm{Im}[e_*: \pi_{64j+57}(F(32j+27)) \to \pi_{64j+57}(S^{32j+27})]$. Then the assertion $8\phi_j = 16\psi_j = 0$ follows from $\pi_{8j+3}(F(4j+1)) \cong \pi_{8j+3}(\Sigma^{4j}\mathbf{P}_{4j+1}^{\infty}) = \mathbf{Z}/8$ and $\pi_{64j+57}(F(32j+27)) \cong \pi_{64j+57}(\Sigma^{32j+26}\mathbf{P}_{32j+27}^{\infty}) \cong \mathbf{Z}/16$.

Remark 2.19.   If n is even and greater than two, Theorem 2.17 implies that spaces $\Omega W_{n+2,2}$ and $\Omega(S^{2n+1} \times S^{2n+3})$ are different as loop spaces (with the natural loop space structures), although they have same homotopy type by Proposition 1.10.   Therefore both $\Omega W_{n+2,2}$ and $\Omega(S^{2n+1} \times S^{2n+3})$ have at least two different loop space structures.   One is the natural one and the other is induced by the homotopy equivalence $\Omega W_{n+2,2} \simeq \Omega(S^{2n+1} \times S^{2n+3})$ in Proposition 1.10.   Hence, if we deloop $\Omega W_{n+2,2}$ and $\Omega(S^{2n+1} \times S^{2n+3})$ with respect to the exotic loop space structures, we have homotopy equivalences $B\Omega W_{n+2,2} \simeq S^{2n+1} \times S^{2n+3}$ and $B\Omega(S^{2n+1} \times S^{2n+3}) \simeq W_{n+2,2}$.   Similarly, if $n \equiv 6$ modulo 8, $\Omega X_{n+2,2}$ and $\Omega(S^{4n+3} \times S^{4n+7})$ have exotic loop space structures.

We also remark that $\Sigma^3 W_{n+2,2}$ (resp. $\Sigma^5 X_{n+2,2}$) is homotopy equivalent to $\Sigma^3(S^{2n+1} \times S^{2n+3})$ (resp. $\Sigma^5(S^{4n+3} \times S^{4n+7})$) for n as above since the attaching of the top cell suspends to zero in the stable range.   Details are left to readers.

§3.   On the Homotopy Fiber of $S^{2n+1} \to QS^{2n+1}$

Throughout this section, p is a fixed odd prime and all spaces, spectra and groups are localized at prime p.   As in the previous section, we denote by $F(n)$ the homotopy fiber of the canonical inclusion $i: S^n \to QS^n$.   The structure of $H_*(QS^n; \mathbf{F}_p)$ is described in [4] and [12].   We review briefly according to [12].

For a sequence $I = (\varepsilon_1, s_1, \ldots, \varepsilon_k, s_k)$ of non-negative integers such that $\varepsilon_j = 0$ or 1 and $s_j \geqq \varepsilon_j$, define the degree, length and excess of I by $d(I) = \sum_{j=1}^{k}(2s_j(p-1) - \varepsilon_j)$, $l(I) = k$ and $e(I) = 2(ps_1 - \varepsilon_1) - d(I)$ respectively.   We denote by Q^I the iterated homology operation $\beta^{\varepsilon_1}Q^{s_1} \cdots \beta^{\varepsilon_k}Q^{s_k}$. I is said to be admissible if $ps_j - \varepsilon_j \geqq s_{j-1}$ for $2 \leqq j \leqq k$.   We also call the empty sequence I admissible and assign $d(I) = l(I) = 0$, $e(I) = \infty$, $Q^I = $ (identity operation).

Theorem 3.1. ([4], [12]).   *For a positive integer n, let $\iota_n \in H_n(QS^n)$ be the image of the canonical generator of $H_n(S^n)$ by $i_*: H_n(S^n) \to H_n(QS^n)$.   And let A_n be the free commutative graded algebra over $\mathbf{F}_p$ generated by $T_n = \{Q^I \iota_n | I: admissible, e(I) \geqq n\}$, that is, A_n is the tensor product of an exterior algebra $E(Q^I \iota_n | I: admissible, e(I) \geqq n, n + d(I): odd)$ and a polynomial algebra $\mathbf{F}_p[Q^I \iota_n | I: admissible, e(I) \geqq n, n + d(I): even]$.   Let B_n be an ideal of A_n generated by $\{Q^s x - x^p | x \in T_n, \deg x = 2s\}$.   Then $H_*(QS^n)$ is isomorphic to A_n/B_n as an algebra and each generator $Q^I \iota_n$ is primitive.*

Corollary 3.2. $i_*: \pi_t(S^{2n+1}) \to \pi_t(QS^{2n+1})$ *is an isomorphism for $t < 2pn + 2p - 3$ and an epimorphism for $t < 2pn + 2p - 2$.   Hence $F(2n+1)$ is $(2pn + 2p - 4)$-connected.*

Proof. Since $H_s(QS^{2n+1}) = 0$ for $2n + 1 < s < 2pn + 2p - 2$, $i_*: H_t(S^{2n+1})$ $\rightarrow H_t(QS^{2n+1})$ is an isomorphism for $t < 2pn + 2p - 2$.

Let us consider the following fibration in order to compute the mod p homology of $F(2n + 1)$.

$$(3.3) \qquad\qquad QS^{2n} \xrightarrow{\delta} F(2n + 1) \xrightarrow{e} S^{2n+1}$$

Note that the above fibration is a principal fibration, hence by applying the Wang sequence we have

Lemma 3.4. $\delta_*: H_*(QS^{2n}) \rightarrow H_*(F(2n + 1))$ *is surjective and its kernel is an ideal generated by* ι_{2n}. *Therefore* $H_*(F(2n + 1)) \cong H_*(QS^{2n})/(\iota_{2n})$.

