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On the Boundary Value of a Solution of the Heat Equation

By

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§0. Introduction

Let U be an open subset of $\mathbf{R}^n$, and let $P(x, D_x)$ be a linear differential operator with analytic coefficients defined on a neighborhood of the closure $[U]$ of U. Suppose that the boundary ∂U of U is smooth (i.e., non-singular and analytic) and that ∂U is non-characteristic with respect to P at each point in ∂U. Then it is well-known ([7], [11]) that the boundary value of a hyperfunction solution of the equation $Pu = 0$ on U is a well-defined hyperfunction. However, little is known about the characterization of a solution whose boundary value determines a hyperfunction near a characteristic boundary point. The purpose of this article is to discuss this problem for one special case, i.e., the pair of the heat operator $\partial/\partial t - \Delta \underset{\text{def}}{=} \partial/\partial t - \sum_{j=1}^{n-1} \partial^2/\partial x_j^2$ and the domain $\{(t, x) \in \mathbf{R}^n;\ t > 0\}$. Our main result (Theorem 1 below) asserts that,

(i) if a C^∞-solution $u(t, x)$ does not behave too wildly as $t \downarrow 0$,

and

(ii) if $u(t, x)$ uniformly tends to zero outside a compact set $K \subset \mathbf{R}_x^{n-1}$ as $t \downarrow 0$,

then we can assign a compactly supported hyperfunction $g(x)$ to $u(t, x)$ so that the vanishing of $g(x)$ entails the vanishing of $u(t, x)$ itself. Furthermore we can find such a tame solution $u(t, x)$ of the heat equation for any compactly supported hyperfunction $g(x)$. [See Theorem 1 for the precise statement. Note also that a hyperfunction supported by a compact set, say L, is an analytic functional with the real carrier L.]

Let us note the following two facts: First, if $u(t, x)$ tends to infinity too rapidly as $t \downarrow 0$, then our procedure will not assign a hyperfunction $g(x)$. (Cf. §2(i)) Second, we know (see [3], for example) that there exists a hyperfunction $e(t, x)$ $(x \in \mathbf{R}^1)$ supported by $\{(t, x) \in \mathbf{R}^2;\ t = 0,\ x \geqq 0\}$ satisfying the equation

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$(\partial/\partial t - \partial^2/\partial x^2)e(t, x) = \delta(t) \otimes \delta(x)$. Putting this differently, we claim that there exists no reasonable assignment of a hyperfunction $g(x)$ to $u(t, x)$ if the condition (ii) is not satisfied.

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§ 1. Main Results

To state our main result (Theorem 1 below) we first introduce the following symbol:

For a compact subset K of $\mathbb{R}^{n-1}$, we denote by $\mathscr{S}_K^{tame}$ the totality of C^∞-solutions of the heat equation $(\partial/\partial t - \Delta)u(t, x) = 0$ on $\{(t, x) \in \mathbb{R}^n;\ t > 0\}$ that satisfy the following condition:

(1) For each $\varepsilon > 0$, there exists a constant C_ε for which

$$|u(t, x)| \leqq C_\varepsilon \exp\left(\frac{\varepsilon}{t} - \frac{\operatorname{dist}(x, K)^2}{4t}\right)$$

holds on $\{(t, x) \in \mathbb{R}^n;\ t > 0\}$.

Here $\operatorname{dist}(x, K)$ denotes the distance of the point $x(\in \mathbb{R}^{n-1})$ and K.

