

Local Existence Theorems and Blow-up of Solutions for Quasi-linear Hyperbolic Cauchy Problems in a Domain with Characteristic Boundary Generated by Initial Data

By

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Introduction

Non-degenerate quasi-linear hyperbolic equations are fairly well studied locally or globally under various boundary conditions. In degenerate case, recently, Y. Ebihara ([1]) showed a local existence theorem on one-dimensional elastic problem:

$$\begin{cases} \partial_t^2 u = \partial_x \{(\partial_x u)^m\} & \text{in } \{0 < t < T, 0 < x < 1\}, \\ \partial_x u = 0 & \text{on } \{0 < t < T, x = 0\} \cup \{0 < t < T, x = 1\}, \\ u = \phi(x), \partial_t u = \psi(x) & \text{on } \{t = 0, 0 < x < 1\}, \end{cases}$$

under some assumptions on $\{\phi(x), \psi(x)\}$ in case of $m \geq 5$, which was extended in case of $m \geq 3$ by Y. Nonaka ([2]). On the other hand, concerning to linear problem, fully degenerate hyperbolic Cauchy problems are studied by the author ([3]). A “fully degenerate” operator in a domain means an operator degenerating only on the boundary which is a characteristic whose order of multiplicity is equal to the order of the operator.

The aim of this paper is to prove a local existence theorem for a quasi-linear hyperbolic Cauchy problem:

$$(P) \begin{cases} A(t, x; \mathfrak{D}^{m-1}u; \partial)u \\ \equiv \partial_t^m u + \sum_{\substack{j+|\nu| \leq m \\ j \leq m-1}} a_{j\nu}(t, x; \mathfrak{D}^{m-1}u) \partial_t^j \partial_x^\nu u \\ = f(t, x; \mathfrak{D}^{m-1}u) & \text{in } (0, T) \times \Omega, \\ \partial_t^j u = \phi_j(x) \quad (j = 0, 1, \dots, m-1) & \text{on } \{t = 0\} \times \Omega, \end{cases}$$

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where we use simple notations:

$$\begin{aligned} \partial^k u &= \{ \partial_t^j \partial_x^\nu u \mid j + |\nu| = k \}, \\ \mathfrak{D}^k u &= \{ \partial_t^j \partial_x^\nu u \mid j + |\nu| \leq k \} = (u, \partial u, \dots, \partial^k u), \end{aligned}$$

when there are no confusions.

The difficulty comes out of non-linearity, because degeneracy of coefficients depends on unknown solutions. This paper maintains that there is a unique classical solution in a short time for Cauchy problems in $(0, T) \times \Omega$ if initial data are “degenerate-structure generating”. Under these situations, the analysis becomes very easy because the boundary values of solutions are determined only by the boundary values of initial data. The existence theorem is proved by the standard iterative method, using the existence theorems for iterative linearized problems ([3]). This paper also gives some examples of equations whose solutions blow-up on the boundary.

Similar results for special systems of equations are known in the fields of fluid dynamics, based on the theories of symmetric hyperbolic systems of first order or second order ([4], [5], [6]).

§1. Linear Cauchy Problems for Fully Degenerate Hyperbolic Equations

Let us consider a linear hyperbolic equation in $I \times \Omega$:

$$Au \equiv \sum_{j+|\nu| \leq m} a_{j\nu}(t, x) \partial_t^j \partial_x^\nu u = f(t, x),$$

where $a_{m0} = 1$ and $a_{j\nu} \in \mathcal{B}^\infty(I \times \Omega)$, where

$$\begin{aligned} I &= (0, T), \quad \Omega = R_+^n = \{x = (x_1, x') \mid x_1 > 0, x' \in R^{n-1}\}, \\ \partial_x &= (\partial_1, \dots, \partial_n), \quad \partial_j = \partial_{x_j} \quad (j = 1, 2, \dots, n). \end{aligned}$$

Let us introduce a function $\rho(x_1) \in \mathcal{B}^\infty(R_+)$ satisfying

$$\rho(x_1) = \begin{cases} x_1 & \text{if } x_1 < 1/2, \\ 1 & \text{if } x_1 > 1, \end{cases}$$

and $0 < \rho(x_1) < 1$ if $1/2 < x_1 < 1$.

Let us say that A is a *hyperbolic equation with degenerate order* $\sigma (\geq 1)$ in $I \times \Omega$, which we denote $A \in RH^\sigma(I \times \Omega; \delta)$ ($\delta > 0$), if

- i) $a_{j\nu}^\wedge(t, x) = \rho(x_1)^{-\sigma|\nu|} a_{j\nu}(t, x)$ is bounded in $I \times \Omega$,
- ii) $A_0^\wedge(t, x; \tau, \zeta) = \sum_{j+|\nu|=m} a_{j\nu}^\wedge(t, x) \tau^j \zeta^\nu = \prod_{k=1}^m (\tau - \tau_k^\wedge(t, x; \zeta))$

is a regularly hyperbolic polynomial in $I \times \Omega$, that is, $\{\tau_k^\wedge(t, x; \zeta)\}$ are real for $(t, x) \in I \times \Omega$ and $\zeta \in S^{n-1}$, and moreover

$$\inf_{\substack{(t,x) \in I \times \Omega \\ \zeta \in S^{n-1}, k \neq k'}} |\tau_k^\wedge(t, x; \zeta) - \tau_{k'}^\wedge(t, x; \zeta)| \geq \delta.$$

We denote $A \in RH^\sigma(I \times \partial\Omega; \delta)$, if $A \in RH^\sigma(I \times \Sigma; \delta)$, where $\Sigma = \Omega \cap \{0 < x_1 < \exists \varepsilon\}$. Moreover, we denote $A \in RH^\sigma(I \times \Omega; \delta; s, K)$ ($s \geq s_0, K > 0$) if $A \in RH^\sigma(I \times \Omega; \delta)$ and $a_{j\nu}$ are represented as $a_{j\nu} = a'_{j\nu} + a''_{j\nu}$ with the estimate

$$\sum_{j,\nu} |a'_{j\nu}|_{\mathcal{B}^s(I \times \Omega)} + \sum_{j,\nu} \| \| a''_{j\nu} \| \|_s \leq K,$$

where

$$\| \| u \| \|_s = \sup_{t \in I} \| \| u(t) \| \|_s = \sup_{t \in I} \sum_{j=0}^s \| \partial_t^j u(t) \| \|_{H^{s-j}(\Omega)},$$

and $s_0 = \langle m\sigma \rangle + [n/2] + 1$, where $\langle k \rangle$ means the least integer not less than k . Let us denote

$$\tilde{a}_{j\nu}(t, x) = \rho(x_1)^{-\langle \delta|\nu| \rangle} a_{j\nu}(t, x).$$

Then we have

$$|\tilde{a}_{j\nu}|_{\mathcal{B}^{s-s_0}(I \times \Omega)} \leq C |a_{j\nu}|_{\mathcal{B}^{s-[n/2]-1}(I \times \Omega)} \leq C'K.$$

Our problem in this section is the Cauchy problem:

$$(P)_{lin} \begin{cases} A(t, x; \partial)u = f(t, x) & \text{in } I \times \Omega, \\ \partial_t^j u = \phi_j(x) \quad (j = 0, 1, \dots, m-1) & \text{on } \{t = 0\} \times \Omega. \end{cases}$$

and basic energy estimates have been obtained in [3]:

Lemma 1.1. *There exists a positive constant C such that*

$$\begin{aligned} & \sum_{j+|\nu| \leq m-1} \| \partial_t^j (\rho^\sigma \partial_x)^\nu u(t) \|^2 \\ & \leq C \left\{ \sum_{j=0}^{m-1} \sum_{|\nu| \leq m-1-j} \| (\rho^\sigma \partial_x)^\nu \phi_j \|^2 + \int_0^t \| f(t) \|^2 dt \right\} \end{aligned}$$

for any $A \in RH^\sigma(I \times \Omega; \delta; s, K)$ and any $u \in H^m(I \times \Omega)$, where $\| \| = \| \|_{L^2(\Omega)}$.

Here we remark a variation of Sobolev’s lemma, which is familiar in the context of non-linear calculations.

Lemma 1.2. *There exists a positive constant C such that*

$$\sum_{|\alpha_1| + \dots + |\alpha_\ell| \leq s} \| \partial^{\alpha_1} v_1 \dots \partial^{\alpha_\ell} v_\ell \| \leq C \prod_{j=1}^\ell \| v_j \|_s$$

for any $v_j \in H^s(\Omega)$ ($j = 1, \dots, \ell$), where $s \geq [n/2] + 1$.

Let us denote

$$\| \| u(t) \| \|_s^2 = \sum_{j+|v|\leq s} \| \partial_t^j \partial_x^v u(t) \| ^2,$$

$$\| \| u(t) \| \|_{(m-1)}^2 = \sum_{j+|v|\leq m-1} \| \partial_t^j (\rho^\sigma \partial_x^v) u(t) \| ^2,$$

and

$$\| \| u(t) \| \|_{(s)}^2 = \sum_{j+|v|\leq s-m+1} \| \partial_t^j \partial_x^v u(t) \| \|_{(m-1)}^2 \quad (s \geq m),$$

then we have

Lemma 1.3. *There exists a positive constant C such that*

$$\sum_{|\mu|\leq s} \| \partial^\mu A u - A \partial^\mu u \| \leq C \| \| u(t) \| \|_{(s+m-1)}$$

for any $A \in RH^\sigma(I \times \Omega; \delta; s, K)$ and any $u \in H^{s+m}(I \times \Omega)$ ($s \geq s_0$).