Since $F(2n + 1)$ has homotopy type of a simply connected CW-complex, we may replace $F(2n + 1)$ by a homotopy equivalent normal CW-complex $\tilde{F}(2n + 1)$. Thus the inclusion map of the $(2np + 2p^2 - 4)$-skeleton of $\tilde{F}(2n + 1)$ induces isomorphism of homology groups through dimension $2np + 2p^2 - 4$. Define a (-1)-connected suspension spectrum L_n to be $\Sigma^{-2np-2p+3}\tilde{F}(2n + 1)^{2np+2p^2-4}$. Let $y_i, z_i \in H_*(L_n)$ be the elements corresponding to $\delta_*(\beta Q^{n+i+1}\iota_{2n})$ and $\delta_*(Q^{n+i+1}\iota_{2n})$ in $H_*(F(2n + 1))$. Hence $\deg y_i = 2i(p - 1)$, $\deg z_i = 2i(p - 1) + 1$ and $\{y_0, z_0, y_1, z_1, \ldots y_{p-1}, z_{p-1}\}$ is a basis of $H_*(L_n)$ if $n \geq p - 1$. So, from now on, we assume $n \geq p - 1$ for simplicity unless otherwise stated. Let A_* be the dual mod p Steenrod algebra, then $A_* = E(\tau_0, \tau_1, \ldots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots]$ as an algebra ([16]).

Lemma 3.5. *The coaction* $\psi: H_*(L_n) \rightarrow A_* \otimes H_*(L_n)$ *is given by*

$$\psi(y_i) = \Sigma_{j=0}^i \binom{n + i + 1}{j} \xi_1^i \otimes y_{i-j}$$

$$\psi(z_i) = \Sigma_{j=0}^i \binom{n + i}{j} \xi_1^j \otimes z_{i-j} + \Sigma_{j=0}^i \binom{n + i + 1}{j} \tau_0 \xi_1^j \otimes y_{i-j}$$

$$- \Sigma_{j=1}^i \binom{n + i}{j - 1} \tau_1 \xi_1^{j-1} \otimes y_{i-j}.$$

Proof. The Nishida relations ([17], see also [12]) determine the action of P_*^j on y_i, z_i as follows.

$$P_*^j y_i = (-1)^j \binom{(n + i - j + 1)(p - 1) - 1}{j} y_{i-j},$$

$$P_*^j z_i = (-1)^j \binom{(n + i - j + 1)(p - 1)}{j} z_{i-j} \quad \text{and} \quad \beta z_i = y_i$$

The result easily follows from the above and congruences

$$(-1)^j \binom{(m-j)(p-1)-1}{j} \equiv \binom{m}{j},$$

$$(-1)^j \binom{(m-j)(p-1)}{j} \equiv \binom{m-1}{j} \text{ modulo } p \text{ for } 0 \leq j \leq p-1.$$

Let $P_* = \mathbb{F}_p[\xi_1, \xi_2, \ldots]$ be the Hopf subalgebra of A_* generated by $\xi_1, \xi_2, \ldots$ and put $E_* = A_*//P_* = E(\tau_0, \tau_1, \ldots)$. Then $P_* \to A_* \to E_*$ is an extension of Hopf algebras. To compute the BP-homology of L_n, we use the Adams spectral sequence $E_2^{s,t} = \operatorname{Ext}_{A_*}^{s,t}(\mathbb{F}_p, H_*(BP \wedge L_n)) \Rightarrow BP_{t-s}(L_n)$. First note that $H_*(BP \wedge L_n) \cong P_* \otimes H_*(L_n)$ is isomorphic to $A_* \square_{E_*} H_*(L_n)$ as an A_*-comodule. In fact, define $f: P_* \otimes H_*(L_n) \to A_* \otimes H_*(L_n)$ by $f(\alpha \otimes x) = \Sigma_i \alpha a_i \otimes x_i$ where $\psi(x) = \Sigma_i a_i \otimes x_i$, then f lifts to $\bar{f}: P_* \otimes H_*(L_n) \to A_* \square_{E_*} H_*(L_n)$ which is an isomorphism of A_*-comodules. We apply the Change-of-Rings isomorphism and have

(3.6) $\operatorname{Ext}_{A_*}(\mathbb{F}_p, H_*(BP \wedge L_n)) \cong \operatorname{Ext}_{A_*}(\mathbb{F}_p, A_* \square_{E_*} H_*(L_n))$

$$\cong \operatorname{Ext}_{E_*}(\mathbb{F}_p, H_*(L_n))$$

Lemma 3.5 gives the coaction $\tilde{\psi}: H_*(L_n) \to E_* \otimes H_*(L_*)$ as follows

(3.7) $\tilde{\psi}(y_i) = 1 \otimes y_i, \; \tilde{\psi}(z_i) = 1 \otimes z_i + \tau_0 \otimes y_i - \tau_1 \otimes y_{i-1}$

$$(\text{Put } y_{-1} = 0)$$

Hence we may regard $H_*(L_n)$ as an $E(\tau_0, \tau_1)$-comodule. Put $E' = E(\tau_2, \tau_3, \ldots)$, then $E_* = E(\tau_0, \tau_1) \otimes E'$. By the Künneth theorem, it is easy to see that the external pairing

(3.8) $\operatorname{Ext}_{E(\tau_0, \tau_1)}(\mathbb{F}_p, H_*(L_n)) \otimes \operatorname{Ext}_{E'}(\mathbb{F}_p, \mathbb{F}_p) \to \operatorname{Ext}_{E_*}(\mathbb{F}_p, H_*(L_n))$

is an isomorphism. Since each τ_i is primitive in E_*, we have

(3.9) $\operatorname{Ext}_{E'}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[v_2, v_3, \ldots, v_i, \ldots], \text{ bideg } v_i = (1, 2p^i - 1)$

v_i is represented by $[\tau_i]$ in the cobar complex.

To compute $\operatorname{Ext}_{E(\tau_0, \tau_1)}(\mathbb{F}_p, H_*(L_n))$, we apply the Cartan-Eilenberg spectral sequence ([18], p. 331) to an extension of Hopf algebras $E(\tau_1) \to E(\tau_0, \tau_1) \to E(\tau_0)$. Clearly, $H_*(L_n)$ is a free $E(\tau_0)$-comodule with basis in degrees $2i(p-1)(i = 0, 1, \ldots, p-1)$. Therefore

(3.10) $\operatorname{Ext}_{E(\tau_0)}^{s,t}(\mathbb{F}_p, H_*(L_n)) \cong \begin{cases} \mathbb{F}_p & s=0, \; t=2i(p-1) \quad 0 \leq i \leq p-1 \\ 0 & \text{otherwise} \end{cases}$