Theorem 1. *Let K be a compact subset of $\mathbb{R}^{n-1}$ and let $\mathscr{B}_K$ denote the space of $(n-1)$-dimensional hyperfunctions supported by K. Then there exists an isomorphism b: $\mathscr{S}_K^{tame} \to \mathscr{B}_K$. Furthermore, for each u in $\mathscr{S}_K^{tame}$, we can find a hyperfunction $\tilde{u}(t, x)$ satisfying $\tilde{u}|_{\{t>0\}} = u$ and* supp $\tilde{u} \subset \{(t, x) \in \mathbb{R}^n;\ t \geqq 0\}$ *so that the following holds*:

(2) $$(\partial/\partial t - \Delta)\tilde{u}(t, x) = \delta(t) \otimes b(u).$$

Remark. Since $\operatorname{supp} \tilde{u} \subset \{t \geqq 0\}$, the relation (2) implies that $b(u)$ describes the "boundary value" of u taken from the side $\{t > 0\}$. Hence Theorem 1 guarantees the existence of the "boundary value" of u as a hyperfunction on the condition that the solution u in question satisfies the condition (1), although the boundary $\{t = 0\}$ is characteristic with respect to the heat equation.

Proof. Our construction of the map b uses a particular extension of u to the entire space $\mathbb{R}^n$. To define such a preferred extension, let us first introduce an auxiliary set $\Omega = \{(t, x) \in \mathbb{R}^n;\ t \neq 0 \text{ or } x \notin K\}$. By using the classical estimate for solutions of heat equations (see [2], for example), we find that the condition (1) implies

(3) $$\lim_{t \downarrow 0} P(\partial/\partial t, \partial/\partial x)u(t, x) = 0 \quad \text{for } x \notin K$$

for any linear differential operator (with constant coefficients) $P(\partial/\partial t, \partial/\partial x)$ of finite order. Hence we can find a C^∞-function $c(t, x)$ which satisfies the following conditions:

(4) $c(t, x) = u(t, x)$ if $t > 0$,

(5) $c(t, x)$, together with all its derivatives, vanishes on $\Omega \backslash \{t, x) \in \mathbf{R}^n;\ t > 0\}$. [What we actually need in the subsequent reasoning is not the C^∞-character of c but its C^2-character.] Since the assumption (1) implies that $u(t, x)$ does not increase faster than $\exp(\varepsilon/t)$ as $t \downarrow 0$ for any $\varepsilon > 0$, a result of Komatsu ([6]), Theorem 2.27) guarantees the existence of an ultradistribution $v(t, x) \in \mathscr{D}^{\{2\}\prime}(\mathbf{R}^n)$ which satisfies the following:

(6) $$v = c \text{ on } \Omega.$$

See [4] for the definition and basic properties of the space $\mathscr{D}^{\{2\}\prime}(\mathbf{R}^n)$, the space of ultra-distributions of Gevrey-type $\{2\}$.

Since $c(t, x)$ vanishes for $t < 0$, we find

(7) $$\operatorname{supp} v \subset \{(t, x) \in \mathbf{R}^n;\ t \geqq 0\}.$$

Further (4) and (5) imply

(8) $$(\partial/\partial t - \Delta)c = 0 \qquad \text{on } \Omega,$$

and hence

(9) $$(\partial/\partial t - \Delta)v = 0 \qquad \text{on } \Omega.$$

In what follows, an ultradistribution $v(t, x)$ thus obtained shall be called a *tame extension* of u for short.

Now, letting $\mu(t, x)$ denote $(\partial/\partial t - \Delta)v$ we find the following:

(10) $$\operatorname{supp} \mu \subset \{(t, x) \in \mathbf{R}^n;\ t = 0,\ x \in K\}.$$

Since μ belongs to $\mathscr{D}^{\{2\}\prime}(\mathbf{R}^n)$, a structure theorem for ultradistributions supported by a submanifold ([5]), entails the following:

(11) $$\mu(t, x) = \sum_{k=0}^{\infty} \delta^{(k)}(t) \otimes \mu_k(x)$$

where $\delta^{(0)}(t) = \delta(t)$, $\delta^{(k)}(t) = \partial^k/\partial t^k \delta(t)$ $(k = 1, 2, \cdots)$ and $\mu_k(x) \in \mathscr{D}^{\{2\}\prime}(\mathbf{R}^{n-1})$ is supported by K and satisfies the following condition:

(12) For any strictly positive constants L, h and δ, there exists a constant C for which

$$|\mu_k(\varphi)| \leqq CL^k(k!)^{-2} \sup_{x \in K_{\delta,\alpha}} \frac{|D^\alpha \varphi(x)|}{h^{|\alpha|}(\alpha!)^2}$$

holds for any φ in $\mathscr{D}^{\{2\}}(\mathbf{R}^{n-1})$.