Proof. Let us assume that $|\mu| \leq s$ and $j + |v| \leq m$. Let us denote

$$\begin{aligned} & \| \partial^\mu a_{jv} \partial_t^j \partial_x^v u - a_{jv} \partial_t^j \partial_x^v \partial^\mu u \| \\ & \leq C \sum_{|\mu'|=1}^{|\mu|} \| (\partial^{\mu'} a_{jv}) (\partial_t^j \partial_x^v \partial^{\mu-\mu'} u) \| = C \sum_{|\mu'|=1}^{|\mu|} I_{\mu'}, \end{aligned}$$

and

$$\begin{aligned} I_{\mu'} & \leq \| (\partial^{\mu'} a'_{jv}) (\partial_t^j \partial_x^v \partial^{\mu-\mu'} u) \| \\ & \quad + \| (\partial^{\mu'} a''_{jv}) (\partial_t^j \partial_x^v \partial^{\mu-\mu'} u) \| = I'_{\mu'} + I''_{\mu'}. \end{aligned}$$

For $|\mu'| \leq \langle \sigma | v \rangle$, we have

$$\begin{aligned} I_{\mu'} & \leq C |a_{jv}|_{\mathcal{B}^{\langle \sigma | v \rangle}} \| \rho^{\langle \sigma | v \rangle - |\mu'|} \partial_t^j \partial_x^v \partial^{\mu-\mu'} u \| \\ & \leq C |a_{jv}|_{\mathcal{B}^{\langle m \sigma \rangle}} \| \mathfrak{F}^s u \|_{(m-1)} \leq C K \| \| u \| \|_{(s+m-1)}. \end{aligned}$$

For $|\mu'| \geq \langle \sigma | v \rangle + 1$ ($\equiv b$), we have

$$\begin{aligned} I'_{\mu'} & \leq |a'_{jv}|_{\mathcal{B}^s} \| \partial_t^j \partial_x^v \partial^{\mu-\mu'} u \| \\ & \leq |a'_{jv}|_{\mathcal{B}^s} \| \partial_t^j u \|_{s-1} \leq K \| \| u \| \|_{(s+m-1)}, \end{aligned}$$

and we have

$$\begin{aligned} I''_{\mu'} & \leq C \| \mathfrak{F}^b a''_{jv} \|_{s-b} \| \partial_t^j \partial_x^v u \|_{s-b} \\ & \leq C \| \| a''_{jv} \| \|_s \| \partial_t^j u \|_{s-1} \leq CK \| \| u \| \|_{(s+m-1)}, \end{aligned}$$

owing to Lemma.1.2. □

Here we have

Proposition 1.4. *There exists a positive number C such that*

$$\begin{aligned} & \| \| u(t) \| \|_{(s+m-1)}^2 \\ & \leq C \{ \| \| u(0) \| \|_{(s+m-1)}^2 + \int_0^t \| \| Au(t') \| \|_s^2 dt' \} \quad (0 < t < T) \end{aligned}$$

for any $A \in RH^\sigma(I \times \Omega; \delta; s, K)$ and any $u \in H^{s+m}(I \times \Omega)$ ($s \geq s_0$).

Proof. Since

$$A\partial^\mu u = \partial^\mu Au - (\partial^\mu Au - A\partial^\mu u) \quad (|\mu| \leq s),$$

we have from Lemma.1.1 and Lemma.1.3

$$\| \| u(t) \| \|_{(s+m-1)}^2 \leq C \| \| u(0) \| \|_{(s+m-1)}^2 + C \int_0^t \{ \| \| Au(t) \| \|_s^2 + \| \| u(t) \| \|_{(s+m-1)}^2 \} dt.$$

Hence we have

$$\partial_t U(t) \leq C U(t) + C F(t),$$

where

$$U(t) = \int_0^t \| \| u(t) \| \|_{(s+m-1)}^2 dt \text{ and } F(t) = \| \| u(0) \| \|_{(s+m-1)}^2 + \int_0^t \| \| Au(t) \| \|_s^2 dt,$$

therefore we have

$$U(t) \leq e^{Ct} F(t).$$

Hence we have

$$\| \| u(t) \| \|_{(s+m-1)}^2 \leq C(1 + e^{Ct})F(t). \quad \square$$

Now, let h be a real number, then we have

$$\rho^{-h} A(t, x; \partial_t, \partial_x)u = A(t, x; \partial_t, \partial_1 + h\rho'/\rho, \partial_2, \dots, \partial_n)(\rho^{-h}u),$$

and we have

Lemma 1.5. *There exists $K' > 0$ such that*

$$A(t, x; \partial_t, \partial_1 + h\rho'/\rho, \partial_2, \dots, \partial_n) \in RH^\sigma(I \times \Omega; \delta; s, K'),$$

if $A(t, x; \partial) \in RH^\sigma(I \times \Omega; \delta; s, K)$.

Here we have from Proposition.1.4 and Lemma.1.5

Proposition 1.4'. *For any real number h and any integer $s (\geq s_0)$, there exists a positive number C such that*

$$\begin{aligned} & \| \rho^{-h}u(t) \|_{(s+m-1)}^2 \\ & \leq C \{ \| \rho^{-h}u(0) \|_{(s+m-1)}^2 + \int_0^t \| \rho^{-h}Au(t') \|_s^2 dt' \} \quad (0 < t < T) \end{aligned}$$

for any $A \in RH^\sigma(I \times \Omega; \delta; s, K)$ and any $u \in H^{s+m,h}(I \times \Omega)$, where $u \in H^{s,h}$ means that

$$\| u \|_{H^{s,h}} = \| \rho^{-h}u \|_{H^s} < +\infty.$$

Let us say that $(P)_{lin}$ is solvable in $H^{\infty,h}$ if there exists a unique solution in $H^{\infty,h}(I \times \Omega)$ for every data $\{f \in H^{\infty,h}(I \times \Omega), \phi_j \in H^{\infty,h}(\Omega)\}$. Then $(P)_{lin}$ is solvable in $H^{\infty,h}$ if $A \in RH^\sigma(I \times \Omega; \delta)([3])$.

Now, denoting

$$\dot{a}_{j\nu,p}(t, x') = (\partial_1^p a_{j\nu})(t, 0, x'),$$

we have

$$\begin{aligned} A(t, x; \partial) - \partial_t^m &= \sum_{\substack{j+|\nu| \leq m \\ j \leq m-1}} a_{j\nu}(t, x) \partial_t^j \partial_x^\nu \\ &\sim \sum_{\substack{j+|\nu| \leq m \\ j \leq m-1}} \sum_{p=|\nu|}^\infty (p!)^{-1} x_1^p \dot{a}_{j\nu,p}(t, x') \partial_t^j \partial_x^\nu \\ &\sim \sum_{p=0}^\infty (p!)^{-1} x_1^p \left\{ \sum_{\substack{j+|\nu| \leq m \\ j \leq m-1 \\ |\nu| \leq p}} p! \{(p+\nu_1)\}^{-1} \dot{a}_{j\nu,p+\nu_1}(t, x') \partial_t^j \partial_x^\nu x_1^{\nu_1} \partial_1^{\nu_1} \right\} \\ &\sim \sum_{p=0}^\infty (p!)^{-1} x_1^p B_p(t, x'; \partial_t, \partial_x; x_1 \partial_1). \end{aligned}$$

Here we remark that

$$B_0(t, x'; \partial_t, \partial_x; x_1 \partial_1)$$

$$= \sum_{\substack{j+k \leq m \\ j \leq m-1}} (k!)^{-1} \dot{a}_{jk0,k}(t, x') \partial_t^j (x_1 \partial_1 - (k-1)) (x_1 \partial_1 - (k-2)) \cdots x_1 \partial_1,$$

which is independent of ∂_x . Let $u(t, x) \in \mathcal{B}^\infty(I \times \Omega)$ and let $\dot{u}(t, x') = (\partial_1^j u)(t, 0, x')$, then we have

$$\begin{aligned} \{A - \partial_t^m\}u(t, x) &\sim \sum_{p=0}^\infty \sum_{q=0}^\infty (p!q!)^{-1} x_1^{p+q} B_p(t, x'; \partial_t, \partial_x; q) \dot{u}_q(t, x') \\ &\sim \sum_{r=0}^\infty (r!)^{-1} x_1^r \left\{ \sum_{p=0}^r \binom{r}{p} B_p(t, x'; \partial_t, \partial_x; r-p) \dot{u}_{r-p}(t, x') \right\}. \end{aligned}$$

Proposition 1.6. *Let us assume that*

$$f(t, x) \in H^\infty(I \times \Omega), \phi_j(x) \in H^\infty(\Omega).$$

For any positive integer h , there exist

$$\{g_j(t, x')\}_{0 \leq j \leq h-1} \subset H^\infty(I \times \partial\Omega) = H^\infty(I \times R^{n-1}),$$

satisfying

$$(Q_h)_{lin} \begin{cases} \{\partial_t^m + B_0(t, x'; \partial_t; r)\}g_r + \binom{r}{1}B_1(t, x'; \partial_t, \partial_x; r-1)g_{r-1} \\ + \dots + B_r(t, x'; \partial_t, \partial_x; 0)g_0 = \dot{f}_r(t, x'), \\ \partial_t^j g_r|_{t=0} = \dot{\phi}_{j,r}(x') \quad (j = 0, 1, \dots, m-1) \end{cases} \text{ for } 0 \leq r \leq h-1,$$

where

$$\dot{f}_p = (\partial_t^p f)(t, 0, x'), \quad \dot{\phi}_{j,p} = (\partial_t^p \phi_j)(t, 0, x').$$

We say that $g(t, x) \in \mathcal{B}^\infty(I \times \Omega)$ is an *approximate function of order h* corresponding to the data $\{f(t, x) \in \mathcal{B}^\infty(I \times \Omega), \phi_j(x) \in \mathcal{B}^\infty(\Omega)\}$, if

$$\begin{cases} (Ag)(t, x) - f(t, x) \in H^{\infty,h}(I \times \Omega), \\ (\partial_t^j g)(0, x) - \phi_j(x) \in H^{\infty,h}(\Omega) \quad (j = 0, 1, \dots, m-1). \end{cases}$$