By dimensional reason, coaction $\operatorname{Ext}_{E(\tau_0)}(\mathbb{F}_p, H_*(L_n)) \to E(\tau_1) \otimes \operatorname{Ext}_{E(\tau_0)}(\mathbb{F}_p, H_*(L_n))$ is trivial. Thus the E_2-term of the Cartan-Eilenberg spectral sequence is given by

(3.11) $\qquad E_2^{s,t} = \mathrm{Ext}_{E(\tau_1)}^s(\mathbb{F}_p,\ \mathrm{Ext}_{E(\tau_0)}^t(\mathbb{F}_p,\ H_*(L_n)))$

$\qquad\qquad\qquad \cong \mathrm{Ext}_{E(\tau_0)}^t(\mathbb{F}_p,\ H_*(L_n)) \otimes \mathrm{Ext}_{E(\tau_1)}^s(\mathbb{F}_p,\ \mathbb{F}_p)$

Therefore $E_2^{s,t} = 0$ if $t \neq 0$ and we see that this spectral sequence collapses. We also denote by y_i the element of $\mathrm{Ext}_{E(\tau_0,\tau_1)}^{0,2i(p-1)}(\mathbb{F}_p, H_*(L_n))$ represented by $y_i \in H_{2i(p-1)}(L_n)$ in the cobar complex. As in (3.9), we have

(3.12) $\qquad \mathrm{Ext}_{E(\tau_1)}(\mathbb{F}_p,\ \mathbb{F}_p) = \mathbb{F}_p[v_1],\ \mathrm{Ext}_{E(\tau_0,\tau_1)}(\mathbb{F}_p,\ \mathbb{F}_p) = \mathbb{F}_p[v_0,\ v_1]$

Hence the E_∞-term is given by

(3.13) $\qquad\qquad E_\infty = \mathbb{F}_p\{y_0,\ y_1,\dots,\ y_{p-1}\} \otimes \mathbb{F}_p[v_1]$

The second formula of (3.7) settles the extension problem on the $\mathrm{Ext}_{E(\tau_0,\tau_1)}(\mathbb{F}_p, \mathbb{F}_p)$-module structure of $\mathrm{Ext}_{E(\tau_0,\tau_1)}(\mathbb{F}_p, H_*(L_n))$, namely, relations $v_0 y_i = v_1 y_{i-1}$ $(i = 0, 1,\dots, p-1)$ hold in $\mathrm{Ext}_{E(\tau_0,\tau_1)}(\mathbb{F}_p, H_*(L_n))$. By iso-morphisms (3.6) and (3.8), we see that the E_2-term of the Adams spectral sequence is generated by $y_i \in \mathrm{Ext}_{A_*}^{0,2i(p-1)}(\mathbb{F}_p, H_*(BP \wedge L_n))$ over $\mathrm{Ext}_{A_*}(\mathbb{F}_p, H_*(BP))$ $= \mathbb{F}_p[v_0, v_1,\dots,v_i,\dots]$ with relations $v_0 y_i = v_1 y_{i-1}$ $(i = 0, 1,\dots, p-1;\ y_{-1} = 0)$. Since $E_2^{s,t} = 0$ if $t - s$ is odd, the Adams spectral sequence collapses. Thus we have shown

Lemma 3.14. *$BP_*(L_n)$ is generated by $y_i \in BP_{2i(p-1)}(L_n)$ $(i = 0, 1,\dots, p-1)$ over BP_* with relations $py_0 = 0$, $py_i = v_1 y_{i-1}$ $(i = 1, 2,\dots, p-1)$ and we can choose y_i such that it is mapped to $y_i \in H_*(L_n)$ by the Thom map $BP_*(L_n) \to H_*(L_n)$.*

Remark 3.15. In $BP_*(L_n)$, equality $pBP_t(L_n) = v_1 BP_{t-2(p-1)}(L_n)$ holds for $t < 2p(p-1)$ and the above y_i's are determined uniquely modulo $pBP_*(L_n)$ so that relations $py_i = v_1 y_{i-1}$ $(0 \leq i \leq p-1)$ hold. These relations imply that $p^{i+1}y_i = 0$ (but $p^i y_i \neq 0$).

Recall that the Thom map $BP_*BP \to A_*$ sends t_i to the conjugate of ξ_i ([24]). It follows from Lemma 3.5 and the relations that there exists $\lambda_{i,j} \in \mathbb{Z}_{(p)}$ such that the coaction $\psi: BP_*(L_n) \to BP_*BP \otimes_{BP_*} BP_*(L_n)$ is of the following form

(3.16) $\qquad \psi(y_i) = \sum_{j=0}^i \left\{(-1)^j \binom{n+i+1}{j} + p\lambda_{i,j}\right\} t_1^j \otimes y_{i-j}$

We note that, since $p^{i+1}y_i = 0$, we may regard $\lambda_{i,j}$ as an element of $\mathbb{Z}/p^{i-j}$, and that each $\lambda_{i,j} \in \mathbb{Z}/p^{i-j}$ does not depend on the choice of generators $y_0, y_1,\dots,$ y_{p-1} which have the properties of Lemma 3.14. We do not need explicit values of $\lambda_{i,j}$'s for our purpose, so we only remark that $\lambda_{i,0} \equiv 0$ modulo p^i by the existence of counit, and that $\lambda_{p-1,1}$ determines every other $\lambda_{i,j}$ by the coassociativity and the BP_*-linearity of ψ.

Define a filtration of $BP_*(L_n)$ by putting $F_t = BP_*\{y_0, y_1,..., y_t\}$, then $0 = F_{-1} \subset F_0 \subset F_1 \subset ... \subset F_{p-1} = BP_*(L_n)$ and the associated graded comodule $E_0^t BP_*(L_n)$ is isomorphic to $BP_*(\Sigma^{2t(p-1)} M_p)$ as a BP_*BP-comodule where M_p is the mod p Moore spectrum. We also filter the cobar complex $\Omega^*(BP_*(L_n))$ by $F_{s,t} = BP_*BP^{\otimes s} \otimes_{BP_*} F_t$. Consider the spectral sequence associated with this filtration. Then, we have $E_1^{s,t,u} = \text{Ext}_{BP_*BP}^{s,u}(BP_*, E_0^t BP_*(L_n))$ and $d_r : E_r^{s,t,u} \to E_r^{s+1,t-r,u}$. Note that this spectral sequence has an $\text{Ext}_{BP_*BP}(BP_*, BP_*)$-module structure. Let $\alpha_1 \in \text{Ext}_{BP_*BP}^{1,2(p-1)}(BP_*, BP_*)$ be the element represented by $[t_1]$ in the cobar complex. The E_1-term is easily obtained by the following results of Miller-Ravenel-Wilson.