Here, and in what follows, $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ is a multi-index with α_j being a non-negative integer, $|\alpha|$ denotes $\sum \alpha_j$, and $K_\delta = \{x \in \mathbf{R}^{n-1},\ \operatorname{dist}(x, K) \leqq \delta\}$.

Let us now set

(13) $$Q_k = \sum_{\ell=0}^{k-1} \partial^\ell/\partial t^\ell \Delta^{k-1-\ell}$$

for $k = 1, 2, \cdots$. Here and in what follows, $\partial^0/\partial t^0$ and Δ^0 denote the identity operator. Then we find

(14) $$(\partial^k/\partial t^k - \Delta^k) = (\partial/\partial t - \Delta)Q_k$$

for $k = 1, 2, \cdots$. It follows from (12) that both $\sum_{k=1}^{\infty} Q_k(\delta(t) \otimes \mu_k(x))$ and $\delta(t) \otimes (\sum_{k=0}^{\infty} \Delta^k \mu_k(x))$ belong to $\mathscr{B}_{\{0\}\times K}$, the space of hyperfunctions supported by $\{0\} \times K$. Denoting $\sum_{k=1}^{\infty} Q_k(\delta(t) \otimes \mu_k(x))$ and $\sum_{k=0}^{\infty} \Delta^k \mu_k$ respectively by $v(t, x)$ and $g(x)$, we obtain the following equality (15) from (14):

(15) $$(\partial/\partial t - \Delta)(v - v) = \delta(t) \otimes g(x).$$

Let us next verify $g(x)$ thus defined is independent of the choice of a tame extension v. Let v_1 and v_2 be two tame extensions of u. Then it follows from the definition of the tame extension that $v_1 - v_2$ vanishes on Ω, *i.e.*,

(16) $$\operatorname{supp}(v_1 - v_2) \subset \{0\} \times K.$$

Since both v_1 and v_2 belong to $\mathscr{D}^{\{2\}\prime}(\mathbb{R}^n)$, we can find

$$h_k(x) \in \mathscr{D}^{\{2\}\prime}(\mathbb{R}^{n-1}) \quad (k = 0, 1, 2, \cdots)$$

which satisfy the following:

(17) $$\operatorname{supp} h_k \subset K \text{ for } k = 0, 1, 2, \cdots$$

(18) $$v_1 - v_2 = \sum_{k=0}^{\infty} \delta^{(k)}(t) \otimes h_k(x)$$

(19) For any strictly positive constants L, h and δ, there exists a constant C for which

$$|h_k(\varphi)| \leqq CL^k(k!)^{-2} \sup_{x \in K_\delta, \alpha} \frac{|D^\alpha \varphi(x)|}{h^{|\alpha|}(\alpha!)^2}$$

holds for any φ in $\mathscr{D}^{\{2\}}(\mathbb{R}^{n-1})$.

Let $\rho_k(x)$ denote the difference of μ_k's determined by v_1 and v_2 respectively, that is,

(20) $$\sum_{k=0}^{\infty} \delta^{(k)}(t) \otimes \rho_k(x) = (\partial/\partial t - \Delta)(v_1 - v_2).$$

Since

(21) $$(\partial/\partial t - \Delta)(v_1 - v_2) = (\partial/\partial t - \Delta)(\sum_{k=0}^{\infty} \delta^{(k)}(t) \otimes h_k(x))$$

$$= \sum_{k=0}^{\infty} (\delta^{(k+1)}(t) \otimes h_k(x) - \delta^{(k)}(t) \otimes \Delta h_k(x)),$$

the comparison of the coefficients of $\delta^{(k)}(t)$ in (20) shows

$$\rho_0(x) = -\Delta h_0(x), \tag{22}$$

$$\rho_k(x) = h_{k-1}(x) - \Delta h_k(x) \qquad \text{for } k = 1, 2, \cdots . \tag{23}$$