Let $\{g_j\}_{0 \leq j \leq h-1}$ satisfy $(Q_h)_{lin}$, and set

$$g(t, x) = \sum_{j=0}^{h-1} (j!)^{-1} \tilde{\rho}(x_1)^j g_j(t, x'),$$

where $\tilde{\rho}(x_1) \in C_0^\infty(R)$ and $\tilde{\rho}(x_1) = x_1$ near $x_1 = 0$. Then $g(t, x)$ is an approximate function of order h corresponding to $\{f(t, x) \in H^\infty(I \times \Omega), \phi_j(x) \in H^\infty(\Omega)\}$. When $g(t, x)$ is an approximate function of order h corresponding to $\{f(t, x) \in H^\infty(I \times \Omega), \phi_j(x) \in H^\infty(\Omega)\}$, set

$$\begin{cases} f'(t, x) = f(t, x) - (Ag)(t, x) \in H^{\infty,h}(I \times \Omega), \\ \phi'_j(x) = \phi_j(x) - \partial_t^j g(0, x) \in H^{\infty,h}(\Omega), \end{cases}$$

then $(P)_{lin}$ is reduced to

$$(P')_{lin} \begin{cases} A(t, x; \partial_t, \partial_x)u' = f'(t, x) \quad \text{in } I \times \Omega, \\ \partial_t^j u' = \phi'_j(x) \quad (j = 0, 1, \dots, m-1) \quad \text{on } \{t = 0\} \times \Omega, \end{cases}$$

where $u' = u - g$.

§2. Local Existence Theorems for Quasi-linear Cauchy Problems with Sufficiently Degenerate Data

Let us consider a quasi-linear Cauchy problem:

$$(P) \begin{cases} A(t, x; \mathfrak{D}^{m-1}u; \partial)u \equiv \sum_{j+|\nu|=m} a_{j\nu}(t, x; \mathfrak{D}^{m-1}u) \partial_t^j \partial_x^\nu u \\ \qquad \qquad \qquad = f(t, x; \mathfrak{D}^{m-1}u) \quad \text{in } I \times \Omega, \\ \partial_t^j u = \phi_j(x) \quad (j = 0, 1, \dots, m-1) \quad \text{on } \{t = 0\} \times \Omega, \end{cases}$$

where we assume (A), (σ -S) and (h -D) as follows. We denote

$$U_k = \mathfrak{D}_t^{m-1-k} \partial_x^k u = \partial_x^k (u, \partial_t u, \dots, \partial_t^{m-1-k} u), \quad U = (U_0, U_1, \dots, U_{m-1}),$$

$$\Phi_k = \partial_x^k (\phi_0, \phi_1, \dots, \phi_{m-1-k}), \quad \Phi = (\Phi_0, \Phi_1, \dots, \Phi_{m-1}),$$

and
$$N = \sum_{k=0}^{m-1} (m-k) \binom{k+n-1}{n-1}.$$

(A): i) $a_{j\nu}(t, x; U) \in C^\infty(\overline{I \times \Omega} \times R^N),$

ii) $A(0, x; \Phi(x); \partial) \in RH^0(I \times \Omega_\varepsilon)$ for any $\varepsilon > 0$, where $\Omega_\varepsilon = \Omega \cap \{x_1 > \varepsilon\},$

iii) $f(t, x; U) \in C^\infty(\overline{I \times \Omega} \times R^N),$

iv) $\phi_j(x) \in H^\infty(I \times \Omega).$

(σ -S): i) $A(t, x; 0; \partial) \in RH^\sigma(I \times \partial\Omega; \delta),$

ii) $f_{U_k}(t, x; 0) \rho(x_1)^{-k\sigma}$ is bounded in $I \times \Omega.$

(h -D): i) $f(t, x; 0) \in H^{\infty,h}(I \times \Omega) \quad (h \geq h_0),$

ii) $\phi_j(x) \in H^{\infty,h}(\Omega) \quad (h \geq h_0),$ where $h_0 = 2 \langle m\sigma \rangle.$

Theorem 1. *Under the assumptions (A), (σ -S) and (h -D), there exists a unique solution of (P) in $H^{s,h}(I_0 \times \Omega)$ ($s \geq s_0$), where $I_0 = (0, T_0)$, if $T_0 = T_0(s, h) (> 0)$ is chosen small enough.*

Taking $v = u - \sum_{j=0}^{m-1} (j!)^{-1} t^j \phi_j(x)$ for u , the Cauchy problem (P) can be reduced to the Cauchy problem with zero initial data:

$$(P-0) \begin{cases} A(t, x; \mathfrak{D}^{m-1}u, \partial)u = f(t, x; \mathfrak{D}^{m-1}u) \quad \text{in } I \times \Omega, \\ \mathfrak{D}_t^{m-1}u = 0 \quad \text{on } \{t = 0\} \times \Omega, \end{cases}$$

where (A), (σ -S), (h -D) are satisfied with $\Phi = 0$. Therefore, Theorem 1 will be shown for (P-0) by using the method of successive approximation. Namely, we shall find $u^{(k)} \in H^{\infty,h}(I_0 \times \Omega)$ satisfying

$$(P^{(k)}-0) \begin{cases} A(t, x; \mathfrak{D}^{m-1}u^{(k-1)}, \partial)u^{(k)} = f(t, x; \mathfrak{D}^{m-1}u^{(k-1)}) \quad \text{in } I \times \Omega, \\ \mathfrak{D}_t^{m-1}u^{(k)} = 0 \quad \text{on } \{t = 0\} \times \Omega, \end{cases}$$

if $u^{(k-1)} \in H^\infty(I_0 \times \Omega)$, and then we shall see that $u^{(k)} \rightarrow u$ in $H^{s-1,h}(I_0 \times \Omega).$

Lemma 2.1. *There exists $C > 0$ such that*

$$\| a(x, u(x))v_1(x) \cdots v_\ell(x) \|_s \leq C |a|_{\mathcal{B}^s(\Omega \times B_r)} (1 + \|u\|_s)^s \prod_{j=1}^{\ell} \|v_j\|_s$$

for any $a(x, u) \in \mathcal{B}^s(\Omega \times \mathbb{R}^N)$ and any $u(x), v(x) \in H^s(\Omega)$, satisfying $u \in B_r = \{u \mid |u(x)| \leq r\}$ ($s \geq [n/2] + 1, r > 0$).

Proof. We only see the case when $N = 1$. Let $|v| \leq s$. We remark that

$$\begin{aligned} & \partial^v \{a(x, u(x))v_1(x) \cdots v_\ell(x)\} \\ &= \sum C_\mu \{ \partial^{v-\mu_1-\dots-\mu_\ell} a(x, u(x)) \} (\partial^{\mu_1} v_1)(x) \cdots (\partial^{\mu_\ell} v_\ell)(x) \\ &= \sum C_{\alpha k \beta} (\partial_x^\alpha \partial_u^k a)(x, u(x)) (\partial^{\beta_1} u)(x) \cdots (\partial^{\beta_k} u)(x) \\ & \quad \times (\partial^{\mu_1} v_1)(x) \cdots (\partial^{\mu_\ell} v_\ell)(x), \end{aligned}$$

where \sum is taken over

$$|\alpha| + |\beta_1| + \cdots + |\beta_k| + |\mu_1| + \cdots + |\mu_\ell| \leq |v|,$$

where $|\beta_j| \geq 1$. Owing to Lemma.1.2, we have

$$\begin{aligned} & \| \partial^v \{a(x, u(x))v_1(x) \cdots v_\ell(x)\} \| \\ & \leq C |a|_{\mathcal{B}^s(\Omega \times B_r)} \sum_{k=0}^s \|u\|_s^k \prod_{j=1}^{\ell} \|v_j\|_s. \end{aligned} \quad \square$$

Let us denote

$$H_{\varepsilon(s, M)}^{\infty, h}(I \times \Omega) = \{u \in H^{\infty, h}(I \times \Omega) \mid \| \rho^{-h} u \|_{s_0} \leq \varepsilon, \| \rho^{-h} u \|_{(s)} \leq M\},$$

then we have a fundamental lemma to get an iterative scheme.

Lemma 2.2. *Let $s \geq s_0$ and $h \geq h_0$, then there exist positive constants ε and K_0 , independent of K , such that*

- i) $A(t, x; \mathfrak{D}^{m-1}u; \partial) \in RH^\sigma(I \times \Omega; \delta/2; s, 2K),$
- ii) $\| \rho^{-h} f(t, x; \mathfrak{D}^{m-1}u_s) \| \leq 2K,$

if

- i) $A(t, x; 0; \partial) \in RH^\sigma(I \times \Omega; \delta; s, K),$
- ii) $\| \rho^{-h} f(t, x; 0) \|_s \leq K,$
- iii) $u \in H_{\varepsilon(s+m-1, M_K)}^{\infty, h}(I \times \Omega),$ where $M_K = (K/K_0)^{1/(s+2)} - 1$

for $K > K_0$.

Proof of i). Let us denote

$$\begin{aligned}
 a_{j\nu}(t, x; U) &= a_{j\nu}(t, x; 0) + \sum_k \int_0^1 a_{j\nu U_k}(t, x; \theta U) d\theta U_k \\
 &= a_{j\nu}(t, x; 0) + \sum_k X_{j\nu k}(t, x; U) U_k,
 \end{aligned}$$

where $X_{j\nu k} \in C^\infty(\overline{I \times \Omega} \times R^N)$, that is, $|X_{j\nu k}|_{\mathcal{B}^s(I \times \Omega \times B_r)} \leq C_{s,r}$.