Theorem 3.17. ([15], see also [18] p.157) $\text{Ext}_{BP_*BP}^{0,*}(BP_*, BP_*(M_P)) = \mathbf{F}_p[v_1]$, $\text{Ext}_{BP_*BP}^{1,*}(BP_*, BP_*(M_P)) = \mathbf{F}_p[v_1] \cdot \alpha_1$ for $* < 2p(p-1)$, and $\text{Ext}_{BP_*BP}^{s,u}(BP_*, BP_*(M_P)) = 0$ for $s \geq 2$ and $u - s < 2p(p-1) - 2$.

Lemma 3.18. $E_1^{0,t,u} \cong \begin{cases} \mathbf{Z}/p & u = 2j(p-1), j \geq t \\ 0 & otherwise \end{cases}$,

$$E_1^{1,t,u} \cong \begin{cases} \mathbf{Z}/p & u = 2j(p-1), t+1 \leq j < p+t \\ 0 & otherwise \text{ for } u < 2(p+t)(p-1) \end{cases}$$

$E_1^{s,t,u} = 0$ for $s \geq 2$, $u - s < 2(t+p)(p-1) - 2$. Moreover, $E_1^{0,t,2(i+t)(p-1)}$ is generated by $v_1^i y_t$ and $E_1^{1,t,2(i+t)(p-1)}$ is generated by $\alpha_1 v_1^{i-1} y_t$ $(i < p)$.

The first differential is given by (3.16).

(3.19) $d_1(v_1^i y_t) = (n + i + t + 1)\alpha_1 v_1^i y_{t-1}$ for $i + t < p$.

Put $-n - 1 = pq_n + r_n$ $(0 \leq r_n < p)$, then $d_1(v_1^i y_t) = (n + i + t + 1)\alpha_1 v_1^i y_{t-1} \neq 0$ if $i + t \neq r_n$, $i + t < p$ and $t > 0$. $d_1(v_1^i y_t) = 0$ if $i + t = r_n$ or $t = 0$. This implies

Lemma 3.20. $E_2^{0,t,u} = 0$ if $u \neq 2r_n(p-1)$, $u < 2p(p-1)$ and $0 < t < p$. $E_2^{1,t,u} = 0$ if $u \neq 2r_n(p-1)$ $u < 2p(p-1)$ and $0 \leq t < p - 1$. On the other hand, $E_2^{0,0,2i(p-1)} \cong \mathbf{Z}/p$ for $0 \leq i < p$, and $E_2^{1,r_n-i,2r_n(p-1)} \cong \mathbf{Z}/p$ for $1 \leq i \leq r_n$ if $r_n > 0$.

Define a filtration on $\text{Ext}_{BP_*BP}^{s,u}(BP_*, BP_*(L_n))$ by $F^{s,t,u} = \text{Im} [\text{Ext}_{BP_*BP}^{s,u}(BP_*, F_t) \to \text{Ext}_{BP_*BP}^{s,u}(BP_*, BP_*(L_n))]$. Then $0 = F^{s,-1,u} \subset F^{s,0,u} \subset ... \subset F^{s,t,u} \subset ... \subset F^{s,p-1,u} = \text{Ext}_{BP_*BP}^{s,u}(BP_*, BP_*(L_n))$ and $F^{s,t,u}/F^{s,t-1,u} \cong E_\infty^{s,t,u}$.

Lemma 3.21. For $j < 2p(p-1) - 2$, $\pi_j^S(L_n) \neq 0$ if and only if one of the following conditions is satisfied ;
i) $j = 2i(p-1)$ for $0 \leq i < p$.
ii) $j = 2r_n(p-1) - 1$ and $r_n > 0$.

Proof. Consider the Adams-Novikov spectral sequence $E_2^{s,u}(L_n) =$

$\mathrm{Ext}^{s,u}_{BP_*BP}(BP_*, BP_*(L_n)) \Rightarrow \pi^S_{t-s}(L_n)$. By Lemma 3.18 and Lemma 3.20, $E^{0,t,u}_\infty = 0$ if $u \not\equiv 0$ modulo $2(p-1)$, and $E^{1,t,u}_\infty = 0$, if $u = 0$ or $u \ne 2r_n(p-1)$, $u < 2p(p-1)$. Hence we have $\mathrm{Ext}^{0,u}_{BP_*BP}(BP_*, BP_*(L_n)) = 0$ if $u \not\equiv 0$ modulo $2(p-1)$, and $\mathrm{Ext}^{1,u}_{BP_*BP}(BP_*, BP_*(L_n)) = 0$ if $u = 0$ or $u \ne 2r_n(p-1)$, $u < 2p(p-1)$. Since $E^{s,t,u}_\infty = E^{s,t,u}_1 = 0$ for $s \geq 2$, $u - s < 2p(p-1) - 2$, we also have $\mathrm{Ext}^{s,u}_{BP_*BP}(BP_*, BP_*(L_n)) = 0$ for $s \geq 2$, $u - s < 2p(p-1) - 2$. It follows that $\pi^S_j(L_n) = 0$ ($j < 2p(p-1) - 2$) if neither i) nor ii) is satisfied. Conversely, since $E^{0,0,2i(p-1)}_\infty = E^{0,0,2i(p-1)}_2 \ne 0$ for $0 \leq i < p$, we see that $\mathrm{Ext}^{0,2i(p-1)}_{BP_*BP}(BP_*, BP_*(L_n)) \ne 0$ for $0 \leq i < p$. By Lemma 3.18 and Lemma 3.20, $E^{1,r_n-1,2r_n(p-1)}_2 \ne 0$ if $r_n \ne 0$, $E^{0,t,2r_n(p-1)}_2 = 0$ if $t > r_n$, and $E^{2,t,u}_2 = 0$ if $u < 2p(p-1)$. Therefore elements of $E^{1,r_n-1,2r_n(p-1)}_2$ survive and $E^{1,r_n-1,2r_n(p-1)}_\infty \ne 0$, thus $\mathrm{Ext}^{1,2r_n(p-1)}_{BP_*BP}(BP_*, BP_*(L_n)) \ne 0$ if $r_n \ne 0$. Recalling that $\mathrm{Ext}^{s,u}_{BP_*BP}(BP_*, BP_*(L_n)) = 0$ for $s \geq 2$, $u - s < 2p(p-1) - 2$, it follows that $E^{0,2i(p-1)}_\infty(L_n) = E^{0,2i(p-1)}_2(L_n) \ne 0$ for $0 \leq i < p$ and $E^{1,2r_n(p-1)}_\infty(L_n) = E^{1,2r_n(p-1)}_2(L_n) \ne 0$ if $r_n \ne 0$. This completes the proof.