Therefore we find

$$\begin{aligned} \rho_0 + \sum_{k=1}^{\infty} \Delta^k \rho_k & \\ &= -\Delta h_0 + \sum_{k=1}^{\infty} \Delta^k h_{k-1} - (\sum_{k=1}^{\infty} \Delta^{k+1} h_k) \\ &= \sum_{k=1}^{\infty} \Delta^k h_{k-1} - (\sum_{\ell=1}^{\infty} \Delta^{\ell} h_{\ell-1}) = 0. \end{aligned} \tag{24}$$

This means that $g(x)$ does not depend on the choice of a tame extension v of u. Hence we define $b(u)$ by $b(u) = g$. Since the sum of tame extensions $v_j (j = 1, 2)$ of u_j ($j = 1, 2$, respectively) in $\mathscr{S}_K^{tame}$ is a tame extension of $u_1 + u_2$, $b(u)$ thus defined is a linear map from $\mathscr{S}_K^{tame}$ into $\mathscr{B}_K$.

We shall now verify that the map b is bijective.

To prove its surjectivity, let us first recall that there exists an elementary solution $E(t, x)$ of the heat operator $(\partial/\partial t - \Delta)$ [i.e., $(\partial/\partial t - \Delta)E(t, x) = \delta(t) \otimes \delta(x)$] in the space $\mathscr{D}^{\{2\}\prime}(\mathbf{R}^n)$ so that it satisfies

$$E(t, x) = \begin{cases} (4\pi t)^{-(n-1)/2} \exp(-x^2/t), & \text{if } t > 0 \\ 0, & \text{if } t < 0. \end{cases} \tag{25}$$

(Cf. [9])

Now, for a hyperfunction $g(x)$ supported by K, we define another hyperfunction $w(t, x)$ by

$$\iint E(t-s, x-y)\,(\delta(s) \otimes g(y))\; dsdy = \int E(t, x-y) g(y) dy.$$

Let $w_+(t, x)$ denote the restriction of w to $\{(t, x) \in \mathbf{R}^n;\ t > 0\}$. Then one can easily verify that w_+ satisfies the condition (1). (Cf. [8], Theorem 1. 2) Hence we may consider $b(w_+)$. We shall prove $b(w_+) = g$. For that purpose let us choose a tame extension v of w_+ and set $\mu = (\partial/\partial t - \Delta)v$. [Needless to say, v does not coincide with w in general.] Then it is known ([8], p.58) that

$$v(t, x) = \iint E(t-s, x-y) \mu(s, y) dsdy. \tag{26}$$

We can further verify that

(27) $$\lim_{t\downarrow 0}\int v(t,x)\chi(x)\varphi(x)dx=\int b(w_+)(x)\varphi(x)dx$$

holds for any entire function $\varphi(x)$, if we choose $\chi(x)$ to be a compactly supported C^∞-function which is equal to 1 on a neighborhood of K. ([8], (1. 22)) On the other hand, the definition of w entails

(28) $$\begin{aligned}&\lim_{t\downarrow 0}\int w_+(t,x)\chi(x)\varphi(x)dx\\ &=\lim_{t\downarrow 0}\int\int E(t,x-y)g(y)\chi(x)\varphi(x)dxdy\\ &=\int g(x)\chi(x)\varphi(x)dx.\end{aligned}$$

(Cf. [8], (1.11)) Since $v(t,x)=w_+(t,x)$ holds for $t>0$, the equality $b(w_+)=g$ follows from the denseness of entire functions in $\mathscr{A}(K)$, the space of real analytic functions on K. Thus we have verified the surjectivity of b.