We remark that

$$\begin{aligned}
 |\rho^{-h+m-1} U| &= |\rho^{-h+m-1} \mathfrak{F}^{m-1} \{ \rho^h(\rho^{-h}u) \}| \\
 &\leq C |\mathfrak{F}^{m-1}(\rho^{-h}u)| \leq C' \| \rho^{-h}u \|_{m+[n/2]} \leq C' \| \rho^{-h}u \|_{s_0},
 \end{aligned}$$

that is, we have

$$|U(t, x)| \leq C \rho^{h-(m-1)} \| \rho^{-h}u \|_{s_0} \leq C \varepsilon \rho^{h-(m-1)}.$$

Hence we have

$$|a_{j\nu}(t, x; U) - a_{j\nu}(t, x; 0)| \leq C \varepsilon \rho^{h-(m-1)} \leq C \varepsilon \rho^{\sigma|\nu|} \quad (h \geq h_0),$$

because $h_0 \geq m - 1 + \sigma m$. Since $A(t, x; 0; \partial) \in RH^\sigma(I \times \Omega; \delta)$, we have $A(t, x; U; \partial) \in RH^\sigma(I \times \Omega; \delta/2)$, taking ε small enough. Moreover, we may assume that $|U(t, x)| \leq 1$, taking ε small enough.

Next, let $s \geq s_0$ and $|\nu| \leq s$. Owing to Lemma.2.1, we have

$$\| X_{j\nu k}(t, x; U) U_k \|_s \leq C(1 + \| u \|_{s+m-1})^s \| u \|_{s+m-1}.$$

We remark that

$$\begin{aligned}
 \| u \|_{s+m-1} &= \| \rho^h(\rho^{-h}u) \|_{s+m-1} \\
 &\leq C \{ \| \rho^h \mathfrak{F}^{s+m-1}(\rho^{-h}u) \| + \| \rho^{h-1} \mathfrak{F}^{s+m-2}(\rho^{-h}u) \| + \dots \\
 &\quad + \| \rho^{h-(m-1)} \mathfrak{F}^s(\rho^{-h}u) \| + \| \rho^{-h}u \|_{s-1} \} \\
 &\leq C' \| \rho^{-h}u \|_{(s+m-1)} \quad (h \geq h_0),
 \end{aligned}$$

because $h_0 \geq (m - 1)\sigma$. Hence we have

$$\begin{aligned}
 \sum_{j,\nu} \| a_{j\nu}(t, x; U) - a_{j\nu}(t, x; 0) \|_s &\leq C_1(1 + \| \rho^{-h}u \|_{(s+m-1)})^{s+1} \\
 &\leq C_1(1 + M_K)^{s+1} \leq K
 \end{aligned}$$

for $K > K_0$, $K_0 \geq C_1$. Here we have

$$a_{j\nu}(t, x; U) = a'_{j\nu}(t, x; 0) + \{ a''_{j\nu}(t, x; 0) + [a_{j\nu}(t, x; U) - a_{j\nu}(t, x; 0)] \},$$

where

$$\begin{aligned} & \sum_{j,v} |a'_{jv}(t, x; 0)|_{\mathcal{B}^s} + \sum_{j,v} \| \| a''_{jv}(t, x; 0) + [a_{jv}(t, x; U) - a_{jv}(t, x; 0)] \| \|_s \\ & \leq \sum_{j,v} |a'_{jv}(t, x; 0)|_{\mathcal{B}^s} + \sum_{j,v} \| \| a''_{jv}(t, x; 0) \| \|_s \\ & \quad + \sum_{j,v} \| \| a_{jv}(t, x; U) - a_{jv}(t, x; 0) \| \|_s \leq 2K. \end{aligned}$$

Hence we have $A(t, x; U; \partial) \in RH^\sigma(I \times \Omega; \delta/2; s, 2K)$. □

Proof of ii). In the same way as in the above, we have

$$f(t, x; U) = f(t, x; 0) + \sum_{k=0}^{m-1} \xi_k(t, x; U) U_k$$

where $\xi_k \in C^\infty(\overline{I \times \Omega} \times R^N)$ and

$$|\xi_k(t, x; 0)| \leq C\rho^{\langle k\sigma \rangle}$$

Moreover, we have

$$\xi_k(t, x; U) = \xi_k(t, x; 0) + \sum_i \eta_{ki}(t, x; U) U_i,$$

where

$$\rho^{-\langle k\sigma \rangle} \xi_k(t, x; 0) \in \mathcal{B}^\infty(I \times \Omega), \quad \eta_{ki} \in C^\infty(\overline{I \times \Omega} \times R^N).$$

Since

$$\begin{aligned} & \rho^{-h} \xi_k(t, x; U) U_k \\ & = \{ \rho^{-\langle k\sigma \rangle} \xi_k(t, x; 0) \} \{ \rho^{-h+\langle k\sigma \rangle} U_k \} \\ & \quad + \sum_i \eta_{ki}(t, x; U) \{ \rho^{-\langle k\sigma \rangle} U_i \} \{ \rho^{-h+\langle k\sigma \rangle} U_k \}, \end{aligned}$$

we have, owing to Lemma.2.1,

$$\begin{aligned} & \| \| \rho^{-h} \xi_k(t, x; U) U_k \| \|_s \\ & \leq C \{ \| \| \rho^{-\langle k\sigma \rangle} \xi_k(t, x; 0) \| \|_s \| \| \rho^{-h+\langle k\sigma \rangle} U_k \| \|_s \\ & \quad + (1 + \| \| U \| \|_s)^s \| \| \rho^{-\langle k\sigma \rangle} U \| \|_s \| \| \rho^{-h+\langle k\sigma \rangle} U_k \| \|_s \} \\ & \leq C' \{ \| \| \rho^{-h+\langle k\sigma \rangle} U_k \| \|_s + (1 + \| \| U \| \|_s)^s \| \| \rho^{-h+\langle k\sigma \rangle} U_k \| \|_s^2 \} \end{aligned}$$

($h \geq h_0$), because $h_0 \geq 2\langle(m-1)\sigma\rangle$. Hence we have

$$\| \| \rho^{-h} \xi_k(t, x; U) U_k \| \|_s \leq C_2 (1 + \| \| \rho^{-hu} \| \|_{(s+m-1)})^{s+2} \leq C_2 (1 + M_K)^{s+2} \leq K,$$

therefore

$$\| \rho^{-h} f(t, x; U) \|_s \leq 2K$$

for $K > K_0$ if $K_0 \geq C_2$. Here we have only to set $K_0 = \max(C_1, C_2)$. \square

Hereafter, we fix K and $M = M_K$ so that K satisfies (i), (ii) and $K > K_0$ in Lemma. 2.2.

Proposition 2.3. *There exists $I_1 = (0, T_1)(T_1 > 0)$ such that there exists a solution $\tilde{u} \in H_{\varepsilon(s+m-1, M)}^{\infty, h}(I_1 \times \Omega)$ of the linearized problem:*

$$(\tilde{P}-0) \begin{cases} A(t, x; \mathfrak{P}^{m-1}u, \partial)\tilde{u} = f(t, x; \mathfrak{P}^{m-1}u) & \text{in } I_1 \times \Omega, \\ \mathfrak{P}_t^{m-1}\tilde{u} = 0 & \text{on } \{t = 0\} \times \Omega, \end{cases}$$

if $u \in H_{\varepsilon(s+m-1, M)}^{\infty, h}(I_1 \times \Omega)$.

Proof. Applying Lemma.2.2, we have

$$A(t, x; \mathfrak{P}^{m-1}u, \partial) \in RH^{\sigma}(I \times \Omega; \delta/2; s, 2K).$$

Hence, applying Proposition.1.4', we can find a unique solution \tilde{u} of $(\tilde{P}-0)$

for any $u \in H_{\varepsilon(s+m-1, M)}^{\infty, h}(I \times \Omega)$, and

$$\begin{aligned} & \| \rho^{-h} \tilde{u}(t) \|_{(s+m-1)}^2 \\ & \leq C \int_0^t \| \rho^{-h} f(t, x; \mathfrak{P}^{m-1}u) \|_s^2 dt \leq C_1 t. \end{aligned}$$

Taking

$$T_1 := \min(\varepsilon^2, M^2)/C_1,$$

we have $\tilde{u} \in H_{\varepsilon(s+m-1, M)}^{\infty, h}(I \times \Omega)$. \square

Following to Proposition.2.3, $u_k \in H_{\varepsilon(s+m-1, M)}^{\infty, h}(I_1 \times \Omega)$ can be defined as a solution of $(P^{(k)}-0)$ successively, taking $u_0 = 0$. Let us denote

$$\begin{aligned} & A(t, x; \mathfrak{P}^{m-1}u^{(k-1)}, \partial)(u^{(k)} - u^{(k-1)}) \\ & = - \{ A(t, x; \mathfrak{P}^{m-1}u^{(k-1)}, \partial) - A(t, x; \mathfrak{P}^{m-1}u^{(k-2)}, \partial) \} u^{(k-1)} \\ & \quad + \{ f(t, x; \mathfrak{P}^{m-1}u^{(k-1)}) - f(t, x; \mathfrak{P}^{m-1}u^{(k-2)}) \} \\ & = g'_k + g''_k = g_k \quad (k = 2, 3, \dots). \end{aligned}$$

Then we have

Lemma 2.4.

$$\| \rho^{-h} g_k \|_{s-1} \leq C \| \rho^{-h}(u^{(k-1)} - u^{(k-2)}) \|_{(s+m-2)} \quad (k = 2, 3, \dots),$$

Proof. Using Lemma.2.1, we have

$$\begin{aligned} \|\rho^{-h}g'_k\|_{s-1} &\leq C(1 + \|u^{(k-1)}\|_{s+m-2} + \|u^{(k-2)}\|_{s+m-2})^{s-1} \\ &\times \|\rho^{-\langle(m-1)\sigma\rangle}(u^{(k-1)} - u^{(k-2)})\|_{s+m-2} \|\rho^{-h+\langle(m-1)\sigma\rangle}\mathfrak{D}^m u^{(k-1)}\|_{s-1} \\ &\leq C(1 + \|\rho^{-h}u^{(k-1)}\|_{(s+m-1)} + \|\rho^{-h}u^{(k-2)}\|_{(s+m-1)})^s \\ &\times \|\rho^{-h}(u^{(k-1)} - u^{(k-2)})\|_{(s+m-2)}, \end{aligned}$$

because $h_0 \geq 2 \langle(m-1)\sigma\rangle$. In the same way, we have

$$\begin{aligned} \|\rho^{-h}g''_k\|_{s-1} &\leq C' (1 + \|\rho^{-h}u^{(k-1)}\|_{(s+m-2)} + \|\rho^{-h}u^{(k-2)}\|_{(s+m-2)})^{s-1} \\ &\times \|\rho^{-h}(u^{(k-1)} - u^{(k-2)})\|_{(s+m-2)}. \end{aligned} \quad \square$$

Let us denote

$$\mathcal{H}^{s,h}(I \times \Omega) = \{u \mid \|\rho^{-h}u\|_{(s)} < +\infty\}.$$

Then we have

Proposition 2.5. *There exists $I' = (0, T')$ ($0 < T' \leq T_1$) such that*

$$u^{(k)} \rightarrow u \quad \text{in } \mathcal{H}^{s+m-2,h}(I' \times \Omega),$$

where $u \in \mathcal{H}^{s+m-1,h}(I' \times \Omega)$ and satisfies $(P - 0)$.