Theorem 3.22. *For $i < 2pn + 2p^2 - 4$ and $n \geq p - 1$, $\pi_i(QS^{2n+1}, S^{2n+1}) \ne 0$ if and only if one of the following conditions is satisfied;*
i) $i = 2pn + 2j(p-1)$ *for* $1 \leq j \leq p$.
ii) $i = 2(r_n + 1)(p-1) + 2pn - 1$ *and* $r_n > 0$.

Proof. $\pi_i(QS^{2n+1}, S^{2n+1})$ is isomorphic to $\pi_{i-1}(F(2n+1))$ and, by Corollary 3.2, $\pi_{i-1}(F(2n+1))$ is isomorphic to the stable homotopy group $\pi^S_{i-1}(F(2n+1))$ if $i < 4pn + 4p - 6$. By the definition of L_n, $\pi^S_{i-1}(F(2n+1)) \cong \pi^S_{i-2pn-2p+2}(L_n)$ for $i < 2pn + 2p^2 - 4$. Note that $2pn + 2p^2 - 4 \leq 4pn + 4p - 6$ since $n \geq p - 1$, and we have an isomorphism $\pi_i(QS^{2n+1}, S^{2n+1}) \cong \pi_{i-2pn-2p+2}(L_n)$ for $i < 2pn + 2p^2 - 4$. Then, the assertion follows from the previous lemma.

In the case $n < p - 1$, we put $m = [(np + p - 2)/(p-1)]$ and define a spectrum L'_n to be $\Sigma^{-2np-2p+3}\tilde{F}(2n+1)^{4np+4p-6}$ instead of L_n. Then $\{y_0, z_0, y_1, z_1, \ldots, y_m, z_m\}$ is a basis of $H_*(L'_n)$, and as in Lemma 3.14, $BP_*(L'_n)$ is generated by $y_i \in BP_*(L'_n)$, we can compute $\mathrm{Ext}_{BP_*BP}(BP_*, BP(L'_n))$ in the same way. Thus we have

Theorem 3.23. *For $i < 4pn + 4p - 6$ and $n < p - 1$, $\pi_i(QS^{2n+1}, S^{2n+1}) \ne 0$ if and only if one of the following conditions is satisfied;*
i) $i = 2pn + 2j(p-1)$ *for* $1 \leq j \leq [(np + p - 2)/(p-1)]$.
ii) $i = 2(r_n + 1)(p-1) + 2pn - 1$ *and* $r_n > 0$.

We also need to know whether $\pi_i(QS^{2n+1}, S^{2n+1})$ is zero or not in the low dimensional unstable range. The following result is due to Toda (see also [2]).

Theorem 3.24. ([22]). *For $1 \leq n < p - 1$,*
$$\pi_i(QS^{2n+1}, S^{2n+1}) \cong \begin{cases} \mathbb{Z}/p & i = 2pn + 2j(p-1), \ 1 \leq j < p - n \\ 0 & \text{otherwise for} \quad i < 2n + 2p(p-1) \end{cases}$$

Summarizing Theorem 3.22, 3.23 and 3.24,

(3.25)   *For $i < \min\{2pn + 2p^2 - 4,\quad \max\{4pn + 4p - 6,\quad 2n + 2p(p - 1)\}\}$,
$\pi_i(QS^{2n+1}, S^{2n+1}) \neq 0$ if and only if one of the following conditions is satisfied.*
i) *$i = 2pn + 2j(p - 1)$ for $j \geqq 1$.*
ii) *$i = 2(r_n + 1)(p - 1) + 2pn - 1$ and $r_n > 0$, where $-n - 1 = pq_n + r_n$, $0 \leqq r_n < p$.*

Remark 3.26.   For an integer i which satisfies the condition i) above, the Hurewicz map $\pi_{i-1}(F(2n + 1)) \to PBP_{i-1}(F(2n + 1))$ is an isomorphism, where $PBP_*(X)$ is the set of primitive elements $\{x \in BP_*(X) | \psi(x) = 1 \otimes x\}$.

§4.   The p-Regularity for an Odd Prime p

As in the previous section, p is a fixed odd prime and all spaces and groups are localized at p throughout this section.   First, we note the following fact which is easily obtained by Corollary 1.2.

Proposition 4.1.   *$W_{n+k,k}$ is not p-regular if $k > p$, or $k = p$ and $p \nmid n$. $X_{n+k,k}$ is not p-regular if $k > (p + 1)/2$, or $k = (p + 1)/2$ and $p \nmid 2n + 1$.*

We recall the conditions (C) and (H) in Section 1.
(C) $k \leqq p - 1$, or $k = p$ and $p|n$.   (H) $k \leqq (p - 1)/2$, or $k = (p + 1)/2$ and $p|2n + 1$.
The following result of James is the starting point of our argument.

Theorem 4.2. ([8]) *Corollary 7. 11) $Q_{n+k,k}(\mathbf{F})$ is a stable retract of $O_{n+k,k}(\mathbf{F})$, where $O_{n+k,k}(\mathbf{F})$ is $V_{n+k,k}$, $W_{n+k,k}$ or $X_{n+k,k}$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ respectively.*

The combination of the above theorem and Theorem 1.8 implies

Lemma 4.3. i) *If (C) is satisfied, there exists a map $f: (W_{n+k,k}, S^{2n+1})$ $\to (QS^{2n+1}, S^{2n+1})$ such that $f|S^{2n+1}: S^{2n+1} \to S^{2n+1}$ is degree 1.*
ii) *If (H) is satisfied, there exists a map $g: (X_{n+k,k}, S^{4n+3}) \to (QS^{4n+3}, S^{4n+3})$ such that $g|S^{4n+3}: S^{4n+3} \to S^{4n+3}$ is degree 1.*