Finally let us prove the injectivity of b. We shall again make use of the elementary solution E. Let v be a tame extension of u satisfying the condition (1) and let $\mu(t,x)=\sum_{k=0}^{\infty}\delta^{(k)}(t)\otimes\mu_k(x)$ be $(\partial/\partial t-\Delta)v$. (Cf. (11)) Suppose now that $b(u)$ vanishes. Then it follows from the definition of $b(u)$ that $\sum_{k=0}^{\infty}\Delta^k\mu_k(x)$ vanishes. Hence (26) entails the following relation for $t>0$:

(29) $$\begin{aligned}u(t,x)&=\sum_{k=0}^{\infty}\frac{\partial^k}{\partial t^k}\int E(t,x-y)\mu_k(y)\,dy\\ &=\sum_{k=0}^{\infty}\left(\frac{\partial^k}{\partial t^k}-\Delta^k\right)\int E(t,x-y)\mu_k(y)\,dy\\ &=\sum_{k=0}^{\infty}Q_k\left(\frac{\partial}{\partial t}-\Delta\right)\int E(t,x-y)\mu_k(y)\,dy\\ &=0.\end{aligned}$$

This proves the injectivity of the map b, completing the proof of Theorem 1.

In the course of the above proof of Theorem 1 (in particular, the part of the proof of the surjectivity of the map b), we have also verified the following results as by-products. As they seem to have their own interests, we present them as theorems.

Theorem 2. *Each function u in $\mathscr{S}_K^{tame}$ is real analytic.*

Theorem 3. *For a compactly supported hyperfunction $f(x)$ on $\mathbb{R}^m$ we can find* *compactly supported ultradistributions* $f_k(x)\,(k=0,1,2,\cdots)$ *in* $\mathscr{E}^{\{2\}}(\mathbf{R}^m)'$ *so that*

$$f(x)=\sum_{k=0}^{\infty}\Delta^k f_k(x)$$

and

$$\mathrm{supp} f_k \subset \mathrm{supp} f$$

holds for any k.

§2. Miscellaneous Remarks

In this section we present some remarks on our main results given in the preceding section.

(i) It follows from Theorem 1 that, for a solution $u(t, x)$ of the heat equation that satisfies the condition (1), its "boundary value" $b(u)$ is a well-defined hyperfunction which has the form $\sum_{k=0}^{\infty}\Delta^k\mu_k(x)$ with $\mu_k(x)$ being an ultradistribution determined by a tame extension of u. Now the following question naturally arises: What if $u(t, x)$ grows faster than $\exp(\varepsilon/t)$ as $t\downarrow 0$? In this case we can still find an extension v of u if we allow u to be a hyperfunction. However the series $\sum_{k=0}^{\infty}\Delta^k\mu_k(x)$ given in an analogous way as in the proof of Theorem 1 does not define a hyperfunction in general. This explains why Aronszajn [1] needed a class of generalized functions that is bigger than the space of hyperfunctions when he discussed the boundary value of a solution of the heat equation.

(ii) In connection with the above remark, we note that we can prove the following

Theorem 4. *Let* K *be a compact set in* $\mathbf{R}^{n-1}$. *Then for each* $\mu(t, x)$ *in* $\mathscr{E}^{\{2\}}_{\{0\}\times K}(\mathbf{R}^n)'$ *we can find* $v(t, x)\in\mathscr{B}_{\{0\}\times K}$ *and* $g(x)\in\mathscr{B}_K$ *so that*

$$\mu=(\partial/\partial t-\Delta)v+\delta(t)\otimes g(x) \tag{30}$$

holds. Furthermore v *and* g *are uniquely determined by* μ.

In fact, the existence of v and g can be verified in exactly the same manner as in the proof of (15). Since the Fourier transform of v and that of g are both entire functions, the uniqueness assertion can be readily verified if we apply the Fourier transformation to the relation (30).

(iii) The same result as Theorem 1 holds if Δ is the Laplace operator on a compact Riemannian manifold M (without boundary) and if we consider the heat equation on $\mathbf{R}_t\times M$ and choose M as K.

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