Proof. Since

$$A(t, x; \mathfrak{D}^{m-1}u^{(k-1)}, \partial)(u^{(k)} - u^{(k-1)}) = g_k,$$

we have from Proposition.1.4' and Lemma.2.4

$$\begin{aligned} \|u^{(k)} - u^{(k-1)}\|_{\mathcal{H}^{s+m-2,h}(I' \times \Omega)}^2 &\leq CT' \|u^{(k-1)} - u^{(k-2)}\|_{\mathcal{H}^{s+m-2,h}(I' \times \Omega)}^2. \end{aligned}$$

Taking T' small enough to satisfy $0 < T' < T_1$ and $CT' < 1$, $\{u^{(k)}\}$ is a Cauchy sequence in $\mathcal{H}^{s+m-2,h}(I' \times \Omega)$. Moreover we have

$$\|u\|_{\mathcal{H}^{s+m-1,h}(I' \times \Omega)} \leq \liminf_k \|u^{(k)}\|_{\mathcal{H}^{s+m-1,h}(I' \times \Omega)} \leq M. \quad \square$$

§ 3. Local Existence Theorems for Quasi-linear Cauchy Problems with Degenerate-Structure-Generating Initial Data

3.1. Condition $(k-*)$. Let us denote

$$\mathcal{U}_k = \mathfrak{D}_t^{m-1} \partial_x^k u \quad (k = 0, 1, \dots), \quad \mathcal{U} = \begin{bmatrix} \mathcal{U}_0 \\ \vdots \\ \mathcal{U}_{m-1} \end{bmatrix},$$

where \mathcal{U}_k is a \mathcal{N}_k -column vector and \mathcal{U} is a \mathcal{N} -column vector, where $\mathcal{N}_k = mN_k$

$= m \binom{k+n-1}{k}$ and $\mathcal{N} = \sum_{k=0}^{m-1} \mathcal{N}_k$. Then we have

Lemma 3.1. *Let p be a positive integer. Then*

$$\begin{aligned} & \partial_x^p \{ \mathcal{F}(t, x; \mathcal{U}(t, x)) \} \\ &= \{ \Gamma_p(\mathcal{U}_0, \dots, \mathcal{U}_{p+m-1}; \partial_x, \partial_{\mathcal{U}}) \mathcal{F} \}(t, x; \mathcal{U}(t, x)), \end{aligned}$$

where

$$\begin{aligned} \Gamma_p &= \sum_{q=0}^p \Gamma_{pq}(\mathcal{U}_0, \dots, \mathcal{U}_{p+m-1}; \partial_x, \partial_{\mathcal{U}}), \\ \Gamma_{pq} &= \sum_{r+s=q} \gamma_{p(r,k_1, \dots, k_s)}(\mathcal{U}_0, \dots, \mathcal{U}_{p+m-1}) \partial_x^r \partial_{\mathcal{U}_{k_1}} \dots \partial_{\mathcal{U}_{k_s}}, \end{aligned}$$

where $\gamma_{p(r,k_1, \dots, k_s)}$ is a $N_p \times (N_r \mathcal{N}_{k_1} \dots \mathcal{N}_{k_s})$ -matrix with homogeneous polynomial entries of degree s with respect to $\{\mathcal{U}_0, \dots, \mathcal{U}_{p+m-1}\}$. Moreover, Γ_p is represented as $\Gamma'_p + \Gamma''_p$, where

$$\begin{aligned} \Gamma'_p &= \sum_{q=0}^p \Gamma'_{pq}, \quad \Gamma''_p = \sum_{q=0}^p \Gamma''_{pq}, \\ \Gamma'_{pq} &= \sum_{\substack{r+s=q \\ \max k_i < q}} \gamma'_{p(r,k_1, \dots, k_s)}(\mathcal{U}_0, \dots, \mathcal{U}_p) \partial_x^r \partial_{\mathcal{U}_{k_1}} \dots \partial_{\mathcal{U}_{k_s}}, \\ \Gamma''_{pq} &= \sum_{\substack{r+s=q \\ \max k_i \geq q}} \gamma''_{p(r,k_1, \dots, k_s)}(\mathcal{U}_0, \dots, \mathcal{U}_{p+m-1}) \partial_x^r \partial_{\mathcal{U}_{k_1}} \dots \partial_{\mathcal{U}_{k_s}}. \end{aligned}$$

Proof. For simplicity, we consider the case when $n = 1$, where \mathcal{U}_k is a scalar valued function. By the differentiation of composite functions, we have

$$\begin{aligned} & \partial_x^p \{ \mathcal{F}(t, x; \mathcal{U}(t, x)) \} \\ &= (\partial_x^p \mathcal{F})(t, x; \mathcal{U}(t, x)) + \sum_{s=1}^p \sum_{\substack{r+\mu_1+\dots+\mu_s=p \\ r \geq 0, \mu_i \geq 1}} \sum_{0 \leq k_i \leq m-1} c_{r\mu_1 \dots \mu_s k_1 \dots k_s} \\ & \quad \times (\partial_x^r \partial_{\mathcal{U}_{k_1}} \dots \partial_{\mathcal{U}_{k_s}} \mathcal{F})(t, x; \mathcal{U}(t, x)) \mathcal{U}_{k_1+\mu_1} \dots \mathcal{U}_{k_s+\mu_s}, \end{aligned}$$

where $c_{r\mu_1 \dots \mu_s k_1 \dots k_s}$ is independent of $\{t, x, \mathcal{U}, \mathcal{F}\}$. We have only to remark that

$$k_i + \mu_i \leq \max k_i + p - r - (s - 1) \leq p \text{ if } \max k_i < r + s. \quad \square$$

Let k be a non-negative integer and let us say that \mathcal{N} boundary functions $\mathcal{B}(t, x')$ satisfies $(k-*)$ with respect to $\mathcal{F}(t, x; \mathcal{U})$ if

$$(\partial_x^r \partial_{\mathcal{U}_{k_1}} \dots \partial_{\mathcal{U}_{k_s}} \mathcal{F})(t, 0, x'; \mathcal{B}(t, x')) = 0$$

if $r + s \leq \max(k - 1, k_1, \dots, k_s)$. Let us denote $\dot{\mathcal{U}}(t, x') = \mathcal{U}(t, 0, x')$. Then we have

Lemma 3.2. *Assume that $\dot{\mathcal{U}}(t, x')$ satisfies (k-*) with respect to \mathcal{F} . Then*

$$\begin{aligned} & \partial_x^p \{ \mathcal{F}(t, x; \mathcal{U}(t, x)) \} |_{x_1=0} \\ &= \sum_{q=k}^p \{ \Gamma'_{pq}(\dot{\mathcal{U}}_0(t, x'), \dots, \dot{\mathcal{U}}_p(t, x'); \partial_x, \partial_{\mathcal{U}}) \mathcal{F} \}(t, 0, x'; \dot{\mathcal{U}}(t, x')). \end{aligned}$$

Proof. Since

$$\begin{aligned} & \partial_x^p \{ \mathcal{F}(t, x; \mathcal{U}(t, x)) \} \\ &= \sum_{q=0}^p \left\{ \sum_{\substack{r+s=q \\ \max k_i < q}} \gamma'_{p(r, k_1, \dots, k_s)}(\mathcal{U}_0, \dots, \mathcal{U}_p) (\partial_x^r \partial_{\mathcal{U}_{k_1}} \dots \partial_{\mathcal{U}_{k_s}} \mathcal{F})(t, x; \mathcal{U}) \right. \\ & \quad \left. + \sum_{\substack{r+s=q \\ \max k_i \geq q}} \gamma''_{p(r, k_1, \dots, k_s)}(\mathcal{U}_0, \dots, \mathcal{U}_{p+m-1}) (\partial_x^r \partial_{\mathcal{U}_{k_1}} \dots \partial_{\mathcal{U}_{k_s}} \mathcal{F})(t, x; \mathcal{U}), \right. \end{aligned}$$

the second part in the right hand side vanishes on $\{x_1 = 0\}$. Moreover, the summations of right hand side over $\{0 \leq q \leq k - 1\}$ also vanish on $\{x_1 = 0\}$. □

3.2. Fully degenerate condition (B). Now let us consider

$$(P) \begin{cases} A(t, x; \mathfrak{D}^{m-1}u; \partial)u = f(t, x; \mathfrak{D}^{m-1}u) & \text{in } I \times \Omega, \\ \partial_t^j u = \phi_j(x) \quad (j = 0, \dots, m - 1) & \text{on } \{t = 0\} \times \Omega, \end{cases}$$

where we assume (A) stated in §2. Denoting

$$\mathcal{A}(t, x; \mathcal{U}; \partial) = A(t, x; U; \partial), \quad \alpha_{j\nu}(t, x; \mathcal{U}) = a_{j\nu}(t, x; U)$$

$$\mathcal{F}(t, x; \mathcal{U}) = f(t, x; U),$$

let us say that \mathcal{N} boundary functions $\mathcal{B}(t, x')$ satisfy (B)—fully degenerate condition—if

- i) \mathcal{B} satisfies ($|\nu|$ -*) with respect to $\alpha_{j\nu}$,
- ii) \mathcal{B} satisfies (0-*) with respect to \mathcal{F} .