Proof.   By Theorem 1.8, there exist maps $g': Q_{n+k,k}(\mathbf{C}) \to S^{2n+1}$ and $g': Q_{n+k,k}(\mathbf{H}) \to S^{4n+3}$ such that the compositions $S^{2n+1} \hookrightarrow Q_{n+k,k}(\mathbf{C}) \xrightarrow{f'} S^{2n+1}$ and $S^{4n+3} \hookrightarrow Q_{n+k,k}(\mathbf{H}) \xrightarrow{g'} S^{4n+3}$ are degree 1.   Let $f'': \Sigma^\infty W_{n+k,k} \to \Sigma^\infty Q_{n+k,k}(\mathbf{C})$ and $g'': \Sigma^\infty X_{n+k,k} \to \Sigma^\infty Q_{n+k,k}(\mathbf{H})$ be stable retractions.   Define f and g to be the adjoints of stable maps $\Sigma^\infty f' \circ f'': \Sigma^\infty W_{n+k,k} \to \Sigma^\infty S^{2n+1}$ and $\Sigma^\infty g' \circ g'': \Sigma^\infty X_{n+k,k} \to \Sigma^\infty S^{4n+3}$, respectively.   Then f and g satisfy the conditions.

We define sets of integers as follows

(4.4) $C_{n+k,k}(\mathbb{F}) = \{i \in \mathbb{Z} \mid i = \sum_{j=1}^{k} \varepsilon_j(dn + dj - 1), \ \varepsilon = 0 \text{ or } 1\}$, where $d = 2$ if $\mathbb{F} = \mathbb{C}$ and $d = 4$ if $\mathbb{F} = \mathbb{H}$.

(4.5) $\Pi(n; p) = \{i \in \mathbb{Z} \mid i = 2pn + 2j(p - 1) \text{ for } j \geq 1, \text{ or } i = 2(r_n + 1)(p - 1) + 2pn - 1 \text{ if } r_n > 0\}$ where r_n is defined by $-n - 1 = pq_n + r_n$, $0 \leq r_n < p$ (See Introduction.).

Note that $C_{n+k,k}(\mathbb{C}) = \{i \in \mathbb{Z} \mid H^i(W_{n+k,k}) \neq 0\}$ and $C_{n+k,k}(\mathbb{H}) = \{i \in \mathbb{Z} \mid H^i(X_{n+k,k}) \neq 0\}$. We also note that obstructions to deform f(resp. g) to S^{2n+1}(resp. S^{4n+3}) lie in $H^i(W_{n+k,k}, S^{2n+1}; \pi_i(QS^{2n+1}, S^{2n+1}))$ (resp. $H^i(X_{n+k,k}, S^{4n+3}; \pi_i(QS^{4n+3}, S^{4n+3}))$).

Theorem 4.6.   i) *Assume $k \leq p - 1$, or $k = p$ and $p \mid n$, then $W_{n+k,k}$ is homotopy equivalent to $S^{2n+1} \times W_{n+k,k-1}$ if $C_{n+k,k}(\mathbb{C}) \cap \Pi(n; p)$ is empty.*
ii) *Assume $k \leq (p - 1)/2$, or $k = (p + 1)/2$ and $p \mid 2n + 1$, then $X_{n+k,k}$ is homotopy equivalent to $S^{4n+3} \times X_{n+k,k-1}$ if $C_{n+k,k}(\mathbb{H}) \cap \Pi(2n + 1; p)$ is empty.*

Proof. First note that $\dim W_{n+k,k} = \max C_{n+k,k}(\mathbb{C}) = 2nk + k^2 < \min\{2pn + 2p^2 - 4, \max\{4pn + 4p - 6, 2n + 2p(p - 1)\}\}$ if $k \leq p - 1$, or $k = p$ and $p \mid n$. Hence, by (3.25), the condition of i) is equivalent to vanishing of $H^i(W_{n+k,k}, S^{2n+1}; \pi_i(QS^{2n+1}, S^{2n+1}))$ for any $i \in \mathbb{Z}$. Therefore there exists a left homotopy inverse of the inclusion $S^{2n+1} \hookrightarrow W_{n+k,k}$, and fibration $S^{2n+1} \to W_{n+k,k} \to W_{n+k,k-1}$ is fiber homotopy equivalent to the product fibration $S^{2n+1} \to S^{2n+1} \times W_{n+k,k-1} \to W_{n+k,k-1}$ ([8], Corollary 4.3). Proof of ii) is similar.

If $n \geq (k^2 - 2p + 3)/(2p - 2k)$ $(k \leq p - 1)$, $\dim W_{n+k,k} \leq 2pn + 2p - 3$. Hence $C_{n+k,k-j}(\mathbb{C}) \cap \Pi(n + j; p)$ is empty for $j = 0, 1, \ldots, k - 1$. And if $m \geq (p - 1)/2$, it is easy to verify that $C_{mp+p,p}(\mathbb{C}) \cap \Pi(mp; p)$ is empty. Similarly, if $n \geq (2k^2 + k - 4p + 3)/(4p - 4k)$ $(k \leq (p - 1)/2)$, $C_{n+k,k-j}(\mathbb{H}) \cap \Pi(2n + 2j + 1; p)$ is empty for $j = 0, 1, \ldots, k - 1$, and if $p \mid 2n + 1$ (hence $n \geq (p - 1)/2$), $C_{n+(p+1)/2,(p+1)/2}(\mathbb{H}) \cap \Pi(2n + 1; p)$ is empty. Thus the preceding theorem implies

Theorem 4.7.   i) *For $k \leq p - 1$, $W_{n+k,k}$ is p-regular if $\bigcup_{j=0}^{k-1}(C_{n+k,k-j}(\mathbb{C}) \cap \Pi(n + j; p))$ is empty. In particular, it is p-regular if $n \geq (k^2 - 2p + + 3)/(2p - 2k)$. And $W_{mp+p,p}$ is p-regular if $m \geq (p - 1)/2$.*
ii) *For $k \leq (p - 1)/2$, $X_{n+k,k}$ is p-regular if $\bigcup_{j=0}^{k-1}(C_{n+k,k-j}(\mathbb{H}) \cap \Pi(2n + 2j + 1; p))$ is empty. In particular, it is p-regular if $n \geq (2k^2 + k - 4p + 3)/(4p - 4k)$. And $X_{n+(p+1)/2,(p+1)/2}$ is p-regular if $p \mid 2n + 1$.*

The fact that $\pi_{2n+1}(BU(n))$ (resp. $\pi_{4n+3}(BSp(n))$) is a trivial group if $n < p$ (resp. $n < (p - 1)/2$) easily implies the p-regularity of $W_{n+k,k}$ (resp. $X_{n+k,k}$) for $n + k \leq p$ (resp. $n + k \leq (p - 1)/2$).

Proposition 4.8 ([6]). *$W_{n+k,k}$ is p-regular if $n + k \leq p$, and $X_{n+k,k}$ is p-regular if $n + k \leq (p - 1)/2$.*

Proof. Consider the fiber bundle $W_{n+k-1,k-1} \to W_{n+k,k} \to S^{2n+2k-1}$ which is associated with the principal bundle $U(n+k-1) \to U(n+k) \to S^{2n+2k-1}$. Then the classifying map is an element of $\pi_{2n+2k-1}(BU(n+k-1))$. If $n+k \le p$, $\pi_{2n+2k-1}(BU(n+k-1)) \cong \mathbf{Z}_{(p)}/(n+k-1)! = 0$. Therefore the above fiber bundle is trivial and $W_{n+k,k}$ is homotopy equivalent to $W_{n+k-1,k-1} \times S^{2n+2k-1}$. Inductively, we can show the p-regularity of $W_{n+k,k}$. Proof of the second statement is similar.

Remark 4.9.   i) It follows from Theorem 4.7 and Proposition 4.8 that $W_{n+k,k}$ is p-regular for any n if $k \le [(2-\sqrt{2})p]+1$ and that $X_{n+k,k}$ is *p-regular for any n* if $k \le (p-1)/2$ for $p = 3, 5, 7, 11$, and $k \le [(3-\sqrt{5})p/2]+1$ for $p \ge 13$.
ii)   Since the obstruction group $\pi_{2pn+2i(p-1)-1}(QS^{2n+1}, S^{2n+1}) \cong \pi_{2pn+2i(p-1)-1}$ $(F(2n+1))$ $(i \ge 1)$ maps isomorphically onto $PBP_{2pn+2i(p-1)-1}(F(2n+1))$ by the BP-Hurewicz map as we noted in Remark 3.26, we may be able to know whether the obstruction in this dimension vanishes or not, if we can describe the map f_*: $BP_*(W_{n+k,k}) \to BP_*(QS^{2n+1})$ explicitly (for this, f: $W_{n+k,k} \to QS^{2n+1}$ should be constructed explicitly). We also remark that if $n > (k^2 - 4p + 4)/(4p - 2k)$, we may assume that the image of f is contained in the $(p+1)$-th May filtration $F_{p+1}C_\infty S^{2n+1}$ of $C_\infty S^{2n+1} \simeq QS^{2n+1}$ ([13]) and $BP_*(F_{p+1}C_\infty S^{2n+1})$ can be computed in the same way as we computed $BP_*(L_n)$.

Examples 4.10.   i) For $p = 3$, $W_{n+k,k}$ is 3-regular if and only if $k \le 2$, or $k = 3$ and $3|n$. Assume that $k \le p-1$, or $k = p$ and $p|n$. $W_{n+k,k}$ is p-regular except for the following undecided cases;
For $p = 5$, $(n,k) = (4,4)$, $(5,5)$
For $p = 7$, $(n,k) = (2,6)$, $(3,6)$, $(6,6)$, $(12,6)$, $(7,7)$, $(14,7)$
For $p = 11$, $(n,k) = (4,8)$, $(i,9)$ for $3 \le i \le 10$, $(i,10)$ for $2 \le i \le 12$,
$\quad\quad\quad\quad\quad (40,10)$, $(11i,11)$ for $i = 1,2,3,4$
For $p = 13$, $(n, k) = (5,9)$, $(i,10)$ for $i = 4,5,6,7$, $(i,11)$ for $3 \le i \le 16$, $(i,12)$
$\quad\quad\quad\quad\quad$ for $2 \le i \le 18$, $(12i,12)$ for $i = 2,3,4,5$,
$\quad\quad\quad\quad\quad (13i,13)$ for $1 \le i \le 5$.
ii)   For $p = 3, 5, 7, 11, 17, X_{n+k,k}$ is p-regular if and only if $k \le (p-1)/2$,
$\quad\quad\quad\quad\quad$ or $k = (p+1)/2$ and $p|2n+1$.
Assume that $k \le (p-1)/2$, or $k = (p+1)/2$ and $p|2n+1$. $X_{n+k,k}$ is p-regular except for the following undecided cases.
For $p = 13$, $(n,k) = (1,6)$
For $p = 19$, $(n,k) = (1,9)$
For $p = 23$, $(n,k) = (1,11)$, $(2,11)$, $(3,11)$.

Appendix.   The Non-2-Regularity of Real Stiefel Manifolds

Recall that real quasiprojective space $Q_{n+k,k}(\mathbf{R})$ is in fact real stunted projective space $\mathbf{P}_n^{n+k-1}$. To begin with we state the following facts without proofs.

Lemma   A.1. $\tilde{H}^*(Q_{n+k,k}(\mathbb{R})) = \mathbb{F}_2\{x_n, x_{n+1}, \dots, x_{n+k-1}\}$, $Sq^i x_j = \binom{j}{i} x_{i+j}$.

Thus the action of the Steenrod operations in trivial if and only if $k = 1$, or $k = 2$ and $2|n$.

Proposition A.2.   *The following three conditions are equivalent.*
i) $k = 1$, or $k = 2$ and $2|n$.   ii) $Q_{n+k,k}(\mathbb{R})$ *is homotopy equivalent to* $\bigvee_{i=1}^{k} S^{n+i-1}$
at prime 2.   iii) $\Omega V_{n+k,k}$ *is homotopy equivalent to* $\prod_{i=1}^{k} \Omega S^{n+i-1}$ *at prime 2.*
Moreover, if n is even, $Q_{n+2,2}(\mathbb{R})$ is globally homotopy equivalent to $S^n \vee S^{n+1}$ and $\Omega V_{n+2,2}$ is globally homotopy equivalent to $\Omega(S^n \times S^{n+1})$.