Let us denote

$$\mathcal{U}_p(t, x) = (\partial_x^p u)(t, x), \quad \alpha_{j\nu, p}(t, x) = (\partial_x^p \alpha_{j\nu})(t, x),$$

where $\alpha_{j\nu}(t, x) = \alpha_{j\nu}(t, x; \mathcal{U}(t, x))$. Then we have

$$\begin{aligned} \alpha_{j\nu, p}(t, x) &= \{ \Gamma_p(\mathcal{U}_0, \dots, \mathcal{U}_{p+m-1}; \partial_x, \partial_{\mathcal{U}}) \alpha_{j\nu} \}(t, x; \mathcal{U}) \\ &= \sum_{k=0}^p \{ \Gamma_{pk}(\mathcal{U}_0, \dots, \mathcal{U}_{p+m-1}; \partial_x, \partial_{\mathcal{U}}) \alpha_{j\nu} \}(t, x; \mathcal{U}) \end{aligned}$$

Let us consider

$$\Gamma_r = (\Gamma_r)_{|r|=r},$$

where $r = (r_1, \dots, r_n)$, then we have

$$\begin{aligned} & \partial_x^r \{ \mathcal{A}(t, x; \mathcal{U}(t, x); \partial)u(t, x) - \partial_t^m u(t, x) \} \\ &= \partial_x^r \left\{ \sum_{\substack{j+|v| \leq m \\ j \leq m-1}} \alpha_{jv}(t, x; \mathcal{U}(t, x)) \partial_t^j \partial_x^v u(t, x) \right\} \\ &= \sum_{\substack{j+|v|=m \\ j \leq m-1}} \sum_{p+q=r} \Gamma!(\mathbf{p}!\mathbf{q}!)^{-1} \partial_t^j \partial_x^{v+q} u(t, x) \alpha_{jv,p}(t, x) \\ &= \sum_{\substack{j+|v|=m \\ j \leq m-1}} \left\{ \sum_{p+q=r} \Gamma!(\mathbf{p}!\mathbf{q}!)^{-1} \partial_t^j \partial_x^{v+q} u(t, x) \right. \\ &\quad \left. \times \Gamma_p(\mathcal{U}_0, \dots, \mathcal{U}_{|p|+m-1}; \partial_x, \partial_{\mathcal{U}}) \right\} \alpha_{jv}(t, x; \mathcal{U}) \\ &= \sum_{\substack{j+|v|=m \\ j \leq m-1}} \Gamma_{r(j,v)}(\mathcal{U}_0, \dots, \mathcal{U}_{|r|+m}; \partial_x, \partial_{\mathcal{U}}) \alpha_{jv}(t, x; \mathcal{U}). \end{aligned}$$

Let us denote

$$\begin{aligned} \Gamma_{r(j,v)} &= (\Gamma_{r(j,v)})_{|r|=r}, \\ \Gamma'_{r(j,v)} &= (\Gamma'_{r(j,v)})_{|r|=r}, \quad \Gamma''_{r(j,v)} = (\Gamma''_{r(j,v)})_{|r|=r} \end{aligned}$$

for $j + |v| \leq m$, where

$$\begin{aligned} \Gamma_{r(j,v)} &= \Gamma_{r(j,v)}(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; \partial_x, \partial_{\mathcal{U}}) \\ &= \sum_{p+q=r} \mathbf{r}!(\mathbf{p}!\mathbf{q}!)^{-1} \partial_t^j \partial_x^{v+q} u \Gamma_p(\mathcal{U}_0, \dots, \mathcal{U}_{|p|+m-1}; \partial_x, \partial_{\mathcal{U}}), \\ \Gamma'_{r(j,v)} &= \Gamma'_{r(j,v)}(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; \partial_x, \partial_{\mathcal{U}}) \\ &= \sum_{p+q=r} \mathbf{r}!(\mathbf{p}!\mathbf{q}!)^{-1} \partial_t^j \partial_x^{v+q} u \Gamma'_p(\mathcal{U}_0, \dots, \mathcal{U}_{|p|}; \partial_x, \partial_{\mathcal{U}}), \\ \Gamma''_{r(j,v)} &= \Gamma''_{r(j,v)}(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; \partial_x, \partial_{\mathcal{U}}) \\ &= \sum_{p+q=r} \mathbf{r}!(\mathbf{p}!\mathbf{q}!)^{-1} \partial_t^j \partial_x^{v+q} u \Gamma''_p(\mathcal{U}_0, \dots, \mathcal{U}_{|p|+m-1}; \partial_x, \partial_{\mathcal{U}}). \end{aligned}$$

Moreover, we denote

$$\Gamma_{r(j,v)}^{\pm} = (\Gamma'_{r(j,v)})_{|r|=r},$$

where

$$\begin{aligned} \Gamma_{r(j,v)}^+ &= \Gamma_{r(j,v)}^+(\mathcal{U}_0, \dots, \mathcal{U}_r; \partial_x, \partial_{\mathcal{U}}) \\ &= \sum_{p+q=r} r!(p!q!)^{-1} \partial_i^j \partial_x^{v+q} u \sum_{k=|v|}^{|p|} \Gamma_{rk}^+(\mathcal{U}_0, \dots, \mathcal{U}_{|p|}; \partial_x, \partial_{\mathcal{U}}), \\ \Gamma_{r(j,v)}^- &= \Gamma_{r(j,v)}^-(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; \partial_x, \partial_{\mathcal{U}}) \\ &= \sum_{p+q=r} r!(p!q!)^{-1} \partial_i^j \partial_x^{v+q} u \sum_{k=0}^{|v|-1} \Gamma_{rk}^-(\mathcal{U}_0, \dots, \mathcal{U}_{|p|}; \partial_x, \partial_{\mathcal{U}}), \end{aligned}$$

Let us denote

$$\begin{aligned} \beta_r(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; t, x; \mathcal{U}) \\ = \sum_{\substack{j+|v|=m \\ j \leq m-1}} \{ \Gamma_{r(j,v)}(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; \partial_x, \partial_{\mathcal{V}}) \alpha_{jv} \}(t, x; \mathcal{V}) \end{aligned}$$

and $\beta'_r, \beta''_r, \beta_r^\pm$ are defined analogously, then we have

Lemma 3.3. *Let us assume that $\mathcal{B}(t, x')$ satisfies (B). Then*

- i) $\beta_r(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; t, 0, x'; \mathcal{B}(t, x'))$
 $= \beta_r^+(\mathcal{U}_0, \dots, \mathcal{U}_r; t, 0, x'; \mathcal{B}(t, x')).$
- ii) $\{ \Gamma_r(\mathcal{U}_0, \dots, \mathcal{U}_{r+m-1}; \partial_x, \partial_{\mathcal{U}}) \mathcal{F} \}(t, 0, x'; \mathcal{B}(t, x'))$
 $= \{ \Gamma'_r(\mathcal{U}_0, \dots, \mathcal{U}_r; \partial_x, \partial_{\mathcal{U}}) \mathcal{F} \}(t, 0, x'; \mathcal{B}(t, x')).$

Let us denote

$$\begin{aligned} \mathcal{F}_r(\mathcal{U}_0, \dots, \mathcal{U}_r; t, x; \mathcal{V}) \\ = -\beta_r^+(\mathcal{U}_0, \dots, \mathcal{U}_r; t, x; \mathcal{V}) + \{ \Gamma'_r(\mathcal{U}_0, \dots, \mathcal{U}_r; \partial_x, \partial_{\mathcal{V}}) \mathcal{F} \}(t, x; \mathcal{V}), \\ \mathcal{H}_r(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; t, x; \mathcal{V}) \\ = -\beta_r^-(\mathcal{U}_0, \dots, \mathcal{V}_{r+m}; t, x; \mathcal{V}) - \beta''_r(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; t, x; \mathcal{V}), \\ + \{ \Gamma'_r(\mathcal{U}_0, \dots, \mathcal{U}_r; \partial_x, \partial_{\mathcal{V}}) \mathcal{F} \}(t, x; \mathcal{V}), \end{aligned}$$

then we have

Corollary. *Let us assume $\mathcal{B}(t, x')$ satisfies (B), then*

$$\mathcal{H}_r(\mathcal{U}_0, \dots, \mathcal{U}_{r+m}; t, 0, x'; \mathcal{B}(t, x')) = 0.$$

Hence we have

Proposition 3.4. *Assume that $u(t, x)$ is a solution of (P) and that*

$$\dot{\mathcal{U}}(t, x') = \mathfrak{D}_t^{m-1} \mathfrak{D}_x^{m-1} u(t, 0, x')$$

satisfies (B), then $\{ \dot{u}_j(t, x') = (\partial_x^j u)(t, 0, x') (j = 0, 1, 2, \dots) \}$ satisfy (Q):

$$(Q) \begin{cases} \partial_t^m \dot{u}_r = \mathcal{F}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_r; t, 0, x'; \mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_{m-1}) \text{ in } I \times R^{n-1}, \\ \partial_t^j \dot{u}_r = \dot{\phi}_{j,r}(x') \quad (j = 0, \dots, m-1) \text{ on } \{t = 0\} \times R^{n-1} \quad (r = 0, 1, 2, \dots), \end{cases}$$

where

$$\dot{\phi}_{j,r}(x') = (\partial_x^r \phi_j)(0, x').$$

Let us denote

$$(Q_h) \begin{cases} \partial_t^m \dot{u}_r = \mathcal{F}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_r; t, 0, x'; \mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_{m-1}) \text{ in } I \times R^{n-1}, \\ \partial_t^j \dot{u}_r = \dot{\phi}_{j,r}(x') \quad (j = 0, \dots, m-1) \text{ on } \{t = 0\} \times R^{n-1} \quad (r = 0, 1, 2, \dots, h-1). \end{cases}$$

Then (Q_h) is a Cauchy problem for a system of ordinary differential equations with respect to t with unknown functions $\{\dot{u}_r \ (r = 0, 1, \dots, h-1)\}$. Therefore, it is uniquely solvable in locale for any $h(\geq m)$.