Proposition A. 3. $V_{n+k,k}$ *cannot be 2-regular unless $k = 1$, or $k = 2$ and $2|n$.*

Let $k_n: S^{n+1} \to BSO(n+1)$ be the classifying map of sphere bundle $S^n \to V_{n+2,2} \to S^{n+1}$, and we also denote by $k_n \in \pi_n(SO(n+1))$ the element corresponding to $k_n \in \pi_{n+1}(BSO(n+1))$ by the natural isomorphism. The following fact is due to Kervaire ([9]).

Lemma A.4.   *Consider principal bundle $SO(n) \to SO(n+1) \xrightarrow{\rho} S^n$. Then*

$$\text{Im}[\rho_*: \pi_n(SO(n+1)) \to \pi_n(S^n)] = \begin{cases} \pi_n(S^n) & \text{for } n = 1, 3, 7 \\ 2\pi_n(S^n) & \text{for odd } n, \ n \neq 1, 3, 7 \\ 0 & \text{for even } n \end{cases}$$

Hence we see that k_2 and k_6 are null homotopic.   This implies

Proposition A. 5.   *There are homeomorphisms $V_{4,2} \cong S^2 \times S^3$ and $V_{8,2} \cong S^6 \times S^7$.*

Note that $k_n \in \pi_n(SO(n+1))$ is the image of generator ι_{n+1} of $\pi_{n+1}(S^{n+1})$ by $\Delta: \pi_{n+1}(S^{n+1}) \to \pi_n(SO(n+1))$.

Lemma A.6.   *For even n, there exists $k'_n \in \pi_n(SO/SO(n))$ which maps to $k_n \in \pi_n(SO(n+1))$ by composition $\pi_{n+1}(SO/SO(n)) \xrightarrow{\partial} \pi_n(SO(n)) \xrightarrow{j_*} \pi_n(SO(n+1))$.*

Proof.   By Lemma A.4, $j_*: \pi_n(SO(n)) \to \pi_n(SO(n+1))$ is surjective.   Choose $k''_n \in \pi_n(SO(n))$ which maps to k_n.   Since $k_n = \Delta(\iota_{n+1})$ is in the kernel of $\pi_n(SO(n+1)) \to \pi_n(SO(n+2)) \cong \pi_n(SO)$, it is easy to see that k''_n is in the image of $\partial: \pi_{n+1}(SO/SO(n)) \to \pi_n(SO(n))$.

A proof of the following lemma is similar to that of Lemma 2.16.

Lemma A.7. $\delta_*: \pi_{2n}(QS^{n-1}) \to \pi_{2n}(F(n))$ *is trivial for even n, $n \neq 2, 6$.*

Lemma A.8.   *For even n, $n \neq 2, 6$, $k''_n = \partial(k'_n) \in \pi_n(SO(n))$ is mapped by J:*

$\pi_n(SO(n)) \to \pi_n(\Omega_1^n S^n)$ to an element which is not in the image of the connecting homomorphism δ': $\pi_{n+1}(S^n) \to \pi_n(\Omega_1^n S^n)$ associated with fibration $\Omega_1^n S^n \to SH(n+1) \to S^n$.

Proof. By the commutativity of (2.12) and the above lemma, composition $\pi_{n+1}(SO/SO(n)) \xrightarrow{\partial} \pi_n(SO(n)) \xrightarrow{J} \pi_n(\Omega_1^n S^n)$ is injective. Suppose $J(k_n'') = \delta'(x)$ for some $x \in \pi_{n+1}(S^n)$. Let j': $S^n \to SO/SO(n)$ be the canonical inclusion, then Δ: $\pi_{n+1}(S^n) \to \pi_n(SO(n))$ factors as $\pi_{n+1}(S^n) \xrightarrow{j'_*} \pi_{n+1}(SO/SO(n)) \xrightarrow{\partial} \pi_n(SO(n))$. Also δ' factors as $J \circ \Delta = J \circ \partial \circ j_*$. Therefore $J \circ \partial(k_n') = J(k_n'') = \delta'(x) = J \circ \partial \circ j'_*(x)$ and we have $k_n' = j'_*(x)$ which maps to zero by composition $\pi_{n+1}(SO/SO(n)) \to \pi_{n+1}(SO/SO(n+1)) \xrightarrow{\partial} \pi_n(SO(n+1))$. This contradicts the choice of k_n' since k_n is not zero.

Theorem A.9. *Except for the cases $n = 0, 2, 6$, $V_{n+2,2}$ is not 2-regular.*

Proof. By the preceding lemma, $k_n \in \pi_n(SO(n+1))$ maps to a non-zero element by J_{H*}: $\pi_n(SO(n+1)) \to \pi_n(H(n+1))$. Then the result follows from Theorem 2.9 and Proposition 2.10.

An analogue of Corollary 2.18 holds.

Corollary A.10. *The attaching map of the top cell of $V_{n+2,2}$ ($n = 2j, j \ne 0, 1, 3$) defines a non-trivial element ξ_j in the 2-component of $\pi_{4j}(S^{2j})$. ξ_j is in the kernel of E^2: $\pi_{4j}(S^{2j}) \to \pi_{4j+2}(S^{2j+2}) \cong \pi_{2j}^S(S^0)$ and has order 2 if j is even and at most 4 if j is odd.*

Proof. Recall that $Q_{n+2,2}(\mathbf{R})$ is a stable summand of $V_{n+2,2}$ (Theorem 4.2) and $Q_{n+2,2}(\mathbf{R}) \cong S^n \vee S^{n+1}$. Hence there exists a map h:$(V_{n+2,2}, S^n) \to (QS^n, S^n)$ such that $h|S^n$: $S^n \to S^n$ is degree 1. In fact, h is an extension of composition $Q_{n+2,2}(\mathbf{R}) \xrightarrow{c} S^n \xrightarrow{i} QS^n$. Thus $i \circ \xi_j = i \circ c \circ \xi_j' = 0$ where ξ_j': $S^{2n} \to Q_{n+2,2}(\mathbf{R})$ is the attaching map and k: $Q_{n+2,2}(\mathbf{R}) \to V_{n+2,2}$ is the inclusion map. This means that ξ_j suspends to zero in the stable range. The last assertion follows from the fact that $\pi_{4j}(F(2j))$ ($\cong \pi_{2j+1}^S \mathbf{P}_{2j}^\infty$)) is isomorphic to $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ if j is even and to $\mathbf{Z}/4$ if j is odd.

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