3.3. σ -structure-generating initial data.

Proposition 3.5. *Let $\{g_j(t, x') \ (j = 0, 1, \dots)\}$ be a solution of (Q) . Assume that $\varphi(t, x') = \mathfrak{D}_t^{m-1}(g_0(t, x'), \dots, g_{m-1}(t, x'))$ satisfies (B) . Let u be a solution of (P) , then we have*

$$\dot{u}_j = g_j \quad (j = 0, 1, 2, \dots).$$

Proof. Since u satisfies (P) , we have

$$\begin{aligned} \partial_t^m \dot{u}_r &= \mathcal{F}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_r; t, 0, x'; \mathcal{U}) \\ &\quad + \mathcal{H}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_{r+m}; t, 0, x'; \mathcal{U}). \end{aligned}$$

On the other hand, since φ satisfies (B) , we have

$$\mathcal{H}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_{r+m}; t, 0, x'; \varphi) = 0,$$

therefore,

$$\begin{aligned} \partial_t^m g_r &= \mathcal{F}_r(\mathfrak{D}_t^{m-1} g_0, \dots, \mathfrak{D}_t^{m-1} g_r; t, 0, x'; \varphi) \\ &\quad + \mathcal{H}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_{r+m}; t, 0, x'; \varphi). \end{aligned}$$

Hence we have

$$\begin{aligned} \partial_t^m (\dot{u}_r - g_r) &= \{ \mathcal{F}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_r; t, 0, x'; \mathcal{U}) \\ &\quad - \mathcal{F}_r(\mathfrak{D}_t^{m-1} g_0, \dots, \mathfrak{D}_t^{m-1} g_r; t, 0, x'; \varphi) \} \\ &\quad + \{ \mathcal{H}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_{r+m}; t, 0, x'; \mathcal{U}) \\ &\quad - \mathcal{H}_r(\mathfrak{D}_t^{m-1} \dot{u}_0, \dots, \mathfrak{D}_t^{m-1} \dot{u}_{r+m}; t, 0, x'; \varphi) \}. \end{aligned}$$

Let us define a system of ordinary differential equations with respect to $v = (v_0, \dots, v_{m-1})$:

$$(\tilde{Q}_m) \left\{ \begin{array}{l} \partial_t^m v_r = \{ \mathcal{F}_r(\partial_t^{m-1}(v_0 + g_0), \dots, \partial_t^{m-1}(v_r + g_r); t, 0, x'; \partial_t^{m-1}v + g) \\ \quad - \mathcal{F}_r(\partial_t^{m-1}g_0, \dots, \partial_t^{m-1}g_r; t, 0, x'; g) \} \\ \quad + \{ \mathcal{H}_r(\partial_t^{m-1}\dot{u}_0, \dots, \partial_t^{m-1}\dot{u}_{r+m}; t, 0, x'; \partial_t^{m-1}v + g) \\ \quad - \mathcal{H}_r(\partial_t^{m-1}\dot{u}_0, \dots, \partial_t^{m-1}\dot{u}_{r+m}; t, 0, x'; g) \} \\ (\partial_t^{m-1}v_r)(0, x') = 0 \quad (r = 0, \dots, m-1), \end{array} \right.$$

where $\{\dot{u}_j, g_j\}$ are considered as given functions, then

$$v = (\dot{u}_0 - g_0, \dots, \dot{u}_{m-1} - g_{m-1})$$

satisfies (\tilde{Q}_m) . On the other hand, it is obvious that $v = 0$ is the unique solution of (\tilde{Q}_m) . Hence we have $\{\dot{u}_r = g_r\}_{r=0,1,\dots,m-1}$. The fact that $\{\dot{u}_r = g_r\}_{r \geq m}$ follows from Proposition.3.4. \square

Set

$$g^{(h)}(t, x) = \sum_{r=0}^{h-1} (r!)^{-1} \tilde{\rho}(x_1)^r g_{(r,0,\dots,0)}(t, x')$$

and

$$G^{(h)}(t, x) = \partial_t^{m-1} g^{(h)}(t, x), \quad \mathcal{G}^{(h)}(t, x) = \partial_t^{m-1} \partial_x^{m-1} g^{(h)}(t, x),$$

where $\{g_r = (g_{r|\alpha|} |_{r=0,1,\dots,h-1})\}$, is a solution of (Q_h) , then we have $\mathcal{G}^{(h)}(t, 0, x') = \mathcal{g}(t, x')$. Let us assume that \mathcal{g} satisfies (B) , then (P) is reduced to

$$(\tilde{P})_{\mathcal{G}^{(h)}} \left\{ \begin{array}{l} \tilde{A}(t, x; \partial_t^{m-1}v; \partial)v = \tilde{f}(t, x; \partial_t^{m-1}v) \quad \text{in } I \times \Omega, \\ \partial_t^j v = \tilde{\phi}_j(x) \quad (j = 0, \dots, m-1) \quad \text{on } \{t = 0\} \times \Omega, \end{array} \right.$$

where

$$\begin{aligned} \tilde{A}(t, x; V; \partial) &= A(t, x; G^{(h)}(t, x) + V; \partial), \\ \tilde{f}(t, x; V) &= -A(t, x; G^{(h)}(t, x) + V; \partial)g^{(h)} + f(t, x; G^{(h)}(t, x) + V), \\ \tilde{\phi}_j(x) &= \phi_j(x) - (\partial_t^j g^{(h)})(0, x). \end{aligned}$$

Moreover, from the definition of (Q_h) , we have

Lemma 3.6. *Let $\{g_r(r = 0, 1, \dots, h-1)\}$ be a solution of (Q_h) , where \mathcal{g} satisfies (B) , then $(\tilde{P})_{\mathcal{G}^{(h)}}$ satisfies $(h-D)$ (defined in §2).*

Let us say that initial data $\{\phi_j\}_{j=0,1,\dots,m-1}$ satisfy $(\sigma - S - G) - \sigma$ -structure-generating condition-, if

i) $\mathcal{g} = \partial_t^{m-1}(g_0, \dots, g_{m-1})$ satisfies (B) , where $\{g_r(r = 0, 1, \dots, m-1)\}$ is a solution

of (Q_m) ,

ii) $(\tilde{P})_{g^{(h)}}$ satisfies $(\sigma - S)$ (defined in §2).

Here we have from Theorem 1

Theorem 2. *Let us assume (A) and assume that $\{\phi_j\}_{j=0,1,\dots,m-1}$ satisfy $(\sigma - S - G)$, then there exists an interval $I_s = (0, T_s)$ for any $s (\geq s_0)$ such that there exists a unique solution of (P) in $H^s(I_s \times \Omega)$. Moreover the boundary functions $\{\dot{u}_r = \partial_x^r u(t, 0, x')\}$ of the solution u satisfy (Q).*

§4. Blow-up of Solutions

Let us consider

$$(P) \quad \begin{cases} \partial_t^2 u - a(u, u_x, u_y)^2 \{\partial_x^2 + \partial_y^2\} u = f(u, u_x, u_y) & \text{in } \{t > 0, x > 0, y \in R\}, \\ \partial_t^j u = \phi_j(x, y) \quad (j = 0, 1) & \text{on } \{t = 0, x > 0, y \in R\}, \end{cases}$$

where $a(u, v, w)$ and $f(u, v, w)$ are homogeneous polynomials of degree $p (\geq 1)$ and $q (\geq 2)$, then the fully degenerate condition is

$$(B) \quad \begin{cases} a(u, v, w) = 0, \\ f_u(u, v, w) = 0, \\ f_w(u, v, w) = 0, \end{cases}$$

and (Q_1) is

$$(\#) \quad \begin{cases} \partial_t^2 u = f(u, v, w), \\ \partial_t^2 v = f_u(u, v, w)v, \\ \partial_t^2 w = f_w(u, v, w)w, \end{cases}$$

in $\{0 < t < T, y \in R\}$ with initial conditions on $\{t = 0, y \in R\}$:

$$\partial_t^j u = \phi_j(0, y), \quad \partial_t^j v = \phi_{jx}(0, y), \quad \partial_t^j w = \phi_{jy}(0, y) \quad (j = 0, 1).$$

Let us define

(B-1): $\{u \equiv 0, v = \alpha u, w = \beta u\}$, where α and β are constants satisfying

$$\{a(1, \alpha, \beta) = 0, f_v(1, \alpha, \beta) = 0, f_w(1, \alpha, \beta) = 0\},$$

(B-2): $\{u = 0, v \equiv 0, w = \gamma v\}$, where γ is a constant satisfying

$$\{a(0, 1, \gamma) = 0, f_v(0, 1, \gamma) = 0, f_w(0, 1, \gamma) = 0\}.$$

Then (B) is satisfied if (B-1) or (B-2) is satisfied.

Let a solution of (Q_1) satisfy (B-1), then $(\#)$ is reduced to

$$\begin{cases} \partial_t^2 u - q^{-1} f_u(1, \alpha, \beta) u^q = 0, \\ \alpha \{ \partial_t^2 u - f_u(1, \alpha, \beta) u^q \} = 0, \\ \beta \{ \partial_t^2 u - f_u(1, \alpha, \beta) u^q \} = 0, \end{cases}$$

because $qf(1, \alpha, \beta) = f_u(1, \alpha, \beta)$. Therefore we have a contradiction, if $f_u(1, \alpha, \beta) \neq 0$, $(\alpha, \beta) \neq (0, 0)$ and $u \neq 0$. If $\alpha = \beta = 0$, (#) is reduced to

$$(\#)' \quad \partial_t^2 u = q^{-1} f_u(1, 0, 0) u^q.$$

Let a solution of (Q_1) satisfy (B-2). Then (#) is reduced to

$$(\#)' \quad \partial_t^2 v = f_u(0, 1, \gamma) v^q.$$

Hereafter, we consider (P) on the assumption

$$(A-1): a(1, 0, 0) = f_v(1, 0, 0) = f_w(1, 0, 0) = 0, f_u(1, 0, 0) > 0,$$

or

$$(A-2): a(0, 1, 0) = f_v(0, 1, 0) = f_w(0, 1, 0) = 0, f_u(0, 1, 0) > 0.$$

Let us consider

$$(\tau) \quad \begin{cases} \tau'' = \lambda \tau^q & \text{in } \{t > 0\}, \\ \tau = \alpha, \tau' = \beta & \text{at } t = 0, \end{cases}$$

where λ is a positive constant. (τ) is solvable by direct integration:

$$t = \pm \int_{\alpha}^{\tau} Q(\tau)^{-1/2} d\tau,$$

where

$$Q(\tau) = 2\lambda/(q + 1)\tau^{q+1} + \beta^2 - 2\lambda/(q + 1)\alpha^{q+1}.$$

First, we consider the case when q is odd. Then let us define

$$\alpha(\beta) = \begin{cases} - \{ (q + 1)/(2\lambda) \beta^2 \}^{1/(q+1)} & \text{if } \beta \geq 0, \\ \{ (q + 1)/(2\lambda) \beta^2 \}^{1/(q+1)} & \text{if } \beta < 0, \end{cases}$$

then the above solution blows up in finite time if $\alpha \neq \alpha(\beta)$. More precisely, let us define

$$t^* = t^*(\alpha, \beta) = \begin{cases} \int_{\alpha}^{\infty} Q(\tau)^{-1/2} d\tau & \text{if } \beta \geq 0, \\ \int_{\tau(\alpha, \beta)}^{\alpha} Q(\tau)^{-1/2} d\tau + \int_{\tau(\alpha, \beta)}^{\infty} Q(\tau)^{-1/2} d\tau & \text{if } \beta < 0, \end{cases}$$

when $\alpha > \alpha(\beta)$, and $t^*(-\alpha, -\beta) = t^*(\alpha, \beta)$ when $\alpha < \alpha(\beta)$, where $\tau(\alpha, \beta)$ is a positive root of $Q(\tau) = 0$. Then we have

Lemma 4.1. *Let q be odd.*

- i) *In case when $\beta > \beta(\alpha)$, there exists a solution of (τ) in $(0, t^*)$ and $\tau(t) \rightarrow +\infty$ as $t \rightarrow t^*$.*
- ii) *In case when $\beta < \beta(\alpha)$, there exists a solution of (τ) in $(0, t^*)$ and $\tau(t) \rightarrow -\infty$ as $t \rightarrow t^*$.*

Next, we consider the case when q is even. Let us define

$$\alpha(\beta) = \{(q + 1)/(2\lambda) \beta^2\}^{1/(q+1)}.$$

Then the above solution blows up in finite time if $\alpha \neq \alpha(\beta)$. More precisely, let us define

$$t^* = t^*(\alpha, \beta) = \begin{cases} \int_{\alpha}^{\infty} Q(\tau)^{-1/2} d\tau & \text{if } \beta \geq 0, \\ \int_{\tau(\alpha, \beta)}^{\alpha} Q(\tau)^{-1/2} d\tau + \int_{\tau(\alpha, \beta)}^{\infty} Q(\tau)^{-1/2} d\tau & \text{if } \beta < 0, \end{cases}$$

where $\tau(\alpha, \beta)$ is a real root of $Q(\tau) = 0$.

Lemma 4.2. *Let q be even and let $\alpha \neq \alpha(\beta)$, then there exists a solution of (τ) in $(0, t^*)$ such that*

$$\tau(t) \rightarrow +\infty \quad \text{as } t \rightarrow t^*.$$

Hence we have

Theorem 3.

- i) *Set $\lambda = q^{-1}f_u(1, 0, 0)$ under the assumption (A. 1). Let us assume that*

$$(C) \quad \phi_0(0, y) \neq \alpha(\phi_1(0, y)).$$

Then there exists no $C^2((0, \infty) \times \bar{\Omega})$ -solution of (P). More precisely, let y_0 satisfy

$$(C_{y_0}) \quad \phi_0(0, y_0) \neq \alpha(\phi_1(0, y_0)).$$

Then

$$|u(t, 0, y_0)| \rightarrow \infty \quad \text{as } t \rightarrow t^*(\phi_0(0, y_0), \phi_1(0, y_0)).$$

- ii) *Set $\lambda = f_u(0, 1, 0)$ under the assumption (A. 2). Let us assume that*

$$(C') \quad \phi_{0x}(0, y) \neq \alpha(\phi_{1x}(0, y)).$$

Then there exists no $C^2((0, \infty) \times \bar{\Omega})$ -solution of (P). More precisely, let y_0 satisfy

$$(C'_{y_0}) \quad \phi_{0x}(0, y_0) \neq \alpha(\phi_{1x}(0, y_0)).$$

Then

$$|u_x(t, 0, y_0)| \rightarrow \infty \quad \text{as } t \rightarrow t^*(\phi_{0x}(0, y_0), \phi_{1x}(0, y_0)).$$

In the following let us consider an example:

$$\begin{cases} u_{tt} - u^2 u_{xx} = \kappa u u_x^2 (\kappa > 0) & \text{in } \{t > 0, 0 < x < 1\}, \\ u(0, x) = \phi_0(x), u_t(0, x) = \phi_1(x), \end{cases}$$

where

$$\begin{aligned} \phi_0(0) = \phi_0(1) = 0, \phi'_0(0) \neq 0, \phi'_0(1) \neq 0, \phi_0(x) \neq 0, (0 < x < 1), \\ \phi_1(0) = \phi_1(1) = 0. \end{aligned}$$

Then we have

$$(B): \{u(t, 0) = u(t, 1) = 0\},$$

$$(\#) \quad \begin{cases} u_{tt}(t, 0) = 0, u_{xxt}(t, 0) = \kappa u_x(t, 0)^3, \\ u_{tt}(t, 1) = 0, u_{xxt}(t, 1) = \kappa u_x(t, 1)^3, \end{cases}$$

and $\{\phi_0, \phi_1\}$ satisfies $(1 - S - G)$. Hence there exists a smooth local solution, which blows-up in finite time.

We remark that this equation has a periodical solution in time, belonging to $C^0(R \times [0, 1]) \cap C^\infty(R \times (0, 1))$:

Proposition 4.3. *Let T be any positive number. Then there exists a solution u of the equation:*

$$u_{tt} - u^2 u_{xx} = \kappa u u_x^2 (\kappa > 0) \quad \text{in } \{t \in R, 0 < x < 1\},$$

satisfying

$$u(t + T, x) = u(t, x), u(t, 1 - x) = u(t, x), u(t, 0) = u(t, 1) = 0.$$

Proof. Set $u(t, x) = \tau(t)\xi(x)$ and

$$\frac{\tau''}{\tau^3} = \xi \xi'' + \kappa \xi'^2 = -\sigma,$$

where σ is an arbitrary positive constant.

First we integrate $\tau'' = -\sigma\tau^3$, and then we have

$$t = \pm \int_0^\tau \{\sigma/2(a^4 - \tau^4)\}^{-1/2} d\tau,$$

where a is an arbitrary positive constant. It defines a periodic function $\tau(t) \in C^\infty(R)$ with period

$$T = 4a^{-1}(\sigma/2)^{-1/2} \int_0^1 (1 - \tau^4)^{-1/2} d\tau.$$

Next, we consider $\xi \xi'' + \kappa \xi'^2 = -\sigma$. Then we have

$$\xi''/\xi' + \kappa \xi'/\xi = -\sigma/(\xi \xi'),$$

and hence,

$$(\log |\xi^\kappa \xi'|)' = -\sigma(\xi^{2\kappa-1} \xi')(\xi^\kappa \xi')^{-2},$$

that is,

$$\eta' = -\sigma(2\kappa)^{-1}(\xi^{2\kappa})' e^{-2\eta},$$

where $\eta = \log |\xi^\kappa \xi'|$. Therefore integrating

$$(e^{2\eta}/2)' = -\sigma(2\kappa)^{-1}(\xi^{2\kappa})',$$

we have

$$e^{2\eta} = (\sigma/\kappa)(b^{2\kappa} - \xi^{2\kappa}),$$

where b is an arbitrary positive constant, that is,

$$(\xi^\kappa \xi')^2 = (\sigma/\kappa)(b^{2\kappa} - \xi^{2\kappa}).$$

Hence we have

$$x = \pm \int_0^\xi \xi^\kappa \{(\sigma/\kappa)(b^{2\kappa} - \xi^{2\kappa})\}^{-1/2} d\xi,$$

where

$$b = (\sigma/\kappa)^{1/2}/2 \left\{ \int_0^1 \xi^\kappa (1 - \xi^{2\kappa}) d\xi \right\}^{-1}.$$

Moreover, we have $\xi(x) \in C^\infty(0, 1)$ and

$$\xi(x)^{\kappa+1}/\{x(1-x)\} \rightarrow (\kappa+1)(\sigma/\kappa b^{2\kappa})^{1/2}$$

as $x \rightarrow 0$ or $x \rightarrow 1$. \square